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Asymptotics of Sobolev Orthogonal Polynomials for Coherent Pairs of Laguerre Type

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Let $\{S_n\}_n$ denote a sequence of polynomials orthogonal with respect to the Sobolev inner product

$$(f,\ g)_S = \int f(x)g(x)d\psi_0(x) + \lambda \int f'(x)g'(x)d\psi_1(x),$$

where $\lambda > 0$ and $\{d\psi_0,\ d\psi_1\}$ is a so-called coherent pair with at least one of the measures $d\psi_0$ or $d\psi_1$ a Laguerre measure. We investigate the asymptotic behaviour of $S_n(x)$ outside the supports of $d\psi_0$ and $d\psi_1$, and $n \to +\infty$.

Key Words: Laguerre polynomials; Sobolev orthogonal polynomials; coherent pairs; asymptotic properties.

1. INTRODUCTION

Consider the Sobolev inner product

$$(f,\ g)_S = \int_a^b f(x)g(x)d\psi_0(x) + \lambda \int_a^b f'(x)g'(x)d\psi_1(x), \tag{1.1}$$

where ψ_0 and ψ_1 are distribution functions on $(a,\ b)$ and $\lambda > 0$.

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In [2] Iserles et al. introduced the notion of a coherent pair of measures for inner products of the form (1.1). This concept proved to be very fruitful (see [3, 7]). We define the notion here as follows. Let $\{P_n\}_n$ and $\{T_n\}_n$ denote orthogonal polynomial sequences with respect to the inner products defined by $d\psi_0$ and $d\psi_1$, respectively. The pair $\{d\psi_0,\ d\psi_1\}$ is called a coherent pair if there exist non–zero constants A_n and B_n such that

$$T_n = A_n P'_{n+1} + B_n P'_n, \qquad n \geq 1. \tag{1.2}$$

Let $\{d\psi_0,\ d\psi_1\}$ denote a coherent pair and let $\{S_n\}_n$ be a sequence of polynomials orthogonal with respect to (1.1). As a direct consequence of (1.2) there exist non–zero constants C_n and D_n such that

$$A_n P_{n+1} + B_n P_n = C_n S_{n+1} + D_n S_n, \qquad n \geq 1. \tag{1.3}$$

The existence of this simple relation between the "Sobolev" polynomials S_n and the "standard" polynomials P_n makes the concept of coherent pair so useful. The relation will play a central part in the present paper. In [8] all coherent pairs of measures have been determined. Especially, it has been proved that at least one of the two measures $d\psi_0$ or $d\psi_1$ has to be a Laguerre or Jacobi measure (apart from a linear change in the variable).

In the present paper, $\{d\psi_0,\ d\psi_1\}$ is a coherent pair where one of the measures is a Laguerre measure $x^\alpha e^{-x}\,dx,\quad \alpha > -1$, on $(0, +\infty)$. We investigate the asymptotic behaviour of $S_n(x)$ outside the supports of $d\psi_0$ and $d\psi_1$, and $n \to +\infty$. In [4] the special case $d\psi_0(x) = d\psi_1(x) = x^\alpha e^{-x}\,dx$ already has been treated. In [6] the similar problem for coherent pairs of Jacobi type has been studied.

In Section 2 we recall some well known results for Laguerre polynomials $\{L_n^{(\alpha)}\}_n$ which will be used in the paper. Especially we give the asymptotic result of Perron for $L_n^{(\alpha)}(x)$ with fixed $x \in \mathbb{C} \setminus [0, +\infty)$ and $n \to \infty$ (Lemma 2.1).

In Section 3 we study coherent pairs $\{d\psi_0,\ d\psi_1\}$ where $d\psi_1$ is a Laguerre measure. We obtain the first two terms of the asymptotic expansion of $S_n(x)$ where $x \in \mathbb{C} \setminus [0, +\infty)$ and $n \to \infty$ (Theorem 3.5). In two special cases our method even gives the complete asymptotic expansion (Theorem 3.6).

In Section 4 the first measure $d\psi_0$ is a Laguerre measure and $\{d\psi_0,\ d\psi_1\}$ is again a coherent pair. The situation is more complicated than in Section 3 and we can only give the first term of the asymptotic expansion of $S_n(x)$ outside the supports of $d\psi_0$ and $d\psi_1$ and $n \to \infty$ (Theorem 4.11).

In both sections we choose the normalization of S_n in such a way that the leading coefficients of S_n and $L_n^{(\alpha)}$ are equal. Then there exist positive constants B_n and D_n such that

$$L_{n+1}^{(\alpha)} - B_n L_n^{(\alpha)} = S_{n+1} - D_n S_n, \qquad n \geq 0, \tag{1.4}$$

see Lemmas 3.1 and 4.7, respectively.

In Lemma 3.1 we have $B_n \equiv 1$ and Lemma 4.7 is just (1.3) in the present normalization. Then it suffices to determine the asymptotic behaviour of the constants in (1.4) in order to obtain the asymptotic behaviour of $S_n(x)$ from Perron's result on the asymptotic behaviour of $L_n^{(\alpha)}$.

We obtain for B_n and D_n in (1.4)

$$\lim_{n\to\infty} B_n = 1,$$

$$\lim_{n\to\infty} D_n = \ell = \frac{\lambda + 2 - \sqrt{\lambda^2 + 4\lambda}}{2} < 1$$

(see Lemmas 3.3 and 4.10). As a consequence the asymptotic expansion of S_n depends on λ.

We remark that in the Jacobi case studied in [6] the asymptotics follow from a relation similar to (1.4), with $\lim B_n$ a non-zero constant and $\lim D_n = 0$. Then, in the Jacobi case, the asymptotic expansion is independent of λ.

2. CLASSICAL LAGUERRE POLYNOMIALS

Laguerre polynomials, for arbitrary real α, are defined by (see Szegő [10, pp. 100–102])

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}, \qquad n = 0, 1, 2, \ldots$$

The leading coefficient, $(-1)^n/n!$, is independent of α and

$$L_n^{(\alpha)}(0) = \binom{n+\alpha}{n}. \tag{2.1}$$

The Rodrigues formula reads

$$L_n^{(\alpha)}(x) = \frac{1}{n!} e^x x^{-\alpha} \left(\frac{d}{dx}\right)^n \left(e^{-x} x^{n+\alpha}\right). \tag{2.2}$$

If $\alpha > -1$, then $\{L_n^{(\alpha)}\}_n$ is orthogonal with respect to the inner product

$$(f, g) = \int_0^{+\infty} f(x) g(x) x^{\alpha} e^{-x}\, dx.$$

Moreover, if $\alpha > -1$, then

$$\int_0^{+\infty} \left(L_n^{(\alpha)}(x)\right)^2 x^{\alpha} e^{-x}\, dx = \frac{\Gamma(n+\alpha+1)}{n!}, \qquad n = 0, 1, 2, \ldots. \tag{2.3}$$

For arbitrary real α the following relations are satisfied

$$L_n^{(\alpha)}(x) - L_{n-1}^{(\alpha)}(x) = L_n^{(\alpha-1)}(x), \tag{2.4}$$

$$\frac{d}{dx} L_n^{(\alpha)}(x) = -L_{n-1}^{(\alpha+1)}(x), \tag{2.5}$$

$$\frac{d}{dx}\left(L_n^{(\alpha)}(x) - L_{n-1}^{(\alpha)}(x)\right) = -L_{n-1}^{(\alpha)}(x). \tag{2.6}$$

The following asymptotic result, due to Perron (see [10, p. 199]), will play a central role in our investigations.

LEMMA 2.1. *Let α be an arbitrary real number. Then*

$$L_n^{(\alpha)}(x) = \frac{1}{2\sqrt{\pi}} e^{x/2}(-x)^{(-(\alpha/2))-1/4} n^{\alpha/2-1/4} e^{2\sqrt{-nx}} \left\{1 + O(n^{-1/2})\right\}.$$

The relation holds if x is in the complex plane cut along the positive part of the real axis; $(-x)^{-(\alpha/2))-1/4}$ and $\sqrt{-x}$ must be taken real and positive if $x < 0$. The bound for the remainder holds uniformly in every closed domain with no points in common with $x \geq 0$.

As a direct consequence of Lemma 2.1 we have for an arbitrary real α

$$\lim_{n\to\infty} \frac{L_n^{(\alpha)}(x)}{L_{n-1}^{(\alpha)}(x)} = 1, \tag{2.7}$$

and

$$\lim_{n\to\infty} \frac{n^{1/2} L_n^{(\alpha-1)}(x)}{L_n^{(\alpha)}(x)} = \sqrt{-x}, \tag{2.8}$$

uniformly on compact subsets of $\mathbb{C} \setminus [0, +\infty)$.

3. COHERENT PAIRS OF LAGUERRE TYPE I

Consider the coherent pair $\{d\psi_0,\ d\psi_1\}$, where $d\psi_1$ is a Laguerre measure on $(0, +\infty)$,

$$d\psi_1(x) = x^\alpha e^{-x}\, dx,$$

with $\alpha > -1$. For $d\psi_0$ there are three different situations (see [8]):

(I a) If $\alpha > 0$, then $d\psi_0(x) = (x - \xi)x^{\alpha-1}e^{-x}\, dx$, with $\xi \leq 0$.

(I b) If $\alpha = 0$, then $d\psi_0(x) = e^{-x}\, dx + M\delta(0)$, with $M \geq 0$.

(I c) If $-1 < \alpha < 0$, then $d\psi_0(x) = x^\alpha e^{-x}\, dx$.

In all cases, the support of $d\psi_0$ is $[0, +\infty)$. We abbreviate to

$$d\psi_0(x) = (x - \xi)x^{\alpha-1}e^{-x}\,dx + M\delta(0),$$

with $\xi = 0$ if $\alpha \le 0$ and $M = 0$ if $\alpha \ne 0$.

Let $\{S_n\}_n$ denote the sequence of polynomials orthogonal with respect to the Sobolev inner product

$$(f, g)_S = \int f(x)g(x)d\psi_0(x) + \lambda \int f'(x)g'(x)d\psi_1(x), \tag{3.1}$$

with $\lambda > 0$, normalized by the condition that S_n and $L_n^{(\alpha)}$ have the same leading coefficient. Then, in particular, $S_0 = L_0^{(\alpha)}$.

LEMMA 3.1. *There exist positive constants a_n such that*

$$L_n^{(\alpha)}(x) - L_{n-1}^{(\alpha)}(x) = S_n(x) - a_{n-1}S_{n-1}(x), \qquad n \ge 1. \tag{3.2}$$

Proof. Put

$$L_n^{(\alpha)}(x) - L_{n-1}^{(\alpha)}(x) = S_n(x) + \sum_{i=0}^{n-1} \gamma_i^{(n)} S_i(x).$$

Then, for $0 \le i \le n-1$, with (2.6)

$$\gamma_i^{(n)}(S_i, S_i)_S = \left(L_n^{(\alpha)} - L_{n-1}^{(\alpha)}, S_i\right)_S = \int \left(L_n^{(\alpha)} - L_{n-1}^{(\alpha)}\right) S_i\,d\psi_0.$$

If $\xi < 0, \alpha > 0$ we apply (2.4) and if $M > 0, \alpha = 0$ we apply (2.1) in order to obtain

$$\int \left(L_n^{(\alpha)} - L_{n-1}^{(\alpha)}\right) S_i\,d\psi_0 = \int_0^{+\infty} \left(L_n^{(\alpha)} - L_{n-1}^{(\alpha)}\right) S_i x^\alpha e^{-x}\,dx.$$

For $0 \le i \le n-2$, the last integral equals zero. For $i = n-1$ we have

$$\begin{aligned}\gamma_{n-1}^{(n)}(S_{n-1}, S_{n-1})_S &= -\int_0^{+\infty} L_{n-1}^{(\alpha)}(x)S_{n-1}(x)x^\alpha e^{-x}\,dx \\ &= -\int_0^{+\infty} \left(L_{n-1}^{(\alpha)}(x)\right)^2 x^\alpha e^{-x}\,dx.\end{aligned}$$

∎

LEMMA 3.2. *The sequence $\{a_n\}_n$ in* (3.2) *satisfies the recurrence relation*

$$a_n = \frac{n+\alpha}{n(2+\lambda) + \alpha - \xi - na_{n-1}}, \qquad n \ge 1, \tag{3.3}$$

with $a_0 = \frac{\alpha}{\alpha-\xi}$ if $\alpha > 0$, $a_0 = \frac{1}{M+1}$ if $\alpha = 0$, and $a_0 = 1$ if $-1 < \alpha < 0$.

Proof. Write

$$R_0 = S_0,$$
$$R_n = S_n - a_{n-1}S_{n-1}, \qquad n \geq 1,$$

then for $n \geq 1$,

$$(R_{n+1}, R_n)_S + a_n(R_n, R_n)_S + a_n a_{n-1}(R_n, R_{n-1})_S = 0.$$

On the other hand, by (3.2), $R_n = L_n^{(\alpha)} - L_{n-1}^{(\alpha)}$ and then computing the Sobolev inner products with (3.1) we obtain

$$(R_{n+1}, R_n)_S = -\frac{\Gamma(n+\alpha+1)}{n!}, \qquad n \geq 0,$$
$$(R_n, R_n)_S = \frac{\Gamma(n+\alpha+1)}{n!} + \frac{\Gamma(n+\alpha)}{(n-1)!} - \xi\frac{\Gamma(n+\alpha)}{n!} + \lambda\frac{\Gamma(n+\alpha)}{(n-1)!}, \qquad n \geq 1.$$

Therefore, the recurrence relation (3.3) follows.

In order to obtain a_0 we use (3.2) with $n = 1$, $S_0 = L_0^{(\alpha)}$, and $(S_1, 1)_S = 0$. Then

$$\int (L_1^{(\alpha)} - L_0^{(\alpha)})\, d\psi_0 = -a_0 \int L_0^{(\alpha)} d\psi_0,$$

and evaluating the integrals as before we obtain a_0. ∎

In order to derive the asymptotic behaviour of S_n we need more information about the sequence $\{a_n\}_n$.

LEMMA 3.3. *The sequence $\{a_n\}_n$ is convergent, and*

$$\ell = \lim_{n\to\infty} a_n = \frac{\lambda + 2 - \sqrt{\lambda^2 + 4\lambda}}{2} < 1.$$

Moreover, for all $p < 1$, we have

$$\lim_{n\to\infty} n^p(a_n - \ell) = 0. \tag{3.4}$$

Proof. First we observe $0 < a_0 \leq 1$, and then a simple induction argument applied on Lemma 3.2 gives $0 < a_n \leq 1$ for all $n \geq 0$.

Suppose that $\ell = \lim_{n\to\infty} a_n$ exists, then (3.3) implies

$$\ell^2 - \ell(2+\lambda) + 1 = 0.$$

Since $0 < a_n \leq 1$ for all $n \geq 0$, we have $0 < \ell \leq 1$. Hence

$$0 < \ell = \frac{\lambda + 2 - \sqrt{\lambda^2 + 4\lambda}}{2} < 1.$$

Now, we prove that (3.4) is satisfied; in particular this implies that $\{a_n\}$ is indeed convergent.

With (3.3) and $\ell(2+\lambda) = \ell^2 + 1$ we have

$$a_n - \ell = \frac{\alpha - \ell(\alpha - \xi) + n\ell(a_{n-1} - \ell)}{n(2+\lambda) + \alpha - \xi - na_{n-1}}.$$

Then, using $a_{n-1} \leq 1$,

$$|a_n - \ell| \leq \frac{|\alpha - \alpha\ell + \ell\xi|}{n(1+\lambda) + \alpha} + \frac{n\ell|a_{n-1} - \ell|}{n(1+\lambda) + \alpha}. \tag{3.5}$$

Put $t_n = n^p|a_n - \ell|$, with $p < 1$, then

$$t_n \leq \frac{n^p|\alpha - \alpha\ell + \ell\xi|}{n(1+\lambda) + \alpha} + \frac{n^{p+1}}{(n-1)^p[n(1+\lambda) + \alpha]}\ell t_{n-1}.$$

Let $\varepsilon > 0$ and $\ell < r < 1$. Then there exists a positive integer number N such that

$$t_n < \varepsilon + rt_{n-1}, \qquad n \geq N + 1.$$

By repeated application, for $k \geq 1$ we deduce

$$t_{N+k} < \varepsilon(1 + r + r^2 + \cdots + r^{k-1}) + r^k t_N < \frac{\varepsilon}{1-r} + r^k t_N.$$

This implies

$$\lim_{n\to\infty} t_n = 0.$$

∎

Remark 3.4. In two special cases,

(i) $d\psi_0(x) = (x - \xi)x^{\alpha-1}e^{-x}\,dx$, with $\alpha > 0$, $\xi < 0$ and $\alpha = -\frac{\ell}{1-\ell}\xi$,

(ii) $d\psi_0(x) = e^{-x}\,dx + M\delta(0)$, with $M \geq 0$,

relation (3.5) reduces to

$$|a_n - \ell| \leq \frac{\ell}{1+\lambda}|a_{n-1} - \ell| \leq \left(\frac{\ell}{1+\lambda}\right)^n |a_0 - \ell|.$$

Then, for every value of p we have

$$\lim_{n\to\infty} n^p(a_n - \ell) = 0.$$

Now, we are able to derive the asymptotic behaviour of the Sobolev polynomials for coherent pairs of Laguerre type I.

THEOREM 3.5. *Let* $d\psi_1(x) = x^\alpha e^{-x}\ dx$ *with* $\alpha > -1$, $d\psi_0(x) = (x-\xi)x^{\alpha-1}e^{-x}\ dx + M\delta(0)$ *with* $\xi = 0$ *if* $\alpha \le 0$, *and* $M = 0$ *if* $\alpha \ne 0$; *the support of the measures is* $[0, +\infty)$. *Let* $\{S_n\}_n$ *denote the sequence of polynomials orthogonal with respect to* (3.1), *where the leading coefficient of* S_n *is equal to the leading coefficient of* $L_n^{(\alpha)}$. *Put*

$$S_n = \frac{1}{1-\ell} L_n^{(\alpha-1)} - \frac{\ell}{1-\ell} F_n.$$

If $x \in \mathbb{C} \setminus [0, +\infty)$, *then*

$$\lim_{n\to\infty} \frac{F_n(x)}{L_n^{(\alpha-2)}(x)} = \frac{1}{1-\ell},$$

uniformly on compact subsets of $\mathbb{C} \setminus [0, +\infty)$, *where* ℓ *is given by Lemma* 3.3.

Before proving Theorem 3.5 we mention two special cases in which our method gives the complete asymptotic expansion of S_n.

THEOREM 3.6. *Suppose*

(i) $d\psi_1(x) = x^\alpha e^{-x}\ dx$, $d\psi_0(x) = (x-\xi)x^{\alpha-1}e^{-x}\ dx$, *with* $\alpha > 0$, $\xi < 0$, *and* $\alpha = -\frac{\ell}{1-\ell}\xi$, *or*

(ii) $d\psi_1(x) = e^{-x}\ dx$, $d\psi_0(x) = e^{-x}\ dx + M\delta(0)$, *with* $M \ge 0$.

The support of the measures is $[0, +\infty)$ *and* $\{S_n\}_n$ *is defined as in Theorem* 3.5. *Then the complete asymptotic expansion of* S_n *is*

$$S_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \ell^k}{(1-\ell)^{k+1}} L_n^{(\alpha-k-1)}(x),$$

uniformly on compact subsets of $\mathbb{C} \setminus [0, +\infty)$.

Proof of Theorem 3.6. For $k \ge 0$, put

$$S_n(x) = \sum_{i=0}^{k} \frac{(-1)^i \ell^i}{(1-\ell)^{i+1}} L_n^{(\alpha-i-1)}(x) + \frac{(-1)^{k+1}\ell^{k+1}}{(1-\ell)^{k+1}} H_n(x). \tag{3.6}$$

We will prove

$$\lim_{n\to\infty} \frac{H_n(x)}{L_n^{(\alpha-k-2)}(x)} = \frac{1}{1-\ell}, \tag{3.7}$$

uniformly on compact subsets of $\mathbb{C} \setminus [0, +\infty)$. This will prove Theorem 3.6.

We start from (3.2) where we write $a_n = \ell + r_n$,

$$L_n^{(\alpha)} - L_{n-1}^{(\alpha)} = S_n - \ell S_{n-1} - r_{n-1} S_{n-1}. \tag{3.8}$$

We have, with (2.4),

$$L_n^{(\alpha)} - L_{n-1}^{(\alpha)} = \frac{1}{1-\ell}\left\{L_n^{(\alpha-1)} - \ell L_{n-1}^{(\alpha-1)}\right\} - \frac{\ell}{1-\ell}\left\{L_n^{(\alpha-1)} - L_{n-1}^{(\alpha-1)}\right\}.$$

By repeated application

$$\begin{aligned} L_n^{(\alpha)} - L_{n-1}^{(\alpha)} &= \sum_{i=0}^{k} \frac{(-1)^i \ell^i}{(1-\ell)^{i+1}} \left\{L_n^{(\alpha-i-1)} - \ell L_{n-1}^{(\alpha-i-1)}\right\} \\ &\quad + \frac{(-1)^{k+1}\ell^{k+1}}{(1-\ell)^{k+1}} \left\{L_n^{(\alpha-k-1)} - L_{n-1}^{(\alpha-k-1)}\right\}. \end{aligned} \tag{3.9}$$

Substitute (3.6) for n and $n-1$ in (3.8) and apply (3.9); then

$$\begin{aligned} L_n^{(\alpha-k-1)} - L_{n-1}^{(\alpha-k-1)} &= H_n - (\ell + r_{n-1})H_{n-1} \\ &\quad + r_{n-1}(-1)^k \frac{(1-\ell)^{k+1}}{\ell^{k+1}} \sum_{i=0}^{k} \frac{(-1)^i \ell^i}{(1-\ell)^{i+1}} L_{n-1}^{(\alpha-i-1)}. \end{aligned}$$

On the left hand side we apply (2.4). Then we rewrite the relation as

$$A_n = 1 + b_{n-1}A_{n-1} + \rho_{n-1}, \tag{3.10}$$

with

$$A_n = \frac{H_n}{L_n^{(\alpha-k-2)}},$$

$$b_{n-1} = (\ell + r_{n-1})\frac{L_{n-1}^{(\alpha-k-2)}}{L_n^{(\alpha-k-2)}},$$

$$\rho_{n-1} = r_{n-1}(-1)^{k+1} \sum_{i=0}^{k} \frac{(-1)^i \ell^{i-k-1}}{(1-\ell)^{i-k}} \frac{L_{n-1}^{(\alpha-i-1)}}{L_n^{(\alpha-k-2)}}.$$

By Remark 3.4, for every value of p, we have

$$\lim_{n\to\infty} n^p r_{n-1} = 0.$$

Then, with (2.7) and (2.8),

$$\lim_{n\to\infty} \rho_{n-1} = 0,$$

uniformly on compact subsets of $\mathbb{C} \setminus [0, +\infty)$. Moreover, by (2.7),

$$\lim_{n\to\infty} b_{n-1} = \ell,$$

again uniformly on compact subsets of $\mathbb{C} \setminus [0, +\infty)$.

It is our intention to prove (3.7), i.e.,

$$\lim_{n\to\infty} A_n = \frac{1}{1-\ell},$$

uniformly on compact subsets of $\mathbb{C}\setminus[0,\infty)$.

Put

$$A_n^* = A_n - \frac{1}{1-\ell}.$$

Then (3.10) implies

$$A_n^* = \frac{b_{n-1}-\ell}{1-\ell} + \rho_{n-1} + b_{n-1}A_{n-1}^*.$$

Let K denote a compact subset of $\mathbb{C}\setminus[0,\infty)$. Let $\varepsilon > 0$ and $\ell < r < 1$. Then there exists an N, such that, if $n \geq N+1$ and $x \in K$, we have

$$\left|A_n^*\right| < \varepsilon + r\left|A_{n-1}^*\right|.$$

By repeated application, for $k \geq 1$ and $x \in K$, we deduce

$$\left|A_{N+k}^*\right| < \varepsilon(1+r+\cdots+r^{k-1}) + r^k\left|A_N^*\right| < \frac{\varepsilon}{1-r} + r^k\left|A_N^*\right|.$$

This implies

$$\lim_{n\to\infty} A_n^* = 0,$$

uniformly on K. This proves (3.7); thus Theorem 3.6 follows. ∎

Proof of Theorem 3.5. The proof of Theorem 3.5 is just the proof of Theorem 3.6 with $k = 0$. Observe that, by Lemma 3.3,

$$\lim_{n\to\infty} n^p r_{n-1} = 0,$$

but only for $p < 1$. Hence

$$\lim_{n\to\infty} \rho_{n-1} = 0,$$

only for $k = 0$, and we can give by our method only two terms of the asymptotic expansion of S_n. ∎

4. COHERENT PAIRS OF LAGUERRE TYPE II

In this section we consider the coherent pair

$$d\psi_0(x) = x^\alpha e^{-x}\,dx, \qquad d\psi_1(x) = \frac{x^{\alpha+1}e^{-x}}{x-\xi}dx + M\delta(\xi),$$

where the absolutely continuous part of the measures are defined on $(0, +\infty)$, $\alpha > -1$, $\xi \le 0$, $M \ge 0$.

Let $\{T_n\}_n$ denote the sequence of polynomials orthogonal with respect to $d\psi_1$, normalized by the condition that the leading coefficients of T_n and $L_n^{(\alpha)}$ are equal. The aim of this section is to derive the asymptotic behaviour of the sequence of Sobolev polynomials $\{S_n\}_n$ orthogonal with respect to the inner product

$$(f, g)_S = \int f(x)g(x)d\psi_0(x) + \lambda \int f'(x)g'(x)d\psi_1(x), \tag{4.1}$$

with $\lambda > 0$, outside the supports of the measures. As before the leading coefficient of S_n is taken equal to the leading coefficient of $L_n^{(\alpha)}$. Observe that

$$T_0 = S_0 = L_0^{(\alpha)}, \qquad S_1 = L_1^{(\alpha)}.$$

We start with a study of $\{T_n\}_n$.

LEMMA 4.1. *The polynomials T_n satisfy*

$$T_n = L_n^{(\alpha+1)} - c_n L_{n-1}^{(\alpha+1)}, \qquad n \ge 1, \tag{4.2}$$

with

$$c_n = \frac{n! \int T_n^2\, d\psi_1}{\Gamma(n+\alpha+1)}. \tag{4.3}$$

Proof. For $0 \le i \le n-1$, we have

$$\int_0^{+\infty} T_n L_i^{(\alpha+1)} x^{\alpha+1} e^{-x}\,dx = \int T_n L_i^{(\alpha+1)}(x-\xi)d\psi_1.$$

For $0 \le i \le n-2$, the last integral is zero. For $i = n-1$ we have

$$\int T_n L_{n-1}^{(\alpha+1)}(x-\xi)d\psi_1 = -n \int T_n^2\, d\psi_1$$

and the Lemma follows from (2.3). ∎

Observe that (4.2) with (2.5) implies that $\{d\psi_0,\ d\psi_1\}$ is indeed a coherent pair as defined by (1.2).

LEMMA 4.2. *Let* $\{c_n\}_n$ *denote the sequence of coefficients in* (4.2). *Put* $d_n = c_n - 1$ $(n \geq 1)$. *Then*

$$d_n = \frac{\xi \int_0^{+\infty} L_n^{(\alpha)} \frac{x^\alpha e^{-x}}{x-\xi} dx + M L_n^{(\alpha)}(\xi)}{\int_0^{+\infty} L_{n-1}^{(\alpha+1)} \frac{x^{\alpha+1} e^{-x}}{x-\xi} dx + M L_{n-1}^{(\alpha+1)}(\xi)}, \qquad n \geq 1. \tag{4.4}$$

Proof. For $n \geq 1$ we have

$$\int T_n \, d\psi_1 = 0.$$

Substituting (4.2) we obtain

$$c_n \left\{ \int_0^{+\infty} L_{n-1}^{(\alpha+1)} \frac{x^{\alpha+1} e^{-x}}{x - \xi} dx + M L_{n-1}^{(\alpha+1)}(\xi) \right\}$$
$$= \int_0^{+\infty} L_n^{(\alpha+1)} \frac{x^{\alpha+1} e^{-x}}{x - \xi} dx + M L_n^{(\alpha+1)}(\xi).$$

Then the lemma follows with (2.4). ∎

LEMMA 4.3. *Let* $\xi < 0$. *Then*

$$I_n^{(\alpha)} = \int_0^{+\infty} L_{n-1}^{(\alpha+1)} \frac{x^{\alpha+1} e^{-x}}{x - \xi} dx \sim (-\xi)^{\alpha/2+1/4} n^{\alpha/2+1/4} e^{-2\sqrt{-n\xi}} e^{(-(1/2)\xi)} \sqrt{\pi}$$

if $n \to \infty$.

Proof. Using Rodrigues' formula (2.2) we obtain

$$I_n^{(\alpha)} = \int_0^{+\infty} L_{n-1}^{(\alpha+1)} \frac{x^{\alpha+1} e^{-x}}{x - \xi} dx = \frac{1}{(n-1)!} \int_0^{+\infty} \frac{1}{x - \xi} D^{n-1}(e^{-x} x^{n+\alpha}) dx.$$

After integration by parts $n - 1$ times, we have

$$I_n^{(\alpha)} = \int_0^{+\infty} \frac{x^{n+\alpha} e^{-x}}{(x - \xi)^n} dx. \tag{4.5}$$

Substitution of $x = -\xi t$ gives

$$I_n^{(\alpha)} = (-\xi)^{\alpha+1} \int_0^{+\infty} e^{\xi t} \left(\frac{t}{t+1} \right)^n t^\alpha \, dt.$$

The function $e^{\xi t}(t/(t+1))^n$ has a maximum in

$$t_m = -\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{n}{\xi}} > 0.$$

Substitute $t = t_m(1+\tau)$. Then

$$I_n^{(\alpha)} = (-\xi)^{\alpha+1} e^{\xi t_m} \frac{t_m^{n+\alpha+1}}{(t_m+1)^n} \int_{-1}^{+\infty} e^{\xi\tau t_m} \left(\frac{1+\tau}{1+\frac{t_m\tau}{t_m+1}} \right)^n (1+\tau)^\alpha \, d\tau.$$

Put

$$h(\tau) = \xi\tau t_m + n \ \log(1+\tau) - n \ \log\left(1 + \frac{t_m\tau}{t_m+1}\right).$$

We divide the interval of integration in three parts: $(-1, -\frac{1}{2})$, $[-\frac{1}{2}, \frac{1}{2}]$ and $(\frac{1}{2}, \infty)$. We determine the asymptotic behaviour of the integral over $[-\frac{1}{2}, \frac{1}{2}]$.

We have

$$\begin{aligned} &\log(1+\tau) - \log\left(1 + \frac{t_m\tau}{t_m+1}\right) \\ &\quad = -\log\left(1 - \frac{\tau}{(1+\tau)(t_m+1)}\right) \\ &\quad = \frac{\tau}{(1+\tau)(t_m+1)} + \frac{1}{2}\frac{\tau^2}{(1+\tau)^2(t_m+1)^2} + \frac{R(\tau)}{(t_m+1)^3}, \end{aligned}$$

where the remainder $R(\tau)$ is uniformly bounded in $[-\frac{1}{2}, \frac{1}{2}]$. Then, with $\xi t_m^2 + \xi t_m + n = 0$,

$$h(\tau) = \xi\tau t_m + \frac{n\tau}{(1+\tau)(t_m+1)} - \frac{1}{2}\frac{\xi\tau^2}{(1+\tau)^2} + \rho_n,$$

where $|\rho_n| < \varepsilon$, if n is sufficiently large, uniformly on $[-\frac{1}{2}, \frac{1}{2}]$.

We obtain

$$h(\tau) = -\frac{n}{t_m+1}\frac{\tau^2}{1+\tau} - \frac{1}{2}\frac{\xi\tau^2}{(1+\tau)^2} + \rho_n.$$

Then Laplace's method (see, e.g., [9, p. 81]) gives

$$\int_{-(1/2)}^{1/2} e^{h(\tau)}(1+\tau)^\alpha \, d\tau \sim \sqrt{\frac{t_m+1}{n}\pi}$$

if $n \to \infty$.

It is easy to see that the other intervals of integration have a contribution of lower order. Hence

$$I_n^{(\alpha)} \sim (-\xi)^{\alpha+1} e^{\xi t_m} t_m^{\alpha+1} \left(\frac{t_m}{t_m+1}\right)^n \sqrt{\frac{t_m}{n}\pi}.$$

From this result the lemma follows. ∎

LEMMA 4.4. *Let $\{d_n\}_n$ denote the constants in* (4.4). *Then*

$$\lim_{n\to\infty} n^{1/2} d_n = \begin{cases} -(-\xi)^{1/2} & \text{if} \quad M = 0, \\ (-\xi)^{1/2} & \text{if} \quad M > 0. \end{cases}$$

Proof. (i) Suppose $M = 0$. If $\xi = 0$, then (4.4) implies $d_n = 0$ for all $n \geq 1$. If $\xi < 0$, then the result follows from Lemma 4.3.

(ii) Suppose $M > 0$ and $\xi < 0$. By Lemma 4.3 and Perron's result (Lemma 2.1), we have

$$\lim_{n\to\infty} \frac{I_n^{(\alpha)}}{L_{n-1}^{(\alpha+1)}(\xi)} = 0.$$

Then the result follows from (2.8).

(iii) Finally suppose $M > 0$, $\xi = 0$. From (4.5) we see

$$\int_0^{+\infty} L_{n-1}^{(\alpha+1)} x^\alpha e^{-x}\, dx = \Gamma(\alpha+1).$$

Then (2.1) gives

$$d_n \sim \frac{\binom{n+\alpha}{n}}{\binom{n+\alpha}{n-1}},$$

and the result follows. ∎

THEOREM 4.5. *Let $\{T_n\}_n$ denote the sequence of polynomials orthogonal with respect to the measure*

$$d\psi_1(x) = \frac{x^{\alpha+1} e^{-x}}{x - \xi} dx + M\delta(\xi), \qquad \alpha > -1, \qquad \xi \leq 0, \qquad M \geq 0,$$

where the support of the absolutely continuous part of the measure is $[0, \infty)$, *and the leading coefficient of T_n is equal to the leading coefficient of $L_n^{(\alpha)}$. Then, if $x \in \mathbb{C} \setminus [0, \infty)$,*

$$\lim_{n\to\infty} \frac{T_n(x)}{L_n^{(\alpha)}(x)} = \begin{cases} 1 + \frac{(-\xi)^{1/2}}{(-x)^{1/2}} & \text{if} \quad M = 0, \\ 1 - \frac{(-\xi)^{1/2}}{(-x)^{1/2}} & \text{if} \quad M > 0; \end{cases}$$

the convergence is uniform on compact subsets of $\mathbb{C} \setminus [0, \infty)$.

Proof. Relation (4.2) with $c_n = 1 + d_n$ reads

$$T_n = L_n^{(\alpha+1)} - L_{n-1}^{(\alpha+1)} - d_n L_{n-1}^{(\alpha+1)}.$$

Then with (2.4),

$$\frac{T_n}{L_n^{(\alpha)}} = 1 - n^{1/2} d_n \frac{L_{n-1}^{(\alpha+1)}}{L_n^{(\alpha)} n^{1/2}}.$$

The theorem follows from (2.7), (2.8), and Lemma 4.4. ∎

Remark 4.6. Observe that the limit in Theorem 4.5 becomes zero if $M > 0$, $x = \xi < 0$. However, in that case, x is in the support of $d\psi_1$. If x is outside the support of $d\psi_1$, then the limit is different from zero.

We now return to the study of $\{S_n\}_n$, the sequence of orthogonal polynomials with respect to the inner product (4.1). Define c_0 by

$$c_0 = \frac{\int T_0^2 \, d\psi_1}{\Gamma(\alpha + 1)},$$

in such a way that (4.3) holds for all $n \geq 0$.

LEMMA 4.7. *There exist positive constants a_n such that*

$$L_n^{(\alpha)} - c_{n-1} L_{n-1}^{(\alpha)} = S_n - a_{n-1} S_{n-1}, \qquad n \geq 1, \tag{4.6}$$

where the coefficients c_n are defined by (4.3).

Proof. By (2.5) and (4.2) we have

$$\frac{d}{dx}\left\{L_n^{(\alpha)}(x) - c_{n-1} L_{n-1}^{(\alpha)}(x)\right\} = -T_{n-1}(x), \qquad n \geq 1.$$

Write

$$L_n^{(\alpha)} - c_{n-1} L_{n-1}^{(\alpha)} = S_n + \sum_{i=0}^{n-1} \gamma_i^{(n)} S_i.$$

Then, for $0 \leq i \leq n - 1$,

$$\gamma_i^{(n)} (S_i, S_i)_S = (L_n^{(\alpha)} - c_{n-1} L_{n-1}^{(\alpha)}, S_i)_S = \int_0^{+\infty} \left(L_n^{(\alpha)} - c_{n-1} L_{n-1}^{(\alpha)}\right) S_i x^\alpha e^{-x} \, dx.$$

For $0 \leq i \leq n - 2$, the last integral is zero. For $i = n - 1$ we have

$$\begin{aligned}\gamma_{n-1}^{(n)} (S_{n-1}, S_{n-1})_S &= \int_0^{+\infty} \left(L_n^{(\alpha)} - c_{n-1} L_{n-1}^{(\alpha)}\right) S_{n-1} x^\alpha e^{-x} \, dx \\ &= -c_{n-1} \int_0^{+\infty} \left(L_{n-1}^{(\alpha)}\right)^2 x^\alpha e^{-x} \, dx,\end{aligned}$$

and the lemma follows. ∎

LEMMA 4.8. *The sequence* $\{a_n\}_n$ *in* (4.6) *satisfies the recurrence relation*

$$a_n = \frac{c_n(n+\alpha)}{n+\alpha+nc_{n-1}^2+\lambda nc_{n-1}-nc_{n-1}a_{n-1}}, \qquad n \geq 1, \tag{4.7}$$

with $a_0 = c_0$.

Proof. As in Section 3, we write $R_0 = S_0$, $R_n = S_n - a_{n-1}S_{n-1}$, $n \geq 1$. Then, for $n \geq 1$,

$$(R_{n+1}, R_n)_S + a_n(R_n, R_n)_S + a_n a_{n-1}(R_n, R_{n-1})_S = 0.$$

With $R_n = L_n^{(\alpha)} - c_{n-1}L_{n-1}^{(\alpha)}$ we evaluate the Sobolev inner products. Then

$$(R_{n+1}, R_n)_S = -c_n \frac{\Gamma(n+\alpha+1)}{n!}, \qquad n \geq 0.$$

With (4.3),

$$(R_n, R_n)_S = \frac{\Gamma(n+\alpha+1)}{n!} + c_{n-1}^2 \frac{\Gamma(n+\alpha)}{(n-1)!} + \lambda c_{n-1} \frac{\Gamma(n+\alpha)}{(n-1)!}, \qquad n \geq 1.$$

Now, the recurrence relation (4.7) follows. With $S_0 = L_0^{(\alpha)}$, $S_1 = L_1^{(\alpha)}$ we obtain from (4.6) $a_0 = c_0$. ∎

Remark 4.9. From $a_{n-1} \leq c_{n-1}$ it follows with (4.7) that $a_n < c_n$. Since $a_0 = c_0$, by induction $a_n < c_n$ for $n \geq 1$. In particular we have for the denominator in (4.7)

$$n+\alpha+nc_{n-1}^2+\lambda nc_{n-1}-nc_{n-1}a_{n-1} > n+\alpha. \tag{4.8}$$

LEMMA 4.10. *The sequence* $\{a_n\}_n$ *in* (4.7) *is convergent, and*

$$\ell = \lim_{n\to\infty} a_n = \frac{\lambda+2-\sqrt{\lambda^2+4\lambda}}{2} < 1.$$

Proof. Write $c_n = 1 + d_n$, with, by Lemma 4.4,

$$\lim_{n\to\infty} d_n = 0.$$

We can rewrite (4.7) as

$$a_n = \frac{n+nr_n}{n(2+\lambda)+ns_n-na_{n-1}},$$

with

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty} s_n = 0.$$

Then, with $\ell(2+\lambda) = \ell^2 + 1$ we have

$$a_n - \ell = \frac{nr_n - n\ell s_n + n\ell(a_{n-1} - \ell)}{n(2+\lambda) + ns_n - na_{n-1}}.$$

With (4.8), we have

$$|a_n - \ell| \le \frac{n}{n+\alpha}|r_n| + \frac{n\ell}{n+\alpha}|s_n| + \frac{n}{n+\alpha}\ell|a_{n-1} - \ell|.$$

Let $\varepsilon > 0$ and $\ell < r < 1$. Then there exists a positive integer number N such that

$$|a_n - \ell| < \varepsilon + r|a_{n-1} - \ell|, \qquad n \ge N+1.$$

Proceeding as in the proof of Lemma 3.3, this implies

$$\lim_{n\to\infty} |a_n - \ell| = 0.$$

∎

THEOREM 4.11. *Let*

$$d\psi_0(x) = x^\alpha e^{-x}\, dx,$$

$$d\psi_1(x) = \frac{x^{\alpha+1}e^{-x}}{x-\xi}dx + M\delta(\xi), \qquad \alpha > -1, \qquad \xi \le 0, \qquad M \ge 0,$$

where the support of the absolutely continuous part of the measures is $[0,\infty)$. *Let* $\{S_n\}_n$ *denote the sequence of polynomials orthogonal with respect to* (4.1), *with the leading coefficient of* S_n *equal to the leading coefficient of* $L_n^{(\alpha)}$.

If $x \in \mathbb{C}\setminus[0,\infty)$, *then*

$$\lim_{n\to\infty}\frac{S_n(x)}{L_n^{(\alpha-1)}(x)} = \begin{cases} \frac{1}{1-\ell}\left\{1 + \frac{(-\xi)^{1/2}}{(-x)^{1/2}}\right\} & \text{if} \quad M = 0, \\ \frac{1}{1-\ell}\left\{1 - \frac{(-\xi)^{1/2}}{(-x)^{1/2}}\right\} & \text{if} \quad M > 0; \end{cases}$$

the convergence is uniform on compact subsets of $\mathbb{C}\setminus[0,\infty)$.

Proof. With $c_n = 1 + d_n$ (4.6) becomes

$$L_n^{(\alpha)} - L_{n-1}^{(\alpha)} - d_{n-1}L_{n-1}^{(\alpha)} = S_n - a_{n-1}S_{n-1}.$$

Then, using (2.4),

$$1 - d_{n-1}n^{1/2}\frac{L_{n-1}^{(\alpha)}}{L_n^{(\alpha-1)}n^{1/2}} = \frac{S_n}{L_n^{(\alpha-1)}} - a_{n-1}\frac{S_{n-1}}{L_{n-1}^{(\alpha-1)}}\frac{L_{n-1}^{(\alpha-1)}}{L_n^{(\alpha-1)}}. \tag{4.9}$$

In the sequel the upper sign in the symbols $\pm$ and $\mp$ is connected with $M > 0$ and the lower one with $M = 0$. With (2.7), (2.8), and Lemma 4.4, we have

$$\lim_{n\to\infty} d_{n-1} n^{1/2} \frac{L_{n-1}^{(\alpha)}}{L_n^{(\alpha-1)} n^{1/2}} = \pm \frac{(-\xi)^{1/2}}{(-x)^{1/2}}.$$

Then we can rewrite (4.9) as

$$A_n = 1 \mp \frac{(-\xi)^{1/2}}{(-x)^{1/2}} + b_{n-1} A_{n-1} + \rho_{n-1},$$

with

$$A_n = \frac{S_n}{L_n^{(\alpha-1)}},$$

$$b_{n-1} = a_{n-1} \frac{L_{n-1}^{(\alpha-1)}}{L_n^{(\alpha-1)}},$$

$$\rho_{n-1} = -d_{n-1} \frac{L_{n-1}^{(\alpha)}}{L_n^{(\alpha-1)}} + \lim_{n\to\infty} d_{n-1} \frac{L_{n-1}^{(\alpha)}}{L_n^{(\alpha-1)}}.$$

Here

$$\lim_{n\to\infty} b_{n-1} = \ell,$$

$$\lim_{n\to\infty} \rho_{n-1} = 0,$$

the convergence is uniform on compact subsets of $\mathbb{C} \setminus [0, \infty)$. Proceeding as in the proof of Theorem 3.6 we arrive at the desired result. ∎

Remark 4.12. If x is outside the support of the measures, then the limit in Theorem 4.11 is not zero. However, if $M > 0$, $x = \xi < 0$, then x is in the support of $d\psi_1$ and the limit becomes zero.

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