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# Inequalities Concerning the $L^p$-Norm of a Polynomial

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In this paper we obtain $L^p$, $p \geq 1$, inequalities for the class of polynomials having no zeros in $|z| < K$, $K \geq 1$. Our result generalizes as well as improves upon some well known results. © 1998 Academic Press

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## 1. INTRODUCTION AND STATEMENT OF RESULTS

Let $p(z) = \sum_{v=0}^{n} a_v z^v$ be a polynomial of degree $n$. Then we have

$$\left(\int_0^{2\pi} |p'(e^{i\theta})|^q \, d\theta\right)^{1/q} \leq n\left(\int_0^{2\pi} |p(e^{i\theta})|^q \, d\theta\right)^{1/q}, \qquad q \geq 1, \quad (1.1)$$



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and

$$\left(\int_0^{2\pi}|p(Re^{i\theta})|^q\,d\theta\right)^{1/q} \le R^n\left(\int_0^{2\pi}|p(e^{i\theta})|^q\,d\theta\right)^{1/q}, \qquad q \ge 1,\ R > 1. \tag{1.2}$$

Inequality (1.1) is due to Zygmund [12] and inequality (1.2) is easy to prove.

For $p(z) \neq 0$ in $|z| < 1$ the inequalities (1.1) and (1.2) have been replaced, respectively, by

$$\left(\int_0^{2\pi}|p'(e^{i\theta})|^q\,d\theta\right)^{1/q} \le n(C_q)^{1/q}\left(\int_0^{2\pi}|p(e^{i\theta})|^q\,d\theta\right)^{1/q}, \qquad q \ge 1, \tag{1.3}$$

where

$$C_q = 2^{-q}\sqrt{\pi}\,\Gamma\left(\tfrac{1}{2}q + 1\right)\Big/\Gamma\left(\tfrac{1}{2}q + \tfrac{1}{2}\right),$$

and

$$\left(\int_0^{2\pi}|p(Re^{i\theta})|^q\,d\theta\right)^{1/q} \le (K_q)^{1/q}\left(\int_0^{2\pi}|p(e^{i\theta})|^q\,d\theta\right)^{1/q},$$
$$R \ge 1,\ q \ge 1, \tag{1.4}$$

where

$$K_q = \int_0^{2\pi}|1 + R^n e^{in\theta}|^q\,d\theta\Big/\int_0^{2\pi}|1 + e^{in\theta}|\,d\theta.$$

Inequality (1.3) was first proved by de-Bruijn [2] (for another proof, see Rahman [11]), and (1.4) is due to Boas and Rahman [1]. The extremal polynomial in each case is $p(z) = \alpha + \beta z^n$, $|\alpha| = |\beta|$.

Dewan and Govil [4] (see also Govil and Jain [7]) considered the class of polynomials $p(z)$ satisfying $p(z) \equiv z^n p(1/z)$ and obtained a sharp inequality analogous to (1.1).

There is greater interest attached to the case when $p(z)$ does not vanish in the circle $|z| < K$, where $K$ is a positive number. The answer to this more general question for the case when $K \le 1$ and $q = 2$ was given by Rahman [10]. For the case when $K \ge 1$, the following result is known [5].

THEOREM A. *If $p(z) = \sum_{v=0}^n a_v z^v$ is a polynomial of degree $n$, having no zeroes in $|z| < K$, $K \ge 1$, then, for $q \ge 1$, we have*

$$\left(\int_0^{2\pi}|p'(e^{i\theta})|^q\,d\theta\right)^{1/q} \le n(E_q)\left(\int_0^{2\pi}|p(e^{i\theta})|^q\,d\theta\right)^{1/q}, \tag{1.5}$$

*where*

$$E_q = \left(2\pi \Big/ \int_0^{2\pi} |K + e^{i\theta}|^q \, d\theta\right)^{1/q}.$$

Theorem A is not sharp and the sharp inequality does not seem to be obtainable even for $q = 2$. In this direction Dewan and Bidkham [3] obtained an inequality for $q = 2$, the bound of which is, in general, better than the bound obtained by (1.5).

In this paper we shall generalize Theorem A as well as improve upon the bound obtained in inequality (1.5) by involving the coefficients $|a_0|$ and $|a_\mu|$, $1 \le \mu \le n$. Besides this, we also prove an inequality analogous to (1.2). We prove

THEOREM 1. *If $p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$ is a polynomial of degree $n$ such that $p(z)$ has no zeros in the disk $|z| < K$, $K \ge 1$, then, for $q \ge 1$,*

$$\left(\int_0^{2\pi} |p'(e^{i\theta})|^q \, d\theta\right)^{1/q} \le nS_q \left(\int_0^{2\pi} |p(e^{i\theta})|^q \, d\theta\right)^{1/q}, \qquad (1.6)$$

*where*

$$S_q = \left(2\pi \Big/ \int_0^{2\pi} |S_{\mu c} + e^{i\theta}|^q \, d\theta\right)^{1/q}$$

*and*

$$S_{\mu c} = \frac{K^{\mu+1}\left(\mu/n |a_\mu/a_0| K^{\mu-1} + 1\right)}{1 + \mu/n |a_\mu/a_0| K^{\mu+1}}.$$

*The result is best possible in the case $K = 1$ and equality holds for the polynomial $p(z) = 1 + z^n$.*

For $K = \mu = 1$, Theorem 1 reduces to (1.3) due to de-Bruijn [2]. For $\mu = 1$, Theorem 1 is, in general, an improvement over Theorem A due to Govil and Rahman [5]. To show this we have to prove that for $q \ge 1$,

$$S_q \le E_q,$$

which is equivalent to

$$\left(2\pi \Big/ \int_0^{2\pi} |S_{\mu c} + e^{i\theta}|^q \, d\theta\right)^{1/q} \le \left(2\pi \Big/ \int_0^{2\pi} |K + e^{i\theta}|^q \, d\theta\right)^{1/q}$$

or

$$\int_0^{2\pi} |K + e^{i\theta}|^q \, d\theta \le \int_0^{2\pi} |S_{\mu c} + e^{i\theta}|^q \, d\theta. \qquad (1.7)$$

Now to prove (1.7) it is sufficient to prove that

$$K \le S_{\mu c}$$

or

$$K \le \frac{K^2(1/n|a_1/a_0| + 1)}{1 + 1/n|a_1/a_0|K^2} \quad \text{for } \mu = 1,$$

which on simplification gives

$$K^2\left(\frac{1}{n}\left|\frac{a_1}{a_0}\right|\right)(K-1) \le K(K-1),$$

which implies

$$\left|\frac{a_1}{a_0}\right| \le \frac{n}{K}. \tag{1.8}$$

Since (1.8) is always true (see [6, pp. 320–321]), hence (1.7) follows.

If we assume $\mu = 2$ in Theorem 1, then we get the following:

COROLLARY 1. *If $p(z) = \sum_{v=0}^{n} a_v z^v$ is a polynomial of degree $n$, having no zeros in $|z| < K$, $K \ge 1$, and $p'(0) = 0$, then, for $q \ge 1$,*

$$\left(\int_0^{2\pi} |p'(e^{i\theta})|^q \, d\theta\right)^{1/q} \le n\left(2\pi\Big/\int_0^{2\pi} |K^2 + e^{i\theta}|^q \, d\theta\right)^{1/q}\left(\int_0^{2\pi} |p(e^{i\theta})|^q \, d\theta\right)^{1/q}. \tag{1.9}$$

If is easy to see that Corollary 1 also provides a generalization and improvement to Theorem A due to Govil and Rahman [5] and to a result proved by Dewan and Bidkham [3].

For $q = 1$, Theorem 1 yields

COROLLARY 2. *Let $p(z)$ be the same as in Theorem 1. Then*

$$\int_0^{2\pi} |p'(e^{1\theta})| \, d\theta \le n\left(2\pi\Big/\int_0^{2\pi} |S_{\mu c} + e^{i\theta}| \, d\theta\right)\int_0^{2\pi} |p(e^{i\theta})| \, d\theta. \tag{1.10}$$

THEOREM 2. *If $p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$ is a polynomial of degree $n$, having no zeroes in $|z| < K$, $K \ge 1$, then, for $R > 1$,*

$$\int_0^{2\pi} |p(Re^{i\theta})| \, d\theta \le \{S_1(R^n - 1) + 1\}\int_0^{2\pi} |p(e^{i\theta})| \, d\theta, \tag{1.11}$$

*where $S_1 = 2\pi/\int_0^{2\pi} |S_{\mu c} + e^{i\theta}| \, d\theta$ and $S_{\mu c}$ is the same as defined in Theorem 1.*

## 2. A LEMMA

We need the following result due to Qazi (see [8, p. 339]) for the proof of the theorems.

LEMMA. *If* $p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$ *has no zeroes in* $|z| < K$, $K \geq 1$, *then, for* $|z| = 1$,

$$|p'(z)| \leq \frac{1 + \mu/n|a_\mu/a_0|K^{\mu+1}}{\mu/n|a_\mu/a_0|K^{\mu-1} + 1} \frac{1}{K^{\mu+1}} |q'(z)|, \tag{2.1}$$

*where*

$$q(z) = z^n \overline{\{p(1/\bar{z})\}}.$$

*Proof of Theorem* 1. By a known result due to de-Bruijn [2, p. 1271], we have for every $q \geq 1$ and real $\alpha$,

$$\int_0^{2\pi} \left| p(e^{i\theta}) - \frac{e^{i\theta} p'(e^{i\theta})}{n} + e^{i(\alpha+\theta)} \frac{p'(e^{i\theta})}{n} \right|^q d\theta \leq \int_0^{2\pi} \left| p(e^{i\theta}) \right|^q d\theta.$$

Integrating both the sides of the above inequality with respect to $\alpha$ from 0 to $2\pi$, we get

$$\begin{aligned} &\int_0^{2\pi} d\alpha \int_0^{2\pi} \left| p(e^{i\theta}) - \frac{e^{i\theta} p'(e^{i\theta})}{n} + e^{i(\alpha+\theta)} \frac{p'(e^{i\theta})}{n} \right|^q d\theta \\ &\qquad \leq 2\pi \int_0^{2\pi} \left| p(e^{i\theta}) \right|^q d\theta, \qquad q \geq 1. \end{aligned} \tag{3.1}$$

Note that $p'(e^{i\theta})$ can be zero only at a countable number of points. Besides, we can clearly invert the order of integration on the left-hand side of (3.1). Therefore,

$$\begin{aligned} &\int_0^{2\pi} d\alpha \int_0^{2\pi} \left| p(e^{i\theta}) - \frac{e^{i\theta} p'(e^{i\theta})}{n} + e^{i(\alpha+\theta)} \frac{p'(e^{i\theta})}{n} \right|^q d\theta \\ &\quad = \int_0^{2\pi} d\alpha \int_0^{2\pi} \left| \frac{p'(e^{i\theta})}{n} \right|^q \left| e^{i\alpha} + \frac{np(e^{i\theta}) - e^{i\theta} p'(e^{i\theta})}{e^{i\theta} p'(e^{i\theta})} \right|^q d\theta \\ &\quad = \int_0^{2\pi} \left| \frac{p'(e^{i\theta})}{n} \right|^q d\theta \int_0^{2\pi} \left| e^{i\alpha} + \frac{np(e^{i\theta}) - e^{i\theta} p'(e^{i\theta})}{e^{i\theta} p'(e^{i\theta})} \right|^q d\alpha. \end{aligned} \tag{3.2}$$

Thus for $0 \le \theta < 2\pi$ and every $q \ge 1$ and using the lemma, we get

$$\begin{aligned}
\int_0^{2\pi} & \left| e^{i\alpha} + \frac{np(e^{i\theta}) - e^{i\theta}p'(e^{i\theta})}{e^{i\theta}p'(e^{i\theta})} \right|^q d\alpha \\
&= \int_0^{2\pi} \left| e^{i\alpha} + \left| \frac{np(e^{i\theta}) - e^{i\theta}p'(e^{i\theta})}{e^{i\theta}p'(e^{i\theta})} \right| \right|^q d\alpha \\
&= \int_0^{2\pi} \left| e^{i\alpha} + \left| \frac{q'(e^{i\theta})}{p'(e^{i\theta})} \right| \right|^q d\alpha \\
&\ge \int_0^{2\pi} \left| e^{i\alpha} + S_{\mu c} \right|^q d\alpha. \qquad (3.3)
\end{aligned}$$

Combining inequalities (3.1), (3.2), and (3.3), we get, for $q \ge 1$,

$$\int_0^{2\pi} \left| \frac{p'(e^{i\theta})}{n} \right|^q d\theta \int_0^{2\pi} \left| e^{i\alpha} + S_{\mu c} \right|^q d\alpha \le 2\pi \int_0^{2\pi} |p(e^{i\theta})|^q \, d\theta,$$

which gives

$$\int_0^{2\pi} |p'(e^{i\theta})|^q \, d\theta \le \left( 2\pi n^q \Big/ \int_0^{2\pi} \left| e^{i\alpha} + S_{\mu c} \right|^q d\alpha \right) \int_0^{2\pi} |p(e^{i\theta})|^q \, d\theta$$

from which the theorem follows.

*Remark*. The proof of the theorem can also be obtained from the arguments used in Rahman [9] for proving de-Bruijn's theorem.

*Proof of Theorem* 2. For each $0 \le \theta < 2\pi$, we have

$$p(Re^{i\theta}) - p(e^{i\theta}) = \int_1^R e^{i\theta} p'(re^{i\theta}) \, dr, \qquad R > 1,$$

which implies

$$|p(Re^{i\theta})| \le \int_1^R |p'(re^{i\theta})| \, dr + |p(e^{i\theta})|, \qquad R > 1.$$

Integrating both sides of the above inequality with respect to $\theta$ from 0 to $2\pi$, we get

$$\int_0^{2\pi} |p(Re^{i\theta})| \, d\theta \le \int_1^R dr \int_0^{2\pi} |p'(re^{i\theta})| \, d\theta + \int_0^{2\pi} |p(e^{i\theta})| \, d\theta. \quad (3.4)$$

Now since the polynomial $p'(z)$ is of degree $(n-1)$, therefore, inequality (1.2), for $q=1$ reduces to

$$\int_0^{2\pi} \left| p'(Re^{i\theta}) \right| d\theta \le R^{n-1} \int_0^{2\pi} \left| p'(e^{i\theta}) \right| d\theta.$$

Applying the above inequality to (3.4), we get

$$\int_0^{2\pi} \left| p(Re^{i\theta}) \right| d\theta \le \int_1^R r^{n-1}\, dr \int_0^{2\pi} \left| p'(e^{i\theta}) \right| d\theta + \int_0^{2\pi} \left| p(e^{i\theta}) \right| d\theta,$$

which on using Corollary 2 gives

$$\int_0^{2\pi} \left| p(Re^{i\theta}) \right| d\theta \le nS_1 \int_0^{2\pi} \left| p(e^{i\theta}) \right| d\theta \int_1^R r^{n-1}\, dr + \int_0^{2\pi} \left| p(e^{i\theta}) \right| d\theta, \tag{3.5}$$

where

$$S_1 = 2\pi \Big/ \int_0^{2\pi} \left| S_{\mu c} + e^{i\theta} \right| d\theta$$

and

$$S_{\mu c} = \frac{K^{\mu+1}\left( \mu/n \left| a_\mu/a_0 \right| K^{\mu-1} + 1 \right)}{1 + \mu/n \left| a_\mu/a_0 \right| K^{\mu+1}}.$$

Thus inequality (3.5) is equivalent to

$$\int_0^{2\pi} \left| p(Re^{i\theta}) \right| d\theta \le \left\{ S_1(R^n - 1) + 1 \right\} \int_0^{2\pi} \left| p(e^{i\theta}) \right| d\theta,$$

which completes the proof of the theorem.

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