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Poincaré and Sobolev inequalities in the Monge-Ampère quasi-metric structure

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ABSTRACT

We establish a number of Poincaré and Sobolev inequalities in the Monge-Ampère quasi-metric structure that are either new or significant improvements upon known ones. As applications we derive a host of new weighted Poincaré and Sobolev inequalities.

1. Introduction and main results

Let $\Omega \subset \mathbb{R}^n$ be open and convex and consider a convex function $\varphi \in C^1(\Omega)$. For $x \in \Omega$ and $t > 0$ the Monge-Ampère section $S_\varphi(x, t)$ associated to φ is defined as the open convex set

$$S_\varphi(x,t) := \{y \in \Omega : \delta_\varphi(x,y) < t\}$$

where

$$\delta_\varphi(x,y) := \varphi(y) - \varphi(x) - \langle \nabla\varphi(x), y - x\rangle \quad \forall x, y \in \Omega. \tag{1.1}$$

In all what follows the Monge-Ampère sections of a convex function φ will be assumed to be bounded sets, which amounts to saying that the graph of φ does not contain half lines.

The Monge-Ampère measure associated to φ, denoted as μ_φ, is the locally-finite, Borel measure defined as

$$\mu_\varphi(F) := |\nabla\varphi(F)| \quad F \subset \Omega, F \text{ Borel set}, \tag{1.2}$$

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where $|E|$ denotes the Lebesgue measure of $E \subset \mathbb{R}^n$.

Through its Monge-Ampère sections and measure the function φ models the geometry and measure theory in the analysis of regularity properties for solutions to the linearized Monge-Ampère equation $L_\varphi(v) = 0$ where

$$L_\varphi(v) := \operatorname{trace}(\mathcal{A}_\varphi D^2 v), \tag{1.3}$$

with $\mathcal{A}_\varphi(x) := \det D^2\varphi(x) D^2\varphi(x)^{-1}$, as well as other singular/degenerate elliptic PDEs, see for instance [3,4,11–13,16,18,19,23].

If φ is three times differentiable at some $x \in \Omega$ with $D^2\varphi(x) > 0$, the fact that the columns of $\mathcal{A}_\varphi$ are divergence free implies that

$$\operatorname{trace}(\mathcal{A}_\varphi(x) D^2 v(x)) = \operatorname{div}(\mathcal{A}_\varphi \nabla v)(x), \tag{1.4}$$

that is, L_φ is a singular/degenerate elliptic operator that takes both the nondivergence and divergence forms. The divergence form of L_φ has naturally led to the study of various properties for the Monge-Ampère sections and measure that guarantee the existence of Sobolev, Poincaré, or other first-order inequalities related to $\mathcal{A}_\varphi$. In turn, such first-order inequalities have been crucial to the regularity theory for solutions to the linearized Monge-Ampère equation in [16,19] as well as to its applications to semi-geostrophic equations and optimal transport in [13,14] and capacitary estimates [17].

Such first-order inequalities are modeled by the function φ through its associated Monge-Ampère gradient ∇^φ defined on a function u differentiable at a point $x \in \Omega$ with $D^2\varphi(x) > 0$ as

$$\nabla^\varphi u(x) := D^2\varphi(x)^{-\frac{1}{2}} \nabla u(x).$$

Always in the context of the linearized Monge-Ampère equation, the first authors to explore and develop a first-order calculus associated to φ (that is, the existence of Sobolev and/or Poincaré inequalities framed by the sections $S_\varphi(x,t)$, the measure μ_φ, and the first-order operator ∇^φ) were G. Tian and J.-X. Wang in [23] who proved

Theorem A *(Theorem 3.1 in [23]). Fix $n > 2$ and $\varphi \in C^2(\Omega)$ with $D^2\varphi > 0$ in Ω such that*

(a) *there exist $C_0 > 0$ and $\theta_0 > 0$ with*

$$\frac{\mu_\varphi(E)}{\mu_\varphi(S)} \le C_0 \left(\frac{|E|}{|S|}\right)^{\theta_0} \tag{1.5}$$

for every section $S := S_\varphi(x,t) \subset\subset \Omega$ and measurable $E \subset S$;

(b) *there exist $\theta \ge 0, \sigma > 0$, $C_1, C_2 > 0$ such that*

$$C_1 |S|^{1+\theta} \le \mu_\varphi(S) \le C_2 |S|^{\frac{1}{n-1}+\sigma} \tag{1.6}$$

for every section $S := S_\varphi(x,t) \subset\subset \Omega$,

then the following Sobolev inequality holds true for every $u \in C_c^\infty(\Omega)$

$$\left(\int_\Omega |u(x)|^p \, d\mu_\varphi(x)\right)^{\frac{1}{p}} \le C \left(\int_\Omega |\nabla^\varphi u(x)|^2 \, d\mu_\varphi(x)\right)^{\frac{1}{2}} \tag{1.7}$$

where $d\mu_\varphi(x) = \det D^2\varphi(x)\,dx$*,* $p := \frac{2n(1+\theta)}{(n-1)(1+\theta)-1} > 2$*, and the constant* $C > 0$ *depends on the constant from the conditions* (1.5) *and* (1.6) *as well as on the diameter of* Ω*.*

In particular, if $\det D^2\varphi \sim 1$, in the sense that there exist constants $0 < \Lambda_1 \leq \Lambda_2$ such that

$$\Lambda_1 \leq \det D^2\varphi(x) \leq \Lambda_2 \quad \forall x \in \Omega, \tag{1.8}$$

it follows (see [23, Example 3]) that the Sobolev inequality (1.7) holds true with $p = 2n/(n-2)$, thus recovering (when $n > 2$) the classical Sobolev inequality in the Euclidean setting from the choice $\varphi(x) = \frac{1}{2}|x|^2$.

The proof of Theorem A relies on a crucial lemma ([23, Lemma 2.1]) establishing a rate of decay for the distribution function of the Green's function associated to the linearized Monge-Ampère operator (1.3) in Ω as a sufficient condition for the Sobolev inequality (1.7).

Notice how the condition (1.5) from Theorem A resembles the classical Coifman-Fefferman characterization Muckenhoupt's A_∞ weights, with the role of the Euclidean balls now being played by the Monge-Ampère sections. It follows from [4, Sections 0 and 5] and [9, Section 3] that the condition (1.5) implies (and it is in general strictly stronger than) the so-called DC-doubling condition defined as follows: A Borel measure μ on Ω is said to satisfy the DC-doubling condition if there exists a constant $C_D \geq 1$ such that for every section $S := S_\varphi(x,t) \subset\subset \Omega$ we have

$$\mu(S) \leq C_D \mu\big(\tfrac{1}{2} \odot S\big), \tag{1.9}$$

where the open and convex subset $\frac{1}{2} \odot S$ denotes the $\frac{1}{2}$-contraction of S with respect to its center of mass. We will denote (1.9) as $\mu \in DC(\Omega, \delta_\varphi)$. Also, for a Borel measure μ on Ω a set $E \subset \Omega$ with $0 < \mu(E) < \infty$ and a μ-measurable function u defined on E, we write

$$\fint_E u(x)\,d\mu(x) := \frac{1}{\mu(E)} \int_E u(x)\,d\mu(x).$$

Now, if instead of (1.5) and (1.6), the Monge-Ampère measure is only assumed to satisfy $\mu_\varphi \in DC(\Omega, \delta_\varphi)$, the first author of this article proved

Theorem B *(Theorem 1 in [15]). Fix* $n > 1$ *and* $\varphi \in C^2(\Omega)$ *with* $D^2\varphi > 0$ *in* Ω *and* $\mu_\varphi \in DC(\Omega, \delta_\varphi)$*. Then the following Sobolev inequality holds true for every section* $S := S_\varphi(x_0, t)$ *with* $S \subset\subset \Omega$ *and every* $u \in C_c^1(S)$

$$\left(\fint_S |u(x)|^{\frac{2n}{n-1}}\,d\mu_\varphi(x)\right)^{\frac{n-1}{2n}} \leq C t^{\frac{1}{2}} \left(\fint_S |\nabla^\varphi u(x)|^2\,d\mu_\varphi(x)\right)^{\frac{1}{2}}, \tag{1.10}$$

where the constant $C > 0$ *depends only on the doubling constant from the condition* $\mu_\varphi \in DC(\Omega, \delta_\varphi)$ *and dimension* n*.*

The Sobolev inequality (1.10) has played a key role in the implementation of Moser's iterations in [19] towards Harnack's inequality for nonnegative solutions of certain singular/degenerate elliptic PDEs.

More recently, when proving Hölder regularity of solutions to the 2D dual semigeostrophic equation by means of the linearized Monge-Ampère equation under the assumption $\det D^2\varphi \sim 1$ in the sense of (1.8) (which, in particular, renders $d\mu_\varphi \sim dx$), N.Q. Le proved

Theorem C *(Proposition 2.6 in [13]). Fix $n = 2$ and $\varphi \in C^2(\Omega)$ with $\det D^2\varphi \sim 1$ in the sense of (1.8). Then, given $q \in (0,\infty)$ there exists a constant $C > 0$, depending only on q and Λ_1, Λ_2 from (1.8), such that the following Sobolev inequality holds true for every section $S := S_\varphi(x_0, t)$ with $S \subset\subset \Omega$ and every $u \in C^1_c(S)$*

$$\left(\fint_S |u(x)|^q \, dx \right)^{\frac{1}{q}} \le C t^{\frac{1}{2}} \left(\fint_S |\nabla^\varphi u(x)|^2 \, dx \right)^{\frac{1}{2}}. \tag{1.11}$$

The proofs of both Theorems B and C rely variations of the aforementioned crucial lemma ([23, Lemma 2.1]).

Regarding Poincaré inequalities, a combination of the Poincaré inequality with respect to Lebesgue measure in [16, Theorem 1.3] and the change of variables from [15, Section 4] yields the following weak (1,2)-Poincaré inequality, which has also been essential to the Harnack inequalities in [19].

Theorem D *([15,16]). Fix $n \ge 2$ and $\varphi \in C^2(\Omega)$ with $D^2\varphi > 0$ in Ω and $\mu_\varphi \in DC(\Omega, \delta_\varphi)$. Then, there exist constants $C_1, C_2 > 1$, depending only on the doubling constant from the condition $\mu_\varphi \in DC(\Omega, \delta_\varphi)$ and dimension n, such that for every section $S := S_\varphi(x_0, t)$ with $S_\varphi(x_0, C_1 t) \subset\subset \Omega$ and every $u \in C^1(S_\varphi(x_0, C_1 t))$ the following Poincaré inequality holds true*

$$\fint_S |u(x) - u_S^{\mu_\varphi}| \, d\mu_\varphi(x) \le C t^{\frac{1}{2}} \left(\fint_{S_\varphi(x_0, C_1 t)} |\nabla^\varphi u(x)|^2 \, d\mu_\varphi(x) \right)^{\frac{1}{2}}, \tag{1.12}$$

where $u_S^{\mu_\varphi} := \fint_S u \, d\mu_\varphi$.

The purpose of this article is to improve upon all the mentioned Sobolev and Poincaré inequality in two ways. Firstly, by increasing the exponents on the left-hand side of the inequalities and by decreasing the ones on the right-hand side, under various assumptions on the Monge-Ampère measure μ. Secondly, by developing approximation arguments to have the condition $\varphi \in C^2(\Omega)$ with $D^2\varphi > 0$ in Ω replaced with $\varphi \in W^{2,n}_{loc}(\Omega, dx)$ with $D^2\varphi > 0$ a.e. in Ω.

We remark that by [20, Theorem 1] the condition $\varphi \in W^{2,n}_{loc}(\Omega, dx)$ with φ convex implies that $\varphi \in C^1(\Omega)$ and that $d\mu_\varphi(x) = \det D^2\varphi(x)\, dx$, that is, $\det D^2\varphi \in L^1_{loc}(\Omega, dx)$ and

$$\mu_\varphi(E) = \int_E \det D^2\varphi(x)\, dx \quad \forall E \subset \Omega, E \text{ Borel}. \tag{1.13}$$

Before stating our main results, let us introduce the four possible conditions on the Monge-Ampère weight $\det D^2\varphi$ that will be involved in their statements.

Let $w \ge 0$ be a weight in $L^1_{loc}(\Omega, dx)$ and, for a Borel set $E \subset \Omega$, define $\mu_w(E) = w(E) := \int_E w(x)\, dx$.

(i) We write $w \in DC(\Omega, \delta_\varphi)$ if $\mu_w \in DC(\Omega, \delta_\varphi)$, as defined in (1.9). If μ_φ satisfies (1.13), we will write interchangeably $\mu_\varphi \in DC(\Omega, \delta_\varphi)$ and $\det D^2\varphi \in DC(\Omega, \delta_\varphi)$. In such case, all the constant depending on the doubling constant from $\mu_\varphi \in DC(\Omega, \delta_\varphi)$ and dimension n will be called *geometric constants.*

(ii) We write $w \in A_\infty(\Omega, \delta_\varphi)$ if there exist constants $C_1, C_2 > 0$ and $\theta > 0$ such that

$$\frac{w(E)}{w(S)} \le C_1 \left(\frac{|E|}{|S|} \right)^\theta \tag{1.14}$$

for every section $S := S_\varphi(x,t)$ with $S_\varphi(x, C_2 t) \subset\subset \Omega$ and every measurable $E \subset S$. As mentioned before, the condition $w \in A_\infty(\Omega, \delta_\varphi)$ implies $w \in DC(\Omega, \delta_\varphi)$, quantitatively (see [4, p. 426]). When $\det D^2\varphi \in A_\infty(\Omega, \delta_\varphi)$ all the constants depending only on the constants C_1, C_2 and θ from (1.14) and dimension n will be called *structural constants*.

(iii) We write $w \in A_1(\Omega, \delta_\varphi)$ if there exist constants $\Theta_1, H_1 \geq 1$ such that

$$\fint_{S_\varphi(x_0,t)} w(x)\, dx \leq H_1 \operatorname*{ess\,inf}_{S_\varphi(x_0,t)} w, \tag{1.15}$$

for every section $S_\varphi(x_0, t)$ with $S_\varphi(x_0, \Theta_1 t) \subset\subset \Omega$.

(iv) We write $w \in RH_\infty(\Omega, \delta_\varphi)$ if there exist constants $\Theta_\infty, H_\infty \geq 1$ such that

$$\operatorname*{ess\,sup}_{S_\varphi(x_0,t)} w \leq H_\infty \fint_{S_\varphi(x_0,t)} w(x)\, dx, \tag{1.16}$$

for every section $S_\varphi(x_0, t)$ with $S_\varphi(x_0, \Theta_\infty t) \subset\subset \Omega$.

It follows from [4, Section 5] that if $w \in A_1(\Omega, \delta_\varphi)$ or $w \in RH_\infty(\Omega, \delta_\varphi)$, then $w \in A_\infty(\Omega, \delta_\varphi)$, quantitatively. Consequently, if $\det D^2\varphi \in A_1(\Omega, \delta_\varphi)$ or $\det D^2\varphi \in RH_\infty(\Omega, \delta_\varphi)$ the constants depending only the corresponding pairs (Θ_1, H_1) or $(\Theta_\infty, H_\infty)$ and dimension n will also be called structural constants.

Finally, given an open set $U \subset \mathbb{R}^n$ we denote by $\mathrm{Lip}(U)$ the class of Lipschitz-continuous functions in U with respect to the Euclidean distance.

We are now in position to state our main results regarding Poincaré inequalities in the Monge-Ampère quasi-metric structure (Ω, δ_φ) with respect to the Monge-Ampère measure μ_φ and to the Lebesgue measure dx, always based on the Monge-Ampère gradient ∇^φ.

1.1. *Poincaré inequalities when* $\det D^2\varphi \in DC(\Omega, \delta_\varphi)$

Theorem 1.1. *Fix $n \geq 2$ and let $\varphi \in W^{2,n}_{loc}(\Omega, dx)$ be a strictly convex function with $D^2\varphi > 0$ a.e. in Ω and $\mu_\varphi \in DC(\Omega, \delta_\varphi)$. Then, there exist geometric constants $K_1, K_2 > 1$ and $\epsilon_1 > 0$ such that for every section $S := S_\varphi(x_0, t)$ with $S_\varphi(x_0, K_1 t) \subset\subset \Omega$ and every $u \in \mathrm{Lip}(K_1 S)$ we have*

$$\left(\fint_S |u(x) - u_S^{\mu_\varphi}|^{q_1}\, d\mu_\varphi(x)\right)^{\frac{1}{q_1}} \leq K_2\, t^{\frac{1}{2}} \left(\fint_{K_1 S} |\nabla^\varphi u(x)|^2\, d\mu_\varphi(x)\right)^{\frac{1}{2}}, \tag{1.17}$$

where $q_1 := \frac{2n}{n-1} + \epsilon_1$ and $u_S^{\mu_\varphi} := \fint_S u(x)\, d\mu_\varphi(x)$.

Remark 1. The strict convexity of φ is not required in Theorem 1.1. However, keep in mind the underlying hypothesis that all the Monge-Ampère sections involved are bounded.

Remark 2. Theorem 1.1 improves upon Theorem D by weakening the hypotheses and by allowing for an exponent $q > 2$ on the left-hand side of (1.12).

Theorem 1.2. *Fix $n \geq 2$ and let $\varphi \in W^{2,n}_{loc}(\Omega, dx)$ be a strictly convex function such that $D^2\varphi > 0$ a.e. in Ω, $\|(D^2\varphi)^{-1}\| \in L^n_{loc}(\Omega, d\mu_\varphi)$, and $\mu_\varphi \in DC(\Omega, \delta_\varphi)$. Then, there exist geometric constants $K_3, K_4 > 1$ and $\epsilon_1 > 0$ such that for every section $S := S_\varphi(x_0, t)$ with $S_\varphi(x_0, K_3 t) \subset\subset \Omega$ and every $h \in \mathrm{Lip}(K_3 S)$ we have*

$$\left(\fint_S |h(x) - h_S|^{q_1}\, dx\right)^{\frac{1}{q_1}} \le K_4\, t^{\frac{1}{2}} \left(\fint_{K_3 S} |\nabla^\varphi h(x)|^2\, dx\right)^{\frac{1}{2}}, \tag{1.18}$$

where $q_1 := \frac{2n}{n-1} + \epsilon_1$ *and* $h_S := \fint_S h(x)\, dx$.

Remark 3. The hypothesis $\|(D^2\varphi)^{-1}\| \in L^n_{loc}(\Omega, d\mu_\varphi)$ will only be used to prove local L^n-integrability of $D^2\psi$, where ψ is the convex conjugate of φ, and it will play no role in the behavior of the constants.

1.2. Poincaré inequalities when $\det D^2\varphi \in A_\infty(\Omega, \delta_\varphi)$

If the assumption $\det D^2\varphi \in DC(\Omega, \delta_\varphi)$ is replaced with the (strictly) stronger $\det D^2\varphi \in A_\infty(\Omega, \delta_\varphi)$, then the exponent on the right-hand sides of the Poincaré inequalities (1.17) and (1.18) can be improved from 2 to $2-\epsilon$ for some structural $0 < \epsilon < 1$. More precisely, we have

Theorem 1.3. *Fix* $n \ge 2$ *and let* $\varphi \in W^{2,n}_{loc}(\Omega, dx)$ *be a strictly convex function with* $D^2\varphi > 0$ *a.e. in* Ω *and* $\det D^2\varphi \in A_\infty(\Omega, \delta_\varphi)$*. Then, there exist structural constants* $K_5, K_6 > 0$ *and* $\epsilon_0 > 0$ *such that for every section* $S := S_\varphi(x_0, t)$ *with* $S_\varphi(x_0, K_5 t) \subset\subset \Omega$ *and every* $u \in \mathrm{Lip}(K_5 S)$ *we have*

$$\left(\fint_S |u(x) - u_S^{\mu_\varphi}|^{q_0}\, d\mu_\varphi(x)\right)^{\frac{1}{q_0}} \le K_6\, t^{\frac{1}{2}} \left(\fint_{K_5 S} |\nabla^\varphi u(x)|^{2-\epsilon_0}\, d\mu_\varphi(x)\, dx\right)^{\frac{1}{2-\epsilon_0}}, \tag{1.19}$$

with $q_0 := \frac{2(n-\epsilon_0)(2-\epsilon_0)}{2(n-\epsilon_0)-(2-\epsilon_0)} > 2$.

Theorem 1.4. *Fix* $n \ge 2$ *and let* $\varphi \in W^{2,n}_{loc}(\Omega, dx)$ *be a strictly convex function such that* $D^2\varphi > 0$ *a.e. in* Ω*,* $\|(D^2\varphi)^{-1}\| \in L^n_{loc}(\Omega, d\mu_\varphi)$ *and* $\det D^2\varphi \in A_\infty(\Omega, \delta_\varphi)$*. Then, there exist structural constants* $K_7, K_8 \ge 1$ *and* $0 < \epsilon_0 < 1$ *such that for every section* $S := S_\varphi(x_0, t)$ *with* $S_\varphi(x_0, K_7 t) \subset\subset \Omega$*, and every* $u \in \mathrm{Lip}(K_7 S)$ *we have*

$$\left(\fint_S |u(x) - u_S|^{q_0}\, dx\right)^{\frac{1}{q_0}} \le K_8\, t^{\frac{1}{2}} \left(\fint_{K_7 S} |\nabla^\varphi u(x)|^{2-\epsilon_0}\, dx\right)^{\frac{1}{2-\epsilon_0}}, \tag{1.20}$$

with $q_0 := \frac{2(n-\epsilon_0)(2-\epsilon_0)}{2(n-\epsilon_0)-(2-\epsilon_0)} > 2$.

1.3. Poincaré inequalities when $\det D^2\varphi \in A_1(\Omega, \delta_\varphi)$

Theorem 1.5. *Fix* $n \ge 3$ *and let* $\varphi \in W^{2,n}_{loc}(\Omega, dx)$ *be a strictly convex function with* $\det D^2\varphi \in A_1(\Omega, \delta_\varphi)$*. Then, there exist structural constants* $K_9, K_{10} \ge 1$ *such that for every section* $S := S_\varphi(x_0, t)$ *with* $S_\varphi(x_0, K_9 t) \subset\subset \Omega$ *and every* $u \in \mathrm{Lip}(K_9 S)$ *we have*

$$\left(\fint_S |u(x) - u_S^{\mu_\varphi}|^{\frac{2n}{n-2}}\, d\mu_\varphi(x)\right)^{\frac{n-2}{2n}} \le K_{10}\, t^{\frac{1}{2}} \left(\fint_{K_9 S} |\nabla^\varphi u(x)|^2\, d\mu_\varphi(x)\right)^{\frac{1}{2}}. \tag{1.21}$$

In addition, there exists a structural constant $\epsilon_0 > 0$ *such that for every* $0 < \epsilon \le \epsilon_0$ *there is a constant* $K_\epsilon > 0$*, depending only on* ϵ *and structural constants, such that*

$$\left(\fint_S |u(x) - u_S^{\mu_\varphi}|^{q_\epsilon}\, d\mu_\varphi(x)\right)^{\frac{1}{q_\epsilon}} \leq K_\epsilon\, t^{\frac{1}{2}} \left(\fint_{K_9 S} |\nabla^\varphi u(x)|^{2-\epsilon}\, d\mu_\varphi(x)\right)^{\frac{1}{2-\epsilon}}, \tag{1.22}$$

with $q_\epsilon := \frac{n(2-\epsilon)}{n-(2-\epsilon)} > 2$.

Theorem 1.6. *Assume* $n = 2$ *and let* $\varphi \in W^{2,2}_{loc}(\Omega)$ *be a strictly convex function with* $\det D^2\varphi \in A_1(\Omega, \delta_\varphi)$. *Then, there exist structural constants* $K_9 \geq 1$ *and* $0 < \epsilon_0 < 1$, *such that for every section* $S := S_\varphi(x_0, t)$ *with* $S_\varphi(x_0, K_9 t) \subset\subset \Omega$, *every* $u \in \mathrm{Lip}(K_9 S)$, *and every* $0 < \epsilon \leq \epsilon_0$ *we have*

$$\left(\fint_S |u(x) - u_S^{\mu_\varphi}|^{q_\epsilon}\, d\mu_\varphi(x)\right)^{\frac{1}{q_\epsilon}} \leq K_\epsilon\, t^{\frac{1}{2}} \left(\fint_{K_9 S} |\nabla^\varphi u(x)|^{2-\epsilon}\, d\mu_\varphi(x)\right)^{\frac{1}{2-\epsilon}}, \tag{1.23}$$

with $q_\epsilon := 2(2-\epsilon)/\epsilon$ *and* $K_\epsilon > 0$ *depends only on* ϵ *and structural constants.*

1.4. Poincaré inequalities when $\det D^2\varphi \in RH_\infty(\Omega, \delta_\varphi)$

Theorem 1.7. *Fix* $n \geq 3$ *and let* $\varphi \in W^{2,n}_{loc}(\Omega, dx)$ *be a strictly convex function with* $\det D^2\varphi \in RH_\infty(\Omega, \delta_\varphi)$ *and* $\|(D^2\varphi)^{-1}\| \in L^1_{loc}(\Omega, dx)$. *Then, there exist structural constants* $K_{11}, K_{12} \geq 1$ *such that for every section* $S := S_\varphi(x_0, t)$ *with* $S_\varphi(x_0, K_{11} t) \subset\subset \Omega$ *and every* $u \in \mathrm{Lip}(K_{11} S)$ *we have*

$$\left(\fint_S |u(x) - u_S|^{\frac{2n}{n-2}}\, dx\right)^{\frac{n-2}{2n}} \leq K_{12}\, t^{\frac{1}{2}} \left(\fint_{K_{11} S} |\nabla^\varphi u(x)|^2\, dx\right)^{\frac{1}{2}}. \tag{1.24}$$

In addition, there exists a structural constant $\epsilon_0 > 0$ *such that for every* $0 < \epsilon \leq \epsilon_0$ *there is a constant* $K_\epsilon > 0$, *depending only on* ϵ *and structural constants, such that*

$$\left(\fint_S |u(x) - u_S|^{q_\epsilon}\, dx\right)^{\frac{1}{q_\epsilon}} \leq K_\epsilon\, t^{\frac{1}{2}} \left(\fint_{K_9 S} |\nabla^\varphi u(x)|^{2-\epsilon}\, dx\right)^{\frac{1}{2-\epsilon}}, \tag{1.25}$$

with $q_\epsilon := \frac{n(2-\epsilon)}{n-(2-\epsilon)} > 2$.

Theorem 1.8. *Assume* $n = 2$ *and let* $\varphi \in W^{2,2}_{loc}(\Omega)$ *be a strictly convex function with* $\det D^2\varphi \in RH_\infty(\Omega, \delta_\varphi)$ *and* $\|(D^2\varphi)^{-1}\| \in L^1_{loc}(\Omega, dx)$. *Then, there exist structural constants* $K_{11} \geq 1$ *and* $0 < \epsilon_0 < 1$, *such that for every section* $S := S_\varphi(x_0, t)$ *with* $S_\varphi(x_0, K_{11} t) \subset\subset \Omega$, *every* $u \in \mathrm{Lip}(K_{11} S)$, *and every* $0 < \epsilon \leq \epsilon_0$ *we have*

$$\left(\fint_S |u(x) - u_S|^{q_\epsilon}\, dx\right)^{\frac{1}{q_\epsilon}} \leq K_\epsilon\, t^{\frac{1}{2}} \left(\fint_{K_{11} S} |\nabla^\varphi u(x)|^{2-\epsilon}\, dx\right)^{\frac{1}{2-\epsilon}}, \tag{1.26}$$

with $q_\epsilon := 2(2-\epsilon)/\epsilon$ *and* $K_\epsilon > 0$ *depends only on* ϵ *and structural constants.*

2. Preliminaries

2.1. Doubling measures

Given $\varphi \in C^1(\Omega)$, a section $S := S_\varphi(x,t)$ and $\lambda > 0$ we will write λS to indicate the section $S_\varphi(x, \lambda t)$. In particular, the contraction $\frac{1}{2}S$ has a different meaning than the contraction $\frac{1}{2} \odot S$ defined in (1.9). Now, given a Borel measure μ on Ω we say that μ is doubling (in the Monge-Ampère quasi-metric structure (Ω, δ_φ)) if there exists a constant $C_\mu \geq 1$ such that

$$\mu(S) \leq C_\mu \mu\big(\tfrac{1}{2}S\big) \tag{2.1}$$

for every section $S := S_\varphi(x,t)$ with $S_\varphi(x,t) \subset\subset \Omega$.

By [4, Lemma 5.2] the Lebesgue measure satisfies the doubling condition (2.1) with $C_\mu = 2^n$. By [11, Corollary 3.3.2], every Borel measure satisfying the DC-doubling condition (1.9) will also satisfy the doubling condition (2.1) with a geometric constant $C_\mu \geq 1$.

2.2. The engulfing property

By [11, Theorem 3.3.7] and [7, Theorem 8], the condition $\mu_\varphi \in DC(\Omega, \delta_\varphi)$ is quantitatively characterized by the so-called engulfing property of the sections of φ; meaning the existence of a geometric constant $\Theta > 1$ such that whenever $x_0 \in \Omega$ and $\tau > 0$ satisfy $S_\varphi(x_0, \Theta^2\tau) \subset\subset \Omega$, then for every $x \in S_\varphi(x_0, \tau)$ following inclusions holds true:

$$S_\varphi(x_0, \tau) \subset S_\varphi(x, \Theta\tau) \subset S_\varphi(x_0, \Theta^2\tau). \tag{2.2}$$

Let us briefly indicate how the inclusions (2.2) amount to a quasi-symmetry and a quasi-triangle inequality for δ_φ. Indeed, given $x, y \in \Omega$ such that $S_\varphi(x, \delta_\varphi(x,y)) \subset\subset \Omega$ and $S_\varphi(y, \Theta\delta_\varphi(x,y)) \subset\subset \Omega$, for $\varepsilon > 0$ sufficiently small we have $y \in S_\varphi(x, \delta_\varphi(x,y) + \varepsilon) \subset\subset \Omega$ and the engulfing property then implies $S_\varphi(x, \delta_\varphi(x,y) + \varepsilon) \subset S_\varphi(y, \Theta(\delta_\varphi(x,y) + \varepsilon))$; in particular, $\delta_\varphi(y,x) < \Theta(\delta_\varphi(x,y) + \varepsilon)$ so that by letting $\varepsilon \to 0$, we get the inequality

$$\delta_\varphi(y,x) \leq \Theta\delta_\varphi(x,y), \tag{2.3}$$

which represents the Θ-quasi symmetry of δ_φ. On the other hand, given $x, y, z \in \Omega$ such that

$$S_\varphi(z, \delta(z,y)), S_\varphi(z, \delta(y,z)), S_\varphi(x, \Theta\delta_\varphi(z,x)) \subset\subset \Omega, \tag{2.4}$$

assume first that $\delta_\varphi(z,x) \leq \delta_\varphi(z,y)$ to write, for $\varepsilon > 0$ small enough,

$$x \in S_\varphi(z, \delta_\varphi(z,x) + \varepsilon) \subset S_\varphi(z, \delta_\varphi(z,y) + \varepsilon) \subset\subset \Omega,$$

so that the engulfing property applied to x and $S_\varphi(z, \delta_\varphi(z,y) + \varepsilon)$ yields $y \in S_\varphi(z, \delta_\varphi(z,y) + \varepsilon) \subset S_\varphi(x, \Theta(\delta_\varphi(z,x) + \varepsilon))$; in particular, $\delta_\varphi(x,y) < \Theta(\delta_\varphi(z,x) + \varepsilon)$ and by letting $\varepsilon \to \infty$, we get

$$\delta_\varphi(x,y) \leq \Theta\delta_\varphi(z,x) \tag{2.5}$$

Next, if $\delta_\varphi(z,x) > \delta_\varphi(z,y)$, we reverse the roles of x and y in the argument above, which requires the inclusions (2.4) with y replaced with x, to obtain

$$\delta_\varphi(y,x) \leq \Theta\delta_\varphi(z,y). \tag{2.6}$$

If, in addition, it holds that $S_\varphi(y,\delta_\varphi(y,x)), S_\varphi(x,\Theta\delta_\varphi(y,x)) \subset\subset \Omega$, the inequalities (2.3) (with x and y interchanged) and (2.6) give

$$\delta_\varphi(x,y) \le \Theta^2\delta_\varphi(z,y). \tag{2.7}$$

Since (2.5) or (2.7) will hold true, it follows that

$$\delta_\varphi(x,y) \le \Theta(\Theta\delta_\varphi(z,y) + \delta_\varphi(z,x)) \le \Theta^2(\delta_\varphi(z,y) + \delta_\varphi(x,z)), \tag{2.8}$$

which effectively represents a Θ^2-quasi triangle inequality for δ_φ.

2.3. Doubling implies reverse doubling in (Ω, δ_φ)

Next, we recall the following result from [21] about reverse-doubling properties of doubling measures in the quasi-metric Monge-Ampère structure.

From now on $\Theta > 1$ will always indicate the geometric constant from the engulfing property (2.2).

Lemma 2.1 *(See [21], Section 2). Fix $\varphi \in C^1(\Omega)$ with $\mu_\varphi \in DC(\Omega,\delta_\varphi)$ and let μ be a Borel measure on Ω which is doubling with respect to the sections of φ. Then, for every $\alpha \in (0,1)$ there exists $\xi \in (0,1)$, depending only on α, the doubling constant of μ, and geometric constants, such that for every section $S_\varphi(x_0,t)$ with $S_\varphi(x_0,\Theta^2 t) \subset\subset \Omega$ we have*

$$\mu(S_\varphi(x_0,\alpha t)) \le \xi\mu(S_\varphi(x_0,t)). \tag{2.9}$$

Using Lemma 2.1, it was proved in [21, Section 2], that if $\mu_\varphi \in DC(\Omega,\delta_\varphi)$ there exist geometric constants $C_D > 0$ and $\varepsilon \in (0,1)$ such that

$$\frac{\mu_\varphi(S_\varphi(x_0,t))}{\mu_\varphi(S_\varphi(x_0,t'))} \le C_D\left(\frac{t}{t'}\right)^{n-\varepsilon} \tag{2.10}$$

for every section $S_\varphi(x_0,t)$ with $S_\varphi(x_0,t) \subset\subset \Omega$ and every $t' \in (0,t)$. Also, Lemma 2.1 will be useful in the proof of Theorem 3.2 and in the proof that every (q,p)-Poincaré inequality implies a corresponding Sobolev inequality in Section 9.

2.4. The convex conjugate

Given a strictly convex $\varphi \in C^1(\Omega)$ let $\psi \in C^1(\nabla\varphi(\Omega))$ denote its convex conjugate, which satisfies

$$\psi(\nabla\varphi(x)) = \langle\nabla\varphi(x), x\rangle - \varphi(x) \quad \forall x \in \Omega,$$

$$\nabla\varphi(\nabla\psi(y)) = y \quad \forall y \in \nabla\varphi(\Omega), \tag{2.11}$$

$$\nabla\psi(\nabla\varphi(x)) = x \quad \forall x \in \Omega, \tag{2.12}$$

since the strict convexity of φ means that $\nabla\varphi$ is one-to-one. Also, by [8, Theorem 12] if $\mu_\varphi \in DC(\Omega,\delta_\varphi)$, then $\mu_\psi \in DC(\nabla\varphi(\Omega),\delta_\psi)$ with a constant depending only on the constant from $\mu_\varphi \in DC(\Omega,\delta_\varphi)$; in addition, there exists a geometric constant $K^* > 1$ such that

$$S_\varphi(z,\tau/K^*) \subset \nabla\psi(S_\psi(\nabla\varphi(z),\tau)) \subset S_\varphi(z,K^*\tau), \tag{2.13}$$

for every section $S_\varphi(z,\tau)$ with $S_\varphi(z,K^*\tau) \subset\subset \Omega$.

Let us now outline the proof of the fact, to be used in Section 6.1, that $\mu_\varphi \in A_\infty(\Omega, \delta_\varphi)$ implies that $\mu_\psi \in A_\infty(\nabla\varphi(\Omega), \delta_\psi)$. That is, the A_∞-property is preserved, quantitatively, under conjugation. Fix $\varphi \in C^1(\Omega)$ such that $\mu_\varphi \in A_\infty(\Omega, \delta_\varphi)$; in particular, $\mu_\varphi \in DC(\Omega, \delta_\varphi)$ and the sections of φ satisfy the engulfing property. Now, since a section $S_\varphi(x,t)$ coincides with the set $\{y \in \Omega : \delta_\varphi(x,y) < t\}$, the quasi-symmetry and quasi-triangle inequality for δ_φ, allows to think of the interior sections (meaning sections with $S_\varphi(x,t) \subset\subset \Omega$) as balls in a space of homogeneous type. Consequently, the usual characterizations of the Muckenhoupt class A_∞ hold true, see for instance [4, Section 5] and [22, Corollary 14]. Thus, the fact that $\mu_\psi \in A_\infty(\nabla\varphi(\Omega), \delta_\psi)$ will be a consequence, for instance, of the existence of structural constants $\alpha_0, \beta_0 \in (0,1)$ and $M_0 \geq 1$ such that for every section $S_\psi(y,t)$ with $S_\psi(y, M_0 t) \subset\subset \nabla\varphi(\Omega)$ and every measurable set $F \subset S_\psi(y,t)$ the implication

$$\mu_\psi(F) \leq \alpha_0 \mu_\psi(S_\psi(y,t)) \Longrightarrow |F| \leq \beta_0 |S_\psi(y,t)| \tag{2.14}$$

holds true. Let us then assume that μ_φ satisfies (1.14) with constants $C_1, C_2 \geq 1$ and $\theta \in (0,1)$ and fix a section $S_\psi := S_\psi(y,t)$ and a measurable set $F \subset S_\psi$. From the second inclusion in (2.13), setting $x := \psi(y)$ we get

$$\nabla\psi(F) \subset \nabla\psi(S_\psi) \subset S_\varphi(x, K^* t). \tag{2.15}$$

Now, setting $E := \nabla\psi(F)$, the fact that $\nabla\varphi$ and $\nabla\psi$ are inverses to each other gives

$$\mu_\varphi(E) = |\nabla\varphi(E)| = |F|$$

and the first inclusion in (2.13) and the doubling property (2.10) for μ_φ imply

$$\begin{aligned}\mu_\varphi(S_\varphi(x, K^* t)) &\leq C_D (K^*)^{2(n-\varepsilon)} \mu_\varphi(S_\varphi(x, t/K^*)) \\ &\leq C_D (K^*)^{2(n-\varepsilon)} \mu_\varphi(\nabla\psi(S_\psi)) = C_D (K^*)^{2(n-\varepsilon)} |S_\psi|.\end{aligned}$$

On the other hand, $|E| = |\nabla\psi(F)| = \mu_\psi(F)$ and, from (2.15), $\mu_\psi(S_\psi) = |\nabla\psi(S_\psi)| \leq |S_\varphi(x, K^* t)|$. Hence, by using (1.14) with $E := \nabla\psi(F)$ and $S_\varphi(x, K^* t)$ (and this requires $S_\varphi(x, K^* C_2 t) \subset\subset \Omega$) it follows that

$$\begin{aligned}\frac{|F|}{C_D (K^*)^{2(n-\varepsilon)} |S_\psi|} &\leq \frac{\mu_\varphi(E)}{\mu_\varphi(S_\varphi(x, K^* t))} \\ &\leq C_1 \left(\frac{|E|}{|S_\varphi(x, K^* t)|} \right)^\theta \leq C_1 \left(\frac{\mu_\psi(F)}{\mu_\psi(S_\psi)} \right)^\theta.\end{aligned}$$

Consequently, by taking $\alpha_0 \in (0,1)$ so that $\beta_0 := C_D (K^*)^{2(n-\varepsilon)} C_1 \alpha_0^\theta \in (0,1)$, the implication (2.14) holds true with structural constants $\alpha_0, \beta_0 \in (0,1)$.

3. Self-improving properties for Poincaré inequalities in the Monge-Ampère quasi-metric structure

Throughout this section μ will denote a Borel measure on Ω absolutely continuous with respect to Lebesgue measure (so that Lebesgue-a.e. implies μ-a.e.) which will later be chosen as dx or as $\det D^2\varphi\, dx$. Also, we assume that the convex function $\varphi \in C^1(\Omega)$ under consideration satisfies $D^2\varphi > 0$ a.e. in Ω and $\|(D^2\varphi)^{-1}\| \in L^1_{loc}(\Omega, d\mu)$. This latter assumption will guarantee the finiteness of some integrals, but will not play into the actual value of the constants involved. Also, $\Theta > 1$ will always indicate the geometric constant from the engulfing property (2.2).

Lemma 3.1. *Let $\varphi \in C^1(\Omega)$ be a convex function with $D^2\varphi > 0$ a.e. in Ω and let μ be a Borel measure on Ω absolutely continuous with respect to Lebesgue measure. Let S, S_0 be sections of φ with $S \subset S_0 \subset\subset \Omega$ and fix $0 < p \le q < \infty$ with $q > 1$. If, for some constant $C_0 > 0$, the inequality*

$$\tau^q \mu(\{x \in S : |u(x) - u_S^\mu| \ge \tau\}) \le C_0 \mu(S) \left(\fint_{S_0} |\nabla^\varphi u|^p \, d\mu \right)^{\frac{q}{p}} \tag{3.1}$$

holds true for every $\tau > 0$ and $u \in \mathrm{Lip}(S_0)$, then

$$\left(\fint_S |u - u_S^\mu|^q \, d\mu \right)^{1/q} \le C_1 \left(\fint_{S_0} |\nabla^\varphi u|^p \, d\mu \right)^{\frac{1}{p}} \quad \forall u \in \mathrm{Lip}(S_0), \tag{3.2}$$

where $C_1 := 16 \left(1 + \left(\frac{q}{q-1}\right)^q\right)^{\frac{1}{q}} C_0^{\frac{1}{q}}$.

Proof. Given $u \in \mathrm{Lip}(S_0)$, without loss of generality we may assume $u_S^\mu = 0$ (otherwise consider $u - u_S^\mu$). Let $k_0 \in \mathbb{Z}$ such that

$$2^{k_0 - 1} \le \fint_S u_+ \, d\mu < 2^{k_0} \tag{3.3}$$

and for $k > k_0$ set

$$u_k := \begin{cases} 0, & u \le 2^k, \\ 2^k, & u \ge 2^{k+1}, \\ u - 2^k, & 2^k < u < 2^{k+1}, \end{cases} \tag{3.4}$$

so that $\nabla u_k = \nabla u \, \chi_{\{2^k < u < 2^{k+1}\}}$ (Lebesgue) a.e. in S_0 (see, for instance [10, Theorem 7.8]) and consequently $\nabla^\varphi u_k = \nabla^\varphi u \, \chi_{\{2^k < u < 2^{k+1}\}}$ (Lebesgue) a.e. in S_0. In particular, $u_k \in \mathrm{Lip}(S_0)$ and, in view of (3.3),

$$(u_k)_S^\mu := \fint_S u_k \, d\mu \le \fint_S u_+ \, d\mu < 2^{k_0} \le 2^{k-1} \quad \forall k > k_0. \tag{3.5}$$

From the definition of u_k and the estimate (3.5) we get

$$\begin{aligned} \{x \in S : u(x) \ge 2^{k+1}\} &\subset \{x \in S : u_k(x) = 2^k\} \\ &= \{x \in S : u_k(x) - (u_k)_S^\mu = 2^k - (u_k)_S^\mu \ge 2^{k-1}\}, \end{aligned}$$

which, along with (3.1) applied to u_k and $\tau = 2^{k-1}$, yields

$$\begin{aligned} I := \int\limits_{\{x \in S : u(x) \ge 2^{k_0+2}\}} u_+^q \, d\mu &\le \sum_{k=k_0+1}^\infty 2^{(k+2)q} \mu(\{x \in S : 2^{k+1} \le u(x) < 2^{k+2}\}) \\ &\le 2^{3q} \sum_{k=k_0+1}^\infty 2^{(k-1)q} \mu(\{x \in S : |u_k(x) - (u_k)_S^\mu| \ge 2^{k-1}\}) \\ &\le 2^{3q} C_0 \mu(S) \sum_{k=k_0+1}^\infty \left(\fint_{S_0} |\nabla^\varphi u_k|^p \, d\mu \right)^{\frac{q}{p}} \le 2^{3q} C_0 \mu(S) \left(\fint_{S_0} |\nabla^\varphi u|^p \, d\mu \right)^{\frac{q}{p}}, \end{aligned}$$

where the last inequality uses the facts that $\nabla^\varphi u_k = \nabla^\varphi u\, \chi_{\{2^k<u<2^{k+1}\}}$ (Lebesgue) a.e. in S_0 for every $k > k_0$ and that $q \ge p$. On the other hand, given $\zeta > 0$, the inequality (3.1) applied to u gives

$$\fint_S u_+\, d\mu \le \zeta + \frac{1}{\mu(S)} \int_\zeta^\infty \mu(\{x \in S : |u(x) - u_S^\mu| \ge \tau\})\, d\tau$$

$$\le \zeta + C_0 \left(\fint_{S_0} |\nabla^\varphi u|^p\, d\mu \right)^{\frac{q}{p}} \int_\zeta^\infty \frac{d\tau}{\tau^q} = \zeta \left(1 + C_0 \left(\fint_{S_0} |\nabla^\varphi u|^p\, d\mu \right)^{\frac{q}{p}} \zeta^{-q} \right).$$

Therefore, by choosing $\zeta := C_0^{1/q} \left(\fint_{S_0} |\nabla^\varphi u|^p\, d\mu \right)^{\frac{1}{p}}$ (see Remark 4) we get

$$\fint_S u_+\, d\mu \le \frac{q C_0^{1/q}}{q-1} \left(\fint_{S_0} |\nabla^\varphi u|^p\, d\mu \right)^{\frac{1}{p}}. \tag{3.6}$$

Notice that (3.6) holds true also in the case $\zeta = 0$. Now, the definition of $k_0 \in \mathbb{Z}$ from (3.3) and (3.6) imply

$$II := \int_{\{x\in S: u(x) < 2^{k_0+2}\}} u_+^q\, d\mu < 2^{(k_0+2)q} \mu(S) \le 2^{3q} \mu(S) \left(\fint_S u_+\, d\mu \right)^q$$

$$\le 2^{3q} C_0 \left(\frac{q}{q-1} \right)^q \mu(S) \left(\fint_{S_0} |\nabla^\varphi u|^p\, d\mu \right)^{\frac{q}{p}}.$$

Finally,

$$\int_S u_+^q d\mu = I + II \le C_1^q \mu(S) \left(\fint_{S_0} |\nabla^\varphi u|^p\, d\mu \right)^{\frac{q}{p}},$$

with $C_1^q := 2^{3q} C_0 \left(1 + \left(\frac{q}{q-1} \right)^q \right)$. Reasoning analogously with u_- finishes the proof. □

Remark 4. When $0 < p \le 2$ (which is the case we will be using), the condition $\|(D^2\varphi)^{-1}\| \in L^1_{loc}(\Omega, d\mu)$ guarantees that $\zeta := C_0^{1/q} \left(\fint_{S_0} |\nabla^\varphi u|^p\, d\mu \right)^{\frac{1}{p}}$ in the proof of Lemma 3.1 is finite. Indeed,

$$\left(\fint_{S_0} |\nabla^\varphi u|^p\, d\mu \right)^{\frac{1}{p}} \le \left(\fint_{S_0} |\nabla^\varphi u|^2\, d\mu \right)^{\frac{1}{2}} = \left(\fint_{S_0} \langle (D^2\varphi)^{-1} \nabla u, \nabla u \rangle\, d\mu \right)^{\frac{1}{2}}$$

$$\le \operatorname*{ess\,sup}_{S_0} |\nabla u| \left(\fint_{S_0} ||(D^2\varphi)^{-1}||\, d\mu \right)^{\frac{1}{2}} < \infty.$$

Theorem 3.2. *Fix $\varphi \in C^1(\Omega)$ with $D^2\varphi > 0$ a.e. in Ω and $\mu_\varphi \in DC(\Omega, \delta_\varphi)$ and let μ be a Borel doubling measure on Ω absolutely continuous with respect to Lebesgue measure satisfying the following conditions:*

(a) *for some* $C_P > 0$, $\lambda \geq 1$, *and* $p > 0$, *the Poincaré inequality*

$$\fint_S |u - u_S^\mu| \, d\mu \leq C_P t^{\frac{1}{2}} \left(\fint_{\lambda S} |\nabla^\varphi u|^p \, d\mu \right)^{\frac{1}{p}}, \tag{3.7}$$

with $u_S^\mu := \fint_S u \, d\mu$, *holds true for every section* $S := S_\varphi(x_0, t)$ *with* $\lambda S \subset\subset \Omega$ *and every* $u \in \mathrm{Lip}(\lambda S)$*;*

(b) *for some* $C_D > 0$ *and* $s > p/2$ *it satisfies the growth condition*

$$\mu(S_\varphi(z, r)) \leq C_D \left(\frac{r}{r'}\right)^s \mu(S_\varphi(z, r')), \tag{3.8}$$

for all $0 < r' \leq r$ *and all sections* $S_\varphi(z, r)$ *with* $S_\varphi(z, r) \subset\subset \Omega$.

Then,

$$\left(\fint_S |u - u_S^\mu|^{\frac{2sp}{2s-p}} \, d\mu \right)^{\frac{2s-p}{2sp}} \leq C_{P,s} t^{\frac{1}{2}} \left(\fint_{\lambda\Theta^2 S} |\nabla^\varphi u|^p \, d\mu \right)^{\frac{1}{p}}, \tag{3.9}$$

for every section $S := S_\varphi(x_0, t)$ *with* $\lambda\Theta^2 S \subset\subset \Omega$ *and every* $u \in \mathrm{Lip}(\lambda\Theta^2 S)$*, where* $C_{P,s} > 0$ *depends only on* s, λ, Θ, C_P*, and* C_D*.*

Proof. Fix $S := S_\varphi(x_0, t)$ such that $S_\varphi(x_0, \lambda\Theta^2 t) \subset\subset \Omega$, $x \in S$, and $u \in \mathrm{Lip}(\lambda\Theta^2 S)$. For $j \in \mathbb{N}$ set $t_j := 2^{-j} t$ and $S_j := S_\varphi(x, t_j)$, for $j = 0$ set $S_0 := S_\varphi(x_0, \Theta^2 t)$ and $t_0 := \Theta^2 t$. Notice that these choices imply $S_{j+1} \subset S_j$ for every $j \in \mathbb{N}_0$ and, for $\lambda \geq 1$, $\lambda S_{j+1} \subset \lambda S_j$ for every $j \in \mathbb{N}_0$. To check this last inclusion when $j = 0$, we use that $x \in S = S_\varphi(x_0, t) \subset S_\varphi(x_0, \lambda t)$ and the second inclusion from (2.2) with "$\tau = \lambda t$" to obtain $\lambda S_1 \subset \lambda S_0$.

Since u is continuous, $x \in S$ is a Lebesgue point of u, then

$$x \in S_\varphi(x_0, t) \subset\subset S_\varphi(x_0, \Theta^2 t) \subset S_\varphi(x_0, \lambda\Theta^2 t) \subset\subset \Omega,$$

we can use (3.8) with $S = S_\varphi(x_0, t)$ and $S_j = S_\varphi(x, t_j)$ to obtain

$$t_j \leq C_D^{1/s} \Theta t \left(\frac{\mu(S_\varphi(x, t_j))}{\mu(S_\varphi(x_0, t))} \right)^{1/s} \qquad \forall j \in \mathbb{N}, \tag{3.10}$$

with $\mu(S_0) = \mu(S_\varphi(x_0, \Theta^2 t)) \leq C_D \Theta^{2s} \mu(S_\varphi(x_0, t))$ from (3.8). Hence,

$$t_j \leq C_D^{2/s} \Theta^3 t \left(\frac{\mu(S_j)}{\mu(S_0)} \right)^{1/s} \qquad \forall j \in \mathbb{N}. \tag{3.11}$$

Notice that (3.11) is obviously true for $j = 0$ because $C_D > 1$ and $\Theta > 1$. In addition, $\mu(S_j) \leq C_D \mu(S_{j+1})$ for every $j \in \mathbb{N}$ and, when $j = 0$, the doubling condition (3.8) and the fact that $S_\varphi(x_0, t) \subset S_\varphi(x, \Theta t)$ give

$$\begin{aligned} \mu(S_0) &\leq C_D \Theta^{2s} \mu(S_\varphi(x_0, t)) \leq C_D \Theta^{2s} \mu(S_\varphi(x, \Theta t)) \\ &\leq 2^s C_D^2 \Theta^{3s} \mu(S_\varphi(x, t/2)) = 2^s C_D^2 \Theta^{3s} \mu(S_1). \end{aligned} \tag{3.12}$$

Consequently, by using the estimates above for $\mu(S_j)/\mu(S_{j+1})$ with $j \in \mathbb{N}_0$, the Poincaré inequality, and (3.11),

$$|u(x) - u^\mu_{S_0}| = \lim_{j\to\infty} |u^\mu_{S_j} - u^\mu_{S_0}| \le \sum_{j=0}^{\infty} \fint_{S_{j+1}} |u - u^\mu_{S_j}|\, d\mu$$

$$\le 2^s C_D^2 \Theta^{3s} \sum_{j=0}^{\infty} \fint_{S_j} |u - u^\mu_{S_j}|\, d\mu \le 2^s C_D^2 \Theta^{3s} C_P \sum_{j=0}^{\infty} t_j^{\frac{1}{2}} \left(\fint_{\lambda S_j} |\nabla^\varphi u|^p \, d\mu \right)^{\frac{1}{p}}$$

$$\le \frac{C_3 t^{\frac{1}{2}}}{\mu(S_0)^{\frac{1}{2s}}} \sum_{j=0}^{\infty} \mu(S_j)^{\frac{1}{2s}} \left(\fint_{\lambda S_j} |\nabla^\varphi u|^p \, d\mu \right)^{\frac{1}{p}},$$

with $C_3 := 2^s C_D^{3/2} \Theta^{3s+3/2} C_P$. On the other hand, recalling that $t_0 := \Theta^2 t$,

$$|u^\mu_S - u^\mu_{S_0}| \le \fint_S |u - u^\mu_{S_0}|\, d\mu \le C_D \Theta^{2s} \fint_{S_0} |u - u^\mu_{S_0}|\, d\mu$$

$$\le C_D \Theta^{2s} C_P t^{\frac{1}{2}} \left(\fint_{\lambda S_0} |\nabla^\varphi u|^p \, d\mu \right)^{\frac{1}{p}}.$$

Therefore, for every $x \in S$ we have

$$|u(x) - u^\mu_S| \le \frac{2C_3 t^{\frac{1}{2}}}{\mu(S_0)^{\frac{1}{2s}}} \sum_{j=0}^{\infty} \mu(S_j)^{\frac{1}{2s}} \left(\fint_{\lambda S_j} |\nabla^\varphi u|^p \, d\mu \right)^{\frac{1}{p}} \tag{3.13}$$

Next, introduce

$$M(x) := \sup_{S'} \fint_{S'} |\nabla^\varphi u|^p \, d\mu > 0$$

where the supremum is taken over all the sections $S' \subset \lambda\Theta^2 S_0$ with $x \in S'$. From [3, Section 5] it follows that $M(x)$ is finite for a.e. $x \in S$ and then $\frac{1}{M(x)} \int_{\lambda S_0} |\nabla^\varphi u|^p \, d\mu > 0$ a.e. $x \in S$. Now, the fact that $\mu(\lambda S_j) \to 0$ as $j \to \infty$ (which follows from Lemma 2.1) implies that, for a.e. $x \in S$, there is a smallest $j \in \mathbb{N}_0$ such that the inequality

$$\mu(\lambda S_j) \le \frac{1}{M(x)} \int_{\lambda S_0} |\nabla^\varphi u|^p \, d\mu \tag{3.14}$$

holds true. Let $j_0 \in \mathbb{N}_0$ denote such integer (which depends on x). Notice that if $j_0 = 0$ then equality occurs in (3.15). In particular, we have

$$\mu(\lambda S_{j_0}) \le \frac{1}{M(x)} \int_{\lambda S_0} |\nabla^\varphi u|^p \, d\mu < \mu(\lambda S_{j_0 - 1}) \le C_D \lambda^s \mu(S_{j_0-1}),$$

with $\mu(S_{j_0-1}) \le C_D \mu(S_{j_0})$ if $j_0 > 1$ and, if $j_0 = 1$, we use (3.12) to obtain $\mu(S_{j_0-1}) = \mu(S_0) \le C_D^2 \Theta^{3s} \mu(S_1)$. Hence,

$$\mu(\lambda S_{j_0}) \leq \frac{1}{M(x)} \int\limits_{\lambda S_0} |\nabla^\varphi u|^p \, d\mu \leq C_D^3 (\lambda\Theta^3)^s \mu(S_{j_0}). \tag{3.15}$$

Let us now split the sum from (3.13) into

$$\sum_{j=0}^{\infty} \mu(S_j)^{\frac{1}{2s}} \left(\fint\limits_{\lambda S_j} |\nabla^\varphi u|^p \, d\mu \right)^{\frac{1}{p}} = \sum_{j=0}^{j_0-1} \cdots + \sum_{j=j_0}^{\infty} \cdots =: \Sigma' + \Sigma''.$$

Let us first consider Σ'. Notice that we can assume $j_0 \in \mathbb{N}$ (that is, $j_0 \geq 1$, because $\Sigma' = 0$ if $j_0 = 0$). Now, for $j_0 > 1$, Lemma 2.1 with $\alpha = 1/2$ implies

$$\mu(S_{j_0}) \leq \xi^{j_0 - j} \mu(S_j) \quad \forall j < j_0, \tag{3.16}$$

where $\xi \in (0,1)$ depends only on C_D and Θ. Now, if $j_0 = 1$ and $j = 0$, from the inclusion $S_1 = S_\varphi(x, t/2) \subset S_\varphi(x_0, \Theta^2 t) = S_0$, we get

$$\mu(S_1) \leq \mu(S_0) = \frac{1}{\xi} \xi^{j_0 - j} \mu(S_0). \tag{3.17}$$

Hence, for each $j_0 \in \mathbb{N}$, we have that $\mu(S_{j_0}) \leq \frac{1}{\xi}\xi^{j_0-j}\mu(S_j)$ for every $j < j_0$, which, in turn, implies (recall that $2s > p$)

$$\begin{aligned}
\Sigma' &:= \sum_{j=0}^{j_0-1} \frac{\mu(S_j)^{\frac{1}{2s}}}{\mu(\lambda S_j)^{\frac{1}{p}}} \left(\int\limits_{\lambda S_j} |\nabla^\varphi u|^p \, d\mu \right)^{\frac{1}{p}} \leq \sum_{j=0}^{j_0-1} \mu(S_j)^{\frac{1}{2s} - \frac{1}{p}} \left(\int\limits_{\lambda S_j} |\nabla^\varphi u|^p \, d\mu \right)^{\frac{1}{p}} \\
&\leq \xi^{\frac{1}{2s} - \frac{1}{p}} \mu(S_{j_0})^{\frac{1}{2s} - \frac{1}{p}} \sum_{j=0}^{j_0-1} \xi^{(\frac{1}{2s} - \frac{1}{p})(j - j_0)} \left(\int\limits_{\lambda S_j} |\nabla^\varphi u|^p \, d\mu \right)^{\frac{1}{p}} \\
&\leq C_4 \mu(S_{j_0})^{\frac{1}{2s} - \frac{1}{p}} \left(\int\limits_{\lambda S_0} |\nabla^\varphi u|^p \, d\mu \right)^{\frac{1}{p}} \leq C_5 M(x)^{\frac{1}{p} - \frac{1}{2s}} \left(\int\limits_{\lambda S_0} |\nabla^\varphi u|^p \, d\mu \right)^{\frac{1}{2s}},
\end{aligned}$$

where $C_4 := \xi^{\frac{1}{2s} - \frac{1}{p}} \sum_{k=1}^{\infty} \xi^{(\frac{1}{p} - \frac{1}{2s})k}$ and $C_5 := C_4 [C_D^3 (\lambda\Theta^3)^s]^{\frac{1}{p} - \frac{1}{2s}}$ and we used the second inequality from (3.15).

We now turn to Σ''. We first use Lemma 2.1 again with $\alpha = 1/2$ to write

$$\mu(S_j) \leq \xi^{j - j_0} \mu(S_{j_0}) \quad \forall j \geq j_0,$$

at least if $j_0 \in \mathbb{N}$. If $j_0 = 0$ the inclusion $S_1 \subset S_0$ and Lemma 2.1 give $\mu(S_j) \leq \xi^{j-1}\mu(S_1) \leq \frac{1}{\xi}\xi^{j-j_0}\mu(S_0) = \frac{1}{\xi}\xi^{j-j_0}\mu(S_{j_0})$. Hence,

$$\mu(S_j) \leq \frac{1}{\xi} \xi^{j - j_0} \mu(S_{j_0}) \quad \forall j \geq j_0 \geq 0. \tag{3.18}$$

Consequently, from (3.18) and (3.15),

$$\Sigma'' := \sum_{j=j_0}^{\infty} \mu(S_j)^{\frac{1}{2s}} \left(\fint_{\lambda S_j} |\nabla^{\varphi} u|^p \, d\mu \right)^{\frac{1}{p}} \leq \xi^{-\frac{1}{2s}} \mu(S_{j_0})^{\frac{1}{2s}} M(x)^{\frac{1}{p}} \sum_{j=j_0}^{\infty} \xi^{\frac{j-j_0}{2s}}$$
$$\leq C_6 M(x)^{\frac{1}{p}-\frac{1}{2s}} \left(\int_{\lambda S_0} |\nabla^{\varphi} u|^p \, d\mu \right)^{\frac{1}{2s}},$$

with $C_6 := \xi^{-\frac{1}{2s}} \sum_{k=0}^{\infty} \xi^{\frac{k}{2s}}$. Coming back (3.13), we now have

$$|u(x) - u_S^{\mu}| \leq \frac{2C_3(C_5 + C_6)t^{\frac{1}{2}}}{\mu(S_0)^{\frac{1}{2s}}} M(x)^{\frac{1}{p}-\frac{1}{2s}} \left(\int_{\lambda S_0} |\nabla^{\varphi} u|^p \, d\mu \right)^{\frac{1}{2s}}$$
$$\leq C_7 t^{\frac{1}{2}} M(x)^{\frac{2s-p}{2sp}} \left(\fint_{\lambda S_0} |\nabla^{\varphi} u|^p \, d\mu \right)^{\frac{1}{2s}}, \tag{3.19}$$

with $C_7 := 2\lambda^{\frac{1}{2}} C_D^{\frac{1}{2s}} C_3(C_5 + C_6)$. Setting $q_s := \frac{2sp}{2s-p} > p$ and given $\tau > 0$, the inequality $|u(x) - u_S^{\mu}| \geq \tau$, the weak (1,1)-type of M with a constant $C_{1,1} > 0$ depending only on C_D and Θ (see [3, Section 5] or [1, Lemma 3.12] along with [18, Lemma 15]), and (3.19) then yield

$$\mu(\{x \in S : |u(x) - u_S^{\mu}| \geq \tau\})$$
$$\leq \mu\left(\left\{ x \in S : M(x) \geq C_7^{-q_s} \tau^{q_s} t^{-q_s/2} \left(\fint_{\lambda S_0} |\nabla^{\varphi} u|^p \, d\mu \right)^{-\frac{q_s}{2s}} \right\} \right)$$
$$\leq C_{1,1} C_7^{q_s} \tau^{-q_s} t^{q_s/2} \left(\fint_{\lambda S_0} |\nabla^{\varphi} u|^p \, d\mu \right)^{\frac{q_s}{2s}} \int_S |\nabla^{\varphi} u|^p \, d\mu$$
$$\leq C_0 \tau^{-q_s} \mu(S) \left(\fint_{\lambda S_0} |\nabla^{\varphi} u|^p \, d\mu \right)^{\frac{q_s}{2s}+1} = C_0 \tau^{-q_s} \mu(S) \left(\fint_{\lambda \Theta^2 S} |\nabla^{\varphi} u|^p \, d\mu \right)^{\frac{q_s}{p}},$$

with $C_0 := C_{1,1} C_7^{q_s} C_D \lambda^s t^{q_s/2}$. Hence, Lemma 3.1 applied with $q = q_s$ and $S_0 = \lambda \Theta^2 S$ imply (3.9). □

4. Proof of Theorem 1.1

Let us start by recalling the following fact from [16].

Theorem E *(Theorem 1.3 in [16]). Given an open convex set $U \subset \mathbb{R}^n$ and $\phi \in C^2(U)$ with $D^2\phi > 0$ in U and $\mu_\phi \in DC(U, \delta_\phi)$ there exists a geometric constant $C_1^* > 0$ such that for every section $S := S_\phi(x_0, t)$ with $S_\phi(x_0, t) \subset\subset U$ and every $h \in C^1(S)$ the following $(1, 2)$-Poincaré holds true in the Monge-Ampère quasi-metric structure with respect to the Lebesgue measure*

$$\fint_S |h(x) - h_S| \, dx \leq C_1^* t^{\frac{1}{2}} \left(\fint_S |\nabla^{\phi} h(x)|^2 \, dx \right)^{\frac{1}{2}}, \tag{4.1}$$

where $h_S := \fint_S h(x) \, dx$.

Theorem 4.1. *Fix an open convex set $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ and $\varphi \in W^{2,n}_{loc}(\Omega)$ such that $D^2\varphi > 0$ a.e. in Ω and $\mu_\varphi \in DC(\Omega, \delta_\varphi)$. Then, there exist geometric constants $C_3^* > 0$ and $K^* \geq 1$ such that for every section $S := S_\varphi(x_0, t)$ with $S_\varphi(x_0, K^*t) \subset\subset \Omega$ and every $h \in C^1(S_\varphi(x_0, 2K^*t))$ the following Poincaré inequality holds true with respect to the Monge-Ampère measure μ_φ*

$$\fint_S |h(x) - h_S^{\mu_\varphi}|\, d\mu_\varphi(x) \leq C_3^*\, t^{\frac{1}{2}} \left(\fint_{S_\varphi(x_0, 2K^*t)} |\nabla^\varphi h(x)|^2\, d\mu_\varphi(x) \right)^{\frac{1}{2}}. \tag{4.2}$$

Proof. Let $\varphi \in W^{2,n}_{loc}(\Omega, dx)$ with $D^2\varphi > 0$ a.e. in Ω and $\mu_\varphi \in DC(\Omega, \delta_\varphi)$. Given a section $S := S_\varphi(x_0, t) \subset\subset \Omega$ let $\Omega_S \subset \mathbb{R}^n$ be an open convex set such that $S \subset\subset \Omega_S \subset\subset \Omega$ set $\varepsilon_0 := \operatorname{dist}(\Omega_S, \partial\Omega)$ and for $0 < \varepsilon < \varepsilon_0$ and $x \in \Omega_S$ define

$$\varphi_\varepsilon(x) := \varphi * \eta_\varepsilon(x) = \int_{\mathbb{R}^n} \varphi(x - y)\eta_\varepsilon(y)\, dy \tag{4.3}$$

where $\eta \in C_c^\infty(\mathbb{R}^n)$, $\eta \geq 0$, $\operatorname{supp}(\eta) \subset B(0, 1)$ and $\|\eta\|_{L^1(\mathbb{R}^n)} = 1$ with $\eta_\varepsilon(y) := \varepsilon^{-n}\eta(\varepsilon^{-1}y)$. Then, for each $\varepsilon > 0$, we have that $\varphi_\varepsilon \in C^\infty(\Omega_S)$ with $D^2\varphi_\varepsilon > 0$ in Ω_S. Indeed, since $D^2\varphi_\varepsilon(x) = \int_{\mathbb{R}^n} D^2\varphi(x-y)\eta_\varepsilon(y)\, dy$, if we had $\langle D^2\varphi_\varepsilon(y_0)v, v\rangle = 0$ for some point $y_0 \in \Omega$ and non-zero vector $v \in \mathbb{R}^n \setminus \{0\}$, then it would follow that $\langle D^2\varphi(y_0 - y)v, v\rangle = 0$ for almost every $|y| < \varepsilon$, contradicting $D^2\varphi > 0$ a.e. in Ω. Also, φ_ε and $\nabla\varphi_\varepsilon$ converge to φ and $\nabla\varphi$, respectively, uniformly over compact subsets of Ω_S. Moreover, from the characterization of $\mu_\varphi \in DC(\Omega, \delta_\varphi)$ in terms of the engulfing property in [8, Theorems 1 and 4] we have that $\mu_{\varphi_\varepsilon} \in DC(\Omega, \delta_\varphi)$ for every $\varepsilon \in (0, \varepsilon_0)$ with constants depending only on the constant from $\mu_\varphi \in DC(\Omega, \delta_\varphi)$ (and, in particular, independent of ε).

Next, for each $0 < \varepsilon < \varepsilon_0$ let $\psi_\varepsilon : \nabla\varphi_\varepsilon(\Omega) \to \mathbb{R}$ denote the convex conjugate to φ_ε, which is smooth, strictly convex, and satisfies

$$\nabla\varphi_\varepsilon(\nabla\psi_\varepsilon(y)) = y \quad \forall y \in \nabla\varphi_\varepsilon(\Omega), \tag{4.4}$$

$$\nabla\psi_\varepsilon(\nabla\varphi_\varepsilon(x)) = x \quad \forall x \in \Omega. \tag{4.5}$$

Moreover, by [8, Theorem 12] we have $\mu_{\psi_\varepsilon} \in DC(\delta_{\psi_\varepsilon}, \nabla\varphi_\varepsilon(\Omega))$ with a constant depending only on the constant from $\mu_\varphi \in DC(\Omega, \delta_\varphi)$. In addition, there exists a constant $K^* > 1$, also depending only on the constant from $\mu_\varphi \in DC(\Omega, \delta_\varphi)$, such that

$$S_{\varphi_\varepsilon}(z, \tau/K^*) \subset \nabla\psi_\varepsilon(S_{\psi_\varepsilon}(\nabla\varphi_\varepsilon(z), \tau)) \subset S_{\varphi_\varepsilon}(z, K^*\tau), \tag{4.6}$$

for every section $S_{\varphi_\varepsilon}(z, \tau)$ with $S_{\varphi_\varepsilon}(z, K^*\tau) \subset\subset \Omega$. At this point, given a section $S_{\varphi_\varepsilon}(x_0, t)$ with $S_{\varphi_\varepsilon}(x_0, 2K^*t) \subset\subset \Omega$, the second inclusion in (4.6) and (4.4) give

$$S_\varepsilon^* := S_{\psi_\varepsilon}(\nabla\varphi_\varepsilon(x_0), t) \subset \nabla\varphi_\varepsilon(S_{\varphi_\varepsilon}(x_0, K^*t)) \subset\subset \nabla\varphi_\varepsilon(\Omega). \tag{4.7}$$

Notice that from the fact that φ_ε and $\nabla\varphi_\varepsilon$ converge to φ and $\nabla\varphi$, respectively, uniformly over compact subsets of Ω we can assume that $\varepsilon > 0$ is small enough so that

$$\nabla\varphi_\varepsilon(S_{\varphi_\varepsilon}(x_0, K^*t)) \subset \nabla\varphi(S_\varphi(x_0, 2K^*t)) \subset\subset \nabla\varphi(\Omega_S). \tag{4.8}$$

The next step is to apply (4.1) with ψ_ε in the section S_ε^*. Given a function $h \in C^1(S_\varphi(x_0, 2K^*t))$ define $u \in C^1(\nabla\varphi(S_\varphi(x_0, 2K^*t)))$ as $u(y) := h(\nabla\psi_\varepsilon(y))$. In particular, the inclusions (4.7) and (4.8) imply $u \in C^1(S_\varepsilon^*)$, so that the Poincaré inequality (4.1) applied with ψ_ε in the section S_ε^* to u reads as

$$\fint_{S_\varepsilon^*} |u(y) - u_{S_\varepsilon^*}|\, dy \le C_2^*\, t^{\frac{1}{2}} \left(\fint_{S_\varepsilon^*} |\nabla^{\psi_\varepsilon} u(y)|^2\, dy \right)^{\frac{1}{2}}. \tag{4.9}$$

By setting $y := \nabla\varphi_\varepsilon(x)$ for $x \in S_\varphi(x_0, 2K^*t))$, and recalling (4.5), we get

$$\nabla h(x) = D^2\varphi_\varepsilon(x)\nabla u(\nabla\varphi_\varepsilon(x)) = D^2\psi_\varepsilon(y)^{-1}\nabla u(y)$$

and then

$$\begin{aligned} |\nabla^{\psi_\varepsilon} u(y)|^2 &= \langle D^2\psi_\varepsilon(y)^{-1}\nabla u(y), \nabla u(y)\rangle \\ &= \langle \nabla h(x), D^2\varphi_\varepsilon(x)^{-1}\nabla h(x)\rangle = |\nabla^{\varphi_\varepsilon} h(x)|^2. \end{aligned} \tag{4.10}$$

Hence, by changing variables $y = \nabla\varphi_\varepsilon(x)$ in (4.9) we get

$$\begin{aligned} &\frac{1}{|S_\varepsilon^*|} \int_{\nabla\psi_\varepsilon(S_\varepsilon^*)} |h(x) - h_{S_\varepsilon}| \det D^2\varphi_\varepsilon(x)\, dx \\ &\le \frac{C_2^*\, t^{\frac{1}{2}}}{|S_\varepsilon^*|^{\frac{1}{2}}} \left(\int_{\nabla\psi_\varepsilon(S_\varepsilon^*)} |\nabla^{\varphi_\varepsilon} h(x)|^2 \det D^2\varphi_\varepsilon(x)\, dx \right)^{\frac{1}{2}} \end{aligned} \tag{4.11}$$

where

$$h_{S_\varepsilon} := \frac{1}{|S_\varepsilon^*|} \int_{\nabla\psi_\varepsilon(S_\varepsilon^*)} h(x) \det D^2\varphi_\varepsilon(x)\, dx.$$

Notice that from the inclusions (4.6), (4.7), and (4.8) it follows that

$$S_\varphi(x_0, t) \subset \nabla\psi_\varepsilon(S_\varepsilon^*) \subset S_\varphi(x_0, 2K^*t), \tag{4.12}$$

so that the integral on the left-hand side of (4.11) can be replaced with the integral over $S_\varphi(x_0, t)$ and the one on its right-hand side by the integral over $S_\varphi(x_0, 2K^*t)$. In addition, the inclusions (4.12), along with the fact that $\nabla\varphi_\varepsilon$ and $\nabla\psi_\varepsilon$ are the inverse of each other, imply

$$\begin{aligned} \mu_{\varphi_\varepsilon}(S_\varphi(x_0, t)) &= |\nabla\varphi_\varepsilon(S_\varphi(x_0, t))| \le |S_\varepsilon^*| \\ &\le |\nabla\varphi_\varepsilon(S_\varphi(x_0, 2K^*t))| = \mu_{\varphi_\varepsilon}(S_\varphi(x_0, 2K^*t)). \end{aligned} \tag{4.13}$$

We are now in position to start taking limits as $\varepsilon \to 0$. From the definition of φ_ε in (4.3) we get that $D^2\varphi_\varepsilon(x)$ (or a subsequence) converges to $D^2\varphi(x)$ for a.e. $x \in \Omega$. In particular, $\det D^2\varphi_\varepsilon(x)$ converges to $\det D^2\varphi(x)$ for a.e. $x \in \Omega$. Let us first show that $\mu_{\varphi_\varepsilon}(F)$ converges to $\mu_\varphi(F)$ for every Borel set $F \subset S_\varphi(x_0, 2K^*t)$. Indeed, since $S_\varphi(x_0, 2K^*t) \subset\subset \Omega$ let S' denote a compact set such that $S_\varphi(x_0, 2K^*t) \subset\subset S' \subset\subset \Omega$ and introduce $H(x) := \Delta\varphi(x)\, \chi_{S'}(x)$. Let us also assume that $\varepsilon < \varepsilon_1 := \operatorname{dist}(S_\varphi(x_0, 2K^*t), \partial S')$ so that, for $x \in S_\varphi(x_0, 2K^*t)$, we get $(\Delta\varphi * \eta_\varepsilon)(x) = (H * \eta_\varepsilon)(x)$. Then, for every $x \in S_\varphi(x_0, 2K^*t)$, the arithmetic-geometric inequality implies

$$0 < \det D^2\varphi_\varepsilon(x) \le \Delta\varphi_\varepsilon(x)^n = (\Delta\varphi * \eta_\varepsilon)(x)^n = (H * \eta_\varepsilon)(x)^n \le \mathcal{M}(H)(x)^n,$$

where $\mathcal{M}$ denotes the Hardy-Littlewood maximal function whose (n, n)-strong type (here is when we use $n \geq 2$) gives

$$\int\limits_{S_\varphi(x_0, 2K^*t)} \mathcal{M}(H)(x)^n \, dx \leq \|\mathcal{M}(H)\|^n_{L^n(\mathbb{R}^n,\, dx)} \leq C_n \|H\|^n_{L^n(\mathbb{R}^n,\, dx)}$$
$$= C_n \int\limits_{S'} \Delta\varphi(x)^n \, dx < \infty,$$

where the hypothesis $\varphi \in W^{2,n}_{loc}(\Omega)$ guarantees the finiteness of the last integral. (Recall that for a convex function ϕ we always have $\frac{1}{n}\Delta\phi \leq \|D^2\phi\| \leq \Delta\phi$ almost everywhere.) Therefore, Lebesgue's dominated convergence theorem implies that $\mu_{\varphi_\varepsilon}(F)$ converges to $\mu_\varphi(F)$ for every Borel set $F \subset S_\varphi(x_0, 2K^*t)$ as claimed. Next, we will use Lebesgue's dominated convergence theorem on the integral

$$\int\limits_{S_\varphi(x_0, 2K^*t)} |\nabla^{\varphi_\varepsilon} h(x)|^2 \det D^2\varphi_\varepsilon(x) \, dx.$$

Given $x \in S_\varphi(x_0, 2K^*t)$ let $0 < \lambda_{1,\varepsilon}(x) \leq \cdots \leq \lambda_{n,\varepsilon}(x)$ denote the eigenvalues of $D^2\varphi_\varepsilon(x)$ and using that $|\nabla^{\varphi_\varepsilon} h(x)|^2 = \langle D^2\varphi_\varepsilon(x)^{-1}\nabla h(x), \nabla h(x)\rangle$ we get

$$|\nabla^{\varphi_\varepsilon} h(x)|^2 \det D^2\varphi_\varepsilon(x) \leq \Big(\sup_{S_\varphi(x_0, 2K^*t)} |\nabla h|\Big)^2 \|D^2\varphi_\varepsilon(x)^{-1}\| \det D^2\varphi_\varepsilon(x)$$

with

$$\|D^2\varphi_\varepsilon(x)^{-1}\| \det D^2\varphi_\varepsilon(x) = \frac{1}{\lambda_{1,\varepsilon}(x)} \prod_{j=1}^n \lambda_{j,\varepsilon}(x) \leq \Big(\sum_{j=2}^n \lambda_{j,\varepsilon}(x)\Big)^{n-1}$$
$$< \Big(\sum_{j=1}^n \lambda_{j,\varepsilon}(x)\Big)^{n-1} = \Delta\varphi_\varepsilon(x)^{n-1} = \Delta\varphi * \eta_\varepsilon(x)^{n-1} \leq \mathcal{M}(H)(x)^{n-1}$$

and, by reasoning as above, in the case $n > 2$ we obtain that $\mathcal{M}(H)^{n-1} \in L^1(S_\varphi(x_0, 2K^*t),\, dx)$. In the case $n = 2$ we just do

$$\int\limits_{S_\varphi(x_0, 2K^*t)} \mathcal{M}(H)(x) \, dx \leq \Bigg(\int\limits_{\mathbb{R}^2} \mathcal{M}(H)(x)^2 \, dx\Bigg)^{\frac{1}{2}} |S_\varphi(x_0, 2K^*t)|^{\frac{1}{2}} < \infty.$$

Finally, by taking limits as $\varepsilon \to 0$ in (4.11) (and we can just use Fatou's lemma on its left-hand side) and by recalling the inequalities (4.13) and the doubling property of μ_φ, the Poincaré inequality (4.2) follows with $K_1 := 2K^*$. $\square$

4.1. Proof of Theorem 1.1

The idea is to use Theorem 3.2 to improve the Poincaré inequality (4.2) from Theorem 4.1. Let us first remark that the condition $\varphi \in W^{2,n}_{loc}(\Omega,\, dx)$ with $D^2\varphi > 0$ a.e. in Ω implies that $\|(D^2\varphi)^{-1}\| \in L^1_{loc}(\Omega,\, d\mu_\varphi)$. In fact, even $\varphi \in W^{2,n-1}_{loc}(\Omega,\, dx)$ with $D^2\varphi > 0$ a.e. in Ω will do so. Indeed, since $D^2\varphi(x) > 0$ for a.e. $x \in \Omega$, let $0 < \lambda_1(x) \leq \cdots \leq \lambda_n(x) < \infty$ denote the eigenvalues of $D^2\varphi(x)$. Then,

$$\|D^2\varphi(x)^{-1}\| \det D^2\varphi(x) = \frac{1}{\lambda_1(x)} \prod_{j=1}^{n} \lambda_j(x) \leq \left(\frac{1}{n-1} \sum_{j=2}^{n} \lambda_j(x) \right)^{n-1}$$
$$\leq \Delta\varphi(x)^{n-1} \in L^1_{loc}(\Omega, dx).$$

Let us now recall the growth condition (2.10) for μ_φ, so that (3.8) from Theorem 3.2 holds true with $s = n - \varepsilon$. Therefore, by using Theorem 3.2 with $p = 2$ and $s = n - \varepsilon$ (notice that $s > p/2$ iff $n - \varepsilon > 1$ iff $n \geq 2$) the Poincaré inequality (4.2) self-improves to (1.17) since from our choices of p and s we get

$$\frac{2sp}{2s-p} = \frac{4(n-\varepsilon)}{2(n-\varepsilon)-2} = \frac{2(n-\varepsilon)}{(n-\varepsilon)-1} = \frac{2n}{n-1} + \varepsilon_1,$$

where $\varepsilon_1 := \frac{2(n-\varepsilon)}{(n-\varepsilon)-1} - \frac{2n}{n-1} > 0$ is a geometric constant. $\square$

5. Proof of Theorem 1.2

The idea of the proof is to apply Theorem 1.1 to ψ, the convex conjugate of φ, and then do a change of variables. In order to see that $\psi \in W^{2,n}_{loc}(\nabla\varphi(\Omega), dy)$, we first notice that (2.12), along with the hypothesis $D^2\varphi > 0$ a.e. in Ω, implies $D^2\psi(\nabla\varphi(x)) = D^2\varphi(x)^{-1} > 0$ for a.e. $x \in \Omega$; therefore, given a compact set $F \subset \Omega$ and changing variables $y := \nabla\varphi(x)$,

$$\int\limits_{\nabla\varphi(F)} \|D^2\psi(y)\|^n \, dy = \int\limits_{F} \|D^2\varphi(x)^{-1}\|^n \det D^2\varphi(x)\, dx < \infty,$$

where the finiteness of the last integral above follows from the hypothesis $\|(D^2\varphi)^{-1}\| \in L^n_{loc}(\Omega, d\mu_\varphi)$. Notice that $y = \nabla\varphi(x)$ is a valid change of variables because $\nabla\varphi$ is one-to-one and $\varphi \in W^{2,n}_{loc}(\Omega)$ (see [20, Section 3]).

Now, given a section $S := S_\varphi(x_0, t)$ with $S_\varphi(x_0, K_1K^*t) \subset\subset \Omega$ and $h \in C^1(S_\varphi(x_0, K_1K^*t))$ (where $K_1 > 1$ is the geometric constant from the Poincaré inequality (1.17) in Theorem 1.1 and $K^* > 1$ is the geometric constant from 2.13), by applying (1.17) to the section $S^* := S_\psi(\nabla\varphi(x_0), t)$ and the function $u(y) := h(\nabla\psi(y))$ we get

$$\left(\fint\limits_{S^*} |u(y) - u^{\mu_\psi}_{S^*}|^q \, d\mu_\psi(y) \right)^{\frac{1}{q}} \leq K_2\, t^{\frac{1}{2}} \left(\fint\limits_{K_1 S^*} |\nabla^\psi u(y)|^2 \, d\mu_\psi(y) \right)^{\frac{1}{2}}, \tag{5.1}$$

where $q = \frac{2n}{n-1} + \epsilon_1$ and $u^{\mu_\psi}_{S^*} := \fint_{S^*} u(y)\, d\mu_\psi(y)$. Now, by changing variables $y = \nabla\varphi(x)$, using the second inclusion in (2.13), reasoning as in (4.10), and noticing that $\det D^2\psi(y) \det D^2\varphi(x) = 1$ for a.e. $x \in \Omega$, the integral on the right-hand side of (5.1) can be controlled by

$$\int\limits_{K_1 S^*} |\nabla^\psi u(y)|^2 \, d\mu_\psi(y) \leq \int\limits_{S_\varphi(x_0, K_1K^*t)} |\nabla^\varphi h(x)|^2 \, dx, \tag{5.2}$$

while, due to the first inclusion in (2.13), the integral on the left-hand side of (5.1) can be bound from below by the integral over $\nabla\varphi(S_\varphi(x_0, t/K^*))$.

On the other hand, the inclusions in (2.13) and the doubling property of the Lebesgue measure give

$$\mu_\psi(S^*) = |\nabla\psi(S^*)| \sim |S_\varphi(x_0, t)|, \tag{5.3}$$

where the implicit constants are geometric constants. Thus, the Poincaré inequality (1.18) follows, with $K_3 := K_1(K^*)^2 > 1$, from (5.1), (5.2), and (5.3). □

6. Proofs of Theorems 1.3 and 1.4

Let us start by proving the following improvement on Theorem E from Section 4.

Theorem 6.1. *Fix an open convex set $U \subset \mathbb{R}^n$ and $\phi \in C^2(U)$ with $D^2\phi > 0$ in U and $\mu_\phi \in A_\infty(U, \delta_\phi)$. Then, there exist constants $N_1, \epsilon > 0$, depending only on the constants from $\mu_\phi \in A_\infty(U, \delta_\phi)$ and dimension n, such that for every section $S := S_\phi(x_0, t)$ with $S_\phi(x_0, t) \subset\subset U$ and every $h \in C^1(S)$ the following $(1, 2-\epsilon)$-Poincaré holds true in the Monge-Ampère quasi-metric structure with respect to the Lebesgue measure*

$$\fint_S |h(x) - h_S|\, dx \le N_1\, t^{\frac{1}{2}} \left(\fint_S |\nabla^\phi h(x)|^{2-\epsilon}\, dx \right)^{\frac{1}{2-\epsilon}}, \tag{6.1}$$

where $h_S := \fint_S h(x)\, dx$.

Proof. The proof of Theorem 6.1 goes along the lines of the proof of [16, Theorem 1.3]. We will follow the notation in [4, Section 1] regarding the normalization technique of a given section $S := S_\varphi(x_0, t)$. Thus, let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an affine transformation such that $B(0, n^{-3/2}) \subset T(S) \subset B(0, 1)$. In particular, $\alpha_n \le |S||\det T| \le \beta_n$ for some positive dimensional constants α_n, β_n. Always as in [4, Section 1], let $\lambda > 0$ and ϕ^* be defined by

$$\lambda^n := \frac{\mu_\phi(S)}{|\det T|} \quad \text{and} \quad \phi^*(y) := \frac{1}{\lambda}\phi(T^{-1}y) - \bar{l}(y) - \frac{t}{\lambda},$$

where $\bar{l}$ is a linear function, so that $\overline{\mu}(y) = \det D^2\phi^*(y)$ with $\overline{\mu}(T(S)) = 1$. We will also use the fact that

$$C_3 t \le \lambda \le C_4 t, \tag{6.2}$$

where $C_3, C_4 > 0$ depend on the doubling constant from $\mu_\phi \in DC(U, \delta_\phi)$ and the dimension n (see Theorem 8 in [7]). From the definition of ϕ^* we get

$$T^t D^2\phi^*(y) T = \frac{1}{\lambda} D^2\phi(T^{-1}y) \tag{6.3}$$

and from the first few lines of the proof of Theorem 2 in [4] or Lemma 3.2.1 in [11] or Lemma 3.2 in [5], there exists a constant $C_5 > 0$, also depending only on the doubling constants from $\mu_\phi \in DC(U, \delta_\phi)$ and dimension n, such that

$$\fint_{T(S)} \Delta\phi^*(y)\, dy \le C_5. \tag{6.4}$$

Now, by $W^{2,1+\epsilon_0}$-estimates in [6, Theorem 2] and [16, Lemma 3.1] when $\mu_\phi \in A_\infty(U, \delta_\phi)$, there exist constants $C_6 > 1$ and $0 < \epsilon_0 < 1$, depending only on the constants from $\mu_\phi \in A_\infty(U, \delta_\phi)$ and dimension n, such that

$$\left(\fint_{T(S)} \Delta\phi^*(y)^{1+\epsilon_0}\, dy \right)^{\frac{1}{1+\epsilon_0}} \le C_6 \fint_{T(S)} \Delta\phi^*(y)\, dy \tag{6.5}$$

Then, given $h \in C^1(S)$ let $\bar{u} \in C^1(T(S))$ be defined as $\bar{u}(y) = h(T^{-1}y)$. Thus, the usual $(1,1)$-Poincaré inequality applied to $\bar{u}$ on the convex set $T(S)$ (recall that $B(0, n^{-3/2}) \subset T(S) \subset B(0,1)$) yields

$$\fint_{T(S)} |\bar{u}(y) - \bar{u}_{T(S)}|\, dy \le C_n \fint_{T(S)} |\nabla\bar{u}(y)|\, dy, \tag{6.6}$$

where $C_n > 0$ is a dimensional constant, and by changing variables $y = Tx$ in (6.6) we obtain

$$\fint_S |h(x) - h_S|\, dx \le C_n \fint_S |(T^{-1})^t \nabla h(x)|\, dx. \tag{6.7}$$

Next, notice that from the identity (6.3) and the fact that $\|D^2\phi\| \le \Delta\phi$ we get

$$\|(T^{-1})^t D^2\phi(x) T^{-1}\| \le \lambda\Delta\phi^*(Tx),$$

which followed by the simple matrix identity

$$\|(T^{-1})^t D^2\phi(x)^{\frac{1}{2}}\|^2 = \|(T^{-1})^t D^2\phi(x) T^{-1}\|,$$

gives $\|(T^{-1})^t D^2\phi(x)^{\frac{1}{2}}\|^2 \le \lambda\Delta\phi^*(Tx)$. Consequently,

$$\begin{aligned} \left(\fint_S \|(T^{-1})^t D^2\phi(x)^{\frac{1}{2}}\|^{2(1+\epsilon_0)}\, dx \right)^{\frac{1}{1+\epsilon_0}} &\le \lambda \left(\fint_{T(S)} \Delta\phi^*(y)^{1+\epsilon_0}\, dy \right)^{\frac{1}{1+\epsilon_0}} \\ &\le C_5 C_6 \lambda, \end{aligned}$$

where the last inequality follows from (6.5) and (6.4). Finally, by setting $p := 2(1+\epsilon_0)$ and recalling that $\nabla^\phi h = D^2\phi^{-\frac{1}{2}}\nabla h$,

$$\begin{aligned} &\fint_S |(T^{-1})^t \nabla h(x)|\, dx = \fint_S |(T^{-1})^t D^2\phi(x)^{\frac{1}{2}} D^2\phi(x)^{-\frac{1}{2}} \nabla h(x)|\, dx \\ &\le \fint_S \|(T^{-1})^t D^2\phi(x)^{\frac{1}{2}}\| |D^2\phi(x)^{-\frac{1}{2}} \nabla h(x)|\, dx \\ &\le \left(\fint_S \|(T^{-1})^t D^2\phi(x)^{\frac{1}{2}}\|^p\, dx \right)^{\frac{1}{p}} \left(\fint_S |\nabla^\phi h(x)|^{p'}\, dx \right)^{\frac{1}{p'}} \\ &\le (C_5 C_6 \lambda)^{\frac{1}{2}} \left(\fint_S |\nabla^\phi h(x)|^{p'}\, dx \right)^{\frac{1}{p'}} \le (C_4 C_5 C_6 t)^{\frac{1}{2}} \left(\fint_S |\nabla^\phi h(x)|^{p'}\, dx \right)^{\frac{1}{p'}}, \end{aligned}$$

where $p' = 2 - \epsilon$ with $\epsilon := 2\epsilon_0/(1+2\epsilon_0) \in (0,1)$, which combined with (6.7) proves (6.1). $\square$

6.1. Proof of Theorem 1.3

The proof of Theorem 1.3 now follows along the lines of the proof of Theorem 1.1. First, Theorem 6.1 (used in lieu of Theorem E from Section 4) implies a version of Theorem 4.1 where the exponent 2 on the right-hand side of (4.2) can be replaced by $2-\epsilon$. Recall that the A_∞ property is qualitatively preserved under convex conjugation (see Section 2.4). It is also quantitatively preserved by the approximations φ_ε due to the fact that φ_ε and $\nabla\varphi_\varepsilon$ converge uniformly on compact sets. Set $\epsilon_0 := \min\{\varepsilon, \epsilon\}$ with $\varepsilon > 0$ the geometric constant from (2.10). Then, just as in Section 4.1, Theorem 3.2 applied with μ as the Monge-Ampère measure, $p = 2 - \epsilon_0$, and $s = n - \epsilon_0$, yields with

$$q_0 := \frac{2(n-\epsilon_0)(2-\epsilon_0)}{2(n-\epsilon_0) - (2-\epsilon_0)},$$

and (1.19) follows. □

6.2. Proof of Theorem 1.4

The proof of Theorem 1.4 goes just like the one of Theorem 1.2, where (instead of using Theorem 1.1) we use Theorem 1.3 with ψ, the convex conjugate of φ, and then change variables $y = \nabla\varphi(x)$. □

7. Proofs of Theorems 1.5 and 1.6

Since $\varphi \in W^{2,n}_{loc}(\Omega, dx)$ is a strictly convex function with $\det D^2\varphi \in A_1(\Omega, \delta_\varphi)$, from [21, Section 4] we have that there exists a structural constant $M_1 > 0$ such that the Monge-Ampère measure satisfies the growth condition

$$\frac{\mu_\varphi(S_\varphi(x_0,t))}{\mu_\varphi(S_\varphi(x_0,t'))} \le M_1 \left(\frac{t}{t'}\right)^{\frac{n}{2}}, \tag{7.1}$$

for every section $S_\varphi(x_0,t)$ with $S_\varphi(x_0,\Theta_1 t) \subset \Omega$ and every $0 < t' < t$. Therefore, Theorem 3.2 applied with $\mu = \mu_\varphi$, $p = 2$ (the right-hand side exponent from the Poincaré inequality from Theorem 4.1), and $s = n/2$ (the growth exponent from (7.1)), yields

$$q = \frac{2sp}{2s-p} = \frac{2n}{n-2},$$

which is finite in the case $n \ge 3$, and (1.21) follows.

On the other hand, let $\epsilon_0 > 0$ be the structural constant from Theorem 1.3 so that for every $0 < \epsilon \le \epsilon_0$, the inequality (1.19) implies (since $q > 2$)

$$\fint_S |u(x) - u_S^{\mu_\varphi}|\, d\mu_\varphi(x) \le K_6\, t^{\frac{1}{2}} \left(\fint_{K_5 S} |\nabla^\varphi u(x)|^{2-\epsilon}\, d\mu_\varphi(x)\, dx\right)^{\frac{1}{2-\epsilon}}. \tag{7.2}$$

Now we use Theorem 3.2 applied with $\mu = \mu_\varphi$, $p = 2 - \epsilon$ (the right-hand side exponent from (7.2)), and $s = n/2$ (the growth exponent from (7.1)) to obtain the inequality (1.22) with

$$q = \frac{2sp}{2s-p} = \frac{n(2-\epsilon)}{n-(2-\epsilon)}. \tag{7.3}$$

Notice that in the case $n = 2$, the expression for q in (7.3) reduces to $q = 2(2-\epsilon)/\epsilon$ and (1.23) follows. □

8. Proofs of Theorems 1.7 and 1.8

Since $\varphi \in W^{2,n}_{loc}(\Omega, dx)$ is a strictly convex function with $\det D^2\varphi \in RH_\infty(\Omega, \delta_\varphi)$, from [21, Section 3] we now have that there exists a structural constant $M_\infty > 0$ such that the Lebesgue measure satisfies the growth condition

$$\frac{|S_\varphi(x_0,t)|}{|S_\varphi(x_0,t')|} \le M_\infty \left(\frac{t}{t'}\right)^{\frac{n}{2}}, \tag{8.1}$$

for every section $S_\varphi(x_0,t)$ with $S_\varphi(x_0,\Theta_\infty t) \subset \Omega$ and every $0 < t' < t$.

Hence, the proofs of Theorems 1.7 and 1.8 follow as the ones of Theorems 1.5 and 1.6. Indeed, the same reasoning from Section 7 but now using Theorem 1.2 instead of Theorem 4.1, and Theorem 1.4 instead of Theorem 1.3, as well as using Theorem 3.2 with the Lebesgue measure instead of μ_φ (but always with $s = n/2$ as in (8.1)) yields (1.24), (1.25), and (1.26). $\square$

9. Sobolev inequalities

In this section we point out that from each one of the Poincaré inequalities in Sections 1.1–1.4 a corresponding Sobolev inequality can be obtained. Indeed, it is a well-known fact (see for instance [1, Theorem 5.51]) that weak (q,p)-Poincaré inequalities with respect to a reverse-doubling measure imply (q,p)-Sobolev ones. For the sake of completeness, we briefly sketch the proof. Given a section $S := S_\varphi(x,t) \subset\subset \Omega$, $u \in \mathrm{Lip}_c(S)$, that is, $u \in \mathrm{Lip}(S)$ with compact support within S, and $q \ge 1$, we have

$$\begin{aligned} |u^\mu_{2S}| &\le \fint_{2S} |u|\chi_S \, d\mu \le \left(\fint_{2S} |u|^q \, d\mu\right)^{\frac{1}{q}} \left(\frac{\mu(S)}{\mu(2S)}\right)^{1-1/q} \\ &\le \left(\fint_{2S} |u|^q \, d\mu\right)^{\frac{1}{q}} \xi^{1-1/q}, \end{aligned}$$

where $\xi \in (0,1)$ is the constant from the reverse-doubling property in Lemma 2.1 corresponding to $\alpha = 1/2$. On the other hand, since

$$\left(\fint_{2S} |u|^q \, d\mu\right)^{\frac{1}{q}} \le \left(\fint_{2S} |u - u^\mu_{2S}|^q \, d\mu\right)^{\frac{1}{q}} + |u^\mu_{2S}|,$$

it then follows that

$$\left(\fint_{2S} |u|^q \, d\mu\right)^{\frac{1}{q}} \le \frac{1}{1-\xi^{1-1/q}} \left(\fint_{2S} |u - u^\mu_{2S}|^q \, d\mu\right)^{\frac{1}{q}},$$

which combined with an arbitrary weak (q,p)-Poincaré inequality

$$\left(\fint_{2S} |u - u^\mu_{2S}|^q \, d\mu\right)^{\frac{1}{q}} \le C_P t^{\frac{1}{2}} \left(\fint_{2\lambda S} |\nabla^\varphi u|^p \, d\mu\right)^{\frac{1}{p}},$$

for some $\lambda \geq 1$, and recalling that u is supported in S, yields the Sobolev inequality

$$\left(\fint_S |u|^q \, d\mu\right)^{\frac{1}{q}} \leq \frac{C_P t^{\frac{1}{2}}}{1-\xi^{1-1/q}} \left(\fint_S |\nabla^\varphi u|^p \, d\mu\right)^{\frac{1}{p}}. \tag{9.1}$$

As an illustration and for future reference, we state the Sobolev inequalities that follow from the Poincaré inequalities in Theorems 1.5 and 1.6 of Section 1.3.

Theorem 9.1. *Fix $n \geq 3$ and let $\varphi \in W^{2,n}_{loc}(\Omega, dx)$ be a strictly convex function with $\det D^2\varphi \in A_1(\Omega, \delta_\varphi)$. Then, there exist structural constants $K_9, K_{10} \geq 1$ such that for every section $S := S_\varphi(x_0, t)$ with $S_\varphi(x_0, K_9 t) \subset\subset \Omega$ and every $u \in \mathrm{Lip}_c(S)$ we have*

$$\left(\fint_S |u(x)|^{\frac{2n}{n-2}} \, d\mu_\varphi(x)\right)^{\frac{n-2}{2n}} \leq K_{10}\, t^{\frac{1}{2}} \left(\fint_S |\nabla^\varphi u(x)|^2 \, d\mu_\varphi(x)\right)^{\frac{1}{2}}. \tag{9.2}$$

In addition, there exists a structural constant $\epsilon_0 > 0$ such that for every $0 < \epsilon \leq \epsilon_0$ there is a constant $K_\epsilon > 0$, depending only on ϵ and structural constants, such that

$$\left(\fint_S |u(x)|^{q_\epsilon} \, d\mu_\varphi(x)\right)^{\frac{1}{q_\epsilon}} \leq K_\epsilon\, t^{\frac{1}{2}} \left(\fint_S |\nabla^\varphi u(x)|^{2-\epsilon} \, d\mu_\varphi(x)\right)^{\frac{1}{2-\epsilon}}, \tag{9.3}$$

with $q_\epsilon := \frac{n(2-\epsilon)}{n-(2-\epsilon)} > 2$.

Theorem 9.2. *Assume $n = 2$ and let $\varphi \in W^{2,2}_{loc}(\Omega)$ be a strictly convex function with $\det D^2\varphi \in A_1(\Omega, \delta_\varphi)$. Then, there exist structural constants $K_9 \geq 1$ and $0 < \epsilon_0 < 1$, such that for every section $S := S_\varphi(x_0, t)$ with $S_\varphi(x_0, K_9 t) \subset\subset \Omega$, every $u \in \mathrm{Lip}_c(S)$, and every $0 < \epsilon \leq \epsilon_0$ we have*

$$\left(\fint_S |u(x)|^{q_\epsilon} \, d\mu_\varphi(x)\right)^{\frac{1}{q_\epsilon}} \leq K_\epsilon\, t^{\frac{1}{2}} \left(\fint_S |\nabla^\varphi u(x)|^{2-\epsilon} \, d\mu_\varphi(x)\right)^{\frac{1}{2-\epsilon}}, \tag{9.4}$$

with $q_\epsilon := 2(2-\epsilon)/\epsilon$ and $K_\epsilon > 0$ depends only on ϵ and structural constants.

Remark 5. Notice that Theorem 9.2 extends Proposition 2.6 in [13], that is, Theorem C from the Introduction, by weakening the assumption $\det D^2\varphi \sim 1$, in the sense of (1.8), to $\det D^2\varphi \in A_1(\Omega, \delta_\varphi)$.

Remark 6. Poincaré and Sobolev inequalities such as the ones in Theorems 1.5, 1.6, and Theorems 9.1, 9.2, respectively, play a central role in the implementation of Moser's iterations for solutions to the linearized Monge-Ampère equation, as described in [19, Section 2.4].

10. Examples and applications

We close this article by recording a list of examples from [21] of convex functions φ with $\det D^2\varphi \in A_1(\Omega, \delta_\varphi)$ or $\det D^2\varphi \in RH_\infty(\Omega, \delta_\varphi)$ and by discussing further applications and connections of Theorems 1.1–1.8 (as well as their corresponding Sobolev inequalities) with related inequalities in the existing literature.

Let us start by listing the following examples from [21].

Examples of $\det D^2\varphi \in A_1(\Omega, \delta_\varphi)$**.**

(A1) The case $\det D^2\varphi \sim 1$ in Ω in the sense of (1.8). Here $\Theta_1 = 1$ and $H_1 = \Lambda_2/\Lambda_1$.
(A2) The case $\det D^2\varphi \sim |q|^{-a}$ in Ω with $q = q(x)$ polynomial and $0 < a < 1/\deg(q)$. Here $\Theta_1 = 1$ and $H_1 \geq 1$ depends only on a, dimension n, and deg(q), the degree of q (and not on its coefficients).
(A3) The case $\varphi_p(x) := \frac{1}{p}|x|^p$, $x \in \mathbb{R}^n$ and $2 - 1/n < p \leq 2$. Here $\Theta_1 = 1$ and $H_1 \geq 1$ depends only on p and n.
(A4) The case $\varphi_P(x) := \sum_{j=1}^{n} \frac{1}{p_j(p_j-1)}|x_j|^{p_j}$ with $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $P := (p_1, \ldots, p_n) \in (1, 2]^n$. Here $\Theta_1 = 1$ and $H_1 \geq 1$ depends only on $p_1, \ldots, p_n$, and n.

Examples of $\det D^2\varphi \in RH_\infty(\Omega, \delta_\varphi)$**.**

(RH1) The case $\det D^2\varphi \sim 1$ in Ω. As before, here $\Theta_\infty = 1$ and $H_\infty = \Lambda_2/\Lambda_1$.
(RH2) The case $\det D^2\varphi \sim |q|^a$ with $q = q(x)$ polynomial and $a > 0$. Here $\Theta_\infty = 1$ and $H_\infty \geq 1$ depends only on a, n, and the degree of q (and not on its coefficients).
(RH3) The case when φ is a convex polynomial in $\mathbb{R}^n$. Here $\Theta_\infty = 1$ and $H_\infty \geq 1$ depends only on n and the degree of φ (and not on its coefficients).
(RH4) The case $\varphi_p(x) := \frac{1}{p}|x|^p$ with $2 \leq p < \infty$. Here $\Theta_\infty = 1$ and $H_\infty \geq 1$ depends only on p and n.
(RH5) The case $\varphi_P(x) := \sum_{j=1}^{n} \frac{1}{p_j(p_j-1)}|x_j|^{p_j}$ with $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $P := (p_1, \ldots, p_n) \in [2, \infty)^n$. Here $\Theta_\infty = 1$ and $H_\infty \geq 1$ depends only on $p_1, \ldots, p_n$, and n.

As mentioned in Remark 6, Theorems 1.1–1.8 will find applications in the implementation of Moser's iterations for certain degenerate/singular PDEs. Also, Remarks 2 and 5 point out how they improve upon a few previously known results. In addition, in view of the examples above, Theorems 1.1–1.8 give rise to a large variety of new or improved Poincaré and Sobolev inequalities some of which complement or extend inequalities from the existing literature. As an illustration, in this section we take a look at just a couple of such inequalities. We start by mentioning the following Sobolev inequality by Tian and Wang in [23] when φ is a strictly convex polynomial in $\mathbb{R}^n$.

Theorem F *(Theorem 1.1 in [23]). Let φ be a strictly convex polynomial in $\mathbb{R}^n$, $n \geq 3$. Then, for any bounded domain $\Omega \subset B_R(0)$ and any function $u \in C_0^\infty(\Omega)$,*

$$\left(\int_\Omega |u(x)|^p \, d\mu_\varphi(x) \right)^{\frac{1}{p}} \leq C \left(\int_\Omega |\nabla^\varphi u(x)|^2 \, d\mu_\varphi(x) \right)^{\frac{1}{2}}, \tag{10.1}$$

where $p > 2$ depends on n and φ and C also depends on R.

Now, Theorem 1.7 and Example (RH3) provide a Poincaré inequality with respect to the Lebesgue measure which, in turn, yields a related Sobolev inequality (as described in Section 9) that complements Theorem F where the Monge-Ampère measure is replaced with Lebegue measure and with a finer tuning on the constants.

Theorem 10.1. *Fix $n \geq 3$ and let φ be a strictly convex polynomial with $\|(D^2\varphi)^{-1}\| \in L^1_{loc}(\mathbb{R}^n, dx)$. Then, there exist constants $K_{11}, K_{12} \geq 1$, depending only on the degree of φ and dimension n, such that for every section $S := S_\varphi(x_0, t)$ we have*

$$\left(\fint_S |u(x)-u_S|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{2n}} \le K_{12}\,t^{\frac12}\left(\fint_{K_{11}S} |\nabla^{\varphi}u(x)|^2\,dx\right)^{\frac12}$$

for every $u \in \mathrm{Lip}(K_{11}S)$*, as well as*

$$\left(\fint_S |u(x)|^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{2n}} \le K_{12}\,t^{\frac12}\left(\fint_S |\nabla^{\varphi}u(x)|^2\,dx\right)^{\frac12}$$

for every $u \in \mathrm{Lip}_c(S)$.

On the other hand, given a vector $A = (a_1, \dots, a_n) \in \mathbb{R}^n$, with $a_j \ge 0$ for every $j = 1, \dots, n$, Cabré and Ros-Oton in [2] proved the following Sobolev inequality.

Theorem G *(Theorem 1.3(a) in [2]). Given* $1 \le p < D := n + a_1 + \cdots + a_n$*, there exists* $C_p > 0$ *such that for every* $u \in C_c^1(\mathbb{R}^n)$

$$\left(\int_{\mathbb{R}^n_*} |u(x)|^{p_*} x^A\,dx\right)^{\frac{1}{p_*}} \le C_p\left(\int_{\mathbb{R}^n_*} |\nabla u(x)|^p x^A\,dx\right)^{\frac1p}, \tag{10.2}$$

where $p_* := \frac{pD}{D-p}$, $x^A := \prod_{j=1}^n |x_j|^{a_j}$*, and*

$$\mathbb{R}^n_* := \{(x_1, \dots, x_n) : \text{ with } x_j > 0 \text{ whenever } a_j > 0\}.$$

By means of the Poincaré inequalities from Section 1.3 and Example (A4) we will next obtain Poincaré and Sobolev inequalities related to the weight x^A as in (10.2) but now in the case $-1/n < a_j \le 0$ for every $j = 1, \dots, n$. Indeed, for $-1/n < a_j \le 0$ set $p_j := 2 + a_j \in (1, 2]$ and

$$\varphi_P(x) := \sum_{j=1}^n \frac{1}{p_j(p_j-1)}|x_j|^{p_j}, x = (x_1, \dots, x_n) \in \mathbb{R}^n, \tag{10.3}$$

as in Example (A4). Then

$$D^2\varphi_P(x) = \begin{bmatrix} |x_1|^{a_1} & 0 & \cdots & 0 \\ 0 & |x_2|^{a_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & |x_n|^{a_n} \end{bmatrix},$$

so that $\det D^2\varphi_P(x) = \prod_{j=1}^n |x_j|^{a_j} = x^A$. Notice that the condition $-1/n < a_j \le 0$ for every $j = 1, \dots, n$ guarantees that $\varphi_P \in W^{2,n}_{loc}(\mathbb{R}^n, dx)$. Also, for a.e. $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $u \in C^1(\mathbb{R}^n)$,

$$\begin{aligned}\nabla^{\varphi_P}u(x) &= D^2\varphi_P(x)^{-\frac12}\nabla u(x) \\ &= (|x_1|^{-\frac{a_1}{2}}u_1(x), \cdots, |x_n|^{-\frac{a_n}{2}}u_n(x))\end{aligned}$$

and consequently

$$|\nabla^{\varphi_P} u(x)| = \left(\sum_{j=1}^{n} |x_j|^{-a_j} |u_j(x)|^2 \right)^{\frac{1}{2}}.$$

Moreover, by [9, Lemma 6] the Monge-Ampère sections of φ_P are related to the ones of $\varphi_{p_j}(x) := \frac{1}{p_j}|x|^{p_j}$, $x \in \mathbb{R}$, by means of the inclusions

$$S_{\varphi_P}(y,t) \subset S_{\varphi_{p_1}}(y_1,t) \times \cdots \times S_{\varphi_{p_n}}(y_n,t) \subset S_{\varphi_P}(y,nt),$$

for every $y = (y_1, \dots, y_n) \in \mathbb{R}^n, y_j \in \mathbb{R}, j = 1, \dots, n$, and $t > 0$.

Therefore, by using, for instance, Theorem 1.5 from Section 1.3, we obtain

Theorem 10.2. *Fix $n \geq 3$ and let φ_P be the strictly convex function defined in (10.3). Then, there exist constants $K_9, K_{10} \geq 1$, depending only on $a_1, \dots, a_n \in (-1/n, 0]$ and dimension n, such that for every section $S := S_{\varphi_P}(x_0, t)$ we have*

$$\left(\fint_S |u(x) - u_S^{\mu_{\varphi_P}}|^{\frac{2n}{n-2}} \, d\mu_{\varphi_P}(x) \right)^{\frac{n-2}{2n}} \leq K_{10} \, t^{\frac{1}{2}} \left(\fint_{K_9 S} |\nabla^{\varphi_P} u(x)|^2 \, d\mu_{\varphi_P}(x) \right)^{\frac{1}{2}}$$

for every $u \in \mathrm{Lip}(K_9 S)$, as well as

$$\left(\fint_S |u(x)|^{\frac{2n}{n-2}} \, d\mu_{\varphi_P}(x) \right)^{\frac{n-2}{2n}} \leq K_{10} \, t^{\frac{1}{2}} \left(\fint_S |\nabla^{\varphi_P} u(x)|^2 \, d\mu_{\varphi_P}(x) \right)^{\frac{1}{2}}$$

for every $u \in \mathrm{Lip}_c(S)$, where $d\mu_{\varphi_P}(x) = x^A \, dx$.

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