Journal of Mathematical Analysis and Applications **236,** 350–369 (1999)
Article ID jmaa.1999.6435, available online at http://www.idealibrary.com on IDEAL®

LaSalle-Type Theorems for Stochastic Differential Delay Equations*

Xuerong Mao

Department of Statistics and Modelling Science, University of Strathclyde, Glasgow G1 1XH, Scotland, United Kingdom

Submitted by V. Lakshmikantham

Received May 20, 1998

The main aim of this paper is to establish the LaSalle-type asymptotic convergence theorems for the solutions of stochastic differential delay equations. These stochastic versions are then applied to establish sufficient criteria for the stochastically asymptotic stability of the delay equations. Several examples are also given for illustration. © 1999 Academic Press

Key Words: Lyapunov function; LaSalle's theorem; supermartingale convergence theorem; Itô's formula.

1. INTRODUCTION

More than 100 years ago Lyapunov introduced the concept of stability of a dynamic system and created a very powerful tool known as the method of Lyapunov functions in the study of stability. The Lyapunov method has been developed and applied by many authors during the past century. One of the important developments in this direction is the LaSalle theorem (cf. LaSalle [10]), from which follow many of the classical Lyapunov results on stability. Another one is Hale's extension to functional differential equations (cf. Hale [3]) with the introduction of the extended dynamical systems (cf. Hale and Infante [4]). On the other hand, since Itô introduced his stochastic calculus about 50 years ago, the theory of stochastic differential equations has been developed very quickly. Especially the Lyapunov method has been developed to deal with stochastic stability by many authors, and we here only mention Arnold [1], Friedman [2], Has'minskii [5], Kushner [7], Kolmanovskii and Myshkis [8], Ladde and Lakshmikan-

* Partially supported by the Royal Society and the EPSRC/BBSRC.

0022-247X/99 $30.00

than [9], Mohammed [15] and myself [12−14]. However, so far there seems no stochastic version of the LaSalle asymptotic convergence theorem for stochastic differential delay equations, and the main aim of this paper is to extend the LaSalle theorem from ordinary differential equations to stochastic differential delay equations. We also apply the LaSalle-type theorems to establish sufficient conditions for the asymptotic stability of stochastic differential delay equations.

2. ASYMPTOTIC CONVERGENCE

Throughout this paper, unless otherwise specified, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $B(t) = (B_1(t), \dots, B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $|\cdot|$ denote the Euclidean norm in R^n. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\operatorname{trace}(A^T A)}$. Let $\tau > 0$ and $C([-\tau, 0]; R^n)$ denote the family of all continuous R^n-valued functions on $[-\tau, 0]$. Let $C^b_{\mathcal{F}_0}([-\tau_0; R^n)$ be the family of all $\mathcal{F}_0$-measurable bounded $C([-\tau, 0]; R^n)$-valued random variables $\xi = \{\xi(\theta): -\tau \leq \theta \leq 0\}$.

Consider an n-dimensional stochastic differential delay equation

$$dx(t) = f(x(t), x(t-\tau), t)\,dt + g(x(t), x(t-\tau), t)\,dB(t), \quad (2.1)$$

on $t \geq 0$ with initial data $\{x(\theta): -\tau \leq \theta \leq 0\} = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$. Here $f: R^n \times R^n \times R_+ \to R^n$ and $g: R^n \times R^n \times R_+ \to R^{n\times m}$. As a standing condition, we impose a hypothesis:

(H1) Both f and g satisfy the local Lipschitz condition and the linear growth condition. That is, for each $k = 1, 2, \dots$, there is a $c_k > 0$ such that

$$|f(x,y,t) - f(\bar{x},\bar{y},t)| \vee |g(x,y,t) - g(\bar{x},\bar{y},t)| \leq c_k\big(|x-\bar{x}| + |y-\bar{y}|\big)$$

for all $t \geq 0$ and those $x, y, \bar{x}, \bar{y} \in R^n$ with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq k$, and there is moreover a $c > 0$ such that

$$|f(x,y,t)| \vee |g(x,y,t)| \leq c(1 + |x| + |y|)$$

for all $(x, y, t) \in R^n \times R^n \times R_+$.

It is known (cf. Mao [13, 14] and Mohammed [15]) that under hypothesis (H1), Eq. (2.1) has a unique continuous solution on $t \geq -\tau$, which is denoted by $x(t;\xi)$ in this paper. Moreover, for every $p > 0$,

$$E\Big[\sup_{-\tau\le s\le t}|x(s;\xi)|^p\Big] < \infty \quad \text{on } t \ge 0.$$

Let $C^{2,1}(R^n \times R_+; R_+)$ denote the family of all nonnegative functions $V(x,t)$ on $R^n \times R_+$ which are continuously twice differentiable in x and once differentiable in t. For each $V \in C^{2,1}(R^n \times R_+; R_+)$, define an operator $\mathscr{L}V$ from $R^n \times R^n \times R_+$ to R by

$$\begin{aligned}\mathscr{L}V(x,y,t) = V_t(x,t) = V_x(x,t)f(x,y,t)&\\ + \tfrac{1}{2}\operatorname{trace}\big[g^T(x,y,t)V_{xx}(x,t)g(x,y,t)\big]&,\end{aligned}$$

where

$$V_t(x,t) = \frac{\partial V(x,t)}{\partial t}, \qquad V_x(x,t) = \left(\frac{\partial V(x,t)}{\partial x_1},\ldots,\frac{\partial V(x,t)}{\partial x_n}\right),$$

$$V_{xx}(x,t) = \left(\frac{\partial^2 V(x,t)}{\partial x_i\,\partial x_j}\right)_{n\times n}.$$

Let us stress that $\mathscr{L}V$ is defined on $R^n \times R^n \times R_+$ while V on $R^n \times R_+$. Moreover, let $\mathscr{K}$ denote the class of continuous (strictly) increasing functions μ from R_+ to R_+ with $\mu(0) = 0$. Let $\mathscr{K}_\infty$ denote the class of functions μ in $\mathscr{K}$ with $\mu(r) \to \infty$ as $r \to \infty$. Functions in $\mathscr{K}$ and $\mathscr{K}_\infty$ are called class $\mathscr{K}$ and $\mathscr{K}_\infty$ functions, respectively. If $\mu \in \mathscr{K}$, its inverse function is denoted by μ^{-1}. We also denote by $L^1(R_+;R_+)$ the family of all functions γ: $R_+ \to R_+$ such that $\int_0^\infty \gamma(t)\,dt < \infty$. Furthermore let $C(R^n;R_+)$ and $C(R^n \times R_+; R_+)$ denote the families of all continuous functions from R^n to R_+ and from $R^n \times R_+$ to R_+, respectively. The following lemma plays an important role in this paper.

LEMMA 2.1. *Let* $(H1)$ *hold. Assume that there are functions* $V \in C^{2,1}(R^n \times R_+; R_+)$, $\gamma \in L^1(R_+;R_+)$, *and* $w_1, w_2 \in C(R^n \times R_+; R_+)$ *such that*

$$\mathscr{L}V(x,y,t) \le \gamma(t) - w_1(x,t) + w_2(y,t), \qquad (x,y,t) \in R^n \times R^n \times R_+, \tag{2.2}$$

and

$$w_1(x,t) \ge w_2(x,t+\tau), \qquad (x,t) \in R^n \times R_+. \tag{2.3}$$

Then, for every $\xi \in C^b_{\mathscr{F}_0}([-\tau, 0]; R^n)$,

$$\lim_{t\to\infty}\left[V(x(t;\xi),t) + \int_{t-\tau}^{t} w_2(x(s;\xi), s+\tau)\,ds\right] < \infty \ a.s., \quad (2.4)$$

and, moreover,

$$\int_0^\infty [w_1(x(t;\xi),t) - w_2(x(t;\xi), t+\tau)]\,dt < \infty \ a.s. \quad (2.5)$$

The proof of this lemma is based on the following semimartingale convergence theorem established by Lipster and Shiryayev [11, Theorem 7 on p. 139].

LEMMA 2.2. *Let* $A(t)$ *and* $U(t)$ *be two continuous adapted increasing processes on* $t \geq 0$ *with* $A(0) = U(0) = 0$ *a.s. Let* $M(t)$ *be a real-valued continuous local martingale with* $M(0) = 0$ *a.s. Let* ζ *be a nonnegative* $\mathscr{F}_0$*-measurable random variable. Define*

$$X(t) = \zeta + A(t) - U(t) + M(t) \quad \text{for } t \geq 0.$$

If $X(t)$ *is nonnegative, then*

$$\left\{\lim_{t\to\infty} A(t) < \infty\right\} \subset \left\{\lim_{t\to\infty} X(t) < \infty\right\} \cap \left\{\lim_{t\to\infty} U(t) < \infty\right\} \ a.s.$$

where $B \subset D$ *a.s. means* $P(B \cap D^c) = 0$. *In particular, if* $\lim_{t\to\infty} A(t) < \infty$ *a.s., then for almost all* $\omega \in \Omega$:

$$\lim_{t\to\infty} X(t,\omega) < \infty \quad \text{and} \quad \lim_{t\to\infty} U(t,\omega) < \infty,$$

that is both $X(t)$ *and* $U(t)$ *converge to finite random variables.*

Proof of Lemma 2.1. Fix initial data $\xi \in C^b_{\mathscr{F}_0}([-\tau, 0]; R^n)$ arbitrarily and write simply $x(t; x_0) = x(t)$. By Itô's formula,

$$V(x(t),t) = V(x(0),0) + \int_0^t \mathscr{L}V(x(s), x(s-\tau), s)\,ds$$
$$+ \int_0^t V_x(x(s),s)\,g(x(s), x(s-\tau), s)\,dB(s).$$

Note that

$$\int_{t-\tau}^{t} w_2(x(s), s+\tau)\,ds = \int_{-\tau}^{0} w_2(x(s), s+\tau)\,ds$$
$$+ \int_0^t [w_2(x(s), s+\tau) - w_2(x(s-\tau), s)]\,ds.$$

Hence

$$
\begin{aligned}
V(x(t),t) &+ \int_{t-\tau}^{t} w_2(x(s), s+\tau)\, ds \\
&= V(\xi(0),0) + \int_{-\tau}^{0} w_2(\xi(\theta), \theta+\tau)\, d\theta + \int_0^t \gamma(s)\, ds \\
&\quad - \int_0^t \big[\gamma(s) - \mathscr{L}V(x(s), x(s-\tau), s) \\
&\qquad\quad - w_2(x(s), s+\tau) + w_2(x(s-\tau), s)\big]\, ds \\
&\quad + \int_0^t V_x(x(s),s) g(x(s), x(s-\tau), s)\, dB(s) \qquad (2.6) \\
&\le V(\xi(0),0) + \int_{-\tau}^{0} w_2(\xi(\theta), \theta+\tau)\, d\theta + \int_0^t \gamma(s)\, ds \\
&\quad - \int_0^t \big[w_1(x(s), s) - w_2(x(s), s+\tau)\big]\, ds \\
&\quad + \int_0^t V_x(x(s),s) g(x(s), x(s-\tau), s)\, dB(s), \qquad (2.7)
\end{aligned}
$$

where we have used the following fact from (2.2) and (2.3) that

$$
\begin{aligned}
\gamma(s) &- \mathscr{L}V(x(s), x(s-\tau), s) - w_2(x(s), s+\tau) + w_2(x(s-\tau), s) \\
&\ge w_1(x(s), s) - w_2(x(s), s+\tau) \ge 0.
\end{aligned}
$$

Note also that $\int_0^\infty \gamma(s)\, ds < \infty$ since $\gamma \in L^1(R_+; R_+)$. Therefore, applying Lemma 2.2 to (2.6) yields the required Assertion (2.4) while applying Lemma 2.2 to (2.7) gives (2.5). The proof is complete.

Let us now employ Lemma 2.1 to establish the asymptotic convergence theorems of LaSalle-type for Eq. (2.1).

THEOREM 2.3. *Let $(H1)$ hold. Assume that there are functions $V \in C^{2,1}(R^n \times R_+; R_+)$, $\gamma \in L^1(R_+; R_+)$, $w \in C(R^n; R_+)$, and a constant $\delta > 1$, such that*

$$
\mathscr{L}V(x,y,t) \le \gamma(t) - \delta w(x) + w(y), \qquad (x,y,t) \in R^n \times R^n \times R_+.
$$

Then, for every $\xi \in C^b_{\mathscr{F}_0}([-\tau, 0]; R^n)$,

$$
\lim_{t\to\infty} V(x(t;\xi), t) < \infty \ a.s., \qquad (2.8)
$$

and

$$\int_0^\infty w(x(t;\xi))\,dt < \infty \ a.s. \tag{2.9}$$

If there is moreover a continuous function $\eta\colon R_+ \to R_+$ *such that*

$$\eta(V(x,t)) \le w(x), \qquad (x,t) \in R^n \times R_+, \tag{2.10}$$

then

$$\lim_{t\to\infty} \eta(V(x(t;\xi),t)) = \eta\Big(\lim_{t\to\infty} V(x(t;\xi),t)\Big) = 0 \ a.s. \tag{2.11}$$

Proof. Letting $w_1(x,t) = \delta w(x)$ and $w_2(x,t) = w(x)$ we obtain (2.9) from (2.5) of Lemma 2.1. Hence we must have

$$\lim_{t\to\infty} \int_{t-\tau}^t w(x(s;\xi))\,ds = 0 \text{ a.s.}$$

This, together with (2.4), yields the required assertion (2.8). If (2.10) holds, it follows from (2.9) that

$$\int_0^\infty \eta(V(x(t;\xi),t))\,dt < \infty \text{ a.s.} \tag{2.12}$$

On the other hand, since $\eta(\cdot)$ is continuous, we see from (2.8) that

$$\lim_{t\to\infty} \eta(V(x(t;\xi),t)) = \eta\Big(\lim_{t\to\infty} V(x(t;\xi),t)\Big). \tag{2.13}$$

The required Assertion (2.11) follows from (2.12) and (2.13) immediately. The proof is complete.

Assertion (2.8) means that $\lim_{t\to\infty} V(x(t;\xi),t)$ is a finite random variable, while (2.11) further shows that $\lim_{t\to\infty} V(x(t;\xi),t)$ takes values in the set $\{z \ge 0 : \eta(z) = 0\}$. Especially, when $\{z \ge 0 : \eta(z) = 0\} = \{c\}$ (i.e., $\eta(z) = 0$ if and only if $z = c$), then $\lim_{t\to\infty} V(x(t;\xi),t) = c$ almost surely.

THEOREM 2.4. *Let* (*H*1) *hold. Assume that there are functions* $V \in C^{2,1}(R^n \times R_+; R_+)$, $\gamma \in L^1(R_+; R_+)$, *and* $w_1, w_2 \in C(R^n; R_+)$ *such that*

$$\mathscr{L}V(x,y,t) \le \gamma(t) - w_1(x) + w_2(y), \qquad (x,y,t) \in R^n \times R^n \times R_+,$$
$$w_1(x) \ge w_2(x), \qquad x \in R^n,$$

and

$$\lim_{|x|\to\infty} \inf_{0\le t<\infty} V(x,t) = \infty. \tag{2.14}$$

Assume also that for each initial data $\xi \in C^b_{\mathscr{F}_0}([-\tau, 0]; R^n)$ *there is a* $p > 2$ *such that*

$$\sup_{-\tau \le t < \infty} E|x(t;\xi)|^p < \infty. \tag{2.15}$$

Then, for every $\xi \in C^b_{\mathscr{F}_0}([-\tau, 0]; R^n)$:

$$\lim_{t\to\infty} \left[w_1(x(t;\xi)) - w_2(x(t;\xi)) \right] = 0 \ a.s. \tag{2.16}$$

The property of (2.14) is known as radially unbounded in the literature (cf. Arnold [1]). If we define $\mathscr{D} = \{x \in R^n : w_1(x) - w_2(x) = 0\}$ and let $d(x, \mathscr{D})$ denote the distance between x and set $\mathscr{D}$, that is $d(x, \mathscr{D}) = \min_{y \in \mathscr{D}} |x - y|$, then (2.16) means

$$\lim_{t\to\infty} d(x(t;\xi), \mathscr{D}) = 0 \quad \text{a.s.} \tag{2.16$'$}$$

In other words, the solutions of Eq. (2.1) asymptotically approach $\mathscr{D}$ with probability 1. To prove the theorem let us present three useful results. The first one is the well-known Kolmogorov–Čentsov theorem on the continuity of a stochastic process driven from the moment property.

LEMMA 2.5. *Suppose that an n-dimensional stochastic process* $X(t)$ *on* $t \ge 0$ *satisfies the condition*

$$E|X(t) - X(s)|^\alpha \le C|t - s|^{1+\beta}, \qquad 0 \le s, t < \infty,$$

for some positive constants α, β, *and* C. *Then there exists a continuous modification* $\tilde{X}(t)$ *of* $X(t)$, *which has the property that for every* $\gamma \in (0, \beta/\alpha)$, *there is a positive random variable* $\delta(\omega)$ *such that*

$$p\left\{ \omega : \sup_{\substack{0 < t - s < \delta(\omega) \\ 0 \le s,\, t < \infty}} \frac{|\tilde{X}(t,\omega) - \tilde{X}(s,\omega)|}{|t - s|^\gamma} \le \frac{2}{1 - 2^{-\gamma}} \right\} = 1.$$

In other words, almost every sample path of $\tilde{X}(t)$ *is locally but uniformly Hölder-continuous with exponent* γ.

The proof of this result can be found in Karatzas and Shreve [6] in the case when the stochastic process $X(t)$ is on the finite interval $[0, T]$ but a little bit of the modification of the proof works for the case when $X(t)$ is on the entire R_+.

LEMMA 2.6. *Let* $(H1)$ *and* (2.15) *hold. Set*

$$y(t) := \int_0^t g(x(s), x(s-\tau), s)\, dB(s) \quad \text{on } t \geq 0,$$

where we write $x(t;\xi) = x(t)$ *simply. Then almost every sample path of* $y(t)$ *is uniformly continuous on* $t \geq 0$.

Proof. By the moment inequality for stochastic integrals (cf. Friedman [2] or Mao [14]) we have that for $0 \leq s < t < \infty$,

$$E|y(t) - y(s)|^p \leq \left[\frac{p(p-1)}{2}\right]^{p/2} (t-s)^{(p-2)/2} \times \int_s^t E|g(x(r), x(r-\tau), r)|^p\, dr.$$

But by hypotheses (H1) and (2.7) we can derive that

$$\begin{aligned} E|g(x(r), x(r-\tau), r)|^p &\leq E\big[c(1 + |x(r)| + |x(r-\tau)|)\big]^p \\ &\leq 3^{p-1}c^p\big(1 + E|x(r)|^p + E|x(r-\tau)|^p\big) \\ &\leq 3^{p-1}c^p(1 + 2K), \end{aligned}$$

where $K := \sup_{-\tau \leq t < \infty} E|x(t)|^p < \infty$. Therefore

$$E|y(t) - y(s)|^p \leq \left[\frac{p(p-1)}{2}\right]^{p/2} 3^{p-1}c^p(1 + 2K)(t-s)^{1+(p-2)/2}.$$

Bearing in mind that $y(t)$ is continuous, we see from Lemma 2.4 that almost every sample path of $y(t)$ is locally but uniformly Hölder continuous with exponent γ for every $\gamma \in (0, (p-2)/2p)$ and therefore almost every sample path of $y(t)$ must be uniformly continuous. The proof is complete.

LEMMA 2.7. *Let* $(H1)$ *hold. Assume the solution of Eq.* (2.1) *has the property that*

$$\sup_{0 \leq t < \infty} |x(t;\xi)| < \infty \; a.s. \tag{2.17}$$

Then almost every sample path of

$$z(t) := \int_0^t f(x(s), x(s-\tau), s)\, ds$$

is uniformly continuous on $t \geq 0$.

Proof. Write $x(t;\xi) = x(t)$ simply. By (2.17) and the boundedness of the initial data on $-\tau \le t \le 0$, we observe that for almost every $\omega \in \Omega$, there is a positive number $h(\omega)$ such that

$$|x(t,\omega)| \le h(\omega) \quad \text{for all } t \ge -\tau.$$

From this and hypothesis (H1) we compute that, for $0 \le s < t < \infty$,

$$\begin{aligned} |z(t,\omega) - z(s,\omega)| &\le \int_s^t |f(x(r,\omega), x(r-\tau,\omega), r)|\, dr \\ &\le c\int_s^t (1 + |x(r,\omega)| + |x(r-\tau,\omega)|)\, dr \\ &\le c(1 + 2h(\omega))(t-s), \end{aligned}$$

which implies that $z(t,\omega)$ is uniformly continuous on $t \ge 0$. The lemma is therefore proven.

We can now easily prove Theorem 2.4.

Proof of Theorem 2.4. Again fix any initial data ξ and write $x(t;\xi) = x(t)$. By Lemma 2.1 and Condition (2.14), we can easily see that (2.17) is satisfied. Hence, by Lemmas 2.1, 2.6, and 2.7, there exists an $\overline{\Omega} \subset \Omega$ with $P(\overline{\Omega}) = 1$ such that for every $\omega \in \overline{\Omega}$, $x(t,\omega)$ is uniformly continuous on $t \ge 0$, and

$$\int_0^\infty [w_1(x(t,\omega)) - w_2(x(t,\omega))]\, dt < \infty. \tag{2.18}$$

In order to prove the theorem, it is clearly enough if we can show that

$$\lim_{t\to\infty} [w_1(x(t,\omega)) - w_2(x(t,\omega))] = 0 \quad \text{for all } \omega \in \overline{\Omega}. \tag{2.19}$$

For convenience, set $w = w_1 - w_2$. If (2.19) is not true, then for some $\hat{\omega} \in \overline{\Omega}$:

$$\limsup_{t\to\infty} w(x(t,\hat{\omega})) > 0.$$

So there is some $\varepsilon > 0$ and a sequence $\{t_k\}_{k\ge 1}$ of positive numbers with $t_k + 1 < t_{k+1}$ such that

$$w(x(t_k,\hat{\omega})) > \varepsilon \quad \text{for all } k \ge 1. \tag{2.20}$$

Set $\overline{S}_h = \{x \in R^n : |x| \le h\}$, where $h = h(\hat{\omega})$ has been defined in the proof of Lemma 2.7 such that $\{x(t,\hat{\omega}) : t \ge -\tau\} \subset \overline{S}_h$. Since it is continuous,

$w(\cdot)$ must be uniformly continuous in $\bar{S}_h$ and there is a $\delta_1 > 0$ such that

$$|w(x) - w(y)| < \frac{\varepsilon}{2} \quad \text{if } x, y \in \bar{S}_h, |x-y| < \delta_1. \tag{2.21}$$

On the other hand, recalling that $x(t, \hat{\omega})$ is uniformly continuous on $t \geq -\tau$, we can find a $\delta_2 \in (0,1)$ such that

$$|x(t, \hat{\omega}) - x(s, \hat{\omega})| < \delta_1 \quad \text{if } -\tau \leq t, s < \infty, |t-s| - \delta_2. \tag{2.22}$$

Combining (2.21) and (2.22) we see that for every $k \geq 1$,

$$\left|w\big(x(t_k, \hat{\omega})\big) - w(x(t, \hat{\omega}))\right| < \frac{\varepsilon}{2} \qquad \text{if } t_k \leq t \leq t_k + \delta_2.$$

This, together with (2.20), yields

$$w(x(t, \hat{\omega})) \geq w\big(x(t_k, \hat{\omega})\big) - \left|w\big(x(t_k, \hat{\omega})\big) - w(x(t, \hat{\omega}))\right| > \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}.$$

Therefore

$$\int_0^\infty w(x(t, \hat{\omega}))\, dt \geq \sum_{k=1}^\infty \int_{t_k}^{t_k + \delta_2} w(x(t, \hat{\omega}))\, dt \geq \sum_{k=1}^\infty \frac{\varepsilon \delta_2}{2} = \infty,$$

which contradicts (2.18). Hence, (2.19) must be true and the theorem has been proven.

Remark 2.8. From the proof above, we see clearly that condition (2.15) is only used to show the uniform continuity of almost every sample path of $\int_0^t g(x(s; \xi), s)\, dB(s)$ on $t \geq 0$. In other words, any condition that guarantees this uniform continuity can replace condition (2.15). For example, condition (2.15) can be replaced by the boundedness of g.

Condition (2.15) means the boundedness of the pth moment of the solution which has its own interest. The following lemma gives a criterion for this boundedness.

LEMMA 2.9. *Assume that there is a convex function $\mu \in \mathscr{K}_\infty$, a constant $p > 0$ and, moreover, functions $U \in C^{2,1}(R^n \times R_+; R_+)$, $\bar{\gamma} \in L^1(R_+; R_+)$, $w \in C(R^n; R_+)$, such that*

$$\mu(|x|^p) \leq U(x,t),$$
$$\mathscr{L}U(x,t) \leq \bar{\gamma}(t) - w(x,t) + w(y,t),$$
$$w(x,t) \geq w(x, t+\tau)$$

for all $(x,t) \in R^n \times R_+$. Then, for every $\xi \in C^b_{\mathscr{F}_0}([-\tau, 0]; R^n)$,

$$\sup_{-\tau \leq t < \infty} E|x(t; \xi)|^p < \infty.$$

Proof. In the same way as in the proof of Lemma 2.1 we can show that

$$
\begin{aligned}
&EU(x(t),t) + E\int_{t-\tau}^{t} w(x(s), s+\tau)\, ds \\
&\quad = EU(\xi(0),0) + E\int_{-\tau}^{0} w(\xi(\theta), \theta+\tau)\, d\theta \\
&\qquad + E\int_{0}^{t}[\mathscr{L}U(x(s), x(s-\tau), s) + w(x(s), s+\tau) \\
&\qquad\qquad -w(x(s-\tau), s)]\, ds \\
&\quad \le EU(\xi(0),0) + E\int_{-\tau}^{0} w(\xi(\theta), \theta+\tau)\, d\theta \\
&\qquad + E\int_{0}^{t}[\bar{\gamma}(s) - w(x(s), s) + w(x(s), s+\tau)]\, ds \\
&\quad \le EU(\xi(0),0) + E\int_{-\tau}^{0} w(\xi(\theta), \theta+\tau)\, d\theta + \int_{0}^{\infty}\bar{\gamma}(s)\, ds \\
&\quad := C,
\end{aligned}
$$

where we simply write $x(t;\xi) = x(t)$ as before. By Jensen's inequality and condition $\mu(|x|^p) \le U(x,t)$, we then have

$$\mu\big(E|x(t)|^p\big) \le E\mu\big(|x(t)|^p\big) \le C,$$

which implies

$$E|x(t)|^p \le \mu^{-1}(C) \quad \text{for all } t \ge 0.$$

Hence the assertion follows since the initial data are bounded. The proof is complete.

3. ASYMPTOTIC STABILITY

The results obtained in the previous section can be applied to establish useful sufficient criteria for the almost surely asymptotic stability of the stochastic differential delay equation (2.1).

COROLLARY 3.1. *Let* $(H1)$ *hold. Assume that there are functions* $V \in C^{2,1}(R^n \times R_+; R_+)$, $\gamma \in L^1(R_+; R_+)$, $w \in C(R^n; R_+)$, *and a constant* $\delta > 1$, *such that*

$$\mathscr{L}V(x,y,t) \le \gamma(t) - \delta w(x) + w(y), \qquad (x,y,t) \in R^n \times R^n \times R_+.$$

Assume furthermore that there are functions $\mu_1, \mu_2 \in \mathscr{K}_\infty$ *and* $\mu_3 \in \mathscr{K}$ *such that*

$$\mu_1(|x|) \le V(x,t) \le \mu_2(|x|) \quad and \quad \mu_3(|x|) \le w(x) \tag{3.1}$$

for $x \in R^n$ *and* ≥ 0. *Then, for every* $\xi \in C^b_{\mathscr{F}_0}([-\tau, 0; R^n)$,

$$\lim_{t \to \infty} x(t;\xi) = 0 \ a.s. \tag{3.2}$$

Proof. Note from (3.1) that

$$\mu_2^{-1}(V(x,t)) \le |x| \le \mu_3^{-1}(w(x)),$$

which yields

$$\mu_3\big(\mu_2^{-1}(V(x,t))\big) \le w(x).$$

Applying Theorem 2.3 with $\eta(\cdot) = \mu_3(\mu_2^{-1}(\cdot))$ we obtain that

$$\lim_{t \to \infty} \mu_3\big(\mu_2^{-1}(V(x(t;\xi),t))\big) = 0 \text{ a.s.},$$

which implies

$$\lim_{t \to \infty} V(x(t;\xi),t) = 0 \text{ a.s.},$$

since $\mu_3(\mu_2^{-1}(\cdot)) \in \mathscr{K}$. Hence, by (3.1) again,

$$\lim_{t \to \infty} \mu_1(|x(t;\xi)|) = 0 \text{ a.s.}$$

Since $\mu_1 \in \mathscr{K}_\infty$, we must have the required assertion (3.2).

The results in the previous section can also be used to discuss the almost surely exponential stability.

COROLLARY 3.2. *Let* $(H1)$ *hold. Assume that there are functions* $U \in C^{2,1}(R^n \times R_+; R_+)$, $U \in C(R^n; R_+)$, *and two constants* $\lambda_1 > \lambda_2 > 0$ *such that*

$$\mathscr{L}U(x,y,t) \le -\lambda_1 w(x) + \lambda_2 w(y), \qquad (x,y,t) \in R^n \times R^n \times R_+, \tag{3.3}$$

and

$$U(x,t) \le w(x), \qquad (x,t) \in R^n \times R_+. \tag{3.4}$$

Then, for every $\xi \in C^b_{\mathscr{F}_0}([-\tau, 0; R^n)$,

$$\limsup_{t\to\infty} \frac{1}{t} \log(U(x(t;\xi), t)) \le -\gamma \ a.s., \tag{3.5}$$

where $\gamma \in (0, \lambda_1 - \lambda_2)$ *is the unique root of*

$$\lambda_1 - \gamma = \lambda_2 e^{\gamma\tau}. \tag{3.6}$$

If furthermore for some positive constants p and c,

$$c|x|^p \le U(x,t), \qquad (x,t) \in R^n \times R_+, \tag{3.7}$$

then

$$\limsup_{t\to\infty} \frac{1}{t} \log(|x(t;\xi)|) \le -\frac{\gamma}{p} \ a.s. \tag{3.8}$$

Proof. Define $V(x,t) = e^{\gamma t}U(x,t)$ for $(x,t) \in R^n \times R_+$. Then, by (3.3) and (3.4),

$$\begin{aligned} \mathscr{L}V(x,y,t) &= e^{\gamma t}[\gamma U(x,t) + \mathscr{L}U(x,y,t)] \\ &< e^{\gamma t}[-(\lambda_1 - \gamma)w(x) - \lambda_2 w(y)]. \end{aligned}$$

Define

$$w_1(x,t) = (\lambda_1 - \gamma)e^{\gamma t}w(x) \quad \text{and} \quad w_2(x,t) = \lambda_2 e^{\gamma t} w(x)$$

for $(x,t) \in R^n \times R_+$. Then

$$\mathscr{L}V(x,y,t) \le -w_1(x,t) + w_2(y,t).$$

Moreover, by (3.6),

$$w_1(x,t) = \lambda_2 e^{\gamma(t+\tau)} w(x) = w_2(x, t+\tau).$$

Applying Lemma 2.1 we obtain that

$$\limsup_{t\to\infty} e^{\gamma t} U(x(t;\xi), t) < \infty \text{ a.s.},$$

which yields (3.5) immediately. Finally (3.8) follows from (3.5) and (3.7) directly. The proof is complete.

COROLLARY 3.3. *Let* $(H1)$ *hold. Assume that there are four positive constants* λ_1–λ_4 *such that*

$$2x^T f(x,0,t) \le -\lambda_1 |x|^2,$$
$$|f(x,y,t) - f(x,0,t)| \le \lambda_2 |y|,$$
$$|g(x,y,t)|^2 \le \lambda_3 |x|^2 + \lambda_4 |y|^2$$

for $x, y \in R^n$ *and* $t \ge 0$. *If*

$$\lambda_1 > 2\lambda_2 + \lambda_3 + \lambda_4, \tag{3.9}$$

then for all $\xi \in C^b_{\mathscr{F}_0}([-\tau,0]; R^n)$,

$$\limsup_{t\to\infty} \frac{1}{t} \log(|x(t;\xi)|) \le -\frac{\gamma}{2} \ a.s., \tag{3.10}$$

where γ *is the unique positive root of*

$$\lambda_1 - \lambda_2 - \lambda_3 - \gamma = (\lambda_2 + \lambda_4) e^{\gamma\tau}. \tag{3.11}$$

Proof. Let $U(x,t) = |x|^2$. Using the conditions, we compute

$$\begin{aligned}
\mathscr{L}U(x,y,t) &= 2x^T f(x,y,t) + |g(x,y,t)|^2 \\
&\le 2x^T f(x,0,t) + 2|x|\,|f(x,y,t) - f(x,0,t)| + |g(x,y,t)|^2 \\
&\le -\lambda_1|x|^2 + 2\lambda_2|x|\,|y| + \lambda_3|x|^2 + \lambda_4|y|^2 \\
&\le -\lambda_1|x|^2 + \lambda_2\big(|x|^2 + |y|^2\big) + \lambda_3|x|^2 + \lambda_4|y|^2 \\
&= -(\lambda_1 - \lambda_2 - \lambda_3)|x|^2 + (\lambda_2 + \lambda_4)|y|^2.
\end{aligned}$$

Now the conclusion follows from Corollary 3.2. The proof is complete.

To close this section, let us show that the results in the previous section can also be applied to deal with the problem of partially asymptotic stability. Let $1 \le \hat{n} \le n$ and $1 \le i_1 < i_2 < \cdots < i_{\hat{n}} \le n$ be all integers. Let $\hat{x} = (x_{i_1}, x_{i_2}, \dots, x_{i_{\hat{n}}})$ be the partial coordinates of x, which can be regarded as in $R^{\hat{n}}$ with the norm $|\hat{x}| = \sqrt{x_{i_1}^2 + \cdots + x_{i_{\hat{n}}}^2}$.

COROLLARY 3.4. *Let* $(H1)$ *hold. Assume that there are functions* $V \in C^{2,1}(R^n \times R_+; R_+)$, $\gamma \in L^1(R_+; R_+)$, *and* $w_1, w_2 \in C(R^n; R_+)$ *and* $\mu_1 \in \mathscr{K}$ *such that*

$$\mathscr{L}V(x,y,t) \le \gamma(t) - w_1(x) + w_2(y), \qquad (x,y,t) \in R^n \times R^n \times R_+,$$

and

$$w_1(x) - w_2(x) \geq \mu_1(|\hat{x}|), \qquad x \in R^{\hat{n}}. \tag{3.12}$$

Moreover, there is a convex function $\mu \in \mathscr{K}_\infty$ *and a constant* $p > 2$ *such that*

$$\mu(|x|^p) < V(x,t), \qquad (x,t) \in R^n \times R_+.$$

Then, for every $\xi \in C^b_{\mathscr{F}_0}([-\tau, 0]; R^n)$:

$$\lim_{t\to\infty} \hat{x}(t;\xi) = 0 \ a.s. \tag{3.13}$$

Proof. To apply Lemma 2.9, let $U(x,t) = V(x,t)$ and $w(x,t) = w_2(x)$. Then

$$LU(x,y,t) \leq \gamma(t) - w_1(x) + w_2(y) \leq \gamma(t) - w(x,t) + w(y,t).$$

Hence, by Lemma 2.9,

$$\sup_{-\tau\leq t<\infty} E|x(t;\xi)|^p < \infty.$$

We can now apply Theorem 2.4 to obtain that

$$\lim_{t\to\infty} \left[w_1(x(t;\xi)) - w_2(x(t;\xi))\right] = 0 \text{ a.s.}$$

This, together with (3.12), yields

$$\lim_{t\to\infty} \mu_1(|\hat{x}(t;\xi)|) = 0 \text{ a.s.}$$

Since $\mu_1 \in \mathscr{K}$, (3.13) must be true.

4. ORDINARY DIFFERENTIAL DELAY EQUATIONS

If $g \equiv 0$, Eq. (2.1) becomes an n-dimensional ordinary differential delay equation

$$\dot{x}(t) = f(x(t), x(t-\tau), t), \tag{4.1}$$

on $t \geq 0$, and the corresponding initial data becomes $\{x(\theta): -\tau \leq \theta \leq 0\} = \xi \in C([-\tau, 0]; R^n)$. Moreover, hypothesis (H1) reduces to:

(H2) The function f satisfies the local Lipschitz condition and the linear growth condition. That is, for each $k = 1, 2, \ldots$, there is a $c_k > 0$ such that

$$|f(x,y,t) - f(\bar{x},\bar{y},t)| \leq c_k(|x-\bar{x}| + |y-\bar{y}|)$$

for all $t \geq 0$ and those $x, y, \bar{x}, \bar{y} \in R^n$ with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq k$, and there is moreover a $c > 0$ such that

$$|f(x,y,t)| \leq c(1 + |x| + |y|),$$

for all $(x, y, t) \in R^n \times R^n \times R_+$.

Under hypothesis (H2), Eq. (4.1) has a unique solution which is still denoted by $x(t;\xi)$. Furthermore, the operator $\mathscr{L}V$ becomes

$$LV(x,y,t) = V_t(x,t) + V_x(x,t)f(x,y,t).$$

Here we use LV instead of $\mathscr{L}V$ to indicate this operator is associated with Eq. (4.1). The following corollaries follow from Theorems 2.3 and 2.4, respectively.

COROLLARY 4.1. *Let (H2) hold. Assume that there are functions* $V \in C^{2,1}(R^n \times R_+; R_+)$, $\gamma \in L^1(R_+; R_+)$, $w \in C(R^n; R_+)$, *and a constant* $\delta > 1$, *such that*

$$LV(x,y,t) \leq \gamma(t) - \delta w(x) + w(y), \qquad (x,y,t) \in R^n \times R^n \times R_+.$$

Then, for every $\xi \in C^b_{\mathscr{F}_0}([-\tau, 0; R^n)$, *the solution of Eq.* (4.1) *has the properties that*

$$\lim_{t\to\infty} V(x(t;\xi),t) < \infty \quad \text{and} \quad \int_0^\infty w(x(t;\xi))\,dt < \infty.$$

If there is moreover a continuous function $\eta\colon R_+ \to R_+$ *such that*

$$\eta(V(x,t)) \leq w(x), \qquad (x,t) \in R^n \times R_+,$$

then

$$\lim_{t\to\infty} \eta(V(x(t;\xi),t)) = \eta\Big(\lim_{t\to\infty} V(x(t;\xi),t)\Big) = 0.$$

COROLLARY 4.2. *Let (H2) hold. Assume that there are functions* $V \in C^{2,1}(R^n \times R_+; R_+)$, $\gamma \in L^1(R_+; R_+)$, *and* $w_1, w_2 \in C(R^n; R_+)$ *such that*

$$LV(x,y,t) \leq \gamma(t) - w_1(x) + w_2(y), \qquad (x,y,t) \in R^n \times R^n \times R_+,$$

$$w_1(x) \geq w_2(x), \qquad x \in R^n,$$

$$\lim_{|x|\to\infty} \inf_{0\leq t<\infty} V(x,t) = \infty.$$

Then, for every $\xi \in C([-\tau, 0]; R^n)$, *the solution of Eq.* (4.1) *has the property that*

$$\lim_{t\to\infty} \big[w_1(x(t;\xi)) - w_2(x(t;\xi))\big] = 0.$$

These results can be used to investigate the asymptotic stability of Eq. (4.1) as we did in the previous section for Eq. (2.1), but the details are left to the reader.

5. EXAMPLES

In this section we discuss a number of examples to illustrate our theory. In the following examples we let $B(t)$ be a scalar Brownian motion. We omit mentioning initial data and write the solutions simply by $x(t)$.

EXAMPLE 5.1. Let α and β be both bounded continuous functions from R_+ to R_+. Consider a one-dimensional stochastic differential delay equation

$$dx(t) = -\alpha(t)x(t)\,dt + \beta(t)x(t-\tau)\,dB(t) \quad \text{on } t \geq 0, \tag{5.1}$$

with initial data $\{x(\theta): -\tau \leq \theta \leq 0\} = \xi \in C^b_{\mathscr{F}_0}([-\tau, 0; R)$. Due to the boundedness of α and β, hypothesis (H1) is satisfied. Assume furthermore that there are two constants $p > 2$ and $\delta > 0$ such that

$$p\alpha(t) \geq \frac{p(p-1)}{2}\beta^2(t), \qquad \beta(t) \geq \beta(t+\tau), \tag{5.2}$$

and

$$2\alpha(t) - \beta^2(t+\tau) \geq \delta \tag{5.3}$$

for all $t \geq 0$. We first show that $E|x(t;\xi)|^p$ is bounded. To apply Lemma 2.9, we define

$$U(x,t) = |x|^p \quad \text{and} \quad w(x,t) = (p-1)\beta^2(t)|x|^p$$

for $(x,t) \in R \times R_+$. Then, for $(x,y,t) \in R \times R \times R_+$, we compute

$$\begin{aligned}
\mathscr{L}U(x,y,t) &\leq -p\alpha(t)|x|^p + \frac{p(p-1)}{2}\beta^2(t)|x|^{p-2}|y|^2 \\
&\leq -p\alpha(t)|x|^p + \frac{p(p-1)}{2}\beta^2(t)\left[\frac{p-2}{p}|x|^p + \frac{2}{p}|y|^p\right] \\
&= -\left[p\alpha(t) - \frac{1}{2}(p-1)(p-2)\beta^2(t)\right]|x|^p \\
&\quad + (p-1)\beta^2(t)|y|^p.
\end{aligned}$$

On the other hand, (5.2) implies

$$p\alpha(t) - \tfrac{1}{2}(p-1)(p-2)\beta^2(t) \geq (p-1)\beta^2(t).$$

Hence

$$\mathscr{L}U(x,y,t) \leq -w(x,t) + w(x,t).$$

Note from the definition of w and (5.2) that

$$w(x,t) = (p-1)\beta^2(t)|x|^p \geq (p-1)\beta^2(t+\tau)|x|^p = w(x,t+\tau).$$

By Lemma 2.9,

$$\sup_{-\tau \leq t < \infty} E|x(t)|^p < \infty. \tag{5.4}$$

To apply Lemma 2.1, we define

$$V(x,t) = |x|^2, \qquad w_1(x,t) = 2\alpha(t)|x|^2, \qquad w_2(x,t) = \beta^2(t)|x|^2$$

for $(x,t) \in R \times R_+$. Then, by (5.3),

$$w_1(x,t) - w_2(x,t+\tau) \geq \delta|x|^2.$$

Moreover, for $(x,y,t) \in R \times \in R \times R_+$, we compute

$$\mathscr{L}V(x,y,t) = -2\alpha(t)|x|^2 + \beta^2(t)|y|^2 = -w_1(x,t) + w_2(x,t).$$

By Lemma 2.1, we have that

$$\sup_{0 \leq t < \infty} |x(t)| < \infty \text{ a.s.}, \tag{5.5}$$

and

$$\int_0^\infty |x(t)|^2\, dt < \infty \text{ a.s.} \tag{5.6}$$

In view of Lemmas 2.6 and 2.7, we observe that almost every sample path of the solution $x(t)$ is uniformly continuous on $t \geq 0$. This and (5.6) implies that

$$\lim_{t\to\infty} x(t) = 0 \text{ a.s.}, \tag{5.7}$$

that is, the solution of Eq. (5.1) asymptotically tends to zero with probability 1. It is useful to point out that if both $\alpha(t) \equiv \alpha$ and $\beta(t) \equiv \beta$ are constants, then (5.2) and (5.3) are guaranteed simply by $2\alpha > \beta^2$.

EXAMPLE 5.2. Consider a stochastic delay oscillator

$$\ddot{z}(t) + 4\dot{z}(t) + 2z(t) = 2\dot{z}(t-\tau)\dot{B}(t), \qquad t \geq 0. \tag{5.8}$$

By introducing a new variable $x = (x_1, x_2)^T = (z, \dot{z})^T$, this oscillator can be written as an Itô delay equation

$$dx(t) = \begin{bmatrix} x_2(t) \\ -2x_1(t) - 4x_2(t) \end{bmatrix} dt + \begin{bmatrix} 0 \\ 2x_2(t-\tau) \end{bmatrix} dB(t). \tag{5.9}$$

Define $V(x,t) = 2x_1^2 + x_2^2$ for $(x,t) \in R^2 \times R_+$. Then, for $(x,y,t) \in R^2 \times R^2 \times R_+$, we compute

$$\mathscr{L}V(x,t) = 4x_1x_2 + 2x_2(-2x_1 - 4x_2) + 4y_2^2 = -8x_2^2 + 4y_2^2.$$

Applying Theorem 2.3 with $w(x) = 4x_2^2$, we conclude that the solution of Eq. (5.8) has the properties that

$$\lim_{t\to\infty}\left[2z^2(t) + \dot{z}^2(t)\right] = \lim_{t\to\infty}\left[2x_1^2(t) + x_2^2(t)\right] < \infty \text{ a.s.},$$

and

$$\int_0^\infty \dot{z}^2(t)\,dt = \int_0^\infty x_2^2(t)\,dt < \infty \text{ a.s.}$$

EXAMPLE 5.3. Consider a three-dimensional stochastic differential delay equation

$$dx(t) = b(x(t),t)\,dt + \begin{bmatrix} e^{-t} \\ [-1 \vee x_1(t-\tau)] \wedge 1 \\ \sin(x_2(t-\tau)) \end{bmatrix} dB(t). \tag{5.10}$$

Here b: $R^3 \times R_+ \to R^n$ satisfies the local Lipschitz condition and the linear growth condition, and has the property that

$$2x^Tb(x,t) \leq -|x|^2, \qquad (x,t) \in R^n \times R_+. \tag{5.11}$$

To apply Theorem 2.4, define

$$V(x,t) = |x|^2, \qquad w_1(x) = |x|^2, \qquad w_2(x) = x_1^2 \wedge 1 + \sin^2(x_2),$$

for $x \in R^n$ and $t \geq 0$. Clearly

$$w_1(x) - w_2(x) \geq 0.$$

Moreover, for $(x, y, t) \in R^3 \times R^3 \times R_+$, we have

$$\mathscr{L}V(x, y, t) = 2x^T b(x, t) + e^{-2t} + y_1^2 \wedge 1 + \sin^2(y_2)$$
$$= e^{-2t} - w_1(x) + w_2(y).$$

By Theorem 2.4 and Remark 2.8 we have that

$$\lim_{t\to\infty} \left[w_1(x(t)) - w_2(x(t)) \right] = 0 \text{ a.s.} \tag{5.12}$$

Note that $w_1(x) - w_2(x) = 0$ if and only if $x_2 = x_3 = 0$ and $x_1^2 \leq 1$. Hence, we can conclude from (5.12) that the solution of Eq. (5.10) has the properties

$$\lim_{t\to\infty} \left[|x_2(t)| + |x_2(t)| \right] = 0 \quad \text{and} \quad \limsup_{t\to\infty} x_1^2(t) \leq 1 \text{ a.s.} \tag{5.13}$$

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