J. Math. Anal. Appl. 488 (2020) 124085

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Hyperbolic polynomials and linear-type generating functions

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ARTICLE INFO

Article history:
Received 21 January 2020
Available online 27 March 2020
Submitted by P. Koskela

Keywords:
Hyperbolic polynomials
Generating functions
Zero distribution

ABSTRACT

In this paper we consider sequences of polynomials $\{H_m(z)\}_{m=0}^{\infty}$ generated by a relation $\sum_{m=0}^{\infty} H_m(z)t^m = \frac{1}{P(t)+zt^rQ(t)}$, where P and Q are real polynomials and $r \in \mathbb{N}$, $r \geq 2$. In the main result of the paper (cf. Theorem 1) we give a necessary conditions on P and Q (and their zeros) to ensure that for all sufficiently large m, the zeros of the polynomials $H_m(z)$ are real. We also show that the set of all zeros of the $H_m(z)$'s for $m \gg 1$ is dense in a real ray.

1. Introduction

The problem of describing the zero distribution of a sequence of polynomials remains an active area of research. From the classical methods of orthogonality to the spectral theory of positive matrices and asymptotic descriptions (i.e. approximate locations) there is a plethora of approaches to the problem. Depending on the approach one takes, the methods employed to investigate the problem can be quite different. In this paper we follow the approach used in the works [4], [3], [10], [11], and [12] as we analyze the zero location of a sequence of polynomials $\{H_m(z)\}_{m=0}^{\infty}$ generated by the relation

$$\sum_{m=0}^{\infty} H_m(z)t^m = \frac{1}{P(t)+zt^rQ(t)}, \qquad (r \in \mathbb{N})$$

where P and Q are real stable polynomials with some restrictions on their zero locus. In two of the authors' recent papers considering such problems the choice of the generating functions was largely motivated by the theory of multiplier sequences (and stability preserving linear operators in general). As such, we considered the (family of) generating functions

$$\frac{1}{(1-t)^n+zt^r} \qquad \text{and} \qquad \frac{1}{P(t)+zt^r}, \qquad (r \in \mathbb{N}) \tag{1.1}$$

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https://doi.org/10.1016/j.jmaa.2020.124085

where $P(t)$ is a polynomial with only positive zeros, and showed that the sequence of polynomials generated by these functions is eventually hyperbolic (see [4, p. 632, Theorem 1] and [3, p. 619, Theorem 1]). Establishing that *all* polynomials generated by functions of the type in (1.1) have only real zeros, not just the ones far enough out in the sequence, remains an open problem. Its resolution (in the positive) is in fact quite desirable, as it would open up the avenue to extending the family of functions that generate hyperbolic polynomials using locally uniform approximation arguments. This paper generalizes the results in [4] and [3] by considering generating functions whose denominators are elements of $\mathbb{R}[t][z]$ with coefficient polynomials that are hyperbolic. The elements of $\mathbb{R}[t][z]$ we consider can be viewed as linear combinations of 1 and z with coefficients that are hyperbolic polynomials. In this light, connections between the properties of the coefficient polynomials and the stability of the generated sequence emerge, very much in the flavor of classical stability theory à la Hermite-Biehler.

The Hermite-Biehler theorem (see for example [9, p. 197]) and the works of R. Ellard and H. Šmigoc, M-T, Ho and A. Datta, and V. Pivovarchik ([2], [6], [5], [8]), among others, address the connections between the stability of a polynomial $f = p(x^2) + xq(x^2)$, and the interlacing of the zeros of its 'constituents' p and q. Some of the cited works study the extent to which one may still be able to draw conclusions about the location of the zeros of the constituents, even if the polynomial f is not Hurwitz-stable. In particular, if n_- (resp. n_+) denote the number of zeros of a polynomial f in the left (resp. right) half plane, then the number of (interlacing) real zeros of its constituents is bounded below in terms of $|n_- - n_+|$.

If we regard $P(t)$ and $Q(t)$ as the 'constituents' of the polynomials $H_m(z)$, it would be reasonable to expect, analogously to the Hermite–Biehler theory, that the interlacing of the zeros of P and Q would imply hyperbolicity (real stability) of the polynomials $H_m(z)$. Alas, somewhat the contrary is true: the more separated the zeros of P and Q are, the 'better' in terms of the hyperbolicity of the generated sequence (cf. Corollary 1 and Remark 1).

The rest of the paper is organized as follows. Section 2 contains the setup and statement of the main result. Section 3 is devoted to the development of two key functions $\tau(\theta)$ and $z(\theta)$, which allow us to identify points in the interval $(0, \pi/r)$ with the zeros of our generated polynomials $H_m(z)$ in a one-to-one fashion. In Section 4 we establish the stability of the polynomials $H_m(z)$ and complete the proof of Theorem 1 modulo three auxiliary lemmas. We prove these lemmas in the concluding section of the paper.

2. The setup and the main result

Let

$$P(t) = \prod_{-p_- < k \le p_+} (t - \tau_k), \qquad \text{and} \qquad Q(t) = \prod_{-q_- < k \le q_+} (t - \gamma_k) \tag{2.1}$$

be two hyperbolic polynomials with p_+, q_+ positive and p_-, q_- negative zeros respectively, and suppose that $P(0), Q(0) \neq 0$. We arrange the zeros of $P(t)$ and $Q(t)$ in an increasing order according to their indices. In particular, τ_0 (γ_0 resp.) is the largest negative zero of $P(t)$ ($Q(t)$ resp.), while τ_1 (γ_1 resp.) is its smallest positive zero.

Definition 1. Given a polynomial $P(z)$, we denote by $\mathcal{Z}(P(z))$ the set of zeros of $P(z)$.

Definition 2. Let P and Q be polynomials. For each $x > 0$, we let $n_+^P(x)$ and $n_+^Q(x)$ be the number of positive zeros of $P(t)$ and $Q(t)$ on $(0, x]$ counting multiplicity. Similarly, for each $x < 0$, we let $n_-^P(x)$ and $n_-^Q(x)$ be the number of negative zeros of $P(t)$ and $Q(t)$ on $[x, 0)$.

In light of our discussion in the preceding section, it is perhaps not surprising that the quantities $n_+^P - n_+^Q$ and $n_-^Q - n_-^P$ appear in what follows, as those controlling the extent to which the zeros of the constituents of the sequence $H_m(z)$ are allowed to intermingle without destroying the hyperbolicity of $H_m(z)$. We formalize this connection in the next Lemma, whose proof we defer momentarily for the sake of a smoother exposition.

Lemma 1. *Let P, Q be as in (2.1), n_+^P and n_+^Q be as in Definition 2, and let*

$$R(t) = r - \frac{tP'(t)}{P(t)} + \frac{tQ'(t)}{Q(t)}, \qquad r \in \mathbb{N},\ r \geq 2. \tag{2.2}$$

Suppose that

(i) $n_+^P(x) - n_+^Q(x) \geq 2\ \forall x \geq \tau_2$, *and* $n_+^Q(x) = 0$, $\forall x \in (0, \tau_2]$ *and*
(ii) $\operatorname{Im} R(t) > 0$ *on the sector* $\{t \mid 0 < |t| < \tau_2, 0 < \operatorname{Arg} t < \pi/r\}$.

If $\tau_1 < \tau_2$, then the lone zero t_a of $P(t)R(t)$ in (τ_1, τ_2) is its smallest positive zero, and its multiplicity is one. If $\tau_1 = \tau_2$, then $t_a = \tau_1 = \tau_2$ is the smallest positive zero of $P(t)R(t)$.

We are now ready to state our main result.

Theorem 1. *Let P, Q be polynomials as in (2.1), and the functions $n_-^P, n_+^P, n_-^Q, n_+^Q$ be as in Definition 2. Consider the sequence of polynomials $\{H_m(z)\}_{m=0}^{\infty}$ generated by the relation*

$$\sum_{m=0}^{+\infty} H_m(z)t^m = \frac{1}{P(t) + zt^rQ(t)} = \frac{1}{D(t,z)}, \qquad r \geq 2. \tag{2.3}$$

If

(1) $n_+^P(x) - n_+^Q(x) \geq 2$, $\forall x \geq \tau_2$, *and* $n_+^Q(x) = 0$, $\forall x \in (0, \tau_2]$,
(2) $n_-^Q(x) - n_-^P(x) \geq 0$, $\forall x < 0$,
(3) $\operatorname{Im} R(t) > 0$ *on the sector* $\{t | 0 < |t| < \tau_2, 0 < \operatorname{Arg} t < \pi/r\}$,
(4) $\operatorname{Im} R(t) > 0$ *on the semi-disk* $\{t | 0 < |t| < t_a, 0 < \operatorname{Arg} t < \pi\}$,

then the zeros of $H_m(z)$ are real and of the same sign $(-1)^{p_+ - q_+}$ for all $m \gg 1$. Moreover, $\bigcup_{m \gg 1} \mathcal{Z}(H_m)$ is dense between

$$a = -\frac{P(t_a)}{t_a^r Q(t_a)} \tag{2.4}$$

and $(-1)^{p_+ - q_+}\infty$.

Example 1. If $P(t) = (t+2)(t-1)(t-2)(t-3)(t-5)$, $Q(t) = (t+1)(t-4))$ and $r = 3$, then $t_a \approx 1.23$, and conditions (3) and (4) are satisfied, as illustrated in Fig. 2.1. We thus conclude that the polynomials generated by $\dfrac{1}{P(t) + zt^3Q(t)}$ are all hyperbolic for $m \gg 1$.

We now provide the proof of Lemma 1.

Proof (of Lemma 1). We note that for any $t \neq \tau_k, \gamma_j$, $-q_- < j \leq q_+, -p_- < k \leq p_+$, we have

$$\operatorname{Im} R(t) = \operatorname{Im}\left(\sum_{-q_- < j \leq q_+} \frac{t}{t - \gamma_j} - \sum_{-p_- < k \leq p_+} \frac{t}{t - \tau_k} \right)$$

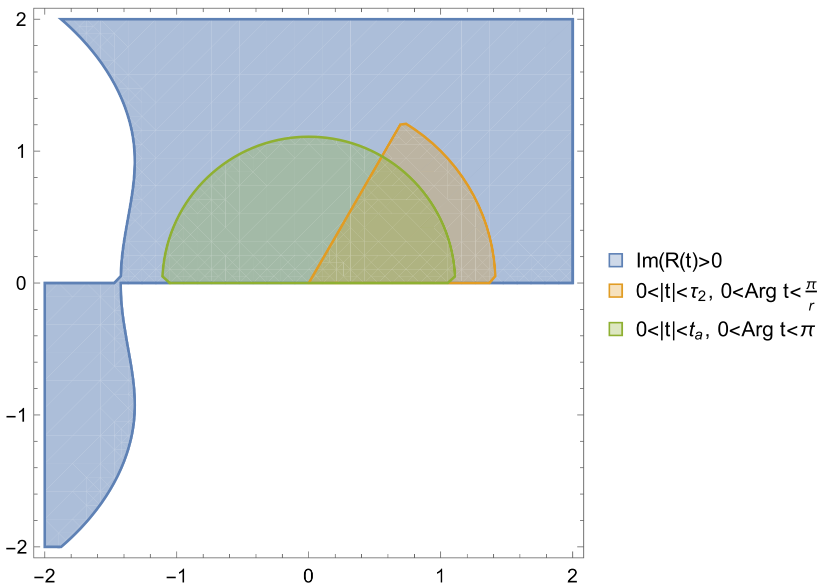

Fig. 2.1. The relevant regions for the applicability of Theorem 1 for the choices $P(t) = (t+2)(t-1)(t-2)(t-3)(t-5)$ and $Q(t) = (t+1)(t-4)$.

$$
\begin{aligned}
&= \operatorname{Im}\left(- \sum_{-q_- < j \le q_+} \frac{t\gamma_j}{|t-\gamma_j|^2} + \sum_{-p_- < k \le p_+} \frac{t\tau_k}{|t-\tau_k|^2} \right) \\
&= \left(\sum_{-p_- < k \le p_+} \frac{\tau_k}{|t-\tau_k|^2} - \sum_{-q_- < j \le q_+} \frac{\gamma_j}{|t-\gamma_j|^2} \right) \operatorname{Im} t. \qquad (2.5)
\end{aligned}
$$

Thus condition (ii) implies that

$$
\sum_{-p_- < k \le p_+} \frac{\tau_k}{|t-\tau_k|^2} - \sum_{-q_- < j \le q_+} \frac{\gamma_j}{|t-\gamma_j|^2} > 0
$$

for all $t \in \{t | 0 < |t| < \tau_2, 0 < \operatorname{Arg} t < \pi/r\}$. We let t approach the x-axis within this sector in order to conclude that

$$
R'(t) = \sum_{-p_- < k \le p_+} \frac{\tau_k}{(t-\tau_k)^2} - \sum_{-q_- < j \le q_+} \frac{\gamma_j}{(t-\gamma_j)^2} \ge 0, \qquad \forall t \in [0, \tau_2) \setminus \{\tau_1\}. \qquad (2.6)
$$

Consider now the case when $\tau_1 < \tau_2$. Since $r \in \mathbb{N}$, we see that $R(0) = r > 0$. In addition, by equation (2.6), $R(t)$ is non-decreasing on $(0, \tau_1) \cup (\tau_1, \tau_2)$. We conclude that R has no zeros on $[0, \tau_1)$. The conditions $\lim_{t \to \tau_1^+} R(t) = -\infty$ and $\lim_{t \to \tau_2^-} R(t) = +\infty$ now imply that R has a unique zero on (τ_1, τ_2) since it is a continuous rational function there. Finally, we argue that t_a is in fact a simple zero of $R(t)$. By condition (ii), $\operatorname{Im} R(t) \ge 0$ on the top half of a sufficiently small circle centered at t_a, and $\operatorname{Im} R(t) \le 0$ on the lower half of the same circle, since $R(t)$ is a rational function with real coefficients. Consequently, the change in argument of $R(t)$ on this circle (oriented counterclockwise) is at most 2π. Using the argument principle we conclude that $R(t)$ has at most one zero (counting multiplicities) in the disk bounded by the circle.

If $\tau_1 = \tau_2$, then $P(\tau_1) = P'(\tau_1) = 0$ and consequently

$$
P(\tau_1)R(\tau_1) = rP(\tau_1) - \tau_1 P'(\tau_1) + \frac{\tau_1 P(\tau_1) Q'(\tau_1)}{Q(\tau_1)} = 0.
$$

Since neither P nor R vanish on $[0, \tau_1)$, we conclude that the smallest positive zero of $P(t)R(t)$ is τ_1. $\square$

Corollary 1 *(to the proof of Lemma 1). Consider the generating relation (2.3). If, in addition to the assumptions of Theorem 1, the zeros of $P(t)$ are positive and those of $Q(t)$ are negative, then the zeros of $H_m(z)$ are real for all $m \gg 1$.*

Proof. It is straightforward that under the assumption of the corollary, conditions (1) and (2) of Theorem 1 are satisfied. In addition, equation (2.5) guarantees that conditions (3) and (4) are also satisfied. The result follows. □

Remark 1. The following observations are immediate:

(i) Since $\tau_0 < 0$, as $t \to \tau_0$ in the upper half plane, the right hand side of (2.5) eventually turns negative. It follows from condition (4) in Theorem 1 that $|\tau_0| > t_a$.

(ii) The conclusion of Theorem 1 is false if we allow the zeros of P and Q to interlace. For example, with $P(t) = (t-1)(t-3)(t-5)$ and $Q(t) = (t-2)(t-4)$ and $r = 3$ we see that $H_{16}(z)$ has a non-real root $z = -0.58844... + i \cdot 0.106817....$

(iii) $\deg H_m(z) \leq \lfloor m/r \rfloor$ for all $m \geq 0$. This is most readily deduced from induction and the identity

$$(P(t) + zt^r Q(t)) \sum_{m=0}^{\infty} H_m(z) t^m = 1,$$

which is equivalent to

$$(P(\Delta) + z\Delta^r Q(\Delta)) [H_m(z)] = \begin{cases} 1 & m = 0 \\ 0 & m \geq 1 \end{cases},$$

where $\Delta[H_m(z)] = H_{m-1}(z)$, and $H_{-k}(z) \equiv 0$ for $k \in \mathbb{N}$.

(iv) With the substitution t by $-t$ in Corollary 1, we see that the zeros of $H_m(z)$ are still real if the zeros of $P(t)$ are negative and $Q(t)$ are positive.

In the remainder of the paper, which is dedicated to the proof of the main result, the notations introduced in this section (in particular in Theorem 1) are in effect even if we do not explicitly repeat them in the statement of a result.

3. The functions $\tau(\theta)$ and $z(\theta)$

In this section we develop two key functions $\tau(\theta)$ and $z(\theta)$, which allow us to identify points in the interval $(0, \pi/r)$ with the zeros of our generated polynomials $H_m(z)$ in a one-to-one fashion.

For each $t = \tau e^{i\theta}$, $0 < \theta < \pi$, we define the angles $0 < \theta_k(t), \eta_j(t) < \pi$ implicitly by

$$\theta_k(t) = \operatorname{Arg}\left(\tau e^{i\theta} - \tau_k\right) \qquad (-p_- < k \leq p_+),$$
$$\eta_j(t) = \operatorname{Arg}\left(\tau e^{i\theta} - \gamma_j\right) \qquad (-q_- < j \leq q_+),$$

or equivalently

$$\frac{\tau e^{i\theta} - \tau_k}{\tau e^{-i\theta} - \tau_k} = e^{2i\theta_k(t)} \qquad (-p_- < k \leq p_+), \tag{3.1}$$

$$\frac{\tau e^{i\theta} - \gamma_j}{\tau e^{-i\theta} - \gamma_j} = e^{2i\eta_j(t)} \qquad (-q_- < j \leq q_+). \tag{3.2}$$

From these equations we obtain

$$\tau = \tau_k \frac{\sin\theta_k(t)}{\sin(\theta_k(t)-\theta)} = \gamma_j \frac{\sin\eta_j(t)}{\sin(\eta_j(t)-\theta)} \qquad (-p_- < k \le p_+, \quad -q_- < j \le q_+). \tag{3.3}$$

Let $\operatorname{Log}(t)$ denote the principal branch of the logarithm. Then the function

$$f(t) = r\operatorname{Log} t + \sum_{-q_- < j \le q_+} \operatorname{Log}(t-\gamma_j) - \sum_{-p_- < k \le p_+} \operatorname{Log}(t-\tau_k) \tag{3.4}$$

is analytic on the region $\operatorname{Im} t > 0$, and hence f satisfies the Cauchy-Riemann equations there:

$$\begin{aligned} \tau\frac{\partial \operatorname{Re} f}{\partial \tau} &= \frac{\partial \operatorname{Im} f}{\partial \theta} = \operatorname{Re} R(t), \\ \frac{\partial \operatorname{Re} f}{\partial \theta} &= -\tau\frac{\partial \operatorname{Im} f}{\partial \tau} = -\operatorname{Im} R(t). \end{aligned} \tag{3.5}$$

On the other hand,

$$\sum_{-p_- < k \le p_+} \theta_k(t) - \sum_{-q_- < k \le q_+} \eta_k(t) - r\theta = -\operatorname{Im} f, \qquad \text{and} \tag{3.6}$$

$$\ln\left|\frac{t^r Q(t)}{P(t)}\right| = \operatorname{Re} f. \tag{3.7}$$

We thus arrive at the following lemmas.

Lemma 2. *Suppose $t = \tau e^{i\theta}$, $0 < \theta < \pi$. The following statements are equivalent*

(1) $\operatorname{Im} R(t) > 0$.
(2) *For any fixed θ, the function*

$$\sum_{-p_- < k \le p_+} \theta_k(t) - \sum_{-q_- < j \le q_+} \eta_j(t)$$

is strictly decreasing in τ.
(3) *For any fixed τ, the function*

$$\left|\frac{t^r Q(t)}{P(t)}\right|$$

is strictly decreasing in $\theta \in (0, \pi)$.

Lemma 3. *Suppose $t = \tau e^{i\theta}$, $0 < \theta < \pi$. The following statements are equivalent*

(1) $\operatorname{Re} R(t) > 0$.
(2) *For any fixed τ, the function*

$$\sum_{-p_- < k \le p_+} \theta_k(t) - \sum_{-q_- < j \le q_+} \eta_j(t) - r\theta$$

is strictly decreasing in θ.

(3) *For any fixed θ, the function*

$$\left|\frac{t^r Q(t)}{P(t)}\right|$$

is strictly increasing in τ.

The next result will allow us to define the function $\tau(\theta)$, which will play a key role in the proof of the main result.

Proposition 1. *Let n and s denote the total number of zeros of P and Q respectively. For each $\theta \in (0, \pi/r)$, there exists a unique $t = \tau e^{i\theta}$ for which*

$$\sum_{-p_-<k\leq p_+} \theta_k(t) - \sum_{-q_-<j\leq q_+} \eta_j(t) - r\theta = (p_+ - q_+ - 1)\pi. \tag{3.8}$$

Proof. We first have the inequalities

$$p_+ - q_+ - 1 = n_+^P(\infty) - n_+^Q(\infty) - 1 > 0$$

and

$$p_+ - q_+ - 1 \geq \frac{n-s}{2}$$

since the second inequality is equivalent to

$$p_+ - q_+ - 2 \geq p_- - q_-.$$

Next, we observe that

$$\sum_{-p_-<k\leq p_+} \theta_k(t) - \sum_{-q_-<k\leq q_+} \eta_k(t) - r\theta$$

approaches $(p_+ - q_+)\pi - r\theta$ as $|t| \to 0$, and $(n - s - r)\theta$ as $|t| \to \infty$ where

$$(n-s-r)\theta < \frac{(n-s-r)\pi}{r} < (p_+ - q_+ - 1)\pi < (p_+ - q_+)\pi - r\theta.$$

By the intermediate value theorem, there is a $\tau \in (0, \infty)$ so that (3.8) holds.

To prove the uniqueness of τ, we will show that

$$\sum_{-p_-<k\leq p_+} \theta_k(t) - \sum_{-q_-<k\leq q_+} \eta_k(t) - r\theta$$

is monotone in τ. Indeed, since $n_-^Q(x) \leq n_-^P(x)$, $\forall x < 0$, we deduce that

$$\sum_{-p_-<k\leq 0} \theta_k - \sum_{-q_-<j\leq 0} \eta_j \leq 0,$$

from which (3.8) implies

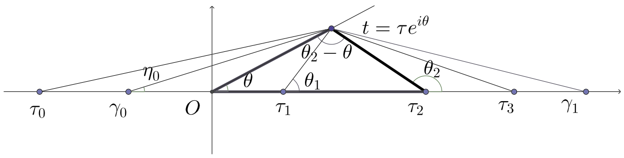

Fig. 3.1. The angles $\theta_k(t)$ and $\eta_j(t)$.

$$(p_+ - q_+ - 1)\pi \leq \sum_{k=1}^{p_+} \theta_k - \sum_{j=1}^{q_+} \eta_j - r\theta. \tag{3.9}$$

The inequality $n_+^P(x) - n_+^Q(x) \geq 2$ implies the existence of angles θ_{k_ℓ} with the following properties:

(i) $k_\ell \in \{1, 2, 3, \dots, p_+\}$ for all $1 \leq \ell \leq p_+ - q_+$,
(ii) $\theta < \theta_{k_\ell} < \pi$, $1 \leq \ell \leq p_+ - q_+$,
(iii) $\theta_1, \theta_2 \in \{\theta_{k_\ell} | 1 \leq \ell \leq p_+ - q_+\}$,
(iv) $\sum_{\ell=1}^{p_+ - q_+} \theta_{k_\ell} \leq \sum_{k=1}^{p_+} \theta_k - \sum_{j=1}^{q_+} \eta_j$.

We thus see that

$$(\theta_1 - \theta) + (\theta_2 - \theta) \geq (p_+ - q_+ - 1)\pi + (r-2)\theta - (p_+ - q_+ - 2)\pi \geq \pi.$$

Since $\theta_2 - \theta \geq \theta_1 - \theta$, we have $\theta_2 - \theta \geq \pi/2$. The inequality

$$\tau_2 > \tau \tag{3.10}$$

follows by noting that in the triangle $\triangle O\tau_2\tau e^{i\theta}$ in Fig. 3.1 the angle at $\tau e^{i\theta}$ (namely $\theta_2 - \theta$) is the largest, and hence the side opposite this vertex is the longest. Condition (3) of Theorem 1 implies that $\operatorname{Im} R(t) > 0$, and consequently

$$\sum_{-p_- < k \leq p_+} \theta_k(t) - \sum_{-q_- < k \leq q_+} \eta_k(t)$$

is monotone in τ by Lemma 2. $\square$

Thus for each $\theta \in (0, \pi/r)$, we can define the functions $\tau(\theta)$, $\theta_k(\theta)$, $-p_- < k \leq p_+$, and $\eta_j(\theta)$, $-q_- < j \leq q_+$, according to (3.1), (3.2), (3.3), and (3.8). To ensure these functions are analytic, we need to make use of the complex version of the Implicit Function Theorem.

Theorem 2 *(Theorem 2.1.2, p. 24 [7]). Let $f_j(w, z)$, $j = 1, \dots, m$, be analytic functions of $(w, z) = (w_1, \dots, w_m, z_1, \dots, z_n)$ in a neighborhood of a point (w^*, z^*) in $\mathbb{C}^m \times \mathbb{C}^n$, and assume that $f_j(w^*, z^*) = 0$, $j = 1, \dots, m$, and that*

$$\det\left(\frac{\partial f_j}{\partial w_k}\right)_{j,k=1}^m \neq 0 \qquad at \quad (w^*, z^*).$$

Then the equations $f_j(w, z) = 0$, $j = 1, \dots, m$ have a uniquely determined analytic solution $w(z)$ in a neighborhood of z^*, such that $w(z^*) = w^*$.

We are now in position to state and prove the following lemma.

Lemma 4. *The functions $\tau(\theta)$, $\theta_k(\theta)$, $-p_- < k \leq p_+$, and $\eta_j(\theta)$, $-q_- < j \leq q_+$ defined by equations (3.1), (3.2), (3.3), and (3.8) are analytic in a neighborhood of $(0, \pi/r)$.*

Proof. Let $\overrightarrow{\theta} = \left(\theta_{-p_-+1}, \theta_{-p_-+2}, \ldots, \theta_{p_+-1}, \theta_{p_+}\right)$ and $\overrightarrow{\eta} = \left(\eta_{-q_-+1}, \eta_{-q_-+2}, \ldots, \eta_{q_+-1}, \eta_{q_+}\right)$. We define the functions $f_k, g_j : \mathbb{C}^{n+s+1} \times \mathbb{C} \to \mathbb{C}$ by

$$\begin{aligned} f_k(\overrightarrow{\theta}, \overrightarrow{\eta}, \tau, \theta) &= \tau_k \frac{\sin\theta_k}{\sin(\theta_k - \theta)} - \tau, \qquad (-p_- < k \leq p_+) \\ g_j(\overrightarrow{\theta}, \overrightarrow{\eta}, \tau, \theta) &= \gamma_j \frac{\sin\eta_j}{\sin(\eta_j - \theta)} - \tau, \qquad (-q_- \leq j \leq q_+) \end{aligned}$$

and

$$h(\overrightarrow{\theta}, \overrightarrow{\eta}, \tau, \theta) = \sum_{-p_- < k \leq p_+} \theta_k - \sum_{-q_- < j \leq q_+} \eta_j - r\theta - (p_+ - q_+ - 1)\pi.$$

Note that for each $\theta \in (0, \pi/r)$, there exist θ_k, η_k, and τ so that

$$\begin{aligned} f_k(\overrightarrow{\theta}, \overrightarrow{\eta}, \tau, \theta) &= 0, \qquad (-p_- < k \leq p_+) \\ g_j(\overrightarrow{\theta}, \overrightarrow{\eta}, \tau, \theta) &= 0, \qquad (-q_- < j \leq q_+) \\ h(\overrightarrow{\theta}, \overrightarrow{\eta}, \tau, \theta) &= 0, \end{aligned}$$

and that there exists a neighborhood $\mathcal{W}$ of $(\overrightarrow{\theta}, \overrightarrow{\eta}\tau, \theta) \in \mathbb{C}^{n+s+1} \times \mathbb{C}$ where each of these function is analytic. We calculate

$$\begin{aligned} \frac{\partial f_k}{\partial \theta_k} &= \frac{-\tau_k \sin\theta}{\sin^2(\theta_k - \theta)} =: c_k, \\ \frac{\partial g_j}{\partial \eta_j} &= \frac{-\gamma_j \sin\theta}{\sin^2(\eta_j - \theta)} =: d_j, \end{aligned}$$

and write the Jacobian matrix at $(\overrightarrow{\theta}, \overrightarrow{\eta}, \tau, \theta)$ as

$$\begin{bmatrix} C & 0 & -1 \\ 0 & D & -1 \\ 1 & -1 & 0 \end{bmatrix},$$

where C and D are two $n \times n$ and $s \times s$ diagonal matrices whose diagonal entries are c_k, $-p_- < k \leq p_+$, and d_j, $-q_- < j \leq q_+$. By expanding along the first row, we find the determinant of this matrix to be

$$\pm \prod_{-p_- < k \leq p_+} c_k \prod_{-q_- < j \leq q_+} d_j \left(\sum_{-p_- < k \leq p_+} \frac{1}{c_k} - \sum_{-q_- < j \leq q_+} \frac{1}{d_j} \right).$$

We now show that this expression is nonzero. To this end note that since $t = \tau e^{i\theta}$, equation (3.3) implies

$$\begin{aligned} t - \tau_k &= \tau e^{i\theta} - \tau_k \\ &\overset{(3.3)}{=} \tau_k \frac{\sin\theta_k}{\sin(\theta_k - \theta)} e^{i\theta} - \tau_k \end{aligned}$$

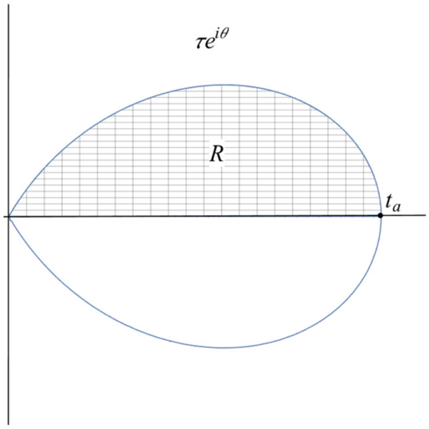

Fig. 3.2. The $\tau(\theta)e^{i\theta}$ curve and the set $\mathcal{R}$ (cf. Lemma 7) for $P(t) = (t-1)(t-2)(t-3)$, $Q(t) = (t+3)(t-4)$ and $r = 3$.

$$\begin{aligned} &= \tau_k \frac{\cos\theta_k \sin\theta + i \sin\theta_k \sin\theta}{\sin(\theta_k - \theta)} \\ &= \tau_k \frac{\sin\theta}{\sin(\theta_k - \theta)} e^{i\theta_k} \\ &= \frac{\tau \sin\theta}{\sin\theta_k} e^{i\theta_k} \end{aligned} \tag{3.11}$$

for $-p_- < k \le p_+$, and similarly

$$t - \gamma_j = \gamma_k \frac{\sin\theta}{\sin(\eta_j - \theta)} e^{i\eta_j} = \frac{\tau \sin\theta}{\sin\eta_j} e^{i\eta_j} \tag{3.12}$$

for $-q_- < j \le q_+$. Together with (2.5) these identities yield

$$\begin{aligned} \frac{\operatorname{Im} R(t)}{\operatorname{Im} t} &= \sum_{-p_- < k \le p_+} \frac{\sin^2(\theta_k - \theta)}{\tau_k \sin^2\theta} - \sum_{-q_- < j \le q_+} \frac{\sin^2(\eta_j - \eta)}{\gamma_j \sin^2\theta} \\ &= -\frac{1}{\sin\theta} \left(\sum_{-p_- < k \le p_+} \frac{1}{c_k} - \sum_{-q_- < j \le q_+} \frac{1}{d_j} \right). \end{aligned}$$

From (3) in Theorem 1 and (3.10), we conclude that the last expression is nonzero, which in turn implies that the functions $\tau(\theta)$, $\theta_k(\theta)$, $-p_- < k \le p_+$, and $\eta_j(\theta)$, $-q_- < j \le q_+$, are analytic in a neighborhood of $(0, \pi/r)$ by the Implicit Function Theorem. $\square$

We now turn our attention to the function $z(\theta)$. Heuristically, we expect from the generating relation (2.3) that the zeros (in z) of the denominator are fundamentally connected to the polynomials $H_m(z)$. This observation motivates the following definition.

Definition 3. We define the function $z(\theta)$ for $\theta \in (0, \pi/r)$ by

$$z(\theta) = -\frac{P(t)}{t^r Q(t)}, \tag{3.13}$$

where $t = \tau(\theta)e^{i\theta}$.

We recall from the definition of $\tau(\theta)$ that equation (3.8) holds for $t = \tau(\theta)e^{i\theta}$.

Lemma 5. *Let $z(\theta)$ be as in Definition 3. Under the hypotheses of Theorem 1 the function $(-1)^{p_+-q_+}z(\theta)$ is real valued, strictly increasing and positive on $(0,\pi/r)$.*

Proof. We write equation (3.13) as

$$z(\theta)=-\frac{\prod_{-p_-<k\le p_+}\left(\tau e^{i\theta}-\tau_k\right)}{\tau^r e^{ir\theta}\prod_{-q_-<j\le q_+}\left(\tau e^{i\theta}-\gamma_j\right)}.$$

With (3.11), (3.12) and (3.8), this equation becomes

$$z(\theta)=(-1)^{p_+-q_+}(\tau\sin\theta)^{n-s}\cdot\frac{\prod_{-q_-<j\le q_+}\sin\eta_j}{\tau^r\prod_{-p_-<k\le p_+}\sin\theta_k},\tag{3.14}$$

from which we deduce that $(-1)^{p_+-q_+}z(\theta)$ is a positive real-valued function on $(0,\pi/r)$. We continue by showing that $\ln|z|$ (and hence $|z|$) is strictly increasing on this interval. Let f be defined as in (3.4). Equations (3.6) and (3.8) imply that $\operatorname{Im} f$ is constant on the τ-curve defined by $\tau(\theta)e^{i\theta}$, $0<\theta<\pi/r$. By computing the derivative of $\operatorname{Im} f$ with respect to θ on the τ-curve we obtain

$$0=\frac{\partial\operatorname{Im} f}{\partial\tau}\frac{d\tau}{d\theta}+\frac{\partial\operatorname{Im} f}{\partial\theta}.\tag{3.15}$$

Using equations (3.7) and (3.13), we see that $\ln|z(\theta)|=-\operatorname{Re} f(t)$ for $t=\tau(\theta)e^{i\theta}$ and hence

$$\frac{d\ln|z(\theta)|}{d\theta}=-\frac{\partial\operatorname{Re} f}{\partial\tau}\frac{d\tau}{d\theta}-\frac{\partial\operatorname{Re} f}{\partial\theta}.$$

With (3.15), we recall from (3.5) that

$$\operatorname{Im} R(\tau e^{i\theta})=\tau\frac{\partial\operatorname{Im} f}{\partial\tau}=-\frac{\partial\operatorname{Re} f}{\partial\theta}$$

and

$$\operatorname{Re} R(\tau e^{i\theta})=\tau\frac{\partial\operatorname{Re} f}{\partial\tau}=\frac{\partial\operatorname{Im} f}{\partial\theta}=-\frac{\partial\operatorname{Im} f}{\partial\tau}\frac{d\tau}{d\theta}.$$

Thus

$$\begin{aligned}\left|R(\tau e^{i\theta})\right|^2&=\left(\operatorname{Re} R(\tau e^{i\theta})\right)^2+\left(\operatorname{Im} R(\tau e^{i\theta})\right)^2\\&=\tau\frac{\partial\operatorname{Re} f}{\partial\tau}\left(-\frac{\partial\operatorname{Im} f}{\partial\tau}\frac{d\tau}{d\theta}\right)+\tau\frac{\partial\operatorname{Im} f}{\partial\tau}\left(-\frac{\partial\operatorname{Re} f}{\partial\theta}\right)\\&=\tau\frac{\partial\operatorname{Im} f}{\partial\tau}\left(-\frac{\partial\operatorname{Re} f}{\partial\tau}\frac{d\tau}{d\theta}-\frac{\partial\operatorname{Re} f}{\partial\theta}\right)\\&=\operatorname{Im} R(\tau e^{i\theta})\frac{d\ln|z(\theta)|}{d\theta}.\end{aligned}$$

By assumption (3) in Theorem 1 and equation (3.10), $\operatorname{Im} R>0$ on the τ-curve, which in turn implies that the right hand side of the above equation is strictly positive there. The result now follows. □

Remark 2. From the definition of $z(\theta)$ in equation (3.13) and the reality of this function given in Lemma 5, we conclude that for each $\theta\in(0,\pi/r)$,

$$t_{0,1} := \tau(\theta)e^{\pm i\theta} \tag{3.16}$$

are two zeros in t of $P(t) + z(\theta)t^r Q(t)$.

Lemma 6. *Let $\tau(\theta)$ and $z(\theta)$ be defined as in (3.3) and (3.13) respectively for $\theta \in (0, \pi/r)$. Then with t_a as defined in Lemma 1, the following equations hold:*

(i) $\lim_{\theta\to 0} \tau(\theta) = t_a$,
(ii) $\lim_{\theta\to 0} z(\theta) = -P(t_a)/t_a^r Q(t_a)$,
(iii) $\lim_{\theta\to \pi/r} \tau(\theta) = 0$, *and*
(iv) $\lim_{\theta\to \pi/r} z(\theta) = (-1)^{p_+ - q_+}\infty$.

Proof. Combining the Cauchy-Riemman equations (3.5) with equation (3.15) we find that along the τ-curve

$$\frac{d\tau}{d\theta} = -\tau \frac{\operatorname{Re} R(t)}{\operatorname{Im} R(t)}.$$

Recall that $R(t)$ is a rational function, and hence the number of critical points of $\tau(\theta)$ on $(0, \pi/r)$ is finite. Since $0 \le \tau(\theta) \le \tau_2$, $\tau(\theta)$ is bounded, and consequently the limits $\lim_{\theta\to 0} \tau(\theta)$ and $\lim_{\theta\to\pi/r} \tau(\theta)$ exist. Consider now the two solutions $t_{0,1} = \tau(\theta)e^{\pm i\theta}$ to the equation $z(\theta) + \frac{P(t)}{t^r Q(t)} = 0$. Lemma 5 implies that $\lim_{\theta\to 0} z(\theta) =: z(0)$ exists, therefore $\lim_{\theta\to 0} \tau(\theta)$ is a double root of $z(0) + P(t)/t^r Q(t)$. As such, it is also a root of

$$\frac{d}{dt}\left(z(0) + \frac{P(t)}{t^r Q(t)}\right) = -P(t)R(t)/t^{r+1}Q(t).$$

Having established these facts,

(i) is now a straightforward consequence of Lemma 1 and equation (3.10) and
(ii) follows from (i) and the definition of $z(\theta)$.

From the definitions of $\theta_k(t)$, $-p_- < k \le p_+$, and $\eta_j(t)$, $-q_- < j \le q_+$, in (3.1) and (3.2), we obtain

$$\lim_{\tau\to 0} \sum_{-p_- < k \le p_+} \theta_k(\tau e^{i\pi/r}) - \sum_{-q_- < j \le 0} \eta_j(\tau e^{i\pi/r}) = (p_+ - q_+)\pi.$$

Also, if $\tau(\pi/r) := \lim_{\theta\to\pi/r} \tau(\theta) \ne 0$, then the limit of (3.8) as $\theta \to \pi/r$ gives

$$\sum_{-p_- < k \le p_+} \theta_k(\tau(\pi/r)e^{i\pi/r}) - \sum_{-q_- < j \le 0} \eta_j(\tau(\pi/r)e^{i\pi/r}) = (p_+ - q_+)\pi,$$

which contradicts to Lemma 2 for $\theta = \pi/r$. We conclude $\tau(\pi/r) = 0$ and (iii) follows. Finally, (iv) is easily seen using (iii) and the definition of $z(\theta)$ to establish that $\lim_{\theta\to\pi/r} |z(\theta)| = +\infty$, and noting that the sign of $z(\theta)$ is $(-1)^{p_+ - q_+}$ by (3.14). $\square$

In light of Lemma 6, we will henceforth understand $\tau(\theta)$ to be a continuous function on $[0, \pi/r]$. Similarly, we define $z(0) = -P(t_a)/t_a^r Q(t_a)$. The next lemma establishes that the function $P(t)/t^r Q(t)$ is real valued on the boundary of the set shown in Fig. 3.2, but nowhere in its interior.

Lemma 7. *If*

$$\mathcal{R} = \left\{t = |t|e^{i\theta} \in \mathbb{C} \mid 0 < \theta < \pi/r \text{ and } 0 < |t| \leq \tau(\theta)\right\} \tag{3.17}$$

and $t = |t|e^{i\theta} \in \mathcal{R}$*, then*

$$\operatorname{Im} \frac{P(t)}{t^r Q(t)} = 0$$

if and only if $t = \tau(\theta)e^{i\theta}$.

Proof. If $t = \tau(\theta)e^{i\theta}$, then by the reality of $z(\theta)$ (cf. Lemma 5)

$$\operatorname{Im} \frac{P(t)}{t^r Q(t)} = \operatorname{Im}(-z(\theta)) = 0.$$

If $t = |t|e^{i\theta} \in \mathcal{R}$ and

$$\operatorname{Im} \frac{P(t)}{t^r Q(t)} = 0,$$

then

$$\frac{P(t)Q(\bar{t})}{P(\bar{t})Q(t)} = e^{2ri\theta}. \tag{3.18}$$

For $-p_- < k \leq p_+$ and $-q_- < j \leq q_+$ we define the angles $\theta_k(t)$ and $\eta_j(t)$ via the equations

$$\frac{|t|e^{i\theta} - \tau_k}{|t|e^{-i\theta} - \tau_k} = e^{2i\theta_k(t)}, \qquad \text{and}$$
$$\frac{|t|e^{i\theta} - \gamma_j}{|t|e^{-i\theta} - \gamma_j} = e^{2i\eta_j(t)}.$$

Substituting these expressions in equations (3.18) and equating exponents yields

$$\sum_{-p_- < k \leq p_+} \theta_k(t) - \sum_{q_- < j \leq q_+} \eta_j(t) = r\theta + l\pi, \qquad \text{for some} \quad l \in \mathbb{Z}.$$

Recall that for each $\theta \in (0, \pi/r)$, the difference of angle sums is decreasing in $|t|$ by Lemma 2 (2). Thus

$$\begin{aligned}(p_+ - q_+)\pi > &\sum_{-p_- < k \leq p_+} \theta_k(t) - \sum_{-q_- < j \leq q_+} \eta_j(t) - r\theta \\ \geq &\sum_{-p_- < k \leq p_+} \theta_k(\tau e^{i\theta}) - \sum_{-q_- < j \leq q_+} \eta_j(\tau e^{i\theta}) - r\theta \\ = &(p_+ - q_+ - 1)\pi.\end{aligned}$$

Since $l \in \mathbb{Z}$, we must have $l = p_+ - q_+ - 1$. By Proposition 1 we conclude that $|t| = \tau$, and the result follows. □

Lemma 8. *Let* $\theta \in [0, \pi/r)$ *be a fixed angle with* $z := z(\theta)$ *and* $\tau := \tau(\theta)$*. The only zeros in* t *of* $P(t) + z(\theta)t^r Q(t)$ *in the closed disk centered at the origin with radius* $\tau(\theta)$ *are* $t_{0,1} := \tau(\theta)e^{\pm i\theta}$.

Proof. Lemma 7 implies that $P(t)+z(\theta)t^rQ(t)$ has no zero in t on the region $\mathcal{R}$ in (3.17) except $\tau(\theta)e^{\pm i\theta}$. Suppose $t\notin\mathcal{R}$, $|t|\le\tau(\theta)$, $\operatorname{Im} t\ge 0$, and $t\neq 0$. We consider four cases.

Case 1: $t\in(0,t_a]$. Recall that t_a as the smallest positive zero of $P(t)R(t)$ where $R(t)$ is given in (2.2). Since $\frac{d}{dt}(P(t)/t^rQ(t))=-\dfrac{P(t)R(t)}{t^{r+1}Q(t)}$, we see that $P(t)/t^rQ(t)$ is monotone on $(0,t_a]$. From (2.1), the sign of the derivative is $(-1)^{p_+-q_++1}$ for $t\ll 1$ and hence

$$(-1)^{p_+-q_++1}\frac{P(t)}{t^rQ(t)}\le(-1)^{p_+-q_++1}\frac{P(t_a)}{t_a^rQ(t_a)}.$$

On the other hand, since $\theta\ge 0$, Lemmas 5 and 6 imply

$$(-1)^{p_+-q_+}z(\theta)\ge(-1)^{p_+-q_++1}\frac{P(t_a)}{t_a^rQ(t_a)}=(-1)^{p_+-q_+}z(0).$$

If $P(t)+z(\theta)t^rQ(t)=0$, then

$$(-1)^{p_+-q_++1}\frac{P(t_a)}{t_a^rQ(t_a)}\ge(-1)^{p_+-q_++1}\frac{P(t)}{t^rQ(t)}=(-1)^{p_+-q_+}z(\theta)\ge(-1)^{p_+-q_++1}\frac{P(t_a)}{t_a^rQ(t_a)},$$

from which we conclude $t=t_a$. If $\theta=0$, then $t=t_a=t_0=t_1$. On the other hand, if $\theta>0$, then the identity $z(0)=z(\theta)$ contradicts to Lemma 5.

Case 2: $0\le\operatorname{Arg} t\le\theta$ and $t\notin(0,t_a]$. Since $t\notin\mathcal{R}$ and $|t|\le\tau(\theta)$, we have $\tau(\theta)\ge|t|>\tau(\operatorname{Arg} t)$. By the intermediate value theorem, there exists $\theta^*\in(\operatorname{Arg} t,\theta]$ such that $\tau(\theta^*)=|t|$. Recall from Lemma 2 that $|t^rQ(t)/P(t)|$ is decreasing in $\operatorname{Arg} t$ for fixed $|t|$. Thus the fact that $\theta^*\in(\operatorname{Arg} t,\theta]$ and $\tau(\theta^*)=|t|$ together with Lemma 5 and (3.13) imply

$$\left|\frac{P(t)}{t^rQ(t)}\right|<(-1)^{p_+-q_+}z(\theta^*)\le(-1)^{p_+-q_+}z(\theta)=|z(\theta)|$$

from which we conclude $P(t)+z(\theta)t^rQ(t)\neq 0$.

Case 3: $\theta<\operatorname{Arg} t<\pi/r$. Similar to the second case, the inequalities $\tau(\theta)\ge|t|>\tau(\operatorname{Arg} t)$ and the intermediate value theorem imply that there exists $\theta^*\in(\theta,\operatorname{Arg} t)$ such that $\tau(\theta^*)=|t|$. Thus from the decreasing of $|t^rQ(t)/P(t)|$ in $\operatorname{Arg} t$ for fixed $|t|$, Lemma 5 and (3.13) imply

$$\left|\frac{P(t)}{t^rQ(t)}\right|>(-1)^{p_+-q_+}z(\theta^*)>(-1)^{p_+-q_+}z(\theta)=|z(\theta)|$$

and consequently $P(t)+z(\theta)t^rQ(t)\neq 0$.

Case 4: $\pi/r\le\operatorname{Arg} t\le\pi$. From Lemma 6, we have $0=\tau(\pi/r)\le|t|\le\tau(\theta)$. By the intermediate value theorem, there exists $\theta^*\in(\theta,\pi/r)$ such that $\tau(\theta^*)=|t|$ and thus, after employing similar arguments as in the previous cases, we conclude

$$\left|\frac{P(t)}{t^rQ(t)}\right|>(-1)^{p_+-q_+}z(\theta^*)>(-1)^{p_+-q_+}z(\theta)=|z(\theta)|$$

and $P(t)+z(\theta)t^rQ(t)\neq 0$.

In all cases we showed that if t is in the closed disk with radius τ and $t\neq\tau e^{\pm i\theta}$, then t cannot be a zero of $P(t)/t^rQ(t)$. The proof is thus complete. □

4. Zeros of $H_m(z)$

Having studied the functions $\tau(\theta)$ and $z(\theta)$, we now turn our attention to studying the zeros of the polynomials $H_m(z)$.

Let $t_\varkappa := t_\varkappa(z)$, $0 \le \varkappa \le L$, be the simple zeros of $P(t) + zt^r Q(t)$ in non-decreasing (in modulus) order according to their indices, so that t_{L+1} is the smallest zero (in modulus) with multiplicity bigger than 1. Then the Cauchy integral formula gives

$$\begin{aligned} H_m(z) &= \frac{1}{2\pi i} \oint\limits_{|t|=\epsilon} \frac{dt}{(P(t) + zt^r Q(t))t^{m+1}} \\ &= -\sum_{0 \le \varkappa \le L} \frac{1}{D_t(t_\varkappa, z) t_\varkappa^{m+1}} + \frac{1}{2\pi i} \oint\limits_{|t| = \frac{|t_L| + |t_{L+1}|}{2}} \frac{dt}{(P(t) + zt^r Q(t))t^{m+1}}, \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} D_t(t_\varkappa, z) &= P'(t_\varkappa) - \frac{P(t_\varkappa)}{t_\varkappa^r Q(t_\varkappa)} \left(r t_\varkappa^{r-1} Q(t_\varkappa) + t_\varkappa^r Q'(t_\varkappa) \right) \\ &= -\frac{P(t_\varkappa) R(t_\varkappa)}{t_\varkappa}. \end{aligned}$$

Substituting this expression into (4.1) we obtain

$$H_m(z) = \sum_{0 \le \varkappa \le L} \frac{1}{P(t_\varkappa) R(t_\varkappa) t_\varkappa^m} + \frac{1}{2\pi i} \oint\limits_{|t| = \frac{|t_L| + |t_{L+1}|}{2}} \frac{dt}{(P(t) + zt^r Q(t))t^{m+1}}. \tag{4.2}$$

The reader will recall that for $\theta \in (0, \pi/r)$, t_0 and t_1 are distinct zeros of $P(t) + z(\theta)t^r Q(t)$ (cf. Lemma 8), hence $L \ge 1$ in this representation.

Let

$$g(t) = P(t) R(t) t^m, \tag{4.3}$$

and denote by $\{\theta_h\}$, $0 \le \theta_h \le \pi/r$, the sequence (possibly finite) of angles which correspond to the points $s_h := \tau(\theta_h) e^{i\theta_h}$ on the τ-curve where[1]

$$\operatorname{Im} g\left(s_h\right) = 0. \tag{4.4}$$

Note that under the assumption (3) of Theorem 1 and the inequality $\tau(\theta) < \tau_2$, for $0 < \theta < \pi/r$, in (3.10), $\operatorname{Im} R(t) > 0$ on the τ-curve, and hence $g(s_h) = 0$ if and only if $s_h = t_a$ or $s_h = 0$. Let $\sigma\left(H_m(z(\theta_h); 0, \pi/r\right)$ denote the number of sign changes of the sequence $\{H_m(z(\theta_h)\}_h$ where $z(0)$ and $z(\pi/r)$ are defined by the limits in Lemma 6. Deferring the proof of $H_m(z(\theta_h)) \ne 0$ for $m \gg 1$ (cf. Lemmas 9 and 11), we now show that

$$\sigma\left(H_m(z(\theta_h); 0, \pi/r\right) \ge \lfloor m/r \rfloor.$$

[1] The function g and the angles θ_h clearly depend on m. In the interest of readability, we suppress this dependence in the notation. When necessary, the dependence will be explicitly emphasized in the text rather in the notation.

With this inequality, Lemmas 5 and 6 show that function $z(\theta)$ is monotone and it maps the interval $(0, \pi/r)$ onto the interval with endpoints $(-1)^{p_+ - q_+}\infty$ and $a = -\dfrac{P(t_a)}{t_a^r Q(t_a)}$ where t_a is defined in Lemma 1. Consequently $H_m(z)$ has at least $\lfloor m/r \rfloor$ real zeros between $-\dfrac{P(t_a)}{t_a^r Q(t_a)}$ and $(-1)^{p_+ - q_+}\infty$ by the intermediate value theorem.

In order to establish a lower bound on the quantity $\sigma\left(H_m(z(\theta_h); 0, \pi/r\right)$, we first study the sign changes in the sequence $\{g(s_h)\}$ omitting the first and last terms (which are 0). To this end, let γ be the counterclockwise loop formed by the curve $\tau(\theta)e^{i\theta}$ and its conjugate (see Fig. 3.2). Denote by γ' the image of γ under the map $g(t) - \xi$, where $\xi \neq 0$ is a small real number chosen so that $\xi P(0)R(0) > 0$. According to the Argument Principle, the winding number of γ' around the origin is equal to the number of zeros of $g(t) - \xi$ inside γ, since this function has no poles there. If $g(t) - \xi = 0$, truncating the Taylor expansion of $P(t)R(t)$ about the origin at the constant term yields

$$P(0)R(0)t^m \left(1 + \mathcal{O}(t)\right) = \xi.$$

Rearranging for t yields

$$t = \omega_k \sqrt[m]{\frac{\xi}{P(0)R(0)}} \left(1 + \mathcal{O}(\xi^{1/m})\right), \tag{4.5}$$

where $\omega_k = e^{2k\pi i/m}$ is an m-th root of 1.

If we truncate the Taylor expansion of $P(t)R(t)$ about the origin at the linear term instead, we obtain

$$P(0)R(0)t^m \left(1 + \left(\frac{P'(0)}{P(0)} + \frac{R'(0)}{R(0)}\right) t + \mathcal{O}(t^2)\right) = \xi.$$

Using the expression of t from (4.5) and rearranging lead to the more precise estimate

$$t = \omega_k \sqrt[m]{\frac{\xi}{P(0)R(0)}} \left(1 - \frac{\omega_k}{m}\left(\frac{P'(0)}{P(0)} + \frac{R'(0)}{R(0)}\right) \sqrt[m]{\frac{\xi}{P(0)R(0)}} + \mathcal{O}(\xi^{2/m})\right). \tag{4.6}$$

Computing the principal argument of both sides gives

$$\operatorname{Arg} t = \frac{2k\pi}{m} - \frac{\sin(2k\pi/m)}{m}\left(\frac{P'(0)}{P(0)} + \frac{R'(0)}{R(0)}\right) \sqrt[m]{\frac{\xi}{P(0)R(0)}} + \mathcal{O}(\xi^{2/m}). \tag{4.7}$$

Equations (4.6) and (4.7) establish that as $\xi \to 0$, $|t| \to 0$, while $\operatorname{Arg}(t) \to 2k\pi/m$. If $|\operatorname{Arg}(t)| \leq \pi/r - \nu$ for some fixed small ν independent of ξ, then for sufficiently small ξ, $|t| < \tau(\operatorname{Arg} t)$, and consequently t lies inside γ. Thus $g(t) - \xi$ has at least

$$2\lfloor m/2r \rfloor + 1$$

zeros inside γ close to the origin if $2r \nmid m$ (namely one for each value of k between $-\lfloor m/2r \rfloor$ and $\lfloor m/2r \rfloor$), while it has at least $m/r - 1$ such zeros in case $2r \mid m$.

In addition to the zeros found by the above asymptotic expansion, $g(t) - \xi$ has an additional zero near t_a. Indeed, $g(0) = g(t_a) = 0$, $g(t) \neq 0$ on $(0, t_a)$ and $g(t)$ is continuous and real valued on $[0, t_a]$. Furthermore, equation (4.6) (with $k = 0$) implies that $g(t) - \xi$ has a simple positive zero close to the origin for sufficiently small ξ. It follows that $g(t) - \xi$ must also have a real zero near (but to the left of) t_a, which therefore lies inside γ. Applying argument principle to $g(t) - \xi$ we conclude that

$$\frac{1}{2\pi i}\oint_{\gamma}\frac{g'(t)}{g(t)-\xi}dt \geq \begin{cases} 2\lfloor m/2r\rfloor + 2 & \text{if } 2r\nmid m \\ m/r & \text{if } 2r|m \end{cases}.$$

Note that we may partition the curve γ into arcs $[s_h, s_{h+1}] \subset \gamma$ and their conjugates. Having done so we obtain

$$\frac{1}{2\pi i}\oint_{\gamma}\frac{g'(t)}{g(t)-\xi}dt = \frac{1}{2\pi i}\sum_{h}\int_{[s_h,s_{h+1}]\cup[\overline{s_{h+1}},\overline{s_h}]}\frac{g'(t)}{g(t)-\xi}dt.$$

By the definition of the points $\{s_h\}$, $g(t)-\xi$ maps $[s_h, s_{h+1}] \cup [\overline{s_{h+1}}, \overline{s_h}]$ to a loop whose winding number around 0 is nonzero if and only if

$$(g(s_{h+1})-\xi)(g(s_h)-\xi) < 0,$$

which, by taking ξ is sufficiently small, implies that $g(s_{h+1})g(s_h) \leq 0$. Since $g(s_h) = 0$ if and only if $\theta_h = 0$ or $\theta_h = \pi/r$, we conclude that

$$\sigma\left(g(s_h), 0, \pi/r\right) \geq \begin{cases} 2\lfloor m/2r\rfloor & \text{if } 2r\nmid m \\ m/r-2 & \text{if } 2r|m \end{cases}.$$

In Section 5 we will prove that

(i) For $m \gg 1$, $H_m(z(\theta_h)) \neq 0$ and the sign of $g(s_h)$ is the same as the sign of $H_m(z(\theta_h))$;
(ii) if $2r|m$ and one endpoint of (θ_h, θ_{h+1}) is π/r then $H_m(z(\theta))$ has a zero on this interval, and
(iii) the sign of $H_m(z(\theta))$ is $(-1)^{p_+}$ and $(-1)^{p_+ + \lfloor m/r\rfloor}$ as $\theta \to 0$ and $\theta \to (\pi/r)^-$ respectively.

Using these three results we now complete the proof of Theorem 1. By the intermediate value theorem $H_m(z(\theta))$ has at least $\lfloor m/r\rfloor - 1$ zeros on $(0, \pi/r)$ each of which gives a distinct zero of $H_m(z)$ between $a = -P(t_a)/(t_a^r Q(t_a))$ and $(-1)^{p_+ - q_+}\infty$ by the monotonicity of $z(\theta)$. Since the degree of $H_m(z)$ is $\lfloor m/r\rfloor$ (see (iii) in Remark 1), its remaining zero must be real, and

$$\operatorname{sgn}\left(\lim_{z\to(-1)^{p_+-q_++1}\infty} H_m(z)\right) = (-1)^{\lfloor m/r\rfloor}\operatorname{sgn}\left(\lim_{z\to(-1)^{p_+-q_+}\infty} H_m(z)\right)$$

Thus by (iii), $H_m(z)$ has the same sign at $z = a$ as it does near $(-1)^{p_+-q_++1}\infty$. We conclude that $H_m(z)$ cannot change sign between a and $(-1)^{p_+-q_++1}\infty$, and must therefore have its remaining zero between a and $(-1)^{p_+-q_+}\infty$.

We finish this section by proving that the union of the zeros of the $H_m(z)$'s ($m \gg 1$) form a dense subset of the interval I with endpoints $a = -\dfrac{P(t_a)}{t_a^r Q(t_a)}$ and $(-1)^{p_+-q_+}\infty$. To this end, note that z (the function defined in (3.13)) is a continuous and monotone map from $(0, \pi/r)$ to I. Therefore, if we can demonstrate that the set of θ_h's (which depend on m) form a dense subset of $(0, \pi/r)$ the claim would follow. To this end, consider an arbitrary open interval or $(\alpha, \beta) \subset (0, \pi/2)$, and the arc

$$\Lambda = \left\{\tau(\theta)e^{i\theta} \mid \theta \in (\alpha, \beta)\right\}$$

associated to it on the τ-curve. Recall that g depends on m, and note that the change in argument of $g(t)$ on Λ

$$\Delta_\Lambda \operatorname{Arg}(g) = \Delta_\Lambda(P(t)R(t)) + m(\beta - \alpha) \to \infty \quad \text{as} \quad m \to \infty.$$

Thus, there exists an m so that $\Delta_\Lambda \operatorname{Arg}(g) > \pi$. Hence $g(t) \in \mathbb{R}$ for some $t \in \Lambda$ (i.e., $\operatorname{Arg}(t) = \theta_h$ for some h), or equivalently, $\operatorname{Im}(g(t)) = 0$.

5. Proofs of the three lemmas

We begin this section with a result concerning exponential polynomials. While we will use it to establish the three claims made at the end of the previous section, the result is interesting in its own right, as exponential polynomials are objects of interest in a number of active research areas. Without striving for completeness, we mention only a few here. Shapiro's 1958 conjecture on the zero distribution of the members of the ring or exponential polynomials motivated D'Aquino, Macintyre and Terzo to study these objects in an algebraic setting in the paper [1], where they attribute the origins of Shapiro's conjecture to complex analytic considerations. Exponential polynomials are also central objects in the study of decomposition of integers into sums of powers of integers (such as Vinogradov's Three primes theorem) and in the methods used in arriving at such theorems (such as the Hardy-Littlewood circle method). Finally, we note that they also appear in Weyl's criterion regarding the equidistribution of sequences. We are unaware of results akin to that of Proposition 2, which shows that certain exponential polynomials have infinitely many real zeros.

Proposition 2. *Let $n \in \mathbb{N}$, $n \geq 2$, and let $\omega_\varkappa = e^{(2\varkappa-1)\pi i/n}$ be the n-th root of -1 for $1 \leq \varkappa \leq n-1$. For any $\ell \in \mathbb{Z}$, and $x \geq 0$ such that $\omega_1^\ell e^{x\omega_1} = \omega_0^\ell e^{x\omega_0} \in \mathbb{R}$, the sign of*

$$\sum_{\varkappa=0}^{n-1} \omega_\varkappa^\ell e^{x\omega_\varkappa} \tag{5.1}$$

is the same as the sign of the first term. In particular, the function in (5.1), as a function of x, has infinitely many real zeros.

Proof. The result is immediate for the cases $n = 2$. We henceforth assume that $n \geq 3$, and (without loss of generality) that $0 \leq \ell < n$. The requirements that $\omega_1^\ell e^{x\omega_1} = \omega_0^\ell e^{x\omega_0} \in \mathbb{R}$ and $x \geq 0$ necessitate that

$$x = \frac{\pi\left(b - \frac{\ell}{n}\right)}{\sin\left(\frac{\pi}{n}\right)}, \qquad \text{for some } b \in \mathbb{N}.$$

With this explicit expression for x we also calculate

$$\omega_0^\ell e^{x\omega_0} = (-1)^b e^{\pi\left(b-\frac{\ell}{n}\right)\cot(\pi/n)}. \tag{5.2}$$

Thus, if x is as required, then

$$\sum_{\varkappa=0}^{n-1} \omega_\varkappa^\ell e^{x\omega_\varkappa} = \omega_0^\ell e^{x\omega_0}\left(2 + \sum_{\varkappa=2}^{n-1}\left(\frac{\omega_\varkappa}{\omega_0}\right)^\ell e^{x(\omega_\varkappa-\omega_0)}\right) \tag{5.3}$$

and consequently, it remains to prove that

$$2 + \sum_{\varkappa=2}^{n-1} \frac{\omega_\varkappa^\ell e^{x\omega_\varkappa}}{\omega_0^\ell e^{x\omega_0}} > 0.$$

We note that

$$\begin{aligned}
\left|\sum_{\varkappa=2}^{n-1}\frac{\omega_\varkappa^\ell e^{x\omega_\varkappa}}{\omega_0^\ell e^{x\omega_0}}\right| &\le \sum_{\varkappa=2}^{n-1}\left|\frac{\omega_\varkappa^\ell e^{x\omega_\varkappa}}{\omega_0^\ell e^{x\omega_0}}\right| \\
&= \sum_{\varkappa=2}^{n-1}\exp^{-1}\left(x\cos\frac{\pi}{n} - x\cos\frac{(2\varkappa-1)\pi}{n}\right) \\
&= \sum_{\varkappa=2}^{n-1}\exp^{-1}\left(2x\sin\frac{\varkappa\pi}{n}\sin\frac{(\varkappa-1)\pi}{n}\right) \\
&\overset{(\star)}{<} 2\sum_{\varkappa=2}^{\lfloor n/2\rfloor}\exp^{-1}\left(\frac{x(\varkappa-1)^2\pi^2}{2n^2}\right) \\
&< 2\exp^{-1}\left(\frac{x\pi^2}{2n^2}\right) + 2\int_1^{+\infty}\exp^{-1}\left(\frac{x\pi^2}{2n^2}t^2\right)dt,
\end{aligned}$$

where inequality $(\star)$ follows from the fact that $2\sin(t) > t$, for all $t < \pi/2$. Observe that

$$\begin{aligned}
\int_1^{+\infty}\exp^{-1}\left(\frac{x\pi^2}{2n^2}t^2\right)dt &= \frac{\sqrt{2}n}{\pi\sqrt{x}}\int_{\pi\sqrt{x}/\sqrt{2}n}^{+\infty} e^{-t^2}dt \\
&\le \frac{\sqrt{2}n}{\pi\sqrt{x}}\int_{\pi\sqrt{x}/\sqrt{2}n}^{+\infty} \frac{t}{\left(\frac{\pi\sqrt{x}}{\sqrt{2}n}\right)}e^{-t^2} \\
&= \frac{2n^2}{\pi^2 x}\int_{\pi\sqrt{x}/\sqrt{2}n}^{+\infty} te^{-t^2}dt \\
&= \frac{n^2}{\pi^2 x}e^{-\pi^2 x/2n^2}.
\end{aligned}$$

We thus deduce that if $x > n^2/8$, then

$$\sum_{\varkappa=2}^{n-1}\exp^{-1}\left(x\cos\frac{\pi}{n} - x\cos\frac{(2\varkappa-1)\pi}{n}\right) < 2. \tag{5.4}$$

We next consider the case when $x \le n^2/8$. It is easy to verify that

$$\begin{aligned}
\frac{\pi}{n} \ge \sin\frac{\pi}{n} \ge \frac{\pi}{n}\left(1-\frac{\pi^2}{6n^2}\right), &\qquad (n \ge 1) \\
1+2t > \frac{1}{1-t}, &\qquad (t \in (0,1/2))
\end{aligned}$$

and consequently

$$nb-\ell \le x = \frac{b-\ell/n}{\sin(\pi/n)}\pi \le (nb-\ell)\left(1+\frac{\pi^2}{3n^2}\right). \tag{5.5}$$

The first inequality in (5.5) implies that

$$nb-\ell \le \frac{n^2}{8}$$

which, when put in the second inequality in (5.5) yields

$$x \le nb - \ell + \frac{\pi^2}{24}.$$

We write $x = nb - \ell + \delta$ where $0 \le \delta \le \pi^2/24$, and expand $e^{x\omega_k}$ in a Maclaurin series to obtain

$$\begin{aligned}\sum_{\varkappa=0}^{n-1} \omega_\varkappa^\ell e^{x\omega_\varkappa} &= \sum_{\varkappa=0}^{n-1} \omega_\varkappa^\ell \sum_{j=0}^{+\infty} \frac{(x\omega_\varkappa)^j}{j!} \\ &= n\sum_{j=1}^{+\infty} (-1)^j \frac{x^{jn-\ell}}{(jn-\ell)!} =: n\sum_{j=1}^{+\infty} a_j.\end{aligned}$$

We note that for any $j \in \mathbb{N}$,

$$\left|\frac{a_{j+1}}{a_j}\right| = \prod_{k=1}^{n} \frac{x}{jn-\ell+k}.$$

Since for any $1 \le k \le n$, $x = nb - \ell + \delta \ge jn - \ell + k$ if $j \le b-1$ and $x \le jn - \ell + k$ if $j \ge b+1$, we conclude that the sequence $|a_j|$ is increasing when $j \le b-1$ and decreasing when $j \ge b+1$. Since the series is alternating, the inequalities

$$\left|\sum_{j \le b-1} a_j\right| \le |a_{b-1}| \qquad \text{and} \qquad \left|\sum_{j \ge b+1} a_j\right| \le |a_{b+1}|$$

are immediate. Thus, the sign of (5.1) is $(-1)^b$, provided that

$$|a_b| > |a_{b-1}| + |a_{b+1}| \tag{5.6}$$

with the convention that $a_0 = 0$. In order to establish (5.6), we observe that if $b > 1$, then

$$\begin{aligned}\ln \frac{|a_{b-1}|}{|a_b|} &= \ln \frac{\prod_{j=1}^n (n(b-1) - \ell + j)}{x^n} \\ &= \ln \prod_{j=1}^n \left(1 - \frac{n-j+\delta}{x}\right) \\ &< -\sum_{j=1}^n \frac{n-j+\delta}{x} \\ &= -\frac{n\delta}{x} - \frac{n^2}{2x}\left(1 - \frac{1}{n}\right) \\ &< -4\left(1 - \frac{1}{n}\right)\end{aligned}$$

where the last inequality follows from the assumption $x \le n^2/8$. Similarly

$$\begin{aligned}\ln \frac{|a_{b+1}|}{|a_b|} &= \ln \frac{x^n}{\prod_{j=1}^n (nb - \ell + j)} \\ &= \ln \prod_{j=1}^n \left(1 + \frac{j-\delta}{x}\right)^{-1}.\end{aligned}$$

The inequalities

$$t/2 < \ln(1+t) \qquad \text{for all} \quad t \in (0,2),$$

and

$$0 < j - \delta \leq n - \delta < 2x$$

imply that

$$\begin{aligned}
\ln \prod_{j=1}^{n} \left(1 + \frac{j-\delta}{x}\right)^{-1} &\leq -\sum_{j=1}^{n} \frac{j-\delta}{2x} \\
&= -\frac{n(n+1)}{4x} + \frac{n\delta}{2x} \\
&< -\frac{n^2}{2x}\left(\frac{1}{2} - \frac{\delta}{n}\right) \\
&< -4\left(\frac{1}{2} - \frac{\pi^2}{24n}\right).
\end{aligned}$$

Thus

$$\frac{|a_{b-1}|}{|a_b|} + \frac{|a_{b+1}|}{|a_b|} < \exp\left(-4 + \frac{4}{n}\right) + \exp\left(-2 + \frac{\pi^2}{6n}\right) < 1$$

since $n \geq 3$. This establishes that the sign of (5.1) is determined by that of its first term.

With this result in hand it is now easy to see that (5.1) has infinitely many zeros, since the right hand side of (5.2) changes signs infinitely many times as x ranges through the real numbers. The proof is thus complete. $\square$

Remark 3. Using equation (5.3) and inequality (5.4) (for $x > n^2/8$), we conclude that there is an $\epsilon > 0$ independent of x so that

$$\left|\sum_{\varkappa=0}^{n-1} \omega_{\varkappa}^{\ell} e^{x\omega_{\varkappa}}\right| > \epsilon e^{x\cos(\pi/n)}$$

for all $x \geq 0$ such that $\omega_1^{\ell} e^{x\omega_1} = \omega_0^{\ell} e^{x\omega_0} \in \mathbb{R}$.

The three lemmas We now prove the three statements preceding the completion of the proof of Theorem 1. The asymptotic $\mathcal{O}$ notation in the proof is used under the assumption that $m \to \infty$. We begin by showing that the function $g(s_h)$ and $H_m(z(\theta_h))$ have the same sign for appropriately chosen values of θ_h.

Lemma 9. *Let $g(t) = P(t)R(t)t^m$ and denote by $\{\theta_h\}$, $0 < \theta_h < \pi/r$, the sequence of angles corresponding to the points $s_h := \tau(\theta_h)e^{i\theta_h}$ on the τ-curve where*

$$\operatorname{Im} g(s_h) = 0.$$

For all $m \gg 1$, $H_m(z(\theta_h)) \neq 0$ and the sign of $g(s_h)$ is the same as the sign of $H_m(z(\theta_h))$ for all values of h under consideration.

Proof. The proof is by cases, based on the asymptotic behavior of the angles θ_h as m tends to infinity. Some of these angles may tend to 0, some may tend to π/r, and some may remain bounded away from both. We begin with the latter, simplest case.

Case 1: $\gamma \le \theta_h \le \pi/r - \gamma$ for some small fixed γ (independent of m).

Lemma 8 implies there exists $\epsilon > 0$ such that if $2 \le k < \max(n, r+s)$, then $|t_k| > \tau(1+2\epsilon)$. Consequently, for angles satisfying $\gamma \le \theta \le \pi/r - \gamma$,

$$\tau^m(\theta)H_m(z(\theta)) = 2\,\mathrm{Re}\,\frac{\tau^m(\theta)}{P(t_1(\theta))R(t_1(\theta))t_1(\theta)^m} + \frac{\tau^m(\theta)}{2\pi i}\oint_{|t|=\tau(\theta)(1+\epsilon)} \frac{dt}{(P(t)+z(\theta)t^rQ(t))t^{m+1}}.$$

Note that on the contour of integration $P(t)+zt^rQ(t)$ is bounded away from 0, hence the integral approaches 0 as $m \to +\infty$. The sign of $H_m(z)$ therefore is the same as the sign of $2\,\mathrm{Re}\,\frac{\tau^m}{P(t_1)R(t_1)t_1^m}$ provided this expression does not also approach 0. We also note that by the definitions of $g(t)$ and s_h in (4.3) and (4.4)

$$0 = \mathrm{Im}(g(s_h)) = \mathrm{Im}(g(t_1(\theta_h))$$

and hence $g(t_1(\theta_h)) \in \mathbb{R}$ for $0 < \theta_h < \pi/r$. Consequently the modulus of the first term is

$$\frac{2}{|P(t_1(\theta_h))R(t_1(\theta_h))|},$$

which is bounded away from zero on the compact set $\gamma \le \theta_h \le \pi/r-\gamma$. It follows that the sign of $H_m(z(\theta_h))$ is the same as the sign of $g(s_h)$ when $\gamma \le \theta_h \le \pi/r - \gamma$.

Case 2: $\theta_h \to 0$ as $m \to +\infty$. Let ρ be the multiplicity of the zero τ_1 of $P(t)$. Suppose first that $\rho = 1$. In this case, the multiplicity of τ_1 (as a zero of $P(t)$) is 1, and by Lemma 1, t_a (the smallest positive zero of $P(t)R(t)$) satisfies $\tau_1 < t_a < \tau_2$. With $a = -P(t_a)/t_a^rQ(t_a)$ (cf. (2.4)), as $\theta \to 0$, the polynomial $P(t) + zt^rQ(t)$ approaches

$$P(t) + at^rQ(t) = \left(\frac{P(t)}{t^rQ(t)} + a\right)t^rQ(t),$$

which has a real zero at $t = t_a$ with multiplicity at least two, as a complex conjugate pair of zeros converge there. This means in particular that

$$\frac{d}{dt}\left(\frac{P(t)}{t^rQ(t)}\right)\Big|_{t=t_a} = 0. \tag{5.7}$$

On the other hand,

$$R(t) = \frac{t_a\dfrac{d}{dt}\left(\dfrac{P(t)}{t^rQ(t)}\right)}{\dfrac{P(t)}{t^rQ(t)}},$$

and hence using equation (5.7) and Lemma 1 we conclude that

$$0 \overset{\text{Lemma 1}}{\neq} R'(t_a) \overset{(5.7)}{=} \frac{t_a\dfrac{d^2}{dt^2}\left(\dfrac{P(t_a)}{t_a^rQ(t_a)}\right)}{\dfrac{P(t_a)}{t_a^rQ(t_a)}}.$$

Consequently, $\frac{d^2}{dt^2}\left(\frac{P(t_a)}{t_a^r Q(t_a)}\right) \neq 0$, and we conclude that the multiplicity of t_a as a zero of $P(t) + zt^r Q(t)$ is exactly two. Thus the remaining zeros $t_\varkappa$, $2 \leq \varkappa \leq \max\{n, r\}$ still satisfy $|t_\varkappa| > (1 + \epsilon)$ for some $\epsilon > 0$, and the argument we gave in Case 1 still applies.

We next consider the case $\rho > 1$. Lemmas 1 and 6 imply that

$$\lim_{\theta\to 0} \tau(\theta)e^{i\theta} = t_a = \tau_1 = \tau_2 = \cdots = \tau_\rho$$

from which (see Fig. 3.1) we conclude as $\theta \to 0$

$$\theta_k \to \begin{cases} 0 & \text{if } -p_- < k \leq 0 \\ \pi & \text{if } \rho < k \leq p_+ \end{cases} \qquad \text{and} \qquad \eta_j \to \begin{cases} 0 & \text{if } -q_- < j \leq 0 \\ \pi & \text{if } 1 \leq j \leq q_+ \end{cases}.$$

We combine these limits, equation (3.8), and the fact that $\theta_1 = \theta_2 = \cdots = \theta_\rho$ to deduce that $\theta_k \to \pi - \pi/\rho$, for $1 \leq k \leq \rho$, as $\theta \to 0$. If we define the angles θ_k^*, $-p_- < k \leq p_+$, and η_j^*, $-q_- < j \leq q_+$, by

$$\theta_k^* = \begin{cases} -\theta_k & \text{if } -p_- < k \leq 0 \\ \pi - \pi/\rho - \theta_k & \text{if } 1 \leq k \leq \rho \\ \pi - \theta_k & \text{if } \rho < k \leq p_+ \end{cases} \qquad \text{and} \qquad \eta_j^* = \begin{cases} -\eta_j & \text{if } -q_- < j \leq 0 \\ \pi - \eta_j & \text{if } 1 \leq j \leq q_+ \end{cases}, \tag{5.8}$$

then $\theta_k^* \to 0$ and $\eta_j^* \to 0$ as $\theta \to 0$. For $\theta_1 = \theta_2 = \cdots = \theta_\rho$, we obtain the following estimate:

$$\begin{aligned} \frac{\sin\theta_1}{\sin(\theta_1 - \theta)} &= \frac{\sin(\pi/\rho + \theta_1^*)}{\sin(\pi/\rho + \theta_1^* + \theta)} \\ &= \frac{\sin(\pi/\rho) + \cos(\pi/\rho)\theta_1^* + \mathcal{O}(\theta_1^{*2})}{\sin(\pi/\rho) + \cos(\pi/\rho)(\theta_1^* + \theta) + \mathcal{O}(\theta_1^{*2} + \theta_1^*\theta + \theta^2)} \\ &= 1 - \left(\cot\frac{\pi}{\rho}\right)\theta + \mathcal{O}(\theta_1^{*2} + \theta_1^*\theta + \theta^2) \end{aligned}$$

The corresponding estimate for the cases $-p_- < k \leq 0$ or $\rho < k$ is given by

$$\frac{\sin\theta_k}{\sin(\theta_k - \theta)} = \frac{\theta_k^* + \mathcal{O}(\theta_k^{*3})}{\theta_k^* + \theta + \mathcal{O}((\theta_k^* + \theta)^3)} = \frac{\theta_k^*}{\theta_k^* + \theta}\left(1 + \mathcal{O}(\theta_k^{*2} + \theta^2 + \theta_k^*\theta)\right).$$

Similarly if $-q_- < j \leq q_+$, then

$$\frac{\sin\eta_j}{\sin(\eta_j - \theta)} = \frac{\eta_j^*}{\eta_j^* + \theta}\left(1 + \mathcal{O}(\eta_j^{*2} + \theta^2 + \eta_j^*\theta)\right).$$

For indices satisfying $-p_- < k \leq 0$ or $\rho < k$, the identity (cf. (3.3))

$$\frac{\tau_1 \sin\theta_1}{\sin(\theta_1 - \theta)} = \frac{\tau_k \sin\theta_k}{\sin(\theta_k - \theta)}$$

gives

$$(\theta_k^* + \theta)\tau_1 - \tau_1\left(\cot\frac{\pi}{\rho}\right)\theta(\theta_k^* + \theta) = \tau_k\theta_k^* + \mathcal{O}((\theta_1^* + \theta_k^* + \theta)^3)$$

which we solve for θ_k^* and obtain

$$\begin{aligned}\theta_k^* &= \frac{\tau_1\left(\theta - \cot(\pi/\rho)\theta^2\right)}{\tau_k - \tau_1 + \tau_1\cot(\pi/\rho)\theta} + \mathcal{O}((\theta_1^* + \theta)^3) \\ &= \left(\frac{\tau_1}{\tau_k - \tau_1}\theta - \tau_1\frac{\cot(\pi/\rho)\theta^2}{\tau_k - \tau_1}\right)\left(1 - \frac{\tau_1\cot(\pi/\rho)\theta}{\tau_k - \tau_1}\right) + \mathcal{O}((\theta_1^* + \theta)^3) \\ &= \frac{\tau_1}{\tau_k - \tau_1}\theta - \frac{\cot(\pi/\rho)\tau_1\tau_k}{(\tau_k - \tau_1)^2}\theta^2 + \mathcal{O}((\theta_1^* + \theta)^3).\end{aligned}$$

With the similar identity

$$\eta_j^* = \frac{\tau_1}{\gamma_j - \tau_1}\theta - \frac{\cot(\pi/\rho)\tau_1\gamma_j}{(\gamma_j - \tau_1)^2}\theta^2 + \mathcal{O}((\theta_1^* + \theta)^3), \qquad -q_- < j \le q_+,$$

and the angle sum identity

$$\sum_{-p_- < k \le p_+} \theta_k^* - \sum_{-q_- < j \le q_+} \eta_j^* + r\theta = 0$$

obtained from (3.8) and definition (5.8), we deduce that

$$\begin{aligned}\rho\theta_1^* = &-\left(\sum_{k\le 0 \text{ or } k>\rho} \frac{\tau_1}{\tau_k - \tau_1} + r - \sum_{-q_- < j \le q_+} \frac{\tau_1}{\gamma_j - \tau_1}\right)\theta \\ &+ \left(\sum_{k<0 \text{ or } k>\rho} \frac{\cot(\pi/\rho)\tau_1\tau_k}{(\tau_k - \tau_1)^2} - \sum_{-q_- < j \le q_+} \frac{\cot(\pi/\rho)\tau_1\gamma_j}{(\gamma_j - \tau_1)^2}\right)\theta^2 + \mathcal{O}(\theta^3).\end{aligned}$$

We now turn our attention to the representation given in (4.2). Note that as $\theta \to 0$, $z \to -P(t_a)/t_a^r Q(t_a)$ by Lemma 6 part (ii). Since $\rho > 1$, Lemma 1 implies that $t_a = \tau_1$ and hence $-P(t_a)/t_a^r Q(t_a) = 0$. It follows that as $\theta \to 0$, the function $P(t) + zt^r Q(t)$ approaches $P(t)$ which has a zero t_a with multiplicity ρ. Lemma 8 implies there exists $\epsilon > 0$ independent of m such that if $\rho \le \varkappa < \max(n, r+s)$, then $|t_\varkappa| > t_a(1 + 2\epsilon)$ and consequently

$$t_a^m H_m(z(\theta)) = \sum_{0 \le \varkappa < \rho} \frac{t_a^m}{P(t_\varkappa)R(t_\varkappa)t_\varkappa^m} + \frac{1}{2\pi i}\oint_{|t| = t_a(1+\epsilon)} \frac{t_a^m dt}{(P(t) + z(\theta)t^r Q(t))t^{m+1}}$$

where the integral approaches 0 as $m \to \infty$. Thus it is sufficient to consider the sign of sum

$$\sum_{0 \le \varkappa < \rho} \frac{1}{P(t_\varkappa)R(t_\varkappa)t_\varkappa^m} \quad \text{as} \quad t_\varkappa \to t_a = \tau_1, \quad (0 \le \varkappa < \rho) \tag{5.9}$$

if after multiplication by t_a^m, the summation does not approach 0 as $m \to +\infty$ (which is the case, as can be seen from equation (5.13) and the expression in (5.16)). Recall that

$$z(\theta) = -\frac{P(t)}{t^r Q(t)},$$

and by the definition of the $t_\varkappa$s,

$$P(t_\varkappa) + zt_\varkappa^r Q(t_\varkappa) = 0.$$

Combining these two equations and rearranging yields the equation

$$\frac{P(t_\varkappa)}{P(\tau e^{i\theta})} - \frac{Q(t_\varkappa)}{Q(\tau e^{i\theta})}\left(\frac{t_\varkappa}{\tau e^{i\theta}}\right)^r = 0, \qquad (1 \le \varkappa < \rho).$$

We let $t_\varkappa = \tau_1 + \tau_1 \epsilon_\varkappa$, $0 \le \varkappa < \rho$, and expand the left hand side in a Taylor series centered at τ_1 using the identity (cf. equation (3.11))

$$\tau e^{i\theta} - \tau_1 = \tau_1 \frac{\sin\theta}{\sin(\theta_1 - \theta)} e^{i\theta_1}.$$

Doing so produces

$$-\frac{\sin^\rho(\pi/\rho)\epsilon_\varkappa^\rho}{(-1)^\rho \theta^\rho}\left(1 + \mathcal{O}\left(\epsilon_\varkappa + \theta\right)\right) - 1 + \mathcal{O}\left(\epsilon_\varkappa + \theta\right) = 0,$$

which we rearrange to get

$$\frac{\sin^\rho(\pi/\rho)\epsilon_\varkappa^\rho}{(-1)^\rho \theta^\rho} = 1 + \mathcal{O}(\epsilon_\varkappa + \theta).$$

We solve for $\epsilon_\varkappa$ to achieve

$$\epsilon_\varkappa = -\frac{\omega_\varkappa}{\sin(\pi/\rho)}\theta\left(1 + \mathcal{O}(\epsilon_\varkappa + \theta)\right) \qquad (0 \le \varkappa < \rho)$$

with $\omega_\varkappa = e^{(2\varkappa - 1)\pi i/\rho}$. We deduce that $\epsilon_\varkappa \asymp \theta$ and the equation above becomes

$$\epsilon_\varkappa = -\frac{\omega_\varkappa}{\sin(\pi/\rho)}\theta + \mathcal{O}(\theta^2) \qquad (0 \le \varkappa < \rho). \tag{5.10}$$

Using the estimate

$$R(t_\varkappa) = r - \sum_{-p_- < k \le p_+} \frac{t_\varkappa}{t_\varkappa - \tau_k} + \sum_{-q_- < j \le q_+} \frac{t_\varkappa}{t_\varkappa - \gamma_j} = \rho \frac{\sin(\pi/\rho)}{\omega_\varkappa \theta} + \mathcal{O}(1) \tag{5.11}$$

together with

$$\begin{aligned} P(t_\varkappa) &= \frac{P^{(\rho)}(\tau_1)}{\rho!}(t_\varkappa - \tau_1)^\rho\left(1 + \mathcal{O}(t_\varkappa - \tau_1)\right) \\ &= \frac{(-1)^\rho P^{(\rho)}(\tau_1)\tau_1^\rho \omega_\varkappa^\rho \theta^\rho \sin(\pi/\rho)^{-\rho}}{\rho!}\left(1 + \mathcal{O}(\theta)\right), \end{aligned} \tag{5.12}$$

we conclude that the main term of the expression in (5.9) is given by

$$(-1)^{\rho+1}\frac{(\rho - 1)!\sin(\pi/\rho)^{\rho-1}}{P^{(\rho)}(\tau_1)\theta^{\rho-1}\tau_1^{m+\rho}} \sum_{0 \le \varkappa < \rho} \frac{\omega_\varkappa}{(1 + \epsilon_\varkappa)^m}. \tag{5.13}$$

If $\theta \ge \delta/\sqrt{m}$ for some small δ, then (5.10) implies that for each $2 \le \varkappa < \rho$,

$$\left|\frac{1 + \epsilon_1}{1 + \epsilon_\varkappa}\right|^m = \left|1 + (\epsilon_1 - \epsilon_\varkappa) + \mathcal{O}(\theta^2)\right|^m$$

$$
\begin{aligned}
&= \left| e^{m(\epsilon_1 - \epsilon_\varkappa) + \mathcal{O}(m\theta^2)} \right| \\
&= \left| \exp\left(-m \frac{\cos(\pi/\rho) - \cos((2\varkappa - 1)\pi/\rho)}{\sin(\pi/\rho)} \theta + \mathcal{O}(m\theta^2) \right) \right| \\
&\leq \exp\left(-m \frac{\cos(\pi/\rho) - \cos((2\varkappa - 1)\pi/\rho)}{2\sin(\pi/\rho)} \theta \right) \\
&\leq \exp\left(-\delta\sqrt{m} \frac{\cos(\pi/\rho) - \cos((2\varkappa - 1)\pi/\rho)}{2\sin(\pi/\rho)} \right) \to 0
\end{aligned}
$$

as $m \to \infty$. Consequently, the sign of the expression in (5.13) when $\theta = \theta_h$ is determined by the sign of the sum of the first two terms of (5.13) (or equivalently of (5.9)) which is the sign of $g(s_h)$.

On the other hand, if $\theta < \delta/\sqrt{m}$ for $\delta \ll 1$, then equations (5.11) and (5.12) imply that

$$
\begin{aligned}
\frac{1}{P(t_1)R(t_1)t_1^m} &= (-1)^{\rho+1} \frac{(\rho-1)! \sin(\pi/\rho)^{\rho-1}}{P^{(\rho)}(\tau_1)\theta^{\rho-1}\tau_1^{m+\rho}} \omega_1 \left(1 + \epsilon_1\right)^{-m} \\
&\overset{(5.10)}{=} (-1)^{\rho+1} \frac{(\rho-1)! \sin(\pi/\rho)^{\rho-1}}{P^{(\rho)}(\tau_1)\theta^{\rho-1}\tau_1^{m+\rho}} \omega_1 \left(1 - \frac{\omega_1}{\sin(\pi/\rho)}\theta + \mathcal{O}(\theta^2)\right)^{-m} \\
&= (-1)^{\rho+1} \frac{(\rho-1)! \sin(\pi/\rho)^{\rho-1}}{P^{(\rho)}(\tau_1)\theta^{\rho-1}\tau_1^{m+\rho}} \omega_1 \exp\left(\frac{m\omega_1\theta}{\sin(\pi/\rho)} + \mathcal{O}(m\theta^2) \right). \qquad (5.14)
\end{aligned}
$$

The condition

$$0 = \mathrm{Im}(g(s_h)) = \mathrm{Im}(g(t_1(\theta_h)))$$

is equivalent to

$$\mathrm{Im}\left(\frac{1}{g(t_1(\theta_h))}\right) = 0,$$

which gives the solutions

$$\theta_h = \frac{h - 1/\rho}{m}\pi + \mathcal{O}(\theta^2).$$

When $\theta = \theta_h$, equation (5.14) shows that $g(s_h)$ and $(-1)^{h+\rho+1}P^{(\rho)}(\tau_1)$ carry the same sign. We claim that the sign of (5.13) is also $(-1)^{h+\rho+1}P^{(\rho)}(\tau_1)$. Writing

$$(1 + \epsilon_\varkappa)^m = \exp\left(-\frac{m\omega_\varkappa\theta}{\sin(\pi/\rho)}\right)\left(1 + \mathcal{O}\left(\frac{h^2}{m}\right)\right),$$

for large m, we see that the sign of (5.13) is same as that of

$$(-1)^{\rho+1}P^{(\rho)}(\tau_1) \sum_{0 \leq \varkappa < \rho} \omega_\varkappa \exp\left(\frac{m\omega_\varkappa\theta}{\sin(\pi/\rho)}\right) \qquad (5.15)$$

or, with $\theta = \theta_h$, the sign of

$$(-1)^{\rho+1}P^{(\rho)}(\tau_1) \sum_{0 \leq \varkappa < \rho} \omega_\varkappa \exp\left(\frac{\omega_\varkappa(h - 1/\rho)\pi}{\sin(\pi/\rho)}\right). \qquad (5.16)$$

Besides the factor $(-1)^{\rho+1}P^{(\rho)}(\tau_1)$, the first summand in (5.16) is real and its sign is $(-1)^h$. The claim now follows from Proposition 2.

Case 3: $\theta \to \pi/r$ **as** $m \to +\infty$**.** If we define the angles $\theta^* = \pi/r - \theta$, and θ_k^*, $-p_- < k \le p_+$, and η_j^*, $-q_- < j \le q_+$, by

$$\theta_k^* = \begin{cases} \pi - \theta_k & \text{if } 0 < k \le p_+ \\ -\theta_k & \text{if } -p_- < k \le 0 \end{cases} \qquad \text{and} \qquad \eta_j^* = \begin{cases} \pi - \eta_j & \text{if } 1 \le j \le q_+ \\ -\eta_j & \text{if } -q_- < j \le 0 \end{cases},$$

then as a consequence of Lemma 6, θ^*, θ_k^* and η_j^* all approach 0 as $\theta \to \pi/r$. Using the equations (3.3) and (3.2) we obtain

$$\begin{aligned} \tau &= \tau_k \frac{\sin\theta_k}{\sin(\theta_k - \theta)} \\ &= \tau_k \frac{\theta_k^* + \mathcal{O}(\theta_k^{*3})}{\sin(\pi/r) + \cos(\pi/r)(\theta_k^* - \theta^*) + \mathcal{O}((\theta_k^* + \theta^*)^2)} \\ &= \frac{\tau_k \theta_k^*}{\sin(\pi/r)} \left(1 - \cot\frac{\pi}{r}(\theta_k^* - \theta^*) + \mathcal{O}((\theta_k^* + \theta^*)^2)\right), \end{aligned}$$

and similarly,

$$\tau = \frac{\gamma_j \eta_j^*}{\sin(\pi/r)} \left(1 - \cot\frac{\pi}{r}(\eta_j^* - \theta^*) + \mathcal{O}((\eta_j^* + \theta^*)^2)\right).$$

Thus for any $-p_- < k \le p_+$ and η_j^*, $-q_- < j \le q_+$, we have $\theta_k^* \asymp \eta_j^* \asymp \tau$, and consequently from the equations above we deduce

$$\theta_k^* = \frac{\tau}{\tau_k} \sin\frac{\pi}{r}\left(1 + \mathcal{O}(\tau + \theta^*)\right) \qquad \text{and} \qquad \eta_j^* = \frac{\tau}{\gamma_j} \sin\frac{\pi}{r}\left(1 + \mathcal{O}(\tau + \theta^*)\right).$$

We combine these identities with the angle sum identity

$$\sum_{-p_- < k \le p_+} \theta_k^* - \sum_{-q_- < j \le q_+} \eta_j^* = r\theta^*$$

obtained from (3.8) to get

$$\tau \sin\frac{\pi}{r} \left(\sum_{-p_- < k \le p_+} \frac{1}{\tau_k} - \sum_{-q_- < j \le q_+} \frac{1}{\gamma_j} \right) = r\theta^*(1 + \mathcal{O}(\tau + \theta^*)).$$

Thus $\theta^* \asymp \tau$ and the equation above becomes

$$\tau \sin\frac{\pi}{r} \left(\sum_{-p_- < k \le p_+} \frac{1}{\tau_k} - \sum_{-q_- < j \le q_+} \frac{1}{\gamma_j} \right) = r\theta^*(1 + \mathcal{O}(\theta^*)). \tag{5.17}$$

We claim that as $\theta \to \pi/r$ (which implies $\tau \to 0$), the polynomial in t

$$\frac{P(t)}{P(\tau e^{i\theta})} - \frac{Q(t)}{Q(\tau e^{i\theta})} \left(\frac{t}{\tau e^{i\theta}}\right)^r$$

has exactly r zeros approaching the circle with radius τ centered at the origin, each of which satisfies $t_\varkappa/\tau \to e_\varkappa = e^{(2\varkappa-1)\pi i/r}$, $0 \le \varkappa < r$. Indeed, if we let $u = t/\tau$, then this polynomial becomes

$$\frac{P(\tau u)}{P(\tau e^{i\theta})} + \frac{Q(\tau u)}{Q(\tau e^{i\theta})}u^r$$

which approaches the polynomial $1 + u^r$ as $\tau \to 0$ and the claim follows.

From the claim above we assume $t_\varkappa$, $0 \le \varkappa < r$, approach $\tau e_\varkappa$ and the remaining zeros $t_\varkappa$, $r \le \varkappa < \max(n, r+s)$, satisfy $|t_\varkappa| > \tau(1+2\epsilon)$ for some fixed $\epsilon > 0$ independent of m. Consequently

$$\tau^m H_m(z(\theta)) = \sum_{0\le\varkappa<r} \frac{\tau^m}{P(t_\varkappa)R(t_\varkappa)t_\varkappa^m} + \frac{1}{2\pi i}\oint\limits_{|t|=\tau(1+\epsilon)} \frac{\tau^m}{(P(t)+z(\theta)t^rQ(t))t^{m+1}}dt$$

where the integral approaches 0 as $m \to \infty$. This leaves us to consider the sign of

$$\sum_{0\le\varkappa<r} \frac{1}{P(t_\varkappa)R(t_\varkappa)t_\varkappa^m} \tag{5.18}$$

if after multiplication by τ^m, the sum does not approach 0 as $m \to \infty$ (which is the case, as can be seen from (5.26) and Remark 3). Writing $t_\varkappa = \tau(e_\varkappa + \epsilon_\varkappa)$, $\epsilon_\varkappa \in \mathbb{C}$, we expand the left hand side of

$$\frac{P(t_\varkappa)Q(\tau e^{i\theta})}{P(\tau e^{i\theta})Q(t_\varkappa)} - \left(\frac{t_\varkappa}{\tau e^{i\theta}}\right)^r = 0$$

in a Taylor series centered at $\tau e^{i\theta}$ to obtain

$$1 + \left(\frac{P'(0)}{P(0)} - \frac{Q'(0)}{Q(0)}\right)\left(e_\varkappa + \epsilon_\varkappa - e^{i\theta}\right)\tau + (e_\varkappa + \epsilon_\varkappa)^r e^{ir\theta^*} = \mathcal{O}(\tau^2), \tag{5.19}$$

where by (5.17)

$$\frac{P'(0)}{P(0)} - \frac{Q'(0)}{Q(0)} = -\sum_{-p_-<k\le p_+}\frac{1}{\tau_k} + \sum_{-q_-<j\le q_+}\frac{1}{\gamma_j} = -\frac{r\theta^*}{\tau\sin(\pi/r)} + \mathcal{O}(\theta^{*2}).$$

With $\mathcal{O}(\tau^2) = \mathcal{O}(\theta^{*2})$ (see (5.17)) and

$$(e_\varkappa + \epsilon_\varkappa)^r e^{ir\theta^*} = -1 - \frac{r\epsilon_\varkappa}{e_\varkappa} - ir\theta^* + \mathcal{O}\left(\epsilon_\varkappa^2 + \theta^{*2}\right),$$

equation (5.19) can be rearranged as

$$-\frac{\theta^*}{\sin(\pi/r)}\left(e_\varkappa + \epsilon_\varkappa - e^{i\pi/r}\right) - \frac{\epsilon_\varkappa}{e_\varkappa} - i\theta^* = \mathcal{O}(\epsilon_\varkappa^2 + \theta^{*2})$$

from which we express $\epsilon_\varkappa$ as

$$\epsilon_\varkappa = \frac{e_\varkappa\theta^*\left(\cos(\pi/r) - e_\varkappa\right)}{\sin(\pi/r)} + \mathcal{O}(\theta^{*2}) \qquad (0 \le \varkappa < r). \tag{5.20}$$

Using the estimates

$$R(t_\varkappa) = r - \sum_{-p_-<k\le p_+}\frac{t_\varkappa}{t_\varkappa - \tau_k} + \sum_{-q_-<j\le q_+}\frac{t_\varkappa}{t_\varkappa - \gamma_j} = r + \mathcal{O}(\theta^*) \tag{5.21}$$

and

$$P(t_\varkappa) = P(0)(1 + \mathcal{O}(\theta^*)), \tag{5.22}$$

we write the main term of (5.18) as

$$\frac{1}{rP(0)\tau^m} \sum_{0\le\varkappa<r} \frac{1}{(e_\varkappa + \epsilon_\varkappa)^m}. \tag{5.23}$$

In the case $\theta^* \ge \delta/\sqrt{m}$ for small δ independent of m, we apply the following computations for $2 \le \varkappa < r$, which are similar to those in Case 2

$$\begin{aligned}\left|\frac{e_1+\epsilon_1}{e_\varkappa+\epsilon_\varkappa}\right|^m &= \left|\frac{1+\epsilon_1/e_1}{1+\epsilon_\varkappa/e_\varkappa}\right|^m \\ &\overset{(5.20)}{=} \left|1 - \frac{e_1 - e_\varkappa}{\sin(\pi/r)}\theta^* + \mathcal{O}(\theta^{*2})\right|^m \\ &\le \exp\left(-\delta\sqrt{m}\frac{\cos(\pi/r) - \cos((2\varkappa-1)\pi/r)}{2\sin(\pi/r)}\right) \to 0\end{aligned}$$

as $m \to \infty$, from which we conclude that when $\theta = \theta_h$ the sign of (5.23) is determined by the sign of the sum of the first two terms of (5.23) (or equivalently of (5.18)) which is the sign of $g(s_h)$.

In the case $\theta^* < \delta/\sqrt{m}$ for small δ, equations (5.22) and (5.21) imply that

$$\begin{aligned}\frac{1}{P(t_1)R(t_1)t_1^m} &= \frac{1}{rP(0)\tau^m}(e_1+\epsilon_1)^{-m} \\ &\overset{(5.20)}{=} \frac{1}{rP(0)\tau^m}e_1^{-m}\left(1 + \frac{\theta^*\left(\cos(\pi/r) - e_1\right)}{\sin(\pi/r)} + \mathcal{O}(\theta^{*2})\right)^{-m} \\ &= \frac{1}{rP(0)\tau^m}e_1^{-m}\exp\left(im\theta^* + \mathcal{O}(m\theta^{*2}\right).\end{aligned} \tag{5.24}$$

The condition $0 = \operatorname{Im} g(s_h) = \operatorname{Im} g(t_1(\theta_h))$ gives the solutions

$$\theta_h^* = \frac{\pi}{r} - \frac{h\pi}{m} + \mathcal{O}(\theta^{*2}). \tag{5.25}$$

When $\theta^* = \theta_h^*$, equation (5.24) shows that the sign of $g(s_h)$ agrees with that of $(-1)^h P(0)$. In order to demonstrate that the sign of (5.18) is the same as that of $(-1)^h P(0)$, we use a calculation analogous to that in (5.24) to write the main term of (5.23) as

$$\frac{e^{-m\theta^*\cot\pi/r}}{rP(0)\tau^m} \sum_{0\le\varkappa<r} e_\varkappa^{-m} \exp\left(\frac{m\theta^* e_\varkappa}{\sin(\pi/r)}\right). \tag{5.26}$$

It thus remains to show that the sign of the sum in (5.26) is $(-1)^h$ when $\theta^* = \theta_h^*$. This claim follows by setting $x = m\theta^*/\sin(\pi/r)$ in Proposition 2. The analysis of Case 3, and the proof the lemma is complete. □

Lemma 10. *Let $H_m(z)$ be as in Theorem 1, and let $z(\theta)$ be as in Definition 3. If $2r \mid m$ then $H_m(z(\theta))$ has a zero on the interval $(\theta_{m/r-1}, \pi/r)$.*

Proof. In case $2r|m$, the inequality $\theta_h^* > 0$ and (5.25) imply that the largest value of h is $m/r-1$. Since the sign of the summation in (5.26) is the same as the sign of

$$\sum_{0\leq\varkappa<r} e_\varkappa^{-m} = r(-1)^{\lfloor m/r\rfloor}$$

as $\theta^* \to 0^+$, and is $(-1)^{m/r-1}$ when $\theta^* = \theta^*_{m/r-1}$ (see the claim after (5.26)), we conclude that $H_m(z(\theta))$ has a zero on the interval $(\theta_{m/r-1}, \pi/r)$. $\square$

Lemma 11. *Let $H_m(z)$ be as in Theorem 1, and let $z(\theta)$ be as in Definition 3. As $\theta \to 0$ and $\theta \to (\pi/r)-$, the limits (possibly infinite) of $H_m(z(\theta))$ are nonzero and their signs are $(-1)^{p_+}$ and $(-1)^{p_+ + \lfloor m/r\rfloor}$ respectively.*

Proof. Since the k-th derivative in θ^* of the sum in (5.26) at $\theta^* = 0$ is 0 for $0 \leq k < (m \mod r)$ and its $(m \mod r)$th derivative at 0 is

$$m^{(m \mod r)} \sum_{0\leq\varkappa<r} e_\varkappa^{-m+(m \mod r)} = rm^{(m \mod r)}(-1)^{\lfloor m/r\rfloor},$$

we conclude that the sign of (5.26) is $(-1)^{\lfloor m/r\rfloor}$ as $\theta^* \to 0^+$. Thus the sign of $H_m(z(\theta))$ as $\theta \to (\pi/r)^-$ is the sign of $(-1)^{\lfloor m/r\rfloor}P(0)$ which is $(-1)^{\lfloor m/r\rfloor + p_+}$ by (2.1).

Similarly, since the $(\rho-1)$st derivative of

$$\sum_{0\leq\varkappa<\rho} \omega_\varkappa \exp\left(\frac{m\omega_\varkappa\theta}{\sin(\pi/\rho)}\right)$$

in θ at $\theta = 0$ is

$$m^{\rho-1}\sum_{0\leq\varkappa<\rho} \omega_\varkappa^\rho = -m^{\rho-1},$$

we conclude this sum is negative when $\theta \to 0$. Thus the sign of (5.15) is the sign of

$$(-1)^\rho P^{(\rho)}(\tau_1) \overset{(2.1)}{=} (-1)^\rho \rho! \prod_{\substack{-p_- < k \leq p_+ \\ k\neq 1}} (\tau_1 - \tau_k)$$

which is $(-1)^\rho(-1)^{p_+-\rho} = (-1)^{p_+}$. $\square$

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