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On stability of nonlinear nonautonomous discrete fractional Caputo systems

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A B S T R A C T

Conditions to establish Mittag-Leffler stability of solutions for nonlinear nonautonomous discrete Caputo-like fractional systems just from the linear associated system is shown. Mittag-Leffler stability for linear systems is tackled pointing out properties the matrix must satisfy. Additionally features on solutions for linear systems are included in order to establish the equivalence between uniform asymptotical stability and exponential stability. Converse-like-Lyapunov Theorem is developed and a particular result to show Mittag-Leffler stability from a condition on a linear part of the full system is worked.

1. Introduction

Fractional calculus have been intensively developed in recent years and there is a remarkable growth of applications to physics, economy, biology, engineering and sciences in general. Particularly the theory of fractional differential equations has been widely studied in recent years; we refer the reader to [13] and [26] as basic monographs to get introduced in the topic. The discrete counterpart, fractional difference equations, also has been studied and applied in many fields recently but unlike the continuous case, there are still many issues to develop in the theory.

A fractional difference equation could be given in different forms, depends on the fractional difference operator. Among the best known are Caputo, Riemman-Liouville and Grüwald-Letnikov-type operators. In papers like [1] there are results on Caputo and Riemman-Liouville-type operators and relations between them. Also it shows solution to one dimensional Caputo-like difference linear equations in terms of Mittag-Leffler functions. In this fashion, paper [15] analyses the stability under the three scheme operators

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mentioned before for any fractional positive order, doing a reduction to a multi-order equation with orders in the interval $(0, 1)$. Solutions to fractional linear difference equations with these operators are discussed in [14], via the classical $\mathcal{Z}$-transform.

This paper focuses on nonlinear Caputo-like fractional difference equations and in this subject F. Chen et al. [8] give results on the existence and uniqueness of the inial value problem. F. Chen [7] studies the asymptotic stability by applying the Schauder fixed point theorem and a discrete version of Arzela-Acoli's theorem. Asymptotic results for 1-dimensional nonautonomous equations are given in [5]. Recent papers [4,3,25] are applications in biology and economics. In [22,17,2] are analyzed synchronization, control methods and higher order equations using the Caputo difference operator and special interest has been the presence of chaos [11,19–21]

As in the classical theory of continuous and discrete dynamical systems that the Lyapunov method gives conditions to establish stability (uniform stability) and asymptotical stability (uniform asymptotical stability) to nonlinear nonautonomous equations, some papers tackle this point. F. Jarad et al. [12] extend the Lyapunov method to Caputo-like fractional difference equations in terms of $\mathcal{K}$-class functions. D. Baleanu et al. [6] gives the Lyapunov method to an implicit version of the Caputo-like fractional difference equation, that is the t-th step depends on itself and on all past steps. M. Wyrwas et al. [23] and Xiang L. et al. [24] modify the Lyapunov method to establish conditions to Mittag-Leffler stability, for h-difference equations.

The goal of this paper is to give conditions to establish uniform stability, uniform asymptotical stability and Mittag-Leffler stability to a nonlinear and nonautonomous Caputo-like fractional difference equation. These conditions are on the linear part of this equation, just as in the classical theory of differential and difference equations of integer order but unknown in the fractional case as far as we have knowledge. A converse-like-Lyapunov Theorem is needed to get the goal, namely asymptotical stability ensures the existence of a Lypaunov function. Also we develop useful results about the behavior of solutions of a nonautonomous linear fractional difference equation with Caputo's operator. Moreover, it is sufficient to tackle a partial part of the associated linear part to state Mittag-Leffler stability. In summary the main result in this manuscript says that linear local behavior does carry over to the full system in suitable cases.

This text is organized by sections, starting with some preliminaries and definitions in order to have a self-containing reading. In section 3 we introduce some results on the stability for linear nonautonomous equations, on asymptotical behavior of solutions and we write a reverse theorem of the Lyapunov direct method, exponential stability implies the existence of a Lyapunov function. In section 4 we present our main result, conditions on the linear part of the equation that ensure Mittag-Leffler stability of the nonlinear equation. Finally we illustrate the power of the main results via an example in section 5.

2. Preliminaries

Let us start introducing some basic definitions about discrete fractional calculus. For $a \in \mathbb{R}$, the functions we deal with are defined in the Banach space $S(\mathbb{N}_a, \mathbb{R}^n)$, the set of all functions $u : \mathbb{N}_a \to \mathbb{R}^n$, whose domain is the discrete set $\mathbb{N}_a := \{a, a+1, a+2, \ldots\}$ and norm $||\cdot||$ [10]. The forward operator $\sigma : \mathbb{N}_a \to \mathbb{N}_a$ given as $\sigma(t) := t+1$ and the corresponding forward difference operator is $\Delta\mathbf{u}(t) = \mathbf{u}(\sigma(t)) - \mathbf{u}(t)$. Iteratively we get

$$\Delta^m \mathbf{u}(t) = \Delta\left(\Delta^{m-1}\mathbf{u}(t)\right) = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k}\mathbf{u}(t+k)\,.$$

Definition 1. Let $\mathbf{u} \in S(\mathbb{N}_a, \mathbb{R}^n)$ and $v > 0$ be given, the v-fractional sum of $\mathbf{u}$ is defined by

$$\Delta_a^{-v}\mathbf{u}(t) = \frac{1}{\Gamma(v)} \sum_{s=a}^{t-v} (t - \sigma(s))^{(v-1)}\mathbf{u}(s)$$

where a is the starting point, Γ is the function gamma, as an extension of the factorial function, and

$$t^{(v)} = \frac{\Gamma(t+1)}{\Gamma(t+1-v)}$$

corresponds to the falling function.

Thus $\Delta_a^{-v} : S(\mathbb{N}_a, \mathbb{R}^n) \to S(\mathbb{N}_{a+v}, \mathbb{R}^n)$. In what follows, the fractional difference we assume is the Caputo derivative in vectorial form.

Definition 2. The v-order Caputo discrete fractional difference for $v > 0$ ($v \notin \mathbb{N}$) and $\mathbf{u}(t) \in S(\mathbb{N}_a, \mathbb{R}^n)$ is given by

$${}^C\Delta_a^v \mathbf{u}(t) = \Delta_a^{-(m-v)} \left(\Delta^m \mathbf{u}(t)\right)$$

where ${}^C\Delta_a^v : S(\mathbb{N}_a, \mathbb{R}^n) \to S(\mathbb{N}_{a+(m-v)}, \mathbb{R}^n)$ is the Caputo delta operator and the order satisfies $m-1 < v < m$. That is

$${}^C\Delta_a^v \mathbf{u}(t) = \frac{1}{\Gamma(m-v)} \sum_{s=a}^{t-(m-v)} (t-\sigma(s))^{(m-v-1)} \Delta^m \mathbf{u}(s)$$

where $t \in \mathbb{N}_{a+m-v}$.

From [8] we have a Taylor's difference formula for the Caputo discrete fractional difference:

$$\mathbf{u}(t) = \sum_{k=0}^{m-1} \frac{(t-a)^{(k)}}{k!} \Delta^k \mathbf{u}(a) + \frac{1}{\Gamma(v)} \sum_{s=a+m-v}^{t-v} (t-\sigma(s))^{(v-1)}\, {}^C\Delta_a^v \mathbf{u}(s)\,,$$

for $v > 0$, v non-integer. In particular for $0 < v < 1$ we know

$$\mathbf{u}(t) = \mathbf{u}(a) + \frac{1}{\Gamma(v)} \sum_{s=a+1-v}^{t-v} (t-\sigma(s))^{(v-1)}\, {}^C\Delta_a^v \mathbf{u}(s)\,.$$

It leads us to a solution for the nonlinear fractional Caputo like difference equation initial value problem

$${}^C\Delta_a^v \mathbf{u}(t) = \mathbf{f}(\mathbf{u}(t+v-1), t+v-1)\,, \quad \mathbf{u}(a) = \mathbf{u}_o\,, \tag{1}$$

with $\mathbf{f} : S(\mathbb{N}_{v-1}, \mathbb{R}^n) \times \mathbb{N}_{v-1} \to \mathbb{R}^n$, $a = v-1$ and $v \in (0,1)$, which corresponds to

$$\mathbf{u}(t) = \mathbf{u}_o + \frac{1}{\Gamma(v)} \sum_{s=0}^{t-v} (t-\sigma(s))^{(v-1)} \mathbf{f}(\mathbf{u}(s+v-1), s+v-1)\,,$$

for $t \in \mathbb{N}_{v-1}$. In the following sections we omit in the notation the initial time for a simpler notation.

A constant solution $\hat{\mathbf{u}} \in \mathbb{R}^n$ of equation (1) is said to be a fixed point if it is such that $\mathbf{f}(\hat{\mathbf{u}}, t) = 0$ for all $t \geq 0$. Without loose of generality we assume from here in after $\hat{\mathbf{u}} = \mathbf{0}$.

We recover the following definitions from [12].

Definition 3. A fixed point $\hat{\mathbf{u}} = \mathbf{0}$ of equation (1) is said to be

1. stable if for every $\varepsilon > 0$ and $t_o \in \mathbb{N}_{1-v}$, there exists $\delta_{\varepsilon,t_o} > 0$ such that every solution $\mathbf{u}(t)$ with initial condition satisfying $||\mathbf{u}_o|| < \delta_{\varepsilon,t_o}$ implies $||\mathbf{u}(t)|| < \varepsilon$, for all $t \in \mathbb{N}_{t_o}$,
2. uniformly stable if it is stable and δ just depend on ε,
3. asymptotically stable if it is stable and for all $t_o \in \mathbb{N}_{1-v}$, there exists $\delta_{t_o} > 0$ such that $\lim_{t\to\infty} \mathbf{u}(t) = \mathbf{0}$ whenever $||\mathbf{u}_o|| < \delta_{t_o}$,
4. uniformly asymptotically stable if it is uniformly stable and, for each $\varepsilon > 0$, there exists $T_\varepsilon \in \mathbb{N}$ and $\delta > 0$ such that $||\mathbf{u}_o|| < \delta$ implies $||\mathbf{u}(t)|| < \varepsilon$ for all $t \in \mathbb{N}_{t_o+T}$ and for all $t_o \in \mathbb{N}_{1-v}$,
5. globally asymptotically stable if it is asymptotically stable for all $\mathbf{u}_o \in \mathbb{R}^n$,
6. globally uniformly asymptotically stable if it is uniformly asymptotically stable for all $\mathbf{u}_o \in \mathbb{R}^n$.

Mittag-Leffler functions, a generalization of exponential ones, are the key to understand behavior of stable solutions in long times. Definition from [14] is introduced.

Definition 4. For $\alpha, \beta, z \in \mathbb{C}$, with $\mathrm{Re}(\alpha) > 0$, the discrete Mittag-Leffler two-parameter function is given as

$$\mathcal{E}_{(\alpha,\beta)}(\lambda, z) := \sum_{k=0}^{\infty} \lambda^k \frac{(z + (k-1)(\alpha-1))^{(k\alpha)}(z + k(\alpha-1))^{(\beta-1)}}{\Gamma(\alpha k + \beta)}.$$

We write $E_{(\alpha)}(\lambda, z) := \mathcal{E}_{(\alpha,1)}(\lambda, z)$. Let us observe $E_{(1)}(\lambda, z) = e^{z \ln(1+\lambda)}$.

In fact, two types of particular stability will be introduced from [23], one generalizes the other; both useful to characterize solutions to linear and nonlinear systems.

Definition 5. A solution to (1) is said to be Mittag-Leffler stable if

$$||\mathbf{u}(t)|| \le \big(m(\mathbf{u}_o))E_{(\alpha)}(-\lambda, t)\big)^b$$

where $\alpha \in (0, 1)$, $\lambda > 0$, $b > 0$, $m(\mathbf{0}) = 0$, $m(\mathbf{u}) \ge 0$, and m locally Lipschitz on $\mathbf{u} \in \mathbb{B} \subset \mathbb{R}^n$ with Lipschitz constant m_o.

For $\alpha = 1$ the solution is said to be exponentially stable.

Also in the following section $\mathcal{K}$-class function definition will be useful from [12].

Definition 6. Let $\phi \in C([0, \rho), \mathbb{R}^+)$ a real-valued function, It is said to be of class $\mathcal{K}$ if it verifies $\phi(0) = 0$ and is strictly monotonically increasing. Moreover, if $\phi \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\phi \in \mathcal{K}$ and such that $\lim_{r\to\infty} \phi(r) = \infty$, it is said to be of class $\mathcal{KR}$.

A Lyapunov function will be the main tool to establish when either a linear or nonlinear fractional discrete system is stable and asymptotically stable.

Definition 7. A scalar function $V : S_\rho^{n-1} \times \mathbb{N}_0 \to \mathbb{R}^+$, where we assume $S_\rho^{n-1} = \{\mathbf{x} \in \mathbb{R}^n \,|\, ||\mathbf{x}|| \le \rho\}$, is said to be Lyapunov function if it is positive definite and decrescent. That is, there exist scalar $\mathcal{K}$-class functions $\phi(r)$ and $\psi(r)$, with $r = ||\mathbf{x}||$, in such a way that V must satisfy the following conditions:

- $V(\mathbf{0}, t) = 0$ for all $t \in \mathbb{N}_0$,
- $\phi(r) \le V(\mathbf{x}, t)$, for $(\mathbf{x}, t) \in S_\rho^{n-1} \times \mathbb{N}_0$ and
- $V(\mathbf{x}, t) \le \psi(r)$ for $(\mathbf{x}, t) \in S_\rho^{n-1} \times \mathbb{N}_0$.

Slightly different to the inequality showed in [6], we add a useful expression in the proves about stability results.

Proposition 1. *For $m = 1$,*

$$^{C}\Delta^{v}\mathbf{x}^{2}(t) \leq 2\mathbf{x}(t+v-1) \cdot {}^{C}\Delta^{v}\mathbf{x}(t)\,.$$

Proof.

$$\begin{aligned}
&^{C}\Delta^{v}\mathbf{x}^{2}(t) - 2\mathbf{x}(t+v-1) \cdot {}^{C}\Delta^{v}\mathbf{x}(t)\\
&= \sum_{s=0}^{t+v-1} \frac{(t-\sigma(s))^{(-v)}}{\Gamma(1-v)} \left(\Delta\mathbf{x}^{2}(s) - 2\mathbf{x}(t+v-1)\cdot\Delta\mathbf{x}(s)\right)\\
&= \frac{1}{\Gamma(1-v)} \sum_{s=0}^{t+v-1} (t-\sigma(s))^{(-v)}\Delta(\mathbf{x}(s)-\mathbf{x}(t+v-1))^{2}\\
&= \frac{1}{\Gamma(1-v)} \sum_{s=0}^{t+v-2} (t-\sigma(s))^{(-v)}\Delta(\mathbf{x}(s)-\mathbf{x}(t+v-1))^{2}
\end{aligned}$$

where the square notation refers to dot product. By using the identity

$$\sum_{s=0}^{t+v-1} g(s+1)\Delta f(s) = g(s+1)f(s+1)\Big|_{0}^{t+v-1} - \sum_{s=0}^{t+v-1} f(s)\Delta g(s)$$

where we identify $g(s+1) = (t-\sigma(s))^{(-v)}$ and $f(s) = (\mathbf{x}(s) - \mathbf{x}(t+v-1))^{2}$, the last equality becomes

$$-(t-1)^{(-v)}(\mathbf{x}(1)-\mathbf{x}(t+v-1))^{2} - \frac{1}{\Gamma(1-v)}\left(\sum_{s=0}^{t+v-1}(\mathbf{x}(s)-2\mathbf{x}(t+v-1))^{2}\Delta(t-s)^{(-v)}\right) \leq 0\,.$$

Therefore we get the proposition as it is claimed. □

The following theorems in [12] using Lyapunov functions show conditions to yield stability and asymptotical stability.

Theorem 1. *If there exists a positive definite and decrescent scalar function $V(\mathbf{x}, t) \in C(S_{\rho}^{n-1} \times \mathbb{N}_a, \mathbb{R}^{+})$ such that $^{C}\Delta_{a}^{v}V(\mathbf{x}, t) \leq \mathbf{0}$ for all $t_o \in \mathbb{N}_a$, then the trivial solution of (1) is uniformly stable.*

Theorem 2. *If there exists a positive definite and decrescent scalar function $V(\mathbf{x}, t) \in C(S_{\rho}^{n-1} \times \mathbb{N}_a, \mathbb{R}^{+})$ such that*

$$^{C}\Delta_{a}^{v}V(\mathbf{x}, t) \leq -\psi(||\mathbf{x}(t+v-1)||)$$

for all $t_0 \in \mathbb{N}_a$, where $\psi \in \mathcal{K}$, then the trivial solution of (1) is uniformly asymptotically stable.

Last theorem in this section indicates, via Lyapunov functions, Mittag Leffler stability from [23].

Theorem 3. *Let $\mathbf{x} = \mathbf{0}$ be an equilibrium point of the system (1). Let $V : S_{\rho}^{n-1} \times \mathbb{N}_a \to \mathbb{R}^{+}$ be a function that is locally Lipschitz with respect to $\mathbf{x}$ and such that*

$$\alpha_1||\mathbf{x}||^{a} \leq V(\mathbf{x}, t) \leq \alpha_2||\mathbf{x}||^{ab}\,,$$
$$^{C}\Delta_{a}^{v}V(\mathbf{x}, t) \leq -\alpha_3||\mathbf{x}||^{ab}\,,$$

where $t \in \mathbb{N}_a$, $\mathbf{x} \in S_{\rho}^{n-1}$, $v \in (0, 1)$, $\alpha_1, \alpha_2, \alpha_3, a, b > 0$. Then $\mathbf{x} = \mathbf{0}$ is Mittag-Leffler stable.

Moreover, we know $||\mathbf{u}(t)|| \leq [mE_{(v)}(-\alpha_3/\alpha_2, t)]^{1/a}$, with m as introduced in Definition 5 and

$$\lim_{t\to\infty} E_{(v)}(-\alpha_3/\alpha_2, t) = 0 \tag{2}$$

sets $-2^v < (-\alpha_3/\alpha_2)$.

3. Nonautonomous linear systems

Let us suppose $\mathbf{A}$ a nonconstant continuous matrix. In this section we consider

$${}^C\Delta^v \mathbf{x}(t) = \mathbf{A}(t+v-1)\mathbf{x}(t+v-1) \tag{3}$$

a linear fractional nonautonomous equation, where $\mathbf{x} = \mathbf{0}$ is the trivial fixed point.

Theorem 4. *Suppose the system (3), with $\mathbf{A}(t)$ a negative semidefinite matrix and suppose a bounded, symmetric and positive definite matrix $\mathbf{P}(t)$. The fixed point $\mathbf{x} = \mathbf{0}$ is uniformly stable.*

Proof. Let us define

$$V(\mathbf{x}, t) = \mathbf{x}^T \mathbf{P}(t)\mathbf{x}\,,$$

therefore $p_1||\mathbf{x}||^2 \leq V(\mathbf{x}, t) \leq p_2||\mathbf{x}||^2$ for some $p_1, p_2 > 0$.

Since $\mathbf{P}(t)$ is symmetric, following spectral theorem, we have the decomposition

$$\mathbf{P}(t) = \mathbf{U}(t)^T \mathbf{D}(t)\mathbf{U}(t)$$

where $\mathbf{U} = [\mathbf{u}_1 \dots \mathbf{u}_n]$ is an orthogonal matrix formed by the vector-columns $(\mathbf{u}_i)$, an orthonormal basis corresponding to the eigenvectors with the eigenvalues $\lambda_i(t)$, which are the ith-entries in the diagonal matrix $\mathbf{D}(t)$. In this way we can write

$$\begin{aligned} V(\mathbf{x}, t) &= \mathbf{x}^T \mathbf{U}^T(t)\mathbf{D}(t)\mathbf{U}(t)\mathbf{x} \\ &= \mathbf{y}^T(\mathbf{x}, t)\mathbf{D}(t)\mathbf{y}(\mathbf{x}, t) \\ &= \sum_i \lambda_i(t) y_i^2(\mathbf{x}, t)\,, \end{aligned}$$

where there was a change of variable $\mathbf{y} = \mathbf{U}(t)\mathbf{x}$ and y_i stands for the i-th component of $\mathbf{y}$. All eigenvalues $\lambda_i(t)$ are bounded as $\mathbf{P}(t)$ so is; in fact it holds

$$\lambda_{min}\mathbf{I} \preceq \mathbf{P}(t) \preceq \lambda_{max}\mathbf{I}$$

in Loewner order, that is $(\lambda_{max}\mathbf{I} - \mathbf{P}(t))$ and $(\mathbf{P}(t) - \lambda_{min}\mathbf{I})$ are symmetric and positive definite matrices. The constants λ_{max} and λ_{min} are defined as $\sup_{t\in\mathbb{R}}\{\lambda_i(t)\,|\,i = 1, \dots, n\}$ and $\inf_{t\in\mathbb{R}}\{\lambda_i(t)\,|\,i = 1, \dots, n\}$ respectively.

Taking into account Proposition 1, we would obtain the following inequality, where we omit dependence on vector $\mathbf{x}$ for a simpler notation,

$${}^C\Delta^v V(\mathbf{x}, t) = {}^C\Delta^v \left(\sum_i \lambda_i(t) y_i^2(t) \right)$$

$$
\begin{aligned}
&\leq \sum_i {}^{C}\Delta^v \left(\lambda_{max} y_i^2(t)\right) , \\
&\leq 2\lambda_{max} \sum_i y_i(t+v-1)^{C}\Delta^v y_i(t) \\
&= 2\lambda_{max} \mathbf{y}^T(t+v-1)^{C}\Delta^v \mathbf{y}(t) \,.
\end{aligned}
$$

From equation (3), it follows

$$
{}^{C}\Delta^v V(\mathbf{x},t) \leq 2\lambda_{max}\mathbf{y}^T(t+v-1)\mathbf{A}(t+v-1)\mathbf{y}(t+v-1) \leq 0 \,, \tag{4}
$$

being true because of the negative semidefinite character of $\mathbf{A}(t)$. All this satisfies Theorem 1. □

Corollary 1. *Under the same hypothesis in Theorem 4, but suppose* $\mathbf{A}(t)$ *being bounded and negative definite matrix, then the fixed point* $\mathbf{x} = \mathbf{0}$ *is uniformly asymptotically stable.*

Proof. It is about showing that the right hand side term in inequality (4) can be bounded by a scalar $\mathcal{K}$-class function ψ, as Theorem 2 points out. We rewrite the quadratic expression as an affine combination of $\mathbf{A}$ and $\mathbf{A}^T$ that yields

$$
\mathbf{y}^T\mathbf{A}\mathbf{y} = \mathbf{y}^T\frac{1}{2}\left(\mathbf{A}+\mathbf{A}^T\right)\mathbf{y}
$$

being $\mathbf{A}+\mathbf{A}^T$ a symmetric matrix, where we have omitted the explicit dependence on t for simplicity. Applying properties of symmetric matrices, the last equality becomes into

$$
\mathbf{y}^T\frac{1}{2}\left(\mathbf{A}+\mathbf{A}^T\right)\mathbf{y} \leq \lambda^*||\mathbf{y}||^2
$$

with $\lambda^* \in \mathbb{R}$ the maximum eigenvalue of the bounded matrix $(\mathbf{A}+\mathbf{A}^T)/2$ for every t. But $\mathbf{A}$ is a negative definite matrix therefore $\lambda^* < 0$. Therefore we set $\psi(||\mathbf{y}||) = 2\lambda_{max}\lambda^*||\mathbf{y}||^2$ in right side of (4). □

Since (3) is linear, uniform stability and uniform asymptotic stability properties stay in global sense.

Remark. Taking into account $\alpha_2 = \lambda_{max}$ and $-\alpha_3 = \lambda_*$ in Theorem 4 and Corollary 1 and that expression (2), it is imposed $\lambda^* > -1/2^{1-v}$.

We realize that inequalities in Theorem 3 are verified and therefore we have the following result.

Corollary 2. *Suppose the system (3), with* $\mathbf{A}(t)$ *a bounded and negative definite matrix and suppose a bounded, symmetric and positive definite matrix* $\mathbf{P}(t)$*. The fixed point* $\mathbf{x} = \mathbf{0}$ *is Mittag-Leffler stable.*

As in linear discrete systems of integer order [18], we introduce the equivalent of the transition matrix $\phi(t,t_o)$, that is for the system (3) we define the transition matrix as $\mathbf{x}(t) = \phi(t,t_o)\mathbf{x}(t_o)$, assuming the condition $\phi(t_o,t_o) = \mathbf{I}$.

Lemma 1. *Suppose the fixed point* $\mathbf{x} = \mathbf{0}$ *of (3) is uniformly asymptotically stable, then there exist* $M > 0$ *such that* $||\mathbf{\Phi}(t,t_o)|| \leq M$*.*

Proof. By definition, there exists $\delta_\varepsilon > 0$ for any given $\varepsilon > 0$ such that $||\mathbf{\Phi}(t,t_o)x(t_o)|| \leq \varepsilon$ provided $||\mathbf{x}(t_o)|| < \delta_\varepsilon$, for all $t \geq t_o$. We can write

$$\begin{aligned}\max_{||\mathbf{x}(t_o)||<\delta_\varepsilon} ||\mathbf{\Phi}(t,t_o)\mathbf{x}(t_o)|| &= \max_{\left(\frac{1}{\delta_\varepsilon}\right)||\mathbf{x}(t_o)||<1} ||\mathbf{\Phi}(t,t_o)\mathbf{x}(t_o)|| \\ &= \delta_\varepsilon\left(\max_{||\mathbf{y}(t_o)||<1} ||\mathbf{\Phi}(t,t_o)\mathbf{y}(t_o)||\right) \\ &\leq \delta_\varepsilon ||\mathbf{\Phi}(t,t_o)|| < \varepsilon\,,\end{aligned}$$

assuming $\mathbf{x}(t_o) = \delta_\varepsilon \mathbf{y}(t_o)$ and the induced norm. Therefore $||\mathbf{\Phi}(t,t_o)|| < M$ with $M := \varepsilon/\delta_\varepsilon$. □

The following lemmas will serve to prove the main result of this section.

Lemma 2. *In relation to the linear system (3), the following statements are equivalent:*

a) $\mathbf{x} = \mathbf{0}$ *is globally uniformly asymptotically stable,*
b) $||\mathbf{\Phi}(t,t_o)|| \to 0$ *as* $t \to \infty$ *uniformly in* t_o*,*
c) $||\mathbf{\Phi}(t,t_o)\mathbf{v}_i|| \to 0$ *as* $t \to \infty$ *uniformly in* t_o*, for* $i = 1,\ldots,n$*, where* $\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ *is a basis of* $\mathbb{R}^n$*.*

Proof. First (a) implies (c). If $||\mathbf{x}(t_o)|| < c$ for some $c > 0$, then $||\mathbf{x}(t)|| \to 0$ as $t \to \infty$, namely $||\mathbf{\Phi}(t,t_o)\mathbf{x}(t_o)|| \to 0$. Given a basis of $\mathbb{R}^n$, $\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$, we compute

$$||\mathbf{\Phi}(t,t_o)\mathbf{v}_i|| = b\left\|\mathbf{\Phi}(t;t_o)\frac{1}{b}\mathbf{v}_i\right\| \to 0$$

as $t \to \infty$ and for a suitable value b such that $||(1/b)\mathbf{v}_i|| < c$ and any $1 \leq i \leq n$.

Second we show (c) implies (b). Without lose of generality we can assume a basis $\{\mathbf{v}_i\}_i$ such that we can write

$$||\mathbf{\Phi}(t,t_o)\mathbf{v}_i|| \to 0 \quad t \to \infty$$

and by the induced norm que obtain

$$||\mathbf{\Phi}(t,t_o)|| = \max_{||\mathbf{x}(t_o)||=1} ||\mathbf{\Phi}(t,t_o)\mathbf{x}(t_o)|| \leq \sum_i c_i||\mathbf{\Phi}(t,t_o)v_i|| \to 0\,, \quad t \to \infty\,,$$

where c_i are the coefficients of the linear combination.

Finally (b) implies (a) directly by definition of the transition matrix. □

Lemma 3. *The fixed point* $\mathbf{x} = \mathbf{0}$ *of (3) is uniformly asymptotically stable if and only if* $||\mathbf{\Phi}(t,t_o)|| \leq ke^{-\lambda(t-t_o)}$ *for all* $t \geq t_o$ *and for some* $k,\lambda > 0$*.*

Proof. On one hand, if the inequality is true, we obtain uniform asymptotical stability in straightforward way.

On the other hand, supposing $\mathbf{x} = \mathbf{0}$ is uniformly asymptotically stable, so

$$||\mathbf{x}(t)|| = ||\mathbf{\Phi}(t,t_o)\mathbf{x}(t_o)|| \to 0\,, \quad t \to \infty$$

when $||\mathbf{x}(t_o)|| < c$ for some $c > 0$, $t \geq t_o$. From Lemma 2, $||\mathbf{\Phi}(t,t_o)|| \to 0$ and we have an increasing sequence $(t_1, t_2, \ldots)$ satisfying

$$
\begin{aligned}
||\mathbf{\Phi}(t,t_o)|| &< 1/2\,, \quad t \geq t_1\,,\\
||\mathbf{\Phi}(t,t_o)|| &< 1/2^2\,, \quad t \geq t_2\,,\\
&\vdots\\
||\mathbf{\Phi}(t,t_o)|| &< 1/2^n\,, \quad t \geq t_n\,,\\
&\vdots
\end{aligned}
$$

thus we define $\eta := \max\{t_{n+1} - t_n \,|\, n \in \mathbb{N} \cup \{0\}\}$ to get

$$
||\mathbf{\Phi}(t,t_o)|| < \frac{1}{2^{\frac{1}{\eta}(t-t_o)}} = e^{-\ln(2)\frac{1}{\eta}(t-t_o)}\,.
$$

Taking $\lambda = \ln(2)/\eta$ and $k = 1$, we complete de proof. $\square$

Thus we realize that asymptotic uniform stability is equivalent to exponential stability, a kind of Mittag-Leffler stability

$$
||\mathbf{x}(t)|| = ||\phi(t,t_o)\mathbf{x}_o|| \leq e^{\frac{\ln 2}{\eta}t_o}||\mathbf{x}_o||E_{(1)}\left((1/2)^{1/\eta}, t\right)\,.
$$

The following result is a converse-like-Lyapunov Theorem for this family of systems.

Theorem 5. *Let $\mathbf{x} = \mathbf{0}$ be a exponentially stable equilibrium of (3) and suppose $\mathbf{A}$ a bounded matrix. Let $\mathbf{Q}(t)$ be a bounded, positive definite and symmetric matrix. Then there is a matrix $\mathbf{P}(t)$ bounded, positive definite and symmetric, such that $V(\mathbf{x},t) = \mathbf{x}^T\mathbf{P}(t)\mathbf{x}$ is a Lyapunov function for the system.*

Proof. Let $\mathbf{P}(t) = \sum_{\tau=t}^{\infty}\mathbf{\Phi}(\tau,t)^T\mathbf{Q}(\tau)\mathbf{\Phi}(\tau,t)$, so it is bounded, positive definite and symmetric as $\mathbf{Q}$ it is, therefore

$$
\begin{aligned}
V(\mathbf{x},t) &= \mathbf{x}(t)^T\mathbf{P}(t)\mathbf{x}(t)\\
&= \mathbf{x}(t)^T\left(\sum_{\tau=t}^{\infty}\mathbf{\Phi}(\tau,t)^T\mathbf{Q}(\tau)\mathbf{\Phi}(\tau,t)\right)\mathbf{x}(t)\\
&= \sum_{\tau=t}^{\infty}\mathbf{x}(t)^T\mathbf{\Phi}(\tau,t)^T\mathbf{Q}(\tau)\mathbf{\Phi}(\tau,t)\mathbf{x}(t)\\
&= \sum_{\tau=t}^{\infty}\mathbf{y}^T(\tau)\mathbf{Q}(\tau)\mathbf{y}(\tau)\\
&\leq \sum_{\tau=t}^{\infty}q_2||\mathbf{y}(\tau)||^2\,,
\end{aligned}
$$

with $\mathbf{y}(\tau) = \mathbf{Q}(\tau,t)\mathbf{x}(t)$ and where q_2 is the supremum eigenvalue of $\mathbf{Q}(\tau)$, since it is bounded, symmetric and positive definite matrix.

Using Lemma 3 and the exponential stability property we get

$$
\begin{aligned}
\sum_{\tau=t}^{\infty}q_2||\mathbf{y}(\tau)||^2 &= \sum_{\tau=t}^{\infty}q_2||\mathbf{\Phi}(\tau,t)\mathbf{x}(t)||^2\\
&\leq q_2||\mathbf{x}(t)||^2\sum_{\tau=t}^{\infty}||\mathbf{\Phi}(\tau,t)||^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq q_2||\mathbf{x}(t)||^2 \sum_{\tau=t}^{\infty} k e^{-\lambda(\tau-t)} \\
&= q_2||\mathbf{x}(t)||^2 \frac{k}{1-e^{-\lambda}} \,.
\end{aligned}
$$

Also by other side,

$$
\begin{aligned}
V(\mathbf{x},t) &= \sum_{\tau=t}^{\infty} \mathbf{x}(t)^T \mathbf{\Phi}(\tau,t)^T \mathbf{Q}(\tau) \mathbf{\Phi}(\tau,t)\mathbf{x}(t) \\
&\geq \mathbf{x}(t)^T \mathbf{\Phi}(t,t)^T \mathbf{Q}(t) \mathbf{\Phi}(t,t)\mathbf{x}(t) \\
&= \mathbf{x}(t)^T \mathbf{Q}(t)\mathbf{x}(t) \\
&\geq q_1||\mathbf{x}(t)||^2 \,,
\end{aligned}
$$

with q_1 the minimum eigenvalue of $\mathbf{Q}(t)$.

Definition (7) indicates $V(\mathbf{x},t)$, as was just defined, is a Lyapunov function.

Finally, ${}^C\Delta^v V(\mathbf{x},t) = {}^C\Delta^v \mathbf{x}^T P(t)\mathbf{x} \leq \lambda^*||\mathbf{y}||^2$, with $\lambda^* < 0$, following the proof of Theorem 4 and Corollary 1. $\square$

Again, we have exponential stability implies Mittag-Leffler stability for linear systems, in concordance with the recent results in [9].

4. Nonlinear nonautonomous systems

Theorems in last section lead us to establish when a fixed point of the nonlinear system (1) is Mittag-Leffler stable just stating if it is in its associated linear system (3).

In order to give sufficient conditions for having Mittag-Leffler stability, let us start recalling Theorem 3, that give us conditions on the Lyapunov function to get Mittag-Leffler stability of the fixed point. What follows to this theorem is the converse, if a nonlinear system is Mittag-Leffler stable then there exists a suitable Lyapunov function satisfying the mentioned theorem. In this way we prove first some important results. Both theorems will help us to establish when a fixed point of the nonlinear system (1) is Mittag-Leffler stable just stating if it is in its associated linear system (3).

In equation (1), let us suppose that $\mathbf{f} : S_r^{n-1} \times \mathbb{N}_{v-1} \to \mathbb{R}^n$ is locally Lipschitz for $\mathbf{x} \in S_r^{n-1}$ and $\mathbf{x} = \mathbf{0}$ a fixed point. Also assume that the Jacobian matrix $\partial\mathbf{f}/\partial\mathbf{x}$ is bounded and locally Lipschitz on S_r^{n-1}, which implies

$$
\left\| \frac{\partial f_i}{\partial \mathbf{x}}(\mathbf{x}_1,t) - \frac{\partial f_i}{\partial \mathbf{x}}(\mathbf{x}_2,t) \right\| \leq L_i||\mathbf{x}_1 - \mathbf{x}_2|| \,,
$$

with Lipschitz constant L_i and $\mathbf{x}_1$, $\mathbf{x}_2 \in S_r^{n-1}$, for every $i = 1,\ldots,n$. By the mean value theorem, there exists $\mathbf{z}_i$ in the line joining the origin and the point $\mathbf{x}$ that verifies

$$
f_i(\mathbf{x},t) - f_i(\mathbf{0},t) = \nabla f_i(\mathbf{z}_i,t)\cdot(\mathbf{x}-\mathbf{0}) \,.
$$

Let us observe de dependence on $\mathbf{0}$ and $\mathbf{x}$ for $\mathbf{z}_i$. From last expression we write

$$
f_i(\mathbf{x},t) = \nabla f_i(\mathbf{0},t)\cdot\mathbf{x} + (\nabla f_i(\mathbf{z}_i,t) - \nabla f_i(\mathbf{0},t))\cdot\mathbf{x}
$$

and we obtain $\mathbf{f}(\mathbf{x},t) = \mathbf{A}(t)\mathbf{x} + \mathbf{g}(\mathbf{x},t)$ being $\mathbf{A}(t)$ the Jacobian matrix evaluated at $\mathbf{x} = \mathbf{0}$ and

$$\mathbf{g}(\mathbf{x},t)=\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{z},t)-\mathbf{A}(t)\right)\mathbf{x}$$

where

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{z},t)=\begin{pmatrix}\nabla \mathbf{f}_1(\mathbf{z}_1,t)\\ \nabla \mathbf{f}_2(\mathbf{z}_2,t)\\ \vdots \\ \nabla \mathbf{f}_n(\mathbf{z}_n,t)\end{pmatrix}$$

underlying that $\mathbf{z}_i$ depends on $\mathbf{x}$ and on $\mathbf{0}$ for all i.

Theorem 6. *Suppose $\mathbf{x}=\mathbf{0}$ is a fixed point of (1), considering the function $\mathbf{f}: S_r^{n-1}\times\mathbb{N}_{v-1}\to\mathbb{R}^n$ locally Lipschitz for $\mathbf{x}\in S_r^{n-1}$. Suppose the Jacobian matrix $\mathbf{A}(t)=\partial\mathbf{f}/\partial\mathbf{x}(\mathbf{0},t)$ is bounded and locally Lipschitz in S_r^{n-1}, uniformly in t. Then $\mathbf{x}=\mathbf{0}$ is Mittag-Leffler stable if $\mathbf{A}(t)$ is negative definite.*

Proof. By Theorem 5, there exists a bounded and positive definite matrix $\mathbf{P}(t)$ such that $V(\mathbf{x},t)=\mathbf{x}^T\mathbf{P}(t)\mathbf{x}$ is a Lyapunov function for (3). Moreover, there exist positive constants k_1,k_2 verifying inequality $0\prec k_1\mathbf{I}\preceq\mathbf{P}(t)\preceq k_2\mathbf{I}$.

The target is to prove that $V(\mathbf{x},t)$, as was just introduced, works well for the fixed point of (1), fulfilling Theorem 3. We recover the expression $\mathbf{f}(\mathbf{x},t)=\mathbf{A}(t)\mathbf{x}+\mathbf{g}(\mathbf{x},t)$. Proceeding as Theorem 4 and omitting explicit dependence on time for $\mathbf{y}$, we have

$$\begin{aligned}{}^C\Delta^v V(\mathbf{x},t) &\le 2k_2\mathbf{y}^T\,{}^C\Delta^v\mathbf{y}(t)\\ &= 2k_2\mathbf{y}^T\left[\mathbf{A}(t+v-1)\mathbf{y}+\mathbf{g}(\mathbf{y},t+v-1)\right]\\ &\le 2k_2\lambda^*||\mathbf{y}||^2+2k_2\mathbf{y}^T\mathbf{g}(\mathbf{y},t+v-1)\\ &\le 2k_2\lambda^*||\mathbf{y}||^2+2k_2\mathbf{y}^T\left(\frac{\partial\mathbf{f}}{\partial\mathbf{y}}(\mathbf{z},t+v-1)-\mathbf{A}(t+v-1)\right)\mathbf{y}\end{aligned}$$

where λ^* is like in Corollary 1 and in order to avoid complicated notation, we write $\mathbf{z}$ as was explained before even though we are calculating respect to vector $\mathbf{y}$. The second term in the right hand side of the last inequality can be rewritten as an affine combination, as in the proof of Corollary 1 and therefore

$${}^C\Delta^v V(\mathbf{x},t)\le 2k_2\lambda^*||\mathbf{y}||^2+2k_2\lambda^{**}||\mathbf{y}||^2$$

being λ^{**} the maximum of the eigenvalues of $(1/2)(B+B^T)$, B being the matrix $(\partial\mathbf{f}/\partial\mathbf{y})(\mathbf{z},t+v-1)-\mathbf{A}(t+v-1)$, as $\mathbf{A}(t)$ is bounded and all partial of $\mathbf{f}$ are bounded and Lipschitz in S_r^{n-1}. Therefore, ${}^C\Delta^v V(\mathbf{x},t)\le\beta||\mathbf{y}||^2$, with $\beta=\max\{2k_2\lambda^*,2k_2\lambda^{**}\}$. □

Finally, a last result says that a condition on a partial linear part of the system (1) is sufficient to ensure Mittag-Leffler stability. Similar but weaker theorem can be found in [13] for autonomous Riemann-Liouville fractional differential equation.

Theorem 7. *Consider the system*

$${}^C\Delta^v\mathbf{x}(t)=\mathbf{A}(t+v-1)\mathbf{x}(t+v-1)+\mathbf{g}(\mathbf{x}(t+v-1),t+v-1)\,,\tag{5}$$

with $\mathbf{g}:\mathbb{R}^n\times\mathbb{R}^+\to\mathbb{R}^n$ such that $\mathbf{g}(\mathbf{0},t)=\mathbf{0}$ for all $t\ge 0$. Let $\mathbf{A}(t)$ be bounded and $\mathbf{g}$ be a continuous Lipschitz function respect to $\mathbf{x}$ uniformly in t, with Lipschitz constant L. If there exists a bounded positive *definite matrix $P(t)$ such that $2\mathbf{A}(t+v-1)+(1+L^2)\mathbf{I}) \prec m\mathbf{I}$ then the fixed point $\mathbf{x}=\mathbf{0}$ is Mittag-Leffler stable.*

Proof. Let us define $V(\mathbf{x},t)=\mathbf{x}^T\mathbf{P}(t)\mathbf{x}$. Since $(1/2)(\mathbf{P}(t)+\mathbf{P}(t)^T)$ is a symmetric and bounded matrix we imitate the proof of Theorem 4,

$$\begin{aligned}{}^C\Delta^v V(\mathbf{x},t) &= {}^C\Delta^v \mathbf{x}^T \frac{1}{2}\left(\mathbf{P}(t)+\mathbf{P}(t)^T\right)\mathbf{x} \\ &= \frac{1}{2}{}^C\Delta^v\mathbf{x}^T\mathbf{P}(t)\mathbf{x}+\frac{1}{2}{}^C\Delta^v\mathbf{x}^T\mathbf{P}^T(t)\mathbf{x} \\ &\leq 2\lambda_{max}\mathbf{y}^T\,{}^C\Delta^v\mathbf{y}\,,\end{aligned}$$

where we have omitted the explicit dependence on the independent variable on the vector $\mathbf{y}$ for simplicity. Assuming (5), the Caputo difference for V as was introduced is

$$ {}^C\Delta^v V(\mathbf{x},t) \leq 2\lambda_{max}\mathbf{y}^T\left(\mathbf{A}(t+v-1)\mathbf{y}+\mathbf{g}(\mathbf{y},t+v-1)\right)\,. \tag{6}$$

From the scalar inequality $2ab \leq a^2+b^2$ and taking into account the Lipschitz property on $\mathbf{g}$, we introduce in (6) the expression $2\mathbf{y}^T\mathbf{g} \leq (1+L^2)\mathbf{y}^T\mathbf{y}$ to obtain

$$\begin{aligned}&2\lambda_{max}\mathbf{y}^T\mathbf{A}(t+v-1)\mathbf{y}+2\lambda_{max}\mathbf{y}^T\mathbf{g}(\mathbf{y},t+v-1) \\ &\quad\leq 2\lambda_{max}\mathbf{y}^T\mathbf{A}(t+v-1)\mathbf{y}+\lambda_{max}(1+L)\mathbf{y}^T\mathbf{y} \\ &\quad= \lambda_{max}\mathbf{y}^T(2\mathbf{A}(t+v-1)+(1+L^2)\mathbf{I})\mathbf{y}\,.\end{aligned}$$

By the hypothesis of boundary, there exists a constant $m<0$ such that we know $2\mathbf{A}(t+v-1)+(1+L^2)\mathbf{I} \prec m\mathbf{I}$ for all t. Finally,

$$ {}^C\Delta^v V(\mathbf{x},t) \leq \bar{m}||\mathbf{y}||^2 $$

with $\bar{m}=\lambda_{max}m$, satisfying conditions in Theorem 3. □

5. Example

The well-known discrete Lotka-Volterra system [16] in its Caputo-like fractional version is

$$\begin{aligned}{}^C\Delta^v x(t) &= ax(t+v-1)-bx(t+v-1)y(t+v-1)-x(t+v-1)\,, \\ {}^C\Delta^v y(t) &= -cy(t+v-1)+dx(t+v-1)y(t+v-1)-y(t+v-1)\,,\end{aligned} \tag{7}$$

with parameters $a,b,c,d \in \mathbb{R}$. There are two fixed points: $(x_1,y_1)=(0,0)$ and $(x_2,y_2)=((1+c)/d,(-1+a)/b)$. We compute the Jacobian matrix at (x_i,y_i) (for $i=1,2$) denoted by A,

$$\begin{pmatrix} a-by_i-1 & bx_i \\ dx_i & -c+dx_i-1 \end{pmatrix}.$$

In order to apply Theorem 6, we consider the domain S_r^1 for some $r>0$. We determine the eigenvalues for the symmetric matrix $(1/2)(\mathbf{A}^T+\mathbf{A})$ at the fixed point (x_1,y_1),

$$\lambda_{1,1}=-1+a\,, \qquad \lambda_{1,2}=-1-c\,,$$

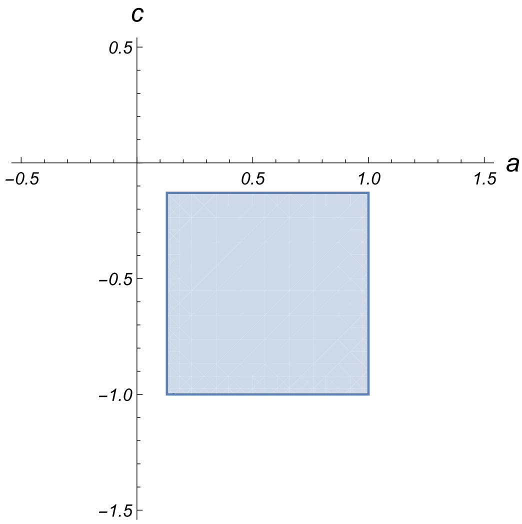

Fig. 1. Mittag-Leffler stability region for the fixed point $(0, 0)$ of (7).

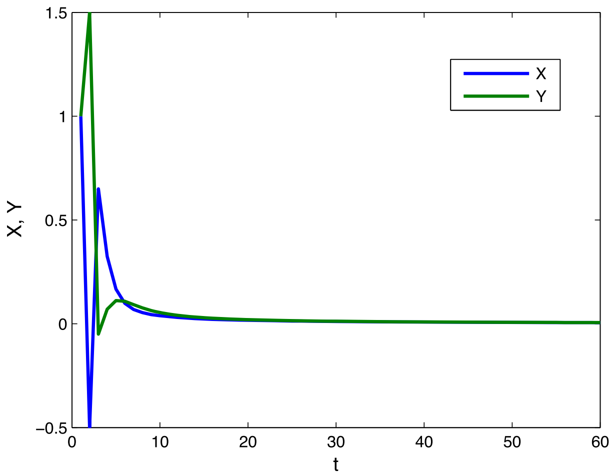

Fig. 2. Time series for x and y, for $a = 0.2$ and $b = -0.2$.

and for the other fixed point (x_2, y_2),

$$\lambda_{2,1} = \frac{1}{2}\left(\frac{b}{d}(1+c) - \frac{d}{b}(-1+a)\right), \quad \lambda_{2,2} = -\frac{1}{2}\left(\frac{b}{d}(1+c) - \frac{d}{b}(-1+a)\right).$$

In Fig. 1 it is plotted the parameters region where Theorem 6 ensures local Mittag-Leffler stability for (7) at (x_1, y_1), with order $v = 0.9$, that is the set $\{(a, c) \in \mathbb{R}^2 \,|\, 1 - (1/2^{1-v}) < a < 1, -1 < c < -1 + (1/2^{1-v})\}$. In Fig. 2 are plotted time series for x and y, with initial condition $x(v-1) = 1.9$, $y(v-1) = 1.9$ and for the values $a = 0.2$, $b = -0.2$, $c = 1$ and $d = 1$.

Also, we consider the system (7) in nonautonomous case, assuming $a(t + v - 1) = 0.5 + 0.1\cos(t + v - 1)$ and $b(t + v - 1) = 0.5 + 0.1\sin(t + v - 1)$. Again, we apply Theorem 6 and we get Mittag-Leffler stability for (7) at (x_1, y_1). In Fig. 3 are plotted time series for x and y, for $v = 0.9$, initial conditions $x(v-1) = 0.8$, $y(v-1) = 0.8$ and coefficients $c = 1$, $d = 1$ and a and b as we introduced before.

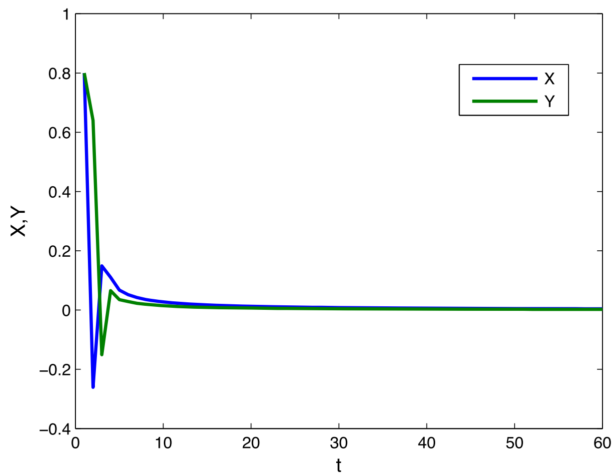

Fig. 3. -Time series for x and y, and coefficients $a(t+v-1) = 0.5 + 0.1\cos(t+v-1)$ and $b(t+v-1) = 0.5 + 0.1\sin(t+v-1)$.

6. Conclusions

We have shown conditions on the associated linear part of a nonlinear nonautonomous discrete Caputo-like fractional system in order to establish Mittag-Leffler stability respect to a fixed point in the full system. First we developed properties on the matrix defining the linear system to have Mittag-Leffler stability and therefore we depicted the behavior of solutions of the linear part. Second we prove a converse version of the Lyapunov direct method, exponential stability implies the existence of a Lyapunov function. Finally we wrote the main results: Mittag-Leffler stability of a fixed point of the full system (nonlinear and nonautonomous) can be inferred either from the associated linear part of the full system.

The results are limited by hypotheses imposed. In order to improve them, we must work in more relaxed conditions that lead us to stability statements in the discrete fractional nonlinear system.

A possible future work is extending this research to stochastic discrete fractional nonlinear systems and control methods of these class of systems.

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