

Lie Symmetry Analysis and Approximate Solutions for Non-linear Radial Oscillations of an Incompressible Mooney–Rivlin Cylindrical Tube

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The non-linear, second-order differential equation derived by Knowles (1960, *Quart. Appl. Math.* **18**, 71–77) which governs the axisymmetric radial oscillations of an infinitely long, hyperelastic cylindrical tube of Mooney–Rivlin material is considered. It is shown that if the boundary conditions are time dependent, then the Knowles equation has no Lie point symmetries, while if the boundary conditions are constant it has one Lie point symmetry corresponding to time translational invariance. The derivation by Knowles (1962, *J. Appl. Mech.* **29**, 283–286) of bounds on the period of the oscillation for the heaviside step loading boundary condition is extended to obtain limiting oscillations that exhibit periods that bound the exact period above and below. The Knowles equation for a Mooney–Rivlin material is expanded in powers of a dimensionless parameter, μ , defined in terms of the thickness of the tube wall. To zero order in μ an Ermakov–Pinney equation is obtained which has three Lie point symmetries. It is shown that the differential equation which is correct to first order in μ also has three Lie point symmetries which disappear at second order in μ . For time independent boundary conditions, the three Lie point symmetries of the order μ equation are derived explicitly and the associated first integrals are obtained. The general solution is derived in terms of the three first integrals and it is illustrated for free oscillations and the heaviside step loading boundary condition. The non-autonomous first order in μ equation is transformed to an autonomous Ermakov–Pinney equation and a non-linear superposition principle for the solution to first order in μ is derived and applied to a blast loaded applied pressure that decays linearly with time. The solutions to first order in μ are compared with numerical solutions of the Knowles equation for a thick-walled cylinder and are found to be more accurate than the zero order solutions described by the Ermakov–Pinney equation. © 2000 Academic Press

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1. INTRODUCTION

The first investigation of a dynamic problem in a bounded medium in finite elasticity was undertaken by Knowles [10, 11]. He considered the non-linear radial oscillations of an infinitely long, circular cylindrical tube of incompressible hyperelastic material and reduced the equation of motion to a second order ordinary differential equation. For a Mooney-Rivlin material, the Knowles equation of motion is

$$\left[\ln \left(1 + \frac{\mu}{x^2} \right) \right] x \ddot{x} + \left[\ln \left(1 + \frac{\mu}{x^2} \right) - \frac{\mu}{x^2 \left(1 + \frac{\mu}{x^2} \right)} \right] \dot{x}^2 + \kappa \left[\ln \left(\frac{1 + \mu}{1 + \frac{\mu}{x^2}} \right) + \frac{\mu(x^2 - 1)}{x^4 \left(1 + \frac{\mu}{x^2} \right)} \right] = \mu P(t), \quad (1.1)$$

where the overhead dot denotes differentiation with respect to time,

$$x(t) = \frac{r_1(t)}{\rho_1}, \quad \mu = \left(\frac{\rho_2}{\rho_1} \right)^2 - 1, \quad (1.2)$$

$$\kappa = \frac{2(C_1 + C_2)}{\rho^* \rho_1^2}, \quad P(t) = \frac{2(P_1(t) - P_2(t))}{\mu \rho^* \rho_1^2},$$

C_1 and C_2 are the Mooney-Rivlin constants which are assumed positive, the constant ρ^* is the density of the homogeneous and elastically isotropic incompressible tube, ρ_1 and ρ_2 are the inner and outer radii of the undeformed tube which deform continuously to give the inner and outer radii $r_1(t)$ and $r_2(t)$ of the deformed tube, and $P_1(t) - P_2(t)$ is the net radial applied pressure, where $P_1(t)$ and $P_2(t)$ are the applied pressures on the inner and outer surfaces of the tube. The dimensionless parameter, μ , which is defined in terms of the thickness of the wall of the tube, is small for a thin-walled tube. Since the left-hand side of Eq. (1.1) is of order μ , $P(t)$ is of order one. This paper is concerned with Eq. (1.1) and its expansion in powers of μ .

In the limiting case of a thin-walled tube, the Knowles equation for a Mooney-Rivlin material (1.1) reduces to the Ermakov-Pinney equation [3, 19]

$$\ddot{x} + (\kappa - P(t))x = \frac{\kappa}{x^3}. \quad (1.3)$$

This was utilized by Nowinski and Wang [17] who considered a vanishing net applied load at the boundary to analyse free oscillations. The general solution of the Ermakov–Pinney equation may be obtained from a non-linear superposition principle [3, 19]. Thus Shahinpoor and Nowinski [22] were able to derive exact solutions for free oscillations, heaviside step loading, blast loading, a harmonically varying load, and a periodic step pulse load at the boundary. Rogers and Baker [21] extended their results to a larger class of strain-energy functions using a generalisation by Burt and Reid [1] of the non-linear superposition principle of Ermakov and Pinney. Their work in this area has been reviewed by Rogers and Ames [20].

The coefficients in the Knowles equation (1.1) can be expanded in powers of the thickness parameter μ and differential equations correct to successive orders of μ can be derived. We will investigate the relationship between the order in μ to which the differential equation is expanded and the Lie point symmetry structure of the equation. The approximation to zero order in μ , which is the Ermakov–Pinney equation (1.3), and the differential equation correct to first order in μ each admit three Lie point symmetries. For the expansion correct to second and higher orders in μ as well as for the exact equation (1.1), symmetry breaking occurs, and the autonomous equations where the net applied pressure at the boundaries is time independent possess only one Lie point symmetry corresponding to time translational invariance, while the non-autonomous equations where the net applied pressure is time dependent possess no Lie point symmetries.

By adopting an approach similar to the one used by Knowles [11] to derive upper and lower bounds on the period we derive limiting oscillations for Eq. (1.1), subject to a heaviside step loaded boundary condition, which have periods which bound the exact period above and below for both net inward and net outward applied pressures.

For the non-linear radial oscillations correct to first order in μ we obtain the first-order correction of the results derived by Shahinpoor and Nowinski [22] for the zero-order equation (1.3). The three Lie point symmetries of the autonomous differential equation correct to first order in μ are obtained explicitly from which three associated first integrals are derived. The general solution for constant $P(t)$ may thus be expressed in terms of the three first integrals. When the boundary conditions are time dependent, the differential equation correct to first order in μ is transformed to an autonomous equation by insisting that the transformed equation possess the Lie point symmetry corresponding to time translational invariance in the transformed time variable. The transformed equation is found to be an autonomous Ermakov–Pinney equation. By starting

from this Ermakov-Pinney equation, a non-linear superposition principle correct to first order in μ is derived.

An outline of the paper is as follows. In Section 2 the Lie point symmetries of the Knowles equation (1.1) for a thick-walled tube are investigated. In Section 3 the heaviside step loading boundary condition is considered and limiting oscillations are obtained which correspond to the upper and lower bounds on the period for both net inward and net outward applied pressures. In Section 4 the Knowles equation is expanded in powers of the dimensionless parameter μ . It is shown that the differential equations correct to zero or first order in μ each have three Lie point symmetries and that the reduction in the number of symmetries from three to one or zero, depending on whether the net applied pressure P is constant or time dependent, occurs at the second order in μ approximation. For the special case in which P is constant the three Lie point symmetries are calculated correct to first order in μ . In Section 5 the autonomous case of constant P is considered. The first integral associated with each of the three Lie point symmetries is derived correct to first order in μ and the general solution for constant P is found in terms of the three first integrals. The general solution is used to obtain explicit solutions correct to first order in μ for free oscillations and for the heaviside step loading boundary condition. In Section 6 the non-autonomous case of time dependent boundary conditions is considered. The differential equation correct to first order in μ is transformed to an autonomous Ermakov-Pinney equation and a non-linear superposition principle, correct to first order in μ , is derived for the equation and used to solve to first order in μ the problem of blast loading with linear decay in time. The new results correct to first order in μ are compared graphically with the zero order solutions obtained by Shahinpoor and Nowinski [22] and with the numerical solution of the exact equation (1.1). Finally, concluding remarks are made in Section 7.

2. LIE POINT SYMMETRIES

In this section we investigate the Lie point symmetries of Eq. (1.1). Extensive treatments of Lie group theory applied to differential equations have been given by Olver [18], Bluman and Kumei [2], Stephani [23], Mahomed and Leach [16], and Ibragimov and Anderson [7].

In general, the second order differential equation

$$\ddot{x} = F(t, x, \dot{x}) \quad (2.1)$$

is said to admit a Lie point symmetry generated by

$$X = \xi(t, x) \frac{\partial}{\partial t} + \eta(t, x) \frac{\partial}{\partial x}, \quad (2.2)$$

if

$$X^{[2]}(\ddot{x} - F(t, x, \dot{x})) = 0 \quad \text{whenever } \ddot{x} = F(t, x, \dot{x}), \quad (2.3)$$

where [7]

$$X^{[2]} = X^{[1]} + \zeta_2(t, x, \dot{x}, \ddot{x}) \frac{\partial}{\partial \ddot{x}} \quad (2.4)$$

is the second prolongation of the generator X ,

$$X^{[1]} = X + \zeta_1(t, x, \dot{x}) \frac{\partial}{\partial \dot{x}} \quad (2.5)$$

is the first prolongation of the generator X ,

$$\zeta_1(t, x, \dot{x}) = D(\eta) - \dot{x}D(\xi), \quad \zeta_2(t, x, \dot{x}, \ddot{x}) = D(\zeta_1) - \ddot{x}D(\xi), \quad (2.6)$$

and

$$D = \frac{\partial}{\partial t} + \dot{x} \frac{\partial}{\partial x} + \ddot{x} \frac{\partial}{\partial \dot{x}}. \quad (2.7)$$

For arbitrarily given $F(t, x, \dot{x})$, the determining equation (2.3) is

$$\begin{aligned} & \frac{\partial^2 \eta}{\partial t^2} + 2\dot{x} \frac{\partial^2 \eta}{\partial t \partial x} + \dot{x}^2 \frac{\partial^2 \eta}{\partial x^2} - \left(\frac{\partial^2 \xi}{\partial t^2} + 2\dot{x} \frac{\partial^2 \xi}{\partial t \partial x} + \dot{x}^2 \frac{\partial^2 \xi}{\partial x^2} \right) \dot{x} \\ & + \left(\frac{\partial \eta}{\partial x} - 2 \frac{\partial \xi}{\partial t} - 3\dot{x} \frac{\partial \xi}{\partial x} \right) F - \xi \frac{\partial F}{\partial t} - \eta \frac{\partial F}{\partial x} \\ & - \left(\frac{\partial \eta}{\partial t} + \dot{x} \left(\frac{\partial \eta}{\partial x} - \frac{\partial \xi}{\partial t} \right) - \dot{x}^2 \frac{\partial \xi}{\partial x} \right) \frac{\partial F}{\partial \dot{x}} = 0. \end{aligned} \quad (2.8)$$

For Eq. (1.1), the expanded determining equation (2.8) yields an equation which can be separated by equating the coefficients of like powers of \dot{x} ,

$$\dot{x}^3: \quad \frac{\partial^2 \xi}{\partial x^2} + \left[-\frac{1}{x} + \frac{\mu}{x^3 \left(1 + \frac{\mu}{x^2} \right) \ln \left(1 + \frac{\mu}{x^2} \right)} \right] \frac{\partial \xi}{\partial x} = 0, \quad (2.9)$$

$$\dot{x}^2: \quad \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial}{\partial x} \left[\left(\frac{1}{x} - \frac{\mu}{x^3 \left(1 + \frac{\mu}{x^2} \right) \ln \left(1 + \frac{\mu}{x^2} \right)} \right) \eta \right] - 2 \frac{\partial^2 \xi}{\partial x \partial t} = 0, \quad (2.10)$$

$$\begin{aligned} \dot{x}^1: \quad & 2 \frac{\partial^2 \eta}{\partial x \partial t} - 2 \left[-\frac{1}{x} + \frac{\mu}{x^3 \left(1 + \frac{\mu}{x^2} \right) \ln \left(1 + \frac{\mu}{x^2} \right)} \right] \frac{\partial \eta}{\partial t} - \frac{\partial^2 \xi}{\partial t^2} \\ & + 3 \left[\kappa \left\{ \frac{\ln(1 + \mu)}{x \ln \left(1 + \frac{\mu}{x^2} \right)} - \frac{1}{x} + \frac{\mu(x^2 - 1)}{x^5 \left(1 + \frac{\mu}{x^2} \right) \ln \left(1 + \frac{\mu}{x^2} \right)} \right\} \right. \\ & \quad \left. - \frac{\mu P(t)}{x \ln \left(1 + \frac{\mu}{x^2} \right)} \right] \frac{\partial \xi}{\partial x} = 0, \end{aligned} \quad (2.11)$$

$$\begin{aligned} \dot{x}^0: \quad & \frac{\partial^2 \eta}{\partial t^2} \\ & + \eta^2 \frac{\partial}{\partial x} \left[\left(\kappa \left\{ \frac{\ln(1 + \mu)}{x \ln \left(1 + \frac{\mu}{x^2} \right)} - \frac{1}{x} + \frac{\mu(x^2 - 1)}{x^5 \left(1 + \frac{\mu}{x^2} \right) \ln \left(1 + \frac{\mu}{x^2} \right)} \right\} \right. \right. \\ & \quad \left. \left. - \frac{\mu P(t)}{x \ln \left(1 + \frac{\mu}{x^2} \right)} \right) \frac{1}{\eta} \right] \\ & + 2 \left[\kappa \left\{ \frac{\ln(1 + \mu)}{x \ln \left(1 + \frac{\mu}{x^2} \right)} - \frac{1}{x} + \frac{\mu(x^2 - 1)}{x^5 \left(1 + \frac{\mu}{x^2} \right) \ln \left(1 + \frac{\mu}{x^2} \right)} \right\} \right. \\ & \quad \left. - \frac{\mu P(t)}{x \ln \left(1 + \frac{\mu}{x^2} \right)} \right] \frac{\partial \xi}{\partial t} - \xi \frac{\mu \dot{P}(t)}{x \ln \left(1 + \frac{\mu}{x^2} \right)} = 0. \end{aligned} \quad (2.12)$$

The aim is to solve Eqs. (2.9)–(2.12) for $\xi(t, x)$ and $\eta(t, x)$.

Consider first Eq. (2.9) for $\xi(t, x)$. It is a first order, homogeneous, linear differential equation for $\partial\xi/\partial x$ with integrating factor $(x(\ln(1 + \mu/x^2))^{1/2})^{-1}$ and can be integrated once with respect to x to obtain

$$\frac{\partial\xi}{\partial x} = \frac{1}{\mu^{1/2}} f_1(t) x \left(\ln \left(1 + \frac{\mu}{x^2} \right) \right)^{1/2}, \quad (2.13)$$

where $f_1(t)$ is an unknown function of t and the factor $\mu^{1/2}$ is introduced in the denominator because the numerator in (2.13) is $O(\mu^{1/2})$ as $\mu \rightarrow 0$. The results derived here will therefore coincide with known results for the Ermakov–Pinney equation (1.3) derived from the Knowles equation for a Mooney–Rivlin material in the thin-shell limit as $\mu \rightarrow 0$ [20]. The integration of (2.13) with respect to x gives

$$\xi(t, x) = f_1(t) H(x) + f_2(t), \quad (2.14)$$

where $f_2(t)$ is an unknown function of t and

$$H(x; \mu) = \frac{1}{\mu^{1/2}} \int^x x \left(\ln \left(1 + \frac{\mu}{x^2} \right) \right)^{1/2} dx. \quad (2.15)$$

Consider next Eq. (2.10) for $\eta(t, x)$. It may be integrated once with respect to x to give

$$\frac{\partial\eta}{\partial x} + \left[\frac{1}{x} - \frac{\mu}{x^3 \left(1 + \frac{\mu}{x^2} \right) \ln \left(1 + \frac{\mu}{x^2} \right)} \right] \eta = 2 \frac{\partial\xi}{\partial t} + k(t), \quad (2.16)$$

where $k(t)$ is an unknown function of t . On substituting (2.14) for $\xi(t, x)$, Eq. (2.16) becomes

$$\frac{\partial\eta}{\partial x} + \left[\frac{1}{x} - \frac{\mu}{x^3 \left(1 + \frac{\mu}{x^2} \right) \ln \left(1 + \frac{\mu}{x^2} \right)} \right] \eta = 2\dot{f}_1(t) H(x) + f_3(t), \quad (2.17)$$

where $f_3(t) = k(t) + 2\dot{f}_2(t)$. Equation (2.17) is a first order differential equation for $\eta(t, x)$ with integrating factor $\mu^{1/2} H'(x)$, where the prime denotes differentiation with respect to x . Hence, Eq. (2.17) may be integrated to give

$$\eta(t, x) = \frac{1}{H'(x)} \left[\dot{f}_1(t) H^2(x) + f_3(t) H(x) + f_4(t) \right], \quad (2.18)$$

where $f_4(t)$ is an unknown function of t .

The substitution of (2.14) for $\xi(t, x)$ and (2.18) for $\eta(t, x)$ into (2.11) yields, after simplification,

$$\left(-\frac{1}{3}\ddot{f}_2(t) + \frac{2}{3}\dot{f}_3(t)\right)y_1(x) + \ddot{f}_1(t)y_2(x) + P(t)f_1(t)y_3(x) + f_1(t)y_4(x) = 0, \quad (2.19)$$

where

$$y_1(x) = 1, \quad (2.20)$$

$$y_2(x) = H(x), \quad (2.21)$$

$$y_3(x) = -\frac{\mu^{1/2}}{\left(\ln\left(1 + \frac{\mu}{x^2}\right)\right)^{1/2}}, \quad (2.22)$$

$$y_4(x) = \kappa \frac{\left[x^4\left(1 + \frac{\mu}{x^2}\right)\ln\left((1 + \mu)/\left(1 + \frac{\mu}{x^2}\right)\right) + \mu(x^2 - 1)\right]}{\mu^{1/2}x^4\left(1 + \frac{\mu}{x^2}\right)\left(\ln\left(1 + \frac{\mu}{x^2}\right)\right)^{1/2}}. \quad (2.23)$$

In order to evaluate the unknown functions in (2.19) we determine the Wronskian determinant $W[y_1, y_2, y_3, y_4]$ [8, 9] on some interval of $x(t)$. If $y_1(x)$, $y_2(x)$, $y_3(x)$, and $y_4(x)$ are defined by (2.20) to (2.23), it can be verified using *Mathematica* that their Wronskian determinant is not identically zero. The expansion of $W[y_1, y_2, y_3, y_4]$ is given in Appendix A. Therefore, for a given radial oscillation there will be an interval of $x(t)$ for which

$$W[y_1, y_2, y_3, y_4] \neq 0. \quad (2.24)$$

Thus, $y_1(x)$, $y_2(x)$, $y_3(x)$, and $y_4(x)$ are linearly independent and we can equate their respective coefficients to zero:

$$2\dot{f}_3(t) - \ddot{f}_2(t) = 0, \ddot{f}_1(t) = 0, P(t)f_1(t) = 0, f_1(t) = 0. \quad (2.25)$$

Hence $f_1(t) = 0$ even for free oscillations for which $P(t) = 0$, and

$$f_3(t) = \frac{1}{2}\dot{f}_2(t) + c_3, \quad (2.26)$$

where c_3 is a constant. Thus, (2.14) and (2.18) reduce to

$$\xi(t, x) = f_2(t), \quad \eta(t, x) = \frac{1}{H'(x)} \left[\left(\frac{1}{2}\dot{f}_2(t) + c_3 \right) H(x) + f_4(t) \right]. \quad (2.27)$$

In order to determine $f_2(t)$, c_3 , and $f_4(t)$, we substitute (2.27) for $\xi(t, x)$ and $\eta(t, x)$ into (2.12) and simplify to obtain the determining equation

$$\begin{aligned} \ddot{f}_2(t)z_1(x) + \dot{f}_2(t)z_2(x) + c_3z_3(x) + \ddot{f}_4(t)z_4(x) + f_4(t)z_5(x) \\ + (\dot{f}_2(t) + 2c_3)P(t)z_6(x) + f_4(t)P(t)z_7(x) \\ + (f_2(t)\dot{P}(t) - 4c_3P(t))z_8(x) = 0, \end{aligned} \quad (2.28)$$

where

$$z_1(x) = \frac{1}{2}H(x), \quad (2.29)$$

$$z_2(x) = \frac{3}{2}K(x) + \frac{1}{2}H(x)G(x), \quad (2.30)$$

$$z_3(x) = -K(x) + H(x)G(x), \quad (2.31)$$

$$z_4(x) = 1, \quad (2.32)$$

$$z_5(x) = G(x), \quad (2.33)$$

$$z_6(x) = -\frac{3}{2} \frac{\mu^{1/2}}{\left(\ln\left(1 + \frac{\mu}{x^2}\right)\right)^{1/2}} - \frac{\mu^2 H(x)}{2x^4 \left(1 + \frac{\mu}{x^2}\right) \left(\ln\left(1 + \frac{\mu}{x^2}\right)\right)^2}, \quad (2.34)$$

$$z_7(x) = -\frac{\mu^2}{x^4 \left(1 + \frac{\mu}{x^2}\right) \left(\ln\left(1 + \frac{\mu}{x^2}\right)\right)^2}, \quad (2.35)$$

$$z_8(x) = -\frac{\mu^{1/2}}{\left(\ln\left(1 + \frac{\mu}{x^2}\right)\right)^{1/2}} \quad (2.36)$$

and where

$$\begin{aligned} K(x; \mu) = \frac{\kappa}{\mu^{1/2}} \left[\frac{\ln(1 + \mu)}{\left(\ln\left(1 + \frac{\mu}{x^2}\right)\right)^{1/2}} - \left(\ln\left(1 + \frac{\mu}{x^2}\right)\right)^{1/2} \right. \\ \left. + \frac{\mu(x^2 - 1)}{x^4 \left(1 + \frac{\mu}{x^2}\right) \left(\ln\left(1 + \frac{\mu}{x^2}\right)\right)^{1/2}} \right] \end{aligned} \quad (2.37)$$

and

$$G(x; \mu) = \kappa \mu \left[\frac{\ln(1 + \mu)}{x^4 \left(1 + \frac{\mu}{x^2}\right) \left(\ln\left(1 + \frac{\mu}{x^2}\right)\right)^2} + \frac{x^2 + 2}{x^6 \left(1 + \frac{\mu}{x^2}\right) \ln\left(1 + \frac{\mu}{x^2}\right)} \right. \\ \left. - \frac{2(x^2 - 1)}{x^6 \left(1 + \frac{\mu}{x^2}\right)^2 \left(\ln\left(1 + \frac{\mu}{x^2}\right)\right)} \right. \\ \left. + \frac{\mu(x^2 - 1)}{x^8 \left(1 + \frac{\mu}{x^2}\right)^2 \left(\ln\left(1 + \frac{\mu}{x^2}\right)\right)^2} \right]. \quad (2.38)$$

The Wronskian determinant $W[z_1, z_2, \dots, z_8]$ with the $z_i(x)$, $i = 1, \dots, 8$, given by (2.29) to (2.36) was evaluated using *Mathematica* but is too long to reproduce here. To check that the Wronskian determinant is not identically zero, the term

$$\frac{\kappa^3 \mu^{47/2}}{x^{47} \left(1 + \frac{\mu}{x^2}\right)^{35} \left(\ln\left(1 + \frac{\mu}{x^2}\right)\right)^{47/2}}, \quad (2.39)$$

which is linearly independent of all other terms in the Wronskian determinant, was considered. The sum of its coefficients was evaluated using *Mathematica* to be 485005423804416. Therefore, the Wronskian determinant is not identically zero for a given radial oscillation and we conclude that $\dot{f}_2(t) = 0$, $\dot{f}_4(t) = 0$, $c_3 = 0$, and $f_2(t)\dot{P}(t) = 0$. Hence $f_2(t) = c_2$ where c_2 is an arbitrary constant and also

$$c_2 \dot{P}(t) = 0. \quad (2.40)$$

Therefore, from (2.27),

$$\xi(t, x) = \begin{cases} c_2, & \dot{P}(t) = 0, \\ 0, & \dot{P}(t) \neq 0, \end{cases} \quad \eta(t, x) = 0 \quad (2.41)$$

and we conclude from (2.2) that the autonomous Knowles equation for a Mooney-Rivlin material possesses one Lie point symmetry,

$$X = \frac{\partial}{\partial t}, \quad (2.42)$$

which corresponds to time translational invariance whereas the time-dependent equation admits no Lie point symmetries.

3. LIMITING OSCILLATIONS

We have seen that it is not possible to solve the Knowles equation for a Mooney–Rivlin material through Lie point symmetry methods when $\dot{P}(t) \neq 0$. When $\dot{P}(t) = 0$, a first integral can be derived [11] because the Knowles equation admits the symmetry $X = \partial/\partial t$ and the problem can be reduced to quadrature. Sometimes, however, the integral cannot be evaluated in closed form in terms of elementary functions and approximate methods for evaluating the integral have to be devised.

In this section we will consider the heaviside step loading boundary condition for which the problem has been reduced to quadrature by Knowles [11]. Knowles derived upper and lower bounds on the period of the oscillation for both net inward and net outward applied pressures. We will extend Knowles' results to obtain the corresponding displacement fields or limiting oscillations for each of the upper and lower bounds on the period. We will compare these limiting oscillations with the numerical solution and with the non-linear superposition solution of the Ermakov–Pinney equation (1.3) for the thin-walled tube approximation obtained by Shahinpoor and Nowinski [22].

Consider the Knowles equation for a Mooney–Rivlin material, (1.1), with the net applied pressure $P(t)$ given by the heaviside function

$$P(t) = \frac{2(P_1(t) - P_2(t))}{\mu \rho^* \rho_1^2} = \begin{cases} 0, & t \leq 0 \\ P_0, & t > 0, \end{cases} \quad (3.1)$$

where P_0 is a constant, and subject to the initial conditions

$$x(0) = 1, \quad \dot{x}(0) = 0. \quad (3.2)$$

The greatest or least value of x during an oscillation is determined from the condition $\dot{x} = 0$. Knowles [11] showed that $\dot{x} = 0$ when either $x = 1$ or $x = a$ where for all values of P_0 satisfying

$$P_0 < \frac{\kappa}{\mu} \ln(1 + \mu), \quad (3.3)$$

a is given uniquely by

$$a = \left[\frac{\mu}{(1 + \mu)e^{-\mu P_0/\kappa} - 1} \right]^{1/2}. \quad (3.4)$$

For a net inward applied pressure, $P_0 < 0$, the inequality (3.3) is always satisfied and $0 < a \leq x \leq 1$. For a net outward applied pressure, $P_0 > 0$, we consider only net applied pressures P_0 for which the inequality (3.3) is satisfied and $1 \leq x \leq a$. It may be shown [11] that the period T of the oscillation is given by

$$T = \operatorname{sgn}(a - 1) \frac{1}{\sqrt{\kappa}} \int_1^{a^2} \frac{(\ln(1 + \mu/z))^{1/2}}{[(z - 1)\ln\{(1 + \mu/z)/(1 + \mu/a^2)\}]^{1/2}} dz, \quad (3.5)$$

where $z = x^2$ and $\operatorname{sgn}(a - 1)$ is the sign of the factor $a - 1$. By a similar argument, the time t taken for the inner radius of the cylinder to reach a particular value of x for the first time by expanding or contracting from rest at $x = 1$ is given by

$$t = \operatorname{sgn}(a - 1) \frac{1}{2\sqrt{\kappa}} \int_1^z \frac{(\ln(1 + \mu/z))^{1/2}}{[(z - 1)\ln\{(1 + \mu/z)/(1 + \mu/a^2)\}]^{1/2}} dz. \quad (3.6)$$

With the aid of the inequalities [11],

$$\frac{r - s}{1 + r} \leq \ln\left(\frac{1 + r}{1 + s}\right) \leq \frac{r - s}{1 + s}, \quad r \geq s, \quad (3.7)$$

$$\frac{r}{1 + r} \leq \ln(1 + r) \leq r, \quad r \geq 0, \quad (3.8)$$

we will briefly rederive Knowles' results for the upper and lower bounds on the periods and then derive the limiting displacement fields corresponding to these bounds on the periods.

3.1. Lower Bound on the Period and Associated Oscillation for Net Outward Applied Pressure

Since $P_0 > 0$ it follows that $a > 1$. Thus, Eq. (3.5) for the period is

$$T = \frac{1}{\sqrt{\kappa}} \int_1^{a^2} \frac{[\ln(1 + \mu/z)]^{1/2}}{[(z - 1)\ln\{(1 + \mu/z)/(1 + \mu/a^2)\}]^{1/2}} dz. \quad (3.9)$$

In order to obtain a lower bound on the period given by (3.9) we use the inequalities (3.7) and (3.8). Since $\mu/z \geq \mu/a^2$ and $\mu/z > 0$ we have

respectively

$$\begin{aligned}\ln\{(1 + \mu/z)/(1 + \mu/a^2)\} &\leq \frac{\mu/z - \mu/a^2}{1 + \mu/a^2}, \\ \ln(1 + \mu/z) &\geq \frac{\mu/z}{1 + \mu/z},\end{aligned}\quad (3.10)$$

whence

$$T \geq \frac{(a^2 + \mu)^{1/2}}{\sqrt{\kappa}} \int_1^{a^2} \left[\frac{z}{(z + \mu)(z - 1)(a^2 - z)} \right]^{1/2} dz. \quad (3.11)$$

Further, since $z \geq 1$ it follows that $z/(z + \mu) \geq 1/(1 + \mu)$, whence

$$T \geq T_L = \frac{(a^2 + \mu)^{1/2}}{\sqrt{\kappa}(1 + \mu)^{1/2}} \int_1^{a^2} \frac{dz}{[(z - 1)(a^2 - z)]^{1/2}}, \quad (3.12)$$

where T_L is the lower bound on the period. Using the substitution [17]

$$u^2 = \frac{z - 1}{a^2 - z}, \quad (3.13)$$

we have

$$\int_1^{a^2} \frac{dz}{[(z - 1)(a^2 - z)]^{1/2}} = 2 \int_0^\infty \frac{du}{1 + u^2} = \pi \quad (3.14)$$

and therefore

$$T_L = \frac{\pi(a^2 + \mu)^{1/2}}{\sqrt{\kappa}(1 + \mu)^{1/2}}, \quad (3.15)$$

which is a better estimate than Knowles' [11] lower bound on the period for a net outward applied pressure,

$$T_L^* = \frac{\pi a}{\sqrt{\kappa}(1 + \mu)^{1/2}}, \quad (3.16)$$

owing to its being a greater lower bound.

We may now derive the displacement field associated with the lower bound on the period (3.15). From Eq. (3.6),

$$t = + \frac{1}{2\sqrt{\kappa}} \int_1^z \frac{[\ln(1 + \mu/z)]^{1/2}}{[(z - 1)\ln\{(1 + \mu/z)/(1 + \mu/a^2)\}]^{1/2}} dz, \quad (3.17)$$

which may be approximated in the same way as was the period to yield the following oscillation associated with the lower bound on the period:

$$t = \frac{(a^2 + \mu)^{1/2}}{2\sqrt{\kappa}(1 + \mu)^{1/2}} \int_1^z \frac{dz}{[(z-1)(a^2-z)]^{1/2}}. \quad (3.18)$$

By making the transformation (3.13) we obtain

$$\int_1^z \frac{dz}{[(z-1)(a^2-z)]^{1/2}} = 2 \int_0^u \frac{du}{1+u^2} = 2 \tan^{-1} \left(\left(\frac{z-1}{a^2-z} \right)^{1/2} \right) \quad (3.19)$$

and hence, expressed in the original variable, x , where $z = x^2$, (3.18) becomes

$$t = \frac{(a^2 + \mu)^{1/2}}{\sqrt{\kappa}(1 + \mu)^{1/2}} \tan^{-1} \left(\left(\frac{x^2 - 1}{a^2 - x^2} \right)^{1/2} \right). \quad (3.20)$$

Equation (3.20) may be solved for the limiting oscillation

$$x(t) = \left[\cos^2 \left(\frac{\sqrt{\kappa}(1 + \mu)^{1/2}}{(a^2 + \mu)^{1/2}} t \right) + a^2 \sin^2 \left(\frac{\sqrt{\kappa}(1 + \mu)^{1/2}}{(a^2 + \mu)^{1/2}} t \right) \right]^{1/2}, \quad (3.21)$$

which is in the form of a non-linear superposition [20] and where a is given by (3.4). The displacement field (3.21) of the approximate solution has period T_L given by (3.15) which is smaller than the exact period, but the maximum and minimum values, a and 1, respectively, of the dimensionless inner radius $x(t)$ are the same as for the exact solution.

3.2. Upper Bound on the Period and Associated Oscillation for Net Outward Applied Pressure

As with the lower bound on the period for net outward applied pressure, $P_0 > 0$ and therefore $a > 1$.

The equation for the period is again given by (3.9), but, in order to obtain an upper bound on the period, we use the inequalities

$$\ln\{(1 + \mu/z)/(1 + \mu/a^2)\} \geq \frac{\mu/z - \mu/a^2}{1 + \mu/z}, \quad \ln(1 + \mu/z) \leq \mu/z, \quad (3.22)$$

since $\mu/z \geq \mu/a^2$ and $\mu/z > 0$. Thus

$$T \leq \frac{a}{\sqrt{\kappa}} \int_1^{a^2} \left[\frac{1 + \mu/z}{(z-1)(a^2-z)} \right]^{1/2} dz. \quad (3.23)$$

Further, since $z \geq 1$ it follows that $1 + \mu/z \leq 1 + \mu$ and therefore

$$T \leq T_U = \frac{a(1 + \mu)^{1/2}}{\sqrt{\kappa}} \int_1^{a^2} \frac{dz}{[(z - 1)(a^2 - z)]^{1/2}}. \quad (3.24)$$

But this integral was evaluated in (3.14). The upper bound on the period is thus

$$T_U = \frac{\pi a(1 + \mu)^{1/2}}{\sqrt{\kappa}}, \quad (3.25)$$

which was first derived by Knowles [11].

The equation of the limiting oscillation with period T_U is obtained from (3.6) by performing the same approximations as were made to derive the period (3.24). This gives

$$t = \frac{a(1 + \mu)^{1/2}}{2\sqrt{\kappa}} \int_1^z \frac{dz}{[(z - 1)(a^2 - z)]^{1/2}} \quad (3.26)$$

and by using (3.19) and transforming back to the original variable $z = x^2$ we obtain

$$t = \frac{a(1 + \mu)^{1/2}}{\sqrt{\kappa}} \tan^{-1} \left(\frac{x^2 - 1}{a^2 - x^2} \right). \quad (3.27)$$

Solving (3.27) for x we obtain the limiting oscillation

$$x(t) = \left[\cos^2 \left(\frac{\sqrt{\kappa}}{a(1 + \mu)^{1/2}} t \right) + a^2 \sin^2 \left(\frac{\sqrt{\kappa}}{a(1 + \mu)^{1/2}} t \right) \right]^{1/2}, \quad (3.28)$$

which is again in the form of a non-linear superposition [20]. It has period T_U given by (3.25) which is greater than the exact period but the same maximum and minimum values for the dimensionless inner radius, a and 1 respectively, as the exact solution.

3.3. Lower Bound on the Period and Associated Oscillation for Net Inward Applied Pressure

We suppose now that $P_0 < 0$, that is, that there is a net inward applied pressure. Then, $a < 1$, whence $a \leq x(t) \leq 1$. By following a similar procedure to that for net outward applied pressure we obtain the lower bound on the period,

$$T_L = \frac{a}{\sqrt{\kappa}} \int_{a^2}^1 \frac{dz}{[(1 - z)(z - a^2)]^{1/2}} = \frac{\pi a}{\sqrt{\kappa}}, \quad (3.29)$$

and the corresponding limiting oscillation,

$$t = \frac{a}{\sqrt{\kappa}} \int_z^1 \frac{dz}{[(1-z)(z-a^2)]^{1/2}}, \quad (3.30)$$

which can be integrated to give

$$x(t) = \left[\cos^2 \left(\frac{\sqrt{\kappa}}{a} t \right) + a^2 \sin^2 \left(\frac{\sqrt{\kappa}}{a} t \right) \right]^{1/2}. \quad (3.31)$$

Equation (3.31) is in the form of a non-linear superposition and $a < 1$ defines the minimum value of the dimensionless inner radius during the oscillation which is equal to the exact minimum value given by (3.4). Clearly, the maximum value of the dimensionless inner radius is unity, in agreement with the initial conditions (3.2).

3.4. Upper Bound on the Period and Associated Oscillation for Net Inward Applied Pressure

Finally, for $P_0 < 0$ and $a \leq x \leq 1$, the upper bound on the period is found to be

$$T_U = \frac{(a^2 + \mu)^{1/2}}{\sqrt{\kappa}} \int_{a^2}^1 \frac{dz}{[(1-z)(z-a^2)]^{1/2}} = \frac{\pi(a^2 + \mu)^{1/2}}{\sqrt{\kappa}} \quad (3.32)$$

and the limiting oscillation with period T_U is given by

$$t = \frac{(a^2 + \mu)^{1/2}}{2\sqrt{\kappa}} \int_z^1 \frac{dz}{[(1-z)(z-a^2)]^{1/2}}, \quad (3.33)$$

which leads to

$$x(t) = \left[\cos^2 \left(\frac{\sqrt{\kappa}}{(a^2 + \mu)^{1/2}} t \right) + a^2 \sin^2 \left(\frac{\sqrt{\kappa}}{(a^2 + \mu)^{1/2}} t \right) \right]^{1/2}. \quad (3.34)$$

As before, $a < 1$ and 1 are the minimum and maximum values of the dimensionless inner radius, respectively, of the exact oscillation and the oscillation, (3.34), corresponding to the upper bound on the period.

For comparison, the displacement field in the thin-shell approximation, obtained by Shahinpoor and Nowinski [22] by solving the Ermakov-Pinney equation (1.3) for the heaviside step loading boundary condition using a

non-linear superposition principle, is

$$x(t) = \left[\cos^2 \left(\frac{\sqrt{\kappa}}{a_0} t \right) + a_0^2 \sin^2 \left(\frac{\sqrt{\kappa}}{a_0} t \right) \right]^{1/2}, \quad (3.35)$$

where

$$a_0 = \frac{1}{(1 - P_0/\kappa)^{1/2}} = \lim_{\mu \rightarrow 0} \left[\frac{\mu}{(1 + \mu)e^{-\mu P_0/\kappa} - 1} \right]^{1/2}. \quad (3.36)$$

The period of the oscillation is

$$T = \frac{\pi a_0}{\sqrt{\kappa}}. \quad (3.37)$$

The results (3.35), (3.36), and (3.37) apply for both net outward and net inward applied pressures, P_0 .

The limiting oscillation (3.31) corresponding to the lower bound on the period for net inward applied pressure has the same form as (3.35). The only difference is a given by (3.4) in (3.31) and a_0 given by (3.36) in (3.35).

The results of this section are illustrated in Fig. 1 where the approximate displacement fields for a net outward applied pressure, (3.21) and (3.28), as well as for a net inward applied pressure, (3.31) and (3.34), the corresponding thin-shell displacement field (3.35) and the numerical solution of the Knowles Equation (1.1) with the heaviside step loading boundary condition, obtained by using the *Mathematica* *NDSolve* routine, are plotted on the same system of axes. We see that for $P_0 > 0$, the maximum displacement of the thin-shell solution of the Ermakov-Pinney equation is a poor approximation of the numerical solution, unlike the maximum displacements of the two limiting oscillations, even for the comparatively small value of $\mu = 0.6$ used in Fig. 1. In addition, the period of the thin-shell oscillation is even less than the lower bound of the period, and the graph of the thin-shell oscillation lags behind the numerical solution even more than the limiting oscillation corresponding to the lower bound of the period. For $P_0 < 0$, we see that the minimum value of the inner radius in the thin-shell approximation is a poor approximation of the numerical solution, unlike the minimum values given by the two limiting oscillations. However, the graph of the thin-shell approximation lags only slightly behind that of the numerical solution and the period of the thin-shell approximation provides a better approximation to the period of the exact solution than do the two limiting oscillations.

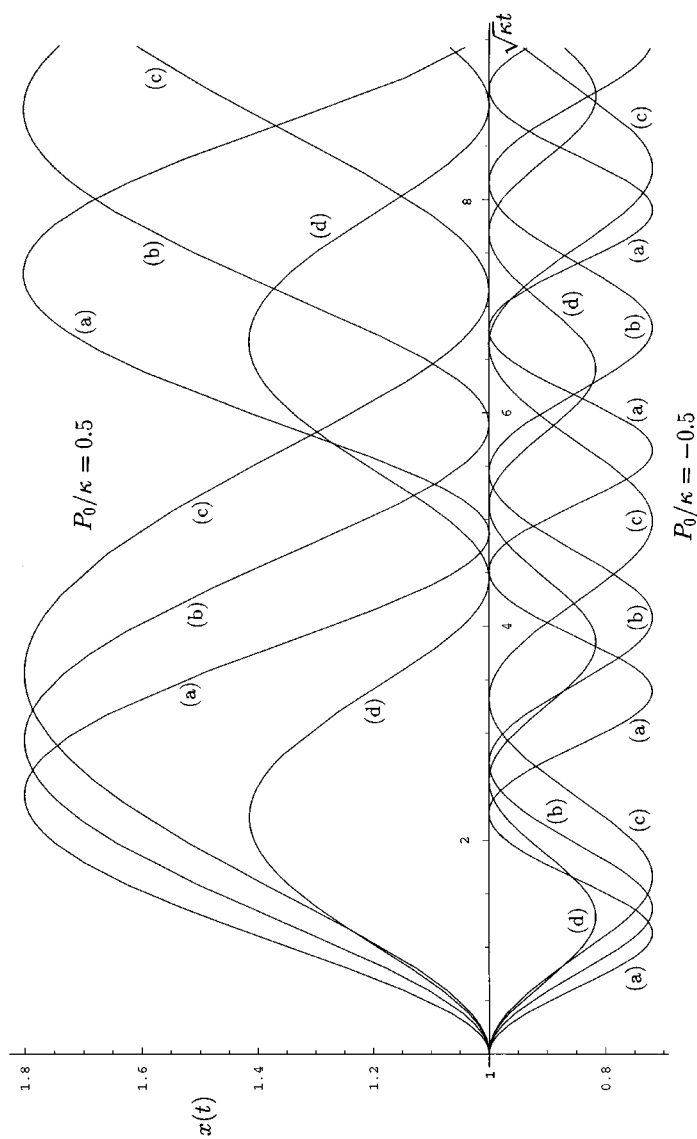


FIG. 1. The nondimensional inner radius, $x(t)$, for heavy-side step loading given by (3.1) plotted against $\sqrt{\kappa}t$ for $P_0/\kappa = \pm 0.5$ and $\mu = 0.6$: (a) oscillations (3.21) for $P_0 > 0$ and (3.31) for $P_0 < 0$ corresponding to the lower bounds on the period, (b) a numerical solution of (1.1), (c) oscillations (3.28) for $P_0 > 0$ and (3.34) for $P_0 < 0$ corresponding to the upper bounds on the period, and (d) thin-shell approximation (3.35).

4. SYMMETRY BREAKING

Nowinski and Wang [17] and Shahinpoor and Nowinski [22] have shown that if the Knowles equation for a Mooney–Rivlin material is expanded in powers of the thickness parameter μ then for a thin-walled tube it reduces to the Ermakov–Pinney equation (1.3) to zero order in μ . But, it is well established [3, 19, 20] that the Ermakov–Pinney equation possesses three Lie point symmetries for both the autonomous and non-autonomous cases. In this section we consider an expansion in powers of μ of the functions in the symmetry-determining equation (2.28) in order to investigate at which order in μ the reduction in the number of Lie point symmetries of the exact Knowles equation for a thick-walled Mooney–Rivlin tube occurs.

From (2.27) the Lie point symmetries are given by

$$X = f_2(t) \frac{\partial}{\partial t} + \frac{1}{H'(x; \mu)} \left[\left(\frac{1}{2} \dot{f}_2(t) + c_3 \right) H(x; \mu) + f_4(t) \right] \frac{\partial}{\partial x}, \quad (4.1)$$

where $H(x; \mu)$, defined by (2.15), may be expanded in a Taylor expansion with respect to μ to give

$$H(x; \mu) = x + \frac{\mu}{4x} - \frac{13\mu^2}{288x^3} + \frac{7\mu^3}{384x^5} - \frac{6271\mu^4}{645120x^7} + O(\mu^5). \quad (4.2)$$

The functions $f_2(t)$ and $f_4(t)$ and the constant c_3 are obtained from the determining equation (2.28). We expand the coefficients $z_1(x)$ to $z_8(x)$ in (2.28) in Taylor expansions with respect to μ correct to order μ^4 because it is only to this order that the expansions of the eight functions become linearly independent. These expansions correct to order μ^4 , with the Taylor expansions of $K(x; \mu)$ and $G(x; \mu)$ given by (2.37) and (2.38) and on which $z_1(x)$ to $z_8(x)$ depend, are given in Appendix B. The expansions are written in such a way that the functions which are linearly dependent and to which order in μ is immediately apparent.

4.1. Zero Order in μ

The Knowles equation (1.1) for a Mooney–Rivlin material when expanded in powers of μ is, to zero order in μ , the Ermakov–Pinney equation

$$\ddot{x} + (\kappa - P(t))x = \frac{\kappa}{x^3}. \quad (4.3)$$

In order to investigate the symmetries of Eq. (4.3), we consider the determining equation (2.28) to zero order in μ :

$$\begin{aligned} \frac{1}{2}x\ddot{f}_2(t) + 2\kappa x\dot{f}_2(t) + 4\frac{\kappa}{x^3}c_3 + \ddot{f}_4(t) + \kappa\left(1 + \frac{3}{x^4}\right)f_4(t) \\ - 2x(\dot{f}_2(t) + 2c_3)P(t) - f_4(t)P(t) \\ - x(f_2(t)\dot{P}(t) - 4c_3P(t)) = 0. \end{aligned} \quad (4.4)$$

We split Eq. (4.4) according to powers of x , starting with the most negative powers. By equating to zero the coefficients of x^{-4} and x^{-3} we find, respectively, that $f_4(t) = 0$ and $c_3 = 0$. The remaining non-zero terms in (4.4) are all proportional to x . By equating the coefficient of x to zero we obtain

$$\ddot{f}_2(t) + 4(\kappa - P(t))\dot{f}_2(t) - 2\dot{P}(t)f_2(t) = 0. \quad (4.5)$$

Equation (4.5) is a third order linear ordinary differential equation for $f_2(t)$ and its solution therefore contains three constants of integration. The Lie point symmetry generators are given by (4.1) with $H(x; \mu)$ given by (4.2) to zero order in μ :

$$X = f_2(t)\frac{\partial}{\partial t} + \frac{1}{2}\dot{f}_2(t)x\frac{\partial}{\partial x}. \quad (4.6)$$

Equation (4.3) therefore admits three Lie point symmetries obtained by setting to zero in turn all constants except one in $f_2(t)$.

4.2. First Order in μ

The Knowles equation (1.1) for a Mooney-Rivlin material expanded to first order in μ is

$$\begin{aligned} \ddot{x} + \frac{\mu}{2x^3}\dot{x}^2 + \left[\kappa\left(1 - \frac{\mu}{2}\right) - P(t)\right]x \\ = -\frac{\mu}{2}(\kappa - P(t))\frac{1}{x} + \kappa\left(1 + \frac{\mu}{2}\right)\frac{1}{x^3} - \frac{\mu}{2}\kappa\frac{1}{x^5}. \end{aligned} \quad (4.7)$$

The determining equation (2.28) correct to first order in μ is

$$\begin{aligned} & \frac{1}{2} \left(x + \frac{\mu}{4x} \right) \ddot{f}_2(t) + 2\kappa \left[x + \mu \left(-\frac{x}{2} + \frac{1}{4x} \right) \right] \dot{f}_2(t) \\ & + 4\kappa \left[\frac{1}{x^3} + \mu \left(\frac{1}{2x^3} - \frac{3}{4x^5} \right) \right] c_3 + \ddot{f}_4(t) \\ & + \kappa \left[1 + \frac{3}{x^4} + \mu \left(-\frac{1}{2} + \frac{3}{2x^4} - \frac{3}{x^6} \right) \right] f_4(t) \\ & - 2 \left(x + \frac{\mu}{4x} \right) (\dot{f}_2(t) + 2c_3) P(t) \\ & + f_4(t) P(t) - \left(x + \frac{\mu}{4x} \right) (f_2(t) \dot{P}(t) - 4c_3 P(t)) = 0. \quad (4.8) \end{aligned}$$

We separate (4.8) according to powers of x , starting with the most negative powers. By equating to zero the coefficients of x^{-6} and x^{-5} it follows that $f_4(t) = 0$ and $c_3 = 0$. By equating to zero the coefficients of the remaining powers, x^{-1} and x , we obtain

$$x^{-1}: \quad \mu \left[\ddot{f}_2(t) + 4(\kappa - P(t)) \dot{f}_2(t) - 2\dot{P}(t) f_2(t) \right] = 0, \quad (4.9)$$

$$x^1: \quad \ddot{f}_2(t) + 4 \left[\kappa \left(1 - \frac{\mu}{2} \right) - P(t) \right] \dot{f}_2(t) - 2\dot{P}(t) f_2(t) = 0. \quad (4.10)$$

Equation (4.9) is consistent with (4.10) because terms of order μ^2 are neglected. Equation (4.10) is a third order, linear ordinary differential equation for $f_2(t)$ and its solution will contain three constants of integration. There are thus three Lie point symmetries of the first order in μ correction to (4.3) with the Lie point symmetry generators given by (4.1) with $H(x; \mu)$ given by (4.2) correct to order μ :

$$X = f_2(t) \frac{\partial}{\partial t} + \frac{1}{2} \dot{f}_2(t) \left(x + \frac{\mu}{2x} \right) \frac{\partial}{\partial x}. \quad (4.11)$$

If

$$\kappa \left(1 - \frac{\mu}{2} \right) - P(t) > 0 \quad (4.12)$$

and we define

$$\omega^2(t) = \kappa \left(1 - \frac{\mu}{2} \right) - P(t), \quad (4.13)$$

then (4.10) can be written as

$$\ddot{f}_2(t) + 4\omega^2(t)\dot{f}_2(t) + 4\omega(t)\dot{\omega}(t)f_2(t) = 0. \quad (4.14)$$

Consider now the special case in which $P(t) = P_0$ where P_0 is a constant satisfying (4.12) and define

$$\omega_0^2 = \kappa(1 - \mu/2) - P_0 > 0. \quad (4.15)$$

We will obtain explicit expressions for the three Lie point symmetries. If we let

$$\dot{f}_2(t) = h(t), \quad (4.16)$$

then (4.14) becomes

$$\ddot{h}(t) + 4\omega_0^2 h(t) = 0. \quad (4.17)$$

Thus

$$h(t) = A_1 \cos(2\omega_0 t) + A_2 \sin(2\omega_0 t), \quad (4.18)$$

where A_1 and A_2 are constants and therefore

$$f_2(t) = \frac{A_1}{2\omega_0} \sin(2\omega_0 t) - \frac{A_2}{2\omega_0} \cos(2\omega_0 t) + A_3, \quad (4.19)$$

where A_3 is a constant. By setting one of the three constants A_1, A_2, A_3 equal to zero in turn, three Lie point symmetries are obtained from (4.11),

$$X_1 = \frac{\partial}{\partial t}, \quad (4.20)$$

$$X_2 = \frac{1}{2\omega_0} \sin(2\omega_0 t) \frac{\partial}{\partial t} + \frac{1}{2} \left(x + \frac{\mu}{2x} \right) \cos(2\omega_0 t) \frac{\partial}{\partial x}, \quad (4.21)$$

$$X_3 = -\frac{1}{2\omega_0} \cos(2\omega_0 t) \frac{\partial}{\partial t} + \frac{1}{2} \left(x + \frac{\mu}{2x} \right) \sin(2\omega_0 t) \frac{\partial}{\partial x}. \quad (4.22)$$

Three first integrals of the autonomous differential equation (4.7) with $P(t)$ a constant will be derived in Section 5 from the three Lie point symmetries (4.20) to (4.22).

Finally, in this subsection, we rewrite (4.10) in a useful alternative form as a second order ordinary differential equation containing an arbitrary constant. If (4.10) is multiplied by $f_2(t)$ then it can be rewritten as

$$\frac{d}{dt} \left(f_2(t) \ddot{f}_2(t) - \frac{1}{2} \dot{f}_2^2(t) \right) + 2 \frac{d}{dt} \left[\left(\kappa \left(1 - \frac{\mu}{2} \right) - P(t) \right) f_2^2(t) \right] = 0 \quad (4.23)$$

and therefore

$$f_2(t)\ddot{f}_2(t) - \frac{1}{2}\dot{f}_2^2(t) + 2\left[\left(\kappa\left(1 - \frac{\mu}{2}\right) - P(t)\right)f_2^2(t)\right] = 2C, \quad (4.24)$$

where C is a constant of integration. If we let $f_2(t) = g^2(t)$, then (4.24) becomes

$$\ddot{g}(t) + \left[\kappa\left(1 - \frac{\mu}{2}\right) - P(t)\right]g(t) = \frac{C}{g^3(t)}, \quad (4.25)$$

which is an Ermakov–Pinney equation. Therefore, the second order differential equation (4.7) admits the Lie point symmetry with generator

$$X = g^2(t)\frac{\partial}{\partial t} + \left(x + \frac{\mu}{2x}\right)g(t)\dot{g}(t)\frac{\partial}{\partial x} \quad (4.26)$$

where $g(t)$ satisfies the second order differential equation (4.25). In Section 6 we will derive, from this result, a non-linear superposition principle for Eq. (4.7) for the general, non-autonomous case in which $P(t)$ can be a function of time.

The results derived in this subsection correct to order μ reduce to the results for the Ermakov–Pinney equation (4.3) by letting $\mu \rightarrow 0$ [13].

4.3. Second Order in μ

The Knowles equation (1.1) for a Mooney–Rivlin material expanded correct to order μ^2 is

$$\begin{aligned} \ddot{x} + \frac{\mu}{2x^3}\left(1 - \frac{5\mu}{6x^2}\right)\dot{x}^2 + \left[\kappa\left(1 - \frac{\mu}{2} + \frac{\mu^2}{3}\right) - P(t)\right]x \\ = -\frac{\mu}{2}\left[\kappa\left(1 - \frac{\mu}{2}\right) - P(t)\right]\frac{1}{x} + \left[\kappa\left(1 + \frac{\mu}{2} + \frac{\mu^2}{12}\right) - \frac{\mu^2}{12}P(t)\right]\frac{1}{x^3} \\ - \frac{\mu}{2}\left(1 + \frac{5\mu}{6}\right)\kappa\frac{1}{x^5} + \frac{5\mu^2}{12}\kappa\frac{1}{x^7}. \end{aligned} \quad (4.27)$$

Now, in the determining equation (2.28) expanded correct to order μ^2 , the highest negative power of x is x^{-8} and it occurs only in the expansion of $z_5(x)$. But the coefficient of $z_5(x)$ in (2.28) is $f_4(t)$ and therefore $f_4(t) = 0$. By equating to zero the coefficients of x^{-7} and x^{-5} in the remaining terms of the determining equation correct to order μ^2 it can be verified

that

$$x^{-7}: \quad \frac{\mu^2}{3} \dot{f}_2(t) + \frac{27\mu^2}{8} c_3 = 0, \quad (4.28)$$

$$x^{-5}: \quad \frac{\mu^2}{6} \dot{f}_2(t) + \left(3\mu + \frac{5\mu^2}{2} \right) c_3 = 0. \quad (4.29)$$

But since the determinant of the coefficients of $\dot{f}_2(t)$ and c_3 in the homogeneous system of equations, (4.28) and (4.29), is non-zero it follows that $\dot{f}_2(t) = 0$ and $c_3 = 0$. Thus $f_2(t) = c_2$ where c_2 is a constant and the determining equation reduces to (2.40) as with the Knowles equation for a thick-walled tube. Thus if $\dot{P}(t) \neq 0$ then $c_2 = 0$ and from (4.1) there are no Lie point symmetries, while if $\dot{P}(t) = 0$ then c_2 is arbitrary and from (4.1) we regain the Lie point symmetry (2.42) which corresponds to time translational invariance.

4.4. Third Order in μ

In the expansion of (2.28) to third order in μ , the largest negative power of x is x^{-10} and it occurs only in $z_5(x)$. The coefficient of $z_5(x)$ in (2.28) is $f_4(t)$ and therefore $f_4(t) = 0$. By considering the coefficients of x^{-9} and x^{-7} which occur only in $z_2(x)$ and $z_3(x)$, it can be verified that $c_3 = 0$ and $\dot{f}_2(t) = 0$. Thus the determining equation reduces again to (2.40) and there are no Lie point symmetries if $\dot{P}(t) \neq 0$ and there is the trivial Lie point symmetry, $X = \partial/\partial t$, if $\dot{P}(t) = 0$.

4.5. Fourth and Higher Orders in μ

If terms of order μ^4 or higher are retained in the determining equation then $z_1(x)$ to $z_8(x)$ are linearly independent. Thus the coefficients of $z_1(x)$ to $z_8(x)$ in (2.28) vanish and the determining equation reduces to (2.40).

We therefore see that the reduction in the number of symmetries from three to one or zero occurs at the second order in μ approximation. It was not necessary for the functions $z_1(x)$ to $z_8(x)$ to be linearly independent for this to occur and indeed $z_1(x)$ and $z_6(x)$ are linearly dependent up to order μ^3 .

5. FIRST INTEGRALS FOR CONSTANT NET APPLIED PRESSURE TO ORDER μ

Throughout this section we will consider the special case for which $P(t) = P_0$ where P_0 is a given constant which satisfies the inequality (4.12). The three Lie point symmetry generators are thus given by (4.20) to

(4.22). We will derive the first integral associated with each of the symmetry generators and obtain the general solution for constant net applied pressure in terms of the three first integrals. The general solution will be used to obtain the particular solutions for free oscillations and for the heaviside step loading boundary condition. All final solutions will be correct to first order in μ .

A function $J(t, x, \dot{x})$ is a first integral associated with the symmetry generator X if [12]

$$X^{[1]}J = 0 \quad \text{and} \quad DJ = 0, \quad (5.1)$$

where $X^{[1]}$ is the first prolongation of X defined by (2.5) and the operator D is defined by (2.7).

Consider first the symmetry generator $X_1 = \partial/\partial t$. From (5.1),

$$\frac{\partial J}{\partial t} + 0 \frac{\partial J}{\partial x} + 0 \frac{\partial J}{\partial \dot{x}} = 0 \quad (5.2)$$

and solving the differential equations of the characteristic curves of (5.2) gives the three independent solutions

$$x = c_1, \quad \dot{x} = c_2, \quad J = c_3, \quad (5.3)$$

where c_1 , c_2 , and c_3 are constants. Hence the general solution of (5.2) is

$$J = J(u, v), \quad (5.4)$$

where $u = x$ and $v = \dot{x}$. Now, from the second relation in (5.1),

$$\dot{u} \frac{\partial J}{\partial u} + \dot{v} \frac{\partial J}{\partial v} = 0 \quad (5.5)$$

and from the differential equations of the characteristic curves of (5.5) it follows that

$$\frac{dv}{du} = \frac{\dot{v}}{\dot{u}}. \quad (5.6)$$

But, $\dot{v} = \ddot{x}$ and $\dot{u} = \dot{x} = v$ and by using (4.7) for \ddot{x} , (5.6) may be written as the first order differential equation for v^2 ,

$$\frac{dv^2}{du} + \frac{\mu}{u^3} v^2 = -2\omega_0^2 u - \mu(\kappa - P_0) \frac{1}{u} + 2\kappa \left(1 + \frac{\mu}{2}\right) \frac{1}{u^3} - \mu\kappa \frac{1}{u^5}, \quad (5.7)$$

which is readily integrated to give, correct to order μ ,

$$\left(1 - \frac{\mu}{2u^2}\right) v^2 = -\omega_0^2 u^2 - \kappa \left(1 + \frac{\mu}{2}\right) \frac{1}{u^2} \left(1 - \frac{\mu}{2u^2}\right) + \hat{f}_1, \quad (5.8)$$

where \hat{J}_1 is a constant. Equation (5.8) may be expressed concisely in terms of the original variables as

$$J_1 = \left(\dot{x}^2 + \omega_0^2 x^2 + \kappa \left(1 + \frac{\mu}{2} \right) \frac{1}{x^2} \right) \left(1 - \frac{\mu}{2x^2} \right), \quad (5.9)$$

where

$$J_1 = \hat{J}_1 - \frac{\mu}{2} (\kappa - P_0). \quad (5.10)$$

The constant J_1 is the required first integral corresponding to the symmetry generator X_1 .

Consider next the symmetry generator X_2 defined by (4.21). This gives, from (5.1), the first order quasi-linear partial differential equation

$$\begin{aligned} \frac{1}{2\omega_0} \sin(2\omega_0 t) \frac{\partial J}{\partial t} + \frac{1}{2} \left(x + \frac{\mu}{2x} \right) \cos(2\omega_0 t) \frac{\partial J}{\partial x} \\ - \left[\frac{1}{2} \left(1 + \frac{\mu}{2x^2} \right) \dot{x} \cos(2\omega_0 t) + \omega_0 \left(x + \frac{\mu}{2x} \right) \sin(2\omega_0 t) \right] \frac{\partial J}{\partial \dot{x}} = 0. \end{aligned} \quad (5.11)$$

The differential equations of the characteristic curves yield the invariants

$$u = \frac{2x^2 + \mu}{\sin(2\omega_0 t)} = c_1, \quad v = x\dot{x} - \frac{1}{2} \omega_0 (2x^2 + \mu) \frac{\cos(2\omega_0 t)}{\sin(2\omega_0 t)} = c_2, \quad (5.12)$$

where c_1 and c_2 are constants. The reduced equation is given by (5.6). Now by using v in (5.12) to eliminate \dot{x} it can be shown that

$$\dot{u} = \frac{2v}{\sin(2\omega_0 t)}. \quad (5.13)$$

Also, using (5.12) and (4.7) to eliminate \dot{x} and \ddot{x} , respectively, and by expressing x in terms of u using (5.12) it can be verified that

$$\dot{v} = \frac{1}{\sin(2\omega_0 t)} \left[\frac{v^2}{u} + \omega_0^2 u + \kappa \left(1 + \frac{\mu}{2} \right) \frac{1}{u} \right]. \quad (5.14)$$

The substitution into (5.6) of (5.13) and (5.14) yields the first order differential equation for v^2 ,

$$\frac{dv^2}{du} - \frac{1}{u} v^2 = \omega_0^2 u + \kappa \left(1 + \frac{\mu}{2} \right) \frac{1}{u}, \quad (5.15)$$

which is readily solved to give

$$J_2 = \frac{v^2}{u} - \omega_0^2 u + \kappa \left(1 + \frac{\mu}{2}\right) \frac{1}{u}, \quad (5.16)$$

where J_2 is a constant. Substituting (5.12) for u and v , the first integral J_2 may be rewritten, correct to order μ , as

$$\begin{aligned} J_2 = & \left(1 - \frac{\mu}{2x^2}\right) \left[\dot{x} - \omega_0 x \left(1 + \frac{\mu}{2x^2}\right) \frac{\cos(2\omega_0 t)}{\sin(2\omega_0 t)} \right]^2 \sin(2\omega_0 t) \\ & + \kappa \left(1 + \frac{\mu}{2}\right) \frac{1}{x^2} \left(1 - \frac{\mu}{2x^2}\right) \sin(2\omega_0 t) - \frac{\omega_0^2 x^2}{\sin(2\omega_0 t)} \left(1 + \frac{\mu}{2x^2}\right). \end{aligned} \quad (5.17)$$

The first integral, J_3 , associated with X_3 given by (4.22), is calculated in the same way as was J_2 . It can be verified that, correct to first order in μ ,

$$\begin{aligned} J_3 = & \left(1 - \frac{\mu}{2x^2}\right) \left[\dot{x} + \omega_0 x \left(1 + \frac{\mu}{2x^2}\right) \frac{\sin(2\omega_0 t)}{\cos(2\omega_0 t)} \right]^2 \cos(2\omega_0 t) \\ & + \kappa \left(1 + \frac{\mu}{2}\right) \frac{1}{x^2} \left(1 - \frac{\mu}{2x^2}\right) \cos(2\omega_0 t) - \frac{\omega_0^2 x^2}{\cos(2\omega_0 t)} \left(1 + \frac{\mu}{2x^2}\right). \end{aligned} \quad (5.18)$$

The three first integrals, J_1 , J_2 , and J_3 , are given correct to first order in μ by (5.9), (5.17), and (5.18). Since (4.7) is a second order ordinary differential equation, only two of the three first integrals are independent. By using any two of the first integrals, the solution for x can be obtained by eliminating \dot{x} . However, the solution can also be written in terms of all three first integrals and in a concise way. It follows directly from (5.9), (5.17), and (5.18) that

$$x(t) = \frac{1}{\sqrt{2}\omega_0} [J_1 - J_2 \sin(2\omega_0 t) - J_3 \cos(2\omega_0 t)]^{1/2}. \quad (5.19)$$

Equation (5.19) can be written alternatively as

$$x(t) = \frac{1}{\sqrt{2}\omega_0} \left[J_1 - \sqrt{J_2^2 + J_3^2} \cos(2\omega_0 t - \alpha) \right]^{1/2}, \quad (5.20)$$

where $\tan \alpha = J_2/J_3$. From (5.20), the maximum and minimum displacements, x_{\max} and x_{\min} , are given by

$$x_{\max} = \frac{1}{\sqrt{2} \omega_0} \left[J_1 + \sqrt{J_2^2 + J_3^2} \right]^{1/2}, \quad x_{\min} = \frac{1}{\sqrt{2} \omega_0} \left[J_1 - \sqrt{J_2^2 + J_3^2} \right]^{1/2}, \quad (5.21)$$

provided μ is such that

$$J_1 > (J_2^2 + J_3^2)^{1/2}. \quad (5.22)$$

Also

$$x_{\max} x_{\min} = \frac{1}{2 \omega_0^2} \left[J_1^2 - J_2^2 - J_3^2 \right]^{1/2}. \quad (5.23)$$

The solution (5.19) may also be rewritten in terms of $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$ as

$$x(t) = \frac{1}{\sqrt{2} \omega_0} \left[(J_1 - J_3) \left(\cos(\omega_0 t) - \frac{J_2}{J_1 - J_3} \sin(\omega_0 t) \right)^2 + \left(\frac{J_1^2 - J_2^2 - J_3^2}{J_1 - J_3} \right) \sin^2(\omega_0 t) \right]^{1/2}. \quad (5.24)$$

We will now use the general results derived in this section to obtain the solution for free oscillations and for heaviside step loading correct to order μ . In both cases we will compare graphically the solution correct to order μ with the solution to zero order in μ obtained from the Ermakov-Pinney equation and with the numerical solution of the exact equation (1.1) calculated using the standard *Mathematica* built-in numerical differentiation function, *NDSolve*.

5.1. Free Oscillations

For free oscillations, $P_0 = 0$ and the inequality (4.12) is identically satisfied since we assume $0 < \mu < 1$. The initial conditions are $x(0) = x_0$,

$\dot{x}(0) = v_0$. It is readily verified from (5.9), (5.17), and (5.18) that

$$J_1 = \kappa \left(x_0^2 + \frac{1}{x_0^2} + \frac{v_0^2}{\kappa} \right) - \frac{\mu}{2} \kappa \left[x_0^2 + \frac{1}{x_0^2} + \left(1 - \frac{1}{x_0^2} \right)^2 + \frac{v_0^2}{\kappa x_0^2} \right] + O(\mu^2), \quad (5.25)$$

$$J_2 = -2x_0 v_0 \sqrt{\kappa} \left(1 - \frac{\mu}{4} \right) + O(\mu^2), \quad (5.26)$$

$$J_3 = \kappa \left(-x_0^2 + \frac{1}{x_0^2} + \frac{v_0^2}{\kappa} \right) - \frac{\mu}{2} \kappa \left[-x_0^2 + \frac{1}{x_0^2} + \left(1 - \frac{1}{x_0^2} \right)^2 + \frac{v_0^2}{\kappa x_0^2} \right] + O(\mu^2), \quad (5.27)$$

as $\mu \rightarrow 0$. It follows directly from (5.24) that

$$x(t) = \left[\left\{ x_0 \cos \left(\sqrt{\kappa} \left(1 - \frac{\mu}{4} \right) t \right) + \frac{v_0}{\sqrt{\kappa}} \left(1 + \frac{\mu}{4} \right) \sin \left(\sqrt{\kappa} \left(1 - \frac{\mu}{4} \right) t \right) \right\}^2 + \frac{1}{x_0^2} \left\{ 1 - \frac{\mu}{2} \left(\frac{v_0^2}{\kappa} + \left(x_0 - \frac{1}{x_0} \right)^2 \right) \right\} \sin^2 \left(\sqrt{\kappa} \left(1 - \frac{\mu}{4} \right) t \right) \right]^{1/2}, \quad (5.28)$$

where terms of $O(\mu^2)$ have been neglected. Equation (5.28) is the correction to first order in μ of the solution for free oscillations given by Shahinpoor and Nowinski [22].

From the relation between x_{\max} and x_{\min} given by (5.23) and by using (5.25) to (5.27) it can be verified that, for free oscillations,

$$x_{\max} x_{\min} = 1 - \frac{\mu}{4} \left(\left(x_0 - \frac{1}{x_0} \right)^2 + \frac{v_0^2}{\kappa} \right) + O(\mu^2), \quad (5.29)$$

as $\mu \rightarrow 0$. For oscillations described by the Ermakov–Pinney equation, in which terms of $O(\mu)$ are neglected, (5.29) reduces to $x_{\max} x_{\min} = 1$, a result first derived by Knowles [10]. It follows from (5.29) that if terms of $O(\mu)$ are included then

$$x_{\max} x_{\min} < 1. \quad (5.30)$$

In Fig. 2, $x(t)$ is plotted against $\sqrt{\kappa}t$ for $x_0 = 2$, $v_0/\sqrt{\kappa} = 2$, and $\mu = 0.2$. For these values of the parameters, the inequality (5.22) is

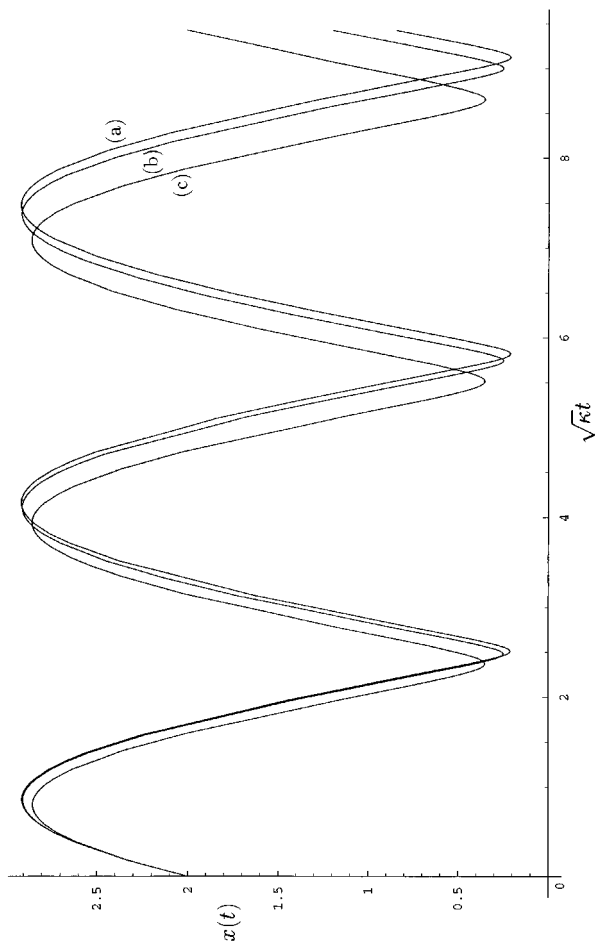


FIG. 2. The nondimensional inner radius, $x(t)$, for free oscillations plotted against $\sqrt{\kappa}t$ for $x_0 = 2$, $v_0/\sqrt{\kappa} = 2$, and $\mu = 0.2$: (a) solution correct to $O(\mu)$ given by (5.28), (b) numerical solution of (1.1) with $P(t) = 0$, (c) zero order approximation given by (5.28) with $\mu = 0$.

satisfied. Three solutions for $x(t)$ are plotted for comparison in Fig. 2, namely, the solution correct to first order in μ given by (5.28), the solution to zero order in μ derived by Shahinpoor and Nowinski [22] which can be obtained from (5.28) by putting $\mu = 0$, and the numerical solution of the exact equation (1.1) with $P(t) = 0$. We see that the approximate solution (5.28) is a better approximation than the zero order in μ solution obtained from the Ermakov–Pinney equation (4.3), especially for early times. The maximum values of the dimensionless inner radius of the order μ approximation and the numerical solution are almost equal and the minimum value of the dimensionless inner radius of the order μ approximation bounds the minimum value attained by the numerical solution from below.

5.2. Heaviside Step Loading

For the heaviside step loading boundary condition given by (3.1) where P_0 is chosen to satisfy (4.12), and subject to the initial conditions (3.2), the first integrals, (5.9), (5.17), and (5.18) take the form

$$J_1 = (2\kappa - P_0)\left(1 - \frac{\mu}{2}\right), \quad J_2 = 0, \quad J_3 = P_0\left(1 + \frac{\mu}{2}\right). \quad (5.31)$$

Thus, from (5.24), to first order in μ ,

$$x(t) = \left[\cos^2(\omega_0 t) + \left(\frac{\kappa\left(1 - \frac{\mu}{2}\right) + \frac{\mu}{2}P_0}{\kappa\left(1 - \frac{\mu}{2}\right) - P_0} \right) \sin^2(\omega_0 t) \right]^{1/2}, \quad (5.32)$$

where ω_0^2 is defined by (4.15).

From (5.32) it can be seen that the amplitude depends on the ratio P_0/κ . We first investigate the range of values of P_0/κ for which the approximate solution exists for given $0 < \mu < 1$. From (4.15),

$$\frac{P_0}{\kappa} < 1 - \frac{\mu}{2}, \quad (5.33)$$

which places an upper bound on P_0/κ when $P_0 > 0$ but which is always satisfied when $P_0 < 0$. Now since $J_2 = 0$, when $P_0 > 0$,

$$x_{\max} = \frac{1}{\sqrt{2}\omega_0} [J_1 + J_3]^{1/2} = \left[\frac{\kappa\left(1 - \frac{\mu}{2}\right) + \frac{\mu}{2}P_0}{\kappa\left(1 - \frac{\mu}{2}\right) - P_0} \right]^{1/2}, \quad (5.34)$$

$$x_{\min} = \frac{1}{\sqrt{2}\omega_0} [J_1 - J_3]^{1/2} = 1, \quad (5.35)$$

and the approximate solution exists provided (5.33) is satisfied. When $P_0 < 0$,

$$x_{\max} = \frac{1}{\sqrt{2} \omega_0} [J_1 - J_3]^{1/2} = 1, \quad (5.36)$$

$$x_{\min} = \frac{1}{\sqrt{2} \omega_0} [J_1 + J_3]^{1/2} = \left[\frac{\kappa \left(1 - \frac{\mu}{2}\right) + \frac{\mu}{2} P_0}{\kappa \left(1 - \frac{\mu}{2}\right) - P_0} \right]^{1/2}, \quad (5.37)$$

and the approximate solution exists provided

$$\frac{P_0}{\kappa} > 1 - \frac{2}{\mu}. \quad (5.38)$$

By combining (5.33) and (5.38) we see that the approximate solution correct to first order in μ exists for given $0 < \mu < 1$ provided

$$1 - \frac{2}{\mu} \leq \frac{P_0}{\kappa} < 1 - \frac{\mu}{2}. \quad (5.39)$$

For $P_0/\kappa = 1 - 2/\mu$, $x_{\min} = 0$ and as $P_0/\kappa \rightarrow 1 - \mu/2$, $x_{\max} \rightarrow \infty$. Graphs of x_{\max} and x_{\min} plotted against P_0/κ for $\mu = 0, 0.25, 0.5$, and 0.75 are presented in Fig. 3. When $\mu > 0$ the approximate solution to order μ gives an upper bound on the magnitude of the net inward applied pressure as well as on the magnitude of the net outward applied pressure. However, since $x_{\min} = 0$ is physically unattainable, the upper bound on the magnitude of the net inward applied pressure is not physical and is a consequence of the approximations made.

In Fig. 4, $x(t)$ is plotted against $\sqrt{\kappa}t$ for $\mu = 0.3$ and $P_0/\kappa = \pm 0.5$. The graphs are of the solution correct to first order in μ given by (5.32), the solution derived by Shahinpoor and Nowinski [22] for zero order in μ obtained from (5.32) by putting $\mu = 0$, and the numerical solution of the exact equation (1.1) with $P(t)$ given by (3.1). The solution to order μ is a better approximation than the zero order solution for both net outward and inward applied pressures. In addition, as with the free oscillations, the maximum value of the dimensionless inner radius, $x(t)$, attained by the numerical solution is bounded above and below by the maximum value of $x(t)$ for the order μ and zero order approximations, respectively.

In the next section, a non-linear superposition principle is derived which provides a systematic way of solving Eq. (4.7) when the net applied force is time dependent.

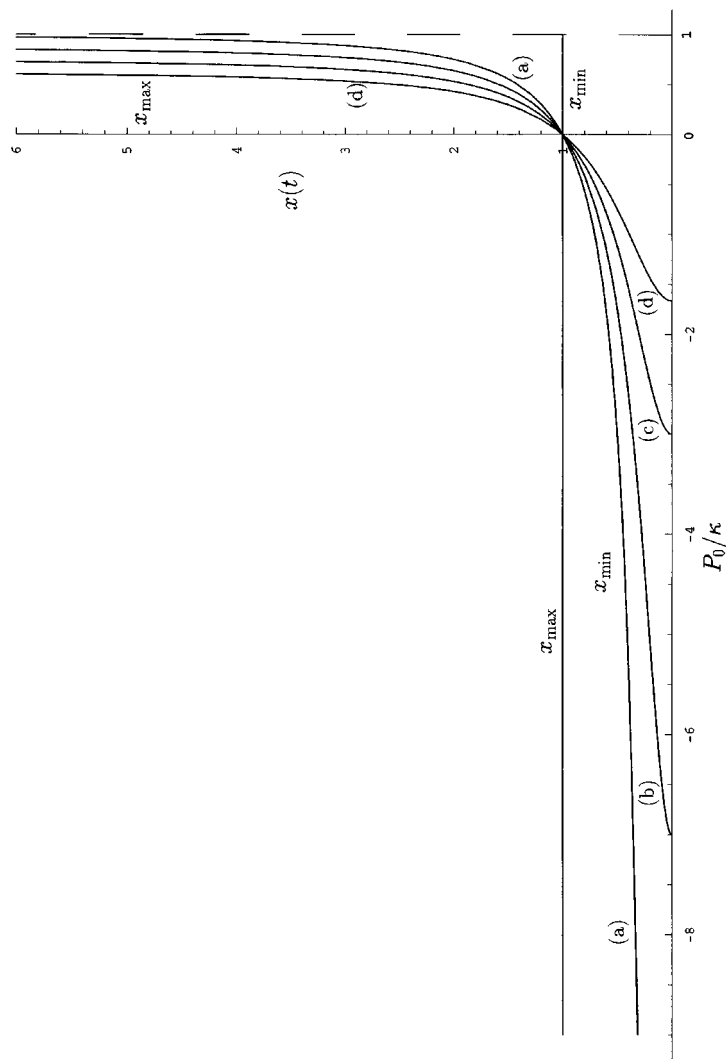


FIG. 3. Maximum and minimum values of the nondimensional inner radius, $x(t)$, for heaviside step loading plotted against P_0/κ for: (a) $\mu = 0$ ($-\infty < P_0/\kappa < 1$), (b) $\mu = 0.25$ ($-7 \leq P_0/\kappa < 7/8$), (c) $\mu = 0.5$ ($-3 \leq P_0/\kappa < 3/4$), and (d) $\mu = 0.75$ ($-5/3 \leq P_0/\kappa < 5/8$). The values $P_0 > 0$ and $P_0 < 0$ correspond to net outward and net inward applied pressures, respectively.

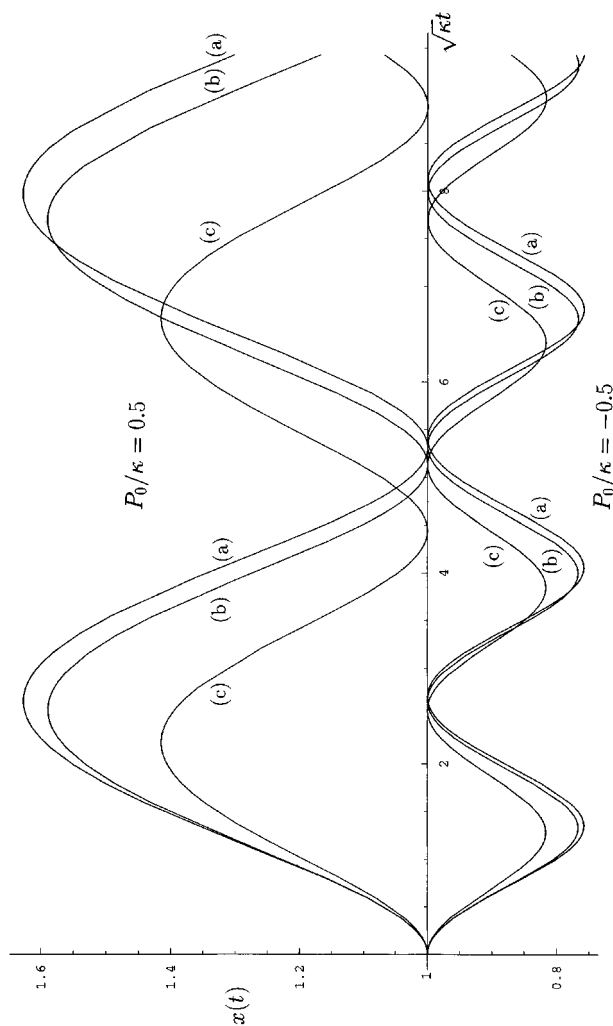


FIG. 4. The nondimensional inner radius, $x(t)$, for heaviside step loading plotted against $\sqrt{\kappa t}$ for $P_0/\kappa = \pm 0.5$ and $\mu = 0.5$: (a) solution correct to $O(\mu)$ given by (5.32), (b) numerical solution of (1.1) with $P(t)$ given by (3.1), and (c) zero order approximation given by (5.32) with $\mu = 0$.

6. TRANSFORMATION TO AN AUTONOMOUS EQUATION AND NON-LINEAR SUPERPOSITION PRINCIPLE FOR ARBITRARY $P(t)$ TO ORDER μ

We consider now the time dependent differential equation (4.7) which admits the Lie point symmetry generator (4.26).

We proceed by transforming Eq. (4.7) into an autonomous equation by insisting that under the transformation $(t, x) \mapsto (t^*, x^*)$ the transformed equation admits the time translation generator $\partial/\partial t^*$. We therefore insist that the symmetry generator (4.26) transform to the generator $X^* = \partial/\partial t^*$; thus

$$X^* = X(t^*) \frac{\partial}{\partial t^*} + X(x^*) \frac{\partial}{\partial x^*} = \frac{\partial}{\partial t^*}. \quad (6.1)$$

Therefore t^* and x^* must satisfy the quasi-linear partial differential equations

$$X(t^*) = 1: \quad g^2(t) \frac{\partial t^*}{\partial t} + \left(x + \frac{\mu}{2x}\right) g(t) \dot{g}(t) \frac{\partial t^*}{\partial x} = 1, \quad (6.2)$$

$$X(x^*) = 0: \quad g^2(t) \frac{\partial x^*}{\partial t} + \left(x + \frac{\mu}{2x}\right) g(t) \dot{g}(t) \frac{\partial x^*}{\partial x} = 0, \quad (6.3)$$

where $g(t)$ satisfies (4.25).

Two independent solutions of the differential equations of the characteristic curves of (6.3) are

$$x^* = c_1, \quad \frac{x}{g(t)} \left(1 + \frac{\mu}{2x^2}\right)^{1/2} = c_2, \quad (6.4)$$

where c_1 and c_2 are constants. A special solution for x^* is therefore

$$x^* = \frac{x}{g(t)} \left(1 + \frac{\mu}{2x^2}\right)^{1/2} = \frac{x}{g(t)} \left(1 + \frac{\mu}{4x^2}\right) + O(\mu^2), \quad (6.5)$$

as $\mu \rightarrow 0$. Also, the differential equations of the characteristic curves of (6.2) yield

$$t^* = \int^t \frac{dt}{g^2(t)} + c_3, \quad (6.6)$$

where c_3 is a constant.

The transformation from (t, x) to (t^*, x^*) , correct to first order in μ , is thus given by (6.5) and (6.6). Under this transformation, (4.7) becomes, neglecting terms of order μ^2 , the autonomous equation

$$\frac{d^2 x^*}{dt^{*2}} + Cx^* = \kappa \left(1 + \frac{\mu}{2}\right) \frac{1}{x^{*3}}, \quad (6.7)$$

where C is the arbitrary constant in (4.25).

Now now derive a non-linear superposition principle for the solution, $x(t)$, of (4.7) using the method described by Rogers and Ames [20]. Since the constant C is arbitrary, we choose $C = 0$. Equations (6.7) and (4.25) thus reduce, respectively, to

$$\frac{d^2 x^*}{dt^{*2}} = \kappa \left(1 + \frac{\mu}{2}\right) \frac{1}{x^{*3}}, \quad (6.8)$$

$$\frac{d^2 g}{dt^2} + \left[\kappa \left(1 - \frac{\mu}{2}\right) - P(t) \right] g = 0. \quad (6.9)$$

Integration of (6.8) with respect to x^* gives

$$\frac{1}{2} \left(\frac{dx^*}{dt^*} \right)^2 + \frac{\kappa}{2} \left(1 + \frac{\mu}{2}\right) \frac{1}{x^{*2}} = J, \quad (6.10)$$

where J is a constant called the Lewis invariant [14, 15]. When expressed in terms of x and t , (6.10) becomes

$$\begin{aligned} 2J = & \left[\dot{x} \left(1 - \frac{\mu}{4x^2}\right) g(t) - x \left(1 + \frac{\mu}{4x^2}\right) \dot{g}(t) \right]^2 \\ & + \kappa \left(1 + \frac{\mu}{2}\right) \frac{1}{x^2} \left(1 - \frac{\mu}{2x^2}\right) g^2(t), \end{aligned} \quad (6.11)$$

where $g(t)$ satisfies the second order differential equation (6.9). Let $g_1(t)$ and $g_2(t)$ be two linearly independent solutions of (6.9) and define

$$\begin{aligned} 2J_1 = & \left[\dot{x} \left(1 - \frac{\mu}{4x^2}\right) g_1(t) - x \left(1 + \frac{\mu}{4x^2}\right) \dot{g}_1(t) \right]^2 \\ & + \kappa \left(1 + \frac{\mu}{2}\right) \frac{1}{x^2} \left(1 - \frac{\mu}{2x^2}\right) g_1^2(t), \end{aligned} \quad (6.12)$$

$$\begin{aligned} 2J_2 = & \left[\dot{x} \left(1 - \frac{\mu}{4x^2}\right) g_2(t) - x \left(1 + \frac{\mu}{4x^2}\right) \dot{g}_2(t) \right]^2 \\ & + \kappa \left(1 + \frac{\mu}{2}\right) \frac{1}{x^2} \left(1 - \frac{\mu}{2x^2}\right) g_2^2(t). \end{aligned} \quad (6.13)$$

Equations (6.12) and (6.13) are two equations for x and \dot{x} . We solve for x by eliminating \dot{x} . This gives the non-linear superposition

$$x(t) = \left[ag_1^2(t) + bg_2^2(t) + 2cg_1(t)g_2(t) - \frac{\mu}{2} \right]^{1/2}, \quad (6.14)$$

where

$$a = \frac{2J_2}{W^2}, \quad b = \frac{2J_1}{W^2}, \quad c = \pm \frac{\left[4J_1J_2 - \kappa \left(1 + \frac{\mu}{2} \right) W^2 \right]^{1/2}}{W^2}, \quad (6.15)$$

and W is the Wronskian determinant of $g_1(t)$ and $g_2(t)$. Since $g_1(t)$ and $g_2(t)$ are linearly independent solutions of (6.9) it can be verified that W is a non-zero constant. The constants a , b , and c are related by

$$ab - c^2 = \frac{\kappa \left(1 + \frac{\mu}{2} \right)}{W^2}. \quad (6.16)$$

We need to impose initial conditions on $g_1(t)$ and $g_2(t)$ such that the initial conditions $x(0) = x_0$, $\dot{x}(0) = v_0$, where $x_0 \neq 0$, are satisfied. We take as initial conditions

$$g_1(0) = x_0, \quad \dot{g}_1(0) = \left(1 - \frac{\mu}{2x_0^2} \right) v_0, \quad (6.17)$$

$$g_2(0) = 0, \quad \dot{g}_2(0) \neq 0, \quad (6.18)$$

where $\dot{g}_2(0)$ is not specified. Then, if terms of order μ^2 are neglected,

$$W = x_0 \dot{g}_2(0) \neq 0, \\ a = 1 + \frac{\mu}{2x_0^2}, \quad b = \frac{1}{x_0^2 (\dot{g}_2(0))^2} \kappa \left(1 + \frac{\mu}{2} \right) \left(1 - \frac{\mu}{2x_0^2} \right), \quad c = 0. \quad (6.19)$$

It can be verified by putting $t = 0$ in (6.14) and by evaluating $\dot{x}(t)$ from (6.14) that if terms of $O(\mu^2)$ are neglected then the initial conditions, $x(0) = x_0$ and $\dot{x}(0) = v_0$, are satisfied by (6.14). We can now state the

following non-linear superposition principle:

Correct to terms of order μ , the solution of the differential equation

$$\ddot{x} + \frac{\mu}{2x^3}\dot{x}^2 + \left[\kappa \left(1 - \frac{\mu}{2} \right) - P(t) \right] x = -\frac{\mu}{2}(\kappa - P(t))\frac{1}{x} + \kappa \left(1 + \frac{\mu}{2} \right) \frac{1}{x^3} - \frac{\mu}{2} \kappa \frac{1}{x^5}, \quad (6.20)$$

subject to the initial conditions

$$x(0) = x_0, \quad \dot{x}(0) = v_0, \quad (6.21)$$

is

$$x(t) = \left[ag_1^2(t) + bg_2^2(t) - \frac{\mu}{2} \right]^{1/2} \quad (6.22)$$

where

$$a = 1 + \frac{\mu}{2x_0^2}, \quad b = \frac{1}{x_0^2(\dot{g}_2(0))^2} \kappa \left(1 + \frac{\mu}{2} \right) \left(1 - \frac{\mu}{2x_0^2} \right), \quad (6.23)$$

and $g_1(t)$ and $g_2(t)$ are linearly independent solutions of the second order differential equation

$$\frac{d^2g}{dt^2} + \left[\kappa \left(1 - \frac{\mu}{2} \right) - P(t) \right] g = 0, \quad (6.24)$$

subject to the initial conditions

$$g_1(0) = x_0, \quad \dot{g}_1(0) = \left(1 - \frac{\mu}{2x_0^2} \right) v_0, \quad (6.25)$$

$$g_2(0) = 0, \quad \dot{g}_2(0) \neq 0. \quad (6.26)$$

The non-linear superposition principle is a generalisation, to first order in μ , of the non-linear superposition principle for the Ermakov-Pinney equation which was used by Shahinpoor and Nowinski [22] to obtain exact solutions to problems with time dependent net applied surface pressures. When the inequality (4.12) is satisfied, (6.24) is the time dependent harmonic oscillator equation. When P is a constant, for example in the cases of free oscillations or heaviside step loading, the non-linear superposition principle gives the same result as may be derived using the first

integrals obtained in Section 5. When P is not constant it gives new results.

Consider the solution of Eq. (4.7) subject to the boundary condition

$$P(t) = \frac{2(P_1(t) - P_2(t))}{\rho^* \rho_1^2 \mu} = \begin{cases} M \left(1 - \frac{t}{T^*}\right), & 0 \leq t \leq T^*, \\ 0, & t > T^*, \end{cases} \quad (6.27)$$

where M and T^* are constants, which describes the response of the tube to a blast loading with linear decay in time. The maximum intensity of the blast is M and T^* is the duration of the blast. The derivation of the solution to first order in μ is similar to that given by Shahinpoor and Nowinski [22] for zero order in μ and therefore only the main results are presented here. Let

$$\eta = \kappa \left(1 - \frac{\mu}{2}\right) - M \left(1 - \frac{t}{T^*}\right) \quad (6.28)$$

and assume that

$$\kappa \left(1 - \frac{\mu}{2}\right) - M > 0. \quad (6.29)$$

Define

$$g = \eta^{1/2} u(y), \quad y = \alpha \eta^{3/2}, \quad \alpha = \frac{2}{3} \frac{T^*}{M}. \quad (6.30)$$

Then the companion equation (6.24) may be transformed to Bessel's equation of order $1/3$,

$$\frac{d^2 u}{dy^2} + \frac{1}{y} \frac{du}{dy} + \left(1 - \frac{1}{9y^2}\right) u = 0. \quad (6.31)$$

The two independent solutions of (6.31) which satisfy the initial conditions (6.17) and (6.18) are

$$\begin{aligned} g_1 &= \eta^{1/2} \left[A_1 J_{1/3}(\alpha \eta^{3/2}) + B_1 Y_{1/3}(\alpha \eta^{3/2}) \right], \\ g_2 &= \eta^{1/2} \left[A_2 J_{1/3}(\alpha \eta^{3/2}) + B_2 Y_{1/3}(\alpha \eta^{3/2}) \right], \end{aligned} \quad (6.32)$$

where $J_{1/3}$ and $Y_{1/3}$ are Bessel functions of order $1/3$ of the first and second kind, respectively, and

$$A_1 = \frac{1}{\gamma} \left[x_0 \left[\kappa \left(1 - \frac{\mu}{2} \right) - M \right] Y_{-2/3}(\beta) - \left(1 - \frac{\mu}{2x_0^2} \right) v_0 \left[\kappa \left(1 - \frac{\mu}{2} \right) - M \right]^{1/2} Y_{1/3}(\beta) \right], \quad (6.33)$$

$$B_1 = \frac{1}{\gamma} \left[-x_0 \left[\kappa \left(1 - \frac{\mu}{2} \right) - M \right] J_{-2/3}(\beta) + \left(1 - \frac{\mu}{2x_0^2} \right) v_0 \left[\kappa \left(1 - \frac{\mu}{2} \right) - M \right]^{1/2} J_{1/3}(\beta) \right], \quad (6.34)$$

$$A_2 = -\frac{1}{\gamma} \left[\kappa \left(1 - \frac{\mu}{2} \right) - M \right]^{1/2} \dot{g}_2(0) Y_{1/3}(\beta) = \dot{g}_2(0) \hat{A}_2, \quad (6.35)$$

$$B_2 = \frac{1}{\gamma} \left[\kappa \left(1 - \frac{\mu}{2} \right) - M \right]^{1/2} \dot{g}_2(0) J_{1/3}(\beta) = \dot{g}_2(0) \hat{B}_2, \quad (6.36)$$

where

$$\beta = \frac{2}{3} \frac{T^*}{M} \left[\kappa \left(1 - \frac{\mu}{2} \right) - M \right]^{3/2}, \quad \gamma = \frac{3M}{\pi T^*} = \frac{2}{\pi \alpha}. \quad (6.37)$$

The non-linear superposition solution (6.22) then gives

$$x_1(t) = \left[\left(1 + \frac{\mu}{2x_0^2} \right) \eta \left\{ A_1 J_{1/3}(\alpha \eta^{3/2}) + B_1 Y_{1/3}(\alpha \eta^{3/2}) \right\}^2 + \frac{\kappa}{x_0^2} \left(1 + \frac{\mu}{2} \right) \times \left(1 - \frac{\mu}{2x_0^2} \right) \eta \left\{ \hat{A}_2 J_{1/3}(\alpha \eta^{3/2}) + \hat{B}_2 Y_{1/3}(\alpha \eta^{3/2}) \right\}^2 - \frac{\mu}{2} \right]^{1/2}, \quad (6.38)$$

where we denote by $x_1(t)$ the solution for $0 \leq t \leq T^*$.

We denote by $x_2(t)$ the solution for $t > T^*$. For $t > T^*$, $P(t) = 0$ and the solution (5.28) for free oscillations applies:

$$\begin{aligned} x_2(t) = & \left[\left\{ x_0^* \cos \left(\sqrt{\kappa} \left(1 - \frac{\mu}{4} \right) (t - T^*) \right) \right. \right. \\ & + \frac{v_0^*}{\sqrt{\kappa}} \left(1 + \frac{\mu}{4} \right) \sin \left(\sqrt{\kappa} \left(1 - \frac{\mu}{4} \right) (t - T^*) \right) \left. \right\}^2 \\ & + \frac{1}{x_0^{*2}} \left\{ 1 - \frac{\mu}{2} \left(\frac{v_0^{*2}}{\kappa} + \left(x_0^* - \frac{1}{x_0^*} \right)^2 \right) \right\} \\ & \times \sin^2 \left(\sqrt{\kappa} \left(1 - \frac{\mu}{4} \right) (t - T^*) \right) \left. \right]^{1/2}, \end{aligned} \quad (6.39)$$

where

$$x_0^* = x_2(0) = x_1(T^*), \quad v_0^* = \dot{x}_2(0) = \dot{x}_1(T^*). \quad (6.40)$$

It can be verified using (6.38) that

$$\begin{aligned} x_0^* = & \left[\kappa \left(1 - \frac{\mu}{2} \right) \left(1 + \frac{\mu}{2x_0^2} \right) \{ A_1 J_{1/3}(\lambda) + B_1 Y_{1/3}(\lambda) \}^2 \right. \\ & + \frac{\kappa^2}{x_0^2} \left(1 - \frac{\mu}{2x_0^2} \right) \{ \hat{A}_2 J_{1/3}(\lambda) + \hat{B}_2 Y_{1/3}(\lambda) \}^2 - \frac{\mu}{2} \left. \right]^{1/2}, \end{aligned} \quad (6.41)$$

$$\begin{aligned} v_0^* = & \frac{1}{x_0^*} \left[\kappa \left(1 - \frac{\mu}{2} \right) \right]^{3/2} \\ & \times \left[\left(1 + \frac{\mu}{2x_0^2} \right) \{ A_1 J_{1/3}(\lambda) + B_1 Y_{1/3}(\lambda) \} \{ A_1 J_{-2/3}(\lambda) + B_1 Y_{-2/3}(\lambda) \} \right. \\ & + \frac{\kappa}{x_0^2} \left(1 + \frac{\mu}{2} \right) \left(1 - \frac{\mu}{2x_0^2} \right) \\ & \times \{ \hat{A}_2 J_{1/3}(\lambda) + \hat{B}_2 Y_{1/3}(\lambda) \} \{ \hat{A}_2 J_{-2/3}(\lambda) + \hat{B}_2 Y_{-2/3}(\lambda) \} \left. \right], \end{aligned} \quad (6.42)$$

where

$$\lambda = \frac{2}{3} \frac{T^*}{M} \left[\kappa \left(1 - \frac{\mu}{2} \right) \right]^{3/2}. \quad (6.43)$$

In Fig. 5, $x_1(t)$ and $x_2(t)$ are plotted against t for $x_0 = 1$, $v_0 = 0$, $T^* = 1$, $M = 1$, and $\mu = 0.2$. We consider the case for which T^* equals the period, $\pi/\sqrt{\kappa}$, of free oscillations to zero order in μ ; thus $\kappa = \pi^2$. At time $t = T^* = 1$, the initial conditions for free oscillations are given by (6.41) and (6.42). The maximum and minimum displacements of the numerical solution of the Knowles equation (1.1) subjected to a blast load with linear decay are bounded above and below by the maximum and minimum displacements of the approximation to order μ , (6.38) for $0 \leq t \leq T^*$ and (6.39) for $t > T^*$, and the zero order approximation obtained by setting $\mu = 0$ in (6.38) and (6.39), respectively. It is seen that the solutions correct to first order in μ are better approximations to the numerical solution of Eq. (1.1) than the corresponding zero order approximations, particularly for $t > T^*$.

7. CONCLUDING REMARKS

Figures 2-5 show that Eq. (4.7) is a better approximation of the Knowles equation (1.1) for a Mooney-Rivlin material than is the Ermakov-Pinney equation (1.3). However, there is the restriction (5.22) on the value of μ for the first order solution to be valid which does not apply to the solution of the Ermakov-Pinney equation. In addition, for the heaviside step loading boundary condition, a bound on the magnitude of the net inward applied pressure exists for the order μ solution which is a result of the approximations made and is not physical.

The general solutions, (5.19), (5.20), and (5.24) of Eq. (4.7) are expressed in terms of the first integrals, J_1 , J_2 , and J_3 , and are therefore invariant solutions. They are invariant solutions because Eq. (4.7) can be transformed to the Ermakov-Pinney equation (6.7) and all solutions of Ermakov-Pinney equations are invariant solutions [5, 6].

The non-linear superposition principle derived in Section 6 is required when solving dynamic boundary value problems with non-constant net applied pressure $P(t)$. The differential equation (4.7) is then non-autonomous and cannot be solved by elementary means, as can the autonomous equation when P is a constant, because then a first integral corresponding to time translational invariance does not exist.

The approximate solutions derived for the displacement with the heaviside step loading boundary condition take the form of non-linear superpo-

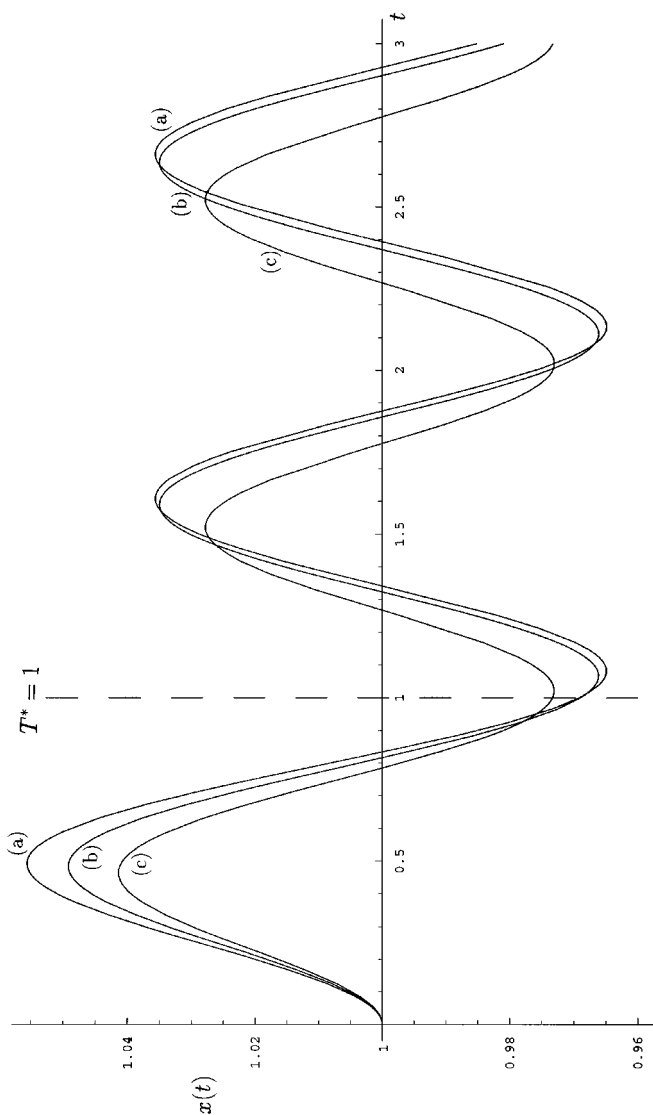


FIG. 5. The nondimensional inner radius, $x(t)$, for blast loading with linear decay plotted against t for $x_0 = 1$, $v_0 = 0$, $T^* = 1$, $M = 1$, $\kappa = \pi^2$, and $\mu = 0.2$; (a) solution correct to $O(\mu)$ given by (6.38) for $0 \leq t \leq T^*$ and (6.39) for $t > T^*$, (b) numerical solution of (1.1) with $P(t)$ given by (6.27), and (c) zero order approximation given by (6.38) with $\mu = 0$ for $0 \leq t \leq T^*$ and (6.39) with $\mu = 0$ for $t > T^*$.

sitions. The maximum and minimum values of the radial displacement are given exactly by the approximate solutions and their periods are upper and lower bounds on the exact period. They give better approximations than the thin-shell solution of the Ermakov-Pinney equation except for the approximation to the period for net inward applied pressure which is given more accurately by the thin-shell solution.

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APPENDIX A: THE WRONSKIAN DETERMINANT $W[y_1, y_2, y_3, y_4]$

$$W[y_1, y_2, y_3, y_4]$$

$$\begin{aligned}
 &= \frac{8\mu^{3/2}\kappa}{x^{18}\left(1 + \frac{\mu}{x^2}\right)^6\left(\ln\left(1 + \frac{\mu}{x^2}\right)\right)^{9/2}} \\
 &\times \left[2\mu^3((2 + \mu)x^2 + x^2\mu + \mu) \right. \\
 &\quad - 2\mu^2\{5(\mu + 2)x^4 + 2\mu(\mu + 5)x^2 + 3\mu^2\}\ln\left(1 + \frac{\mu}{x^2}\right) \\
 &\quad - \mu\{4(\mu + 2)x^6 + \mu(12 - 5\mu)x^4 + \mu^2(20 - \mu)x^2 + 8\mu^3\} \\
 &\quad \times \left(\ln\left(1 + \frac{\mu}{x^2}\right)\right)^2 \\
 &\quad + 4\{3(\mu + 2)x^8 + 2\mu(6 + \mu)x^6 + 12\mu^2x^4 + 6\mu^3x^2 + \mu^4\} \\
 &\quad \left. \times \left(\ln\left(1 + \frac{\mu}{x^2}\right)\right)^3 \right].
 \end{aligned}$$

APPENDIX B:
TAYLOR EXPANSION OF $K(x; \mu)$, $G(x; \mu)$,
AND $z_1(x; \mu)$ TO $z_8(x; \mu)$

$$\begin{aligned}
 K(x; \mu) = & \kappa \left[x - \frac{1}{x^3} + \mu \left(-\frac{x}{2} + \frac{1}{4x} - \frac{1}{2x^3} + \frac{3}{4x^5} \right) \right. \\
 & + \mu^2 \left(\frac{x}{3} - \frac{1}{8x} - \frac{7}{96x^3} + \frac{13}{24x^5} - \frac{65}{96x^7} \right) \\
 & + \mu^3 \left(-\frac{x}{4} + \frac{1}{12x} + \frac{7}{192x^3} + \frac{5}{128x^5} - \frac{35}{64x^7} + \frac{245}{384x^9} \right) \\
 & + \mu^4 \left(\frac{x}{5} - \frac{1}{16x} - \frac{7}{288x^3} - \frac{5}{256x^5} - \frac{787}{30720x^7} \right. \\
 & \quad \left. + \frac{6271}{11520x^9} - \frac{6271}{10240x^{11}} \right) \left. \right] + O(\mu^5),
 \end{aligned}$$

$$\begin{aligned}
 G(x; \mu) = & \kappa \left[1 + \frac{3}{x^4} + \mu \left(-\frac{1}{2} + \frac{3}{2x^4} - \frac{3}{x^6} \right) \right. \\
 & + \mu^2 \left(\frac{1}{3} + \frac{1}{12x^4} - \frac{7}{3x^6} + \frac{43}{12x^8} \right) \\
 & + \mu^3 \left(-\frac{1}{4} - \frac{1}{24x^4} - \frac{1}{12x^6} + \frac{73}{24x^8} - \frac{25}{6x^{10}} \right) \\
 & + \mu^4 \left(\frac{1}{5} + \frac{1}{36x^4} + \frac{1}{24x^6} + \frac{19}{240x^8} - \frac{1327}{360x^{10}} + \frac{3407}{720x^{12}} \right) \left. \right] \\
 & + O(\mu^5).
 \end{aligned}$$

$$z_1(x; \mu) = \frac{1}{2} \left[x + \frac{\mu}{4x} - \frac{13\mu^2}{288x^3} + \frac{7\mu^3}{384x^5} - \frac{6271\mu^4}{645120x^7} \right] + O(\mu^5),$$

$$\begin{aligned}
 z_2(x; \mu) = & 2\kappa \left[x + \mu \left(-\frac{x}{2} + \frac{1}{4x} \right) \right. \\
 & + \mu^2 \left(\frac{x}{3} - \frac{1}{8x} - \frac{13}{288x^3} - \frac{1}{12x^5} + \frac{1}{6x^7} \right) \\
 & + \mu^3 \left(-\frac{x}{4} + \frac{1}{12x} + \frac{13}{576x^3} + \frac{7}{384x^5} + \frac{3}{16x^7} - \frac{7}{24x^9} \right)
 \end{aligned}$$

$$\begin{aligned}
& + \mu^4 \left(\frac{x}{5} - \frac{1}{16x} - \frac{13}{864x^3} - \frac{7}{768x^5} - \frac{3097}{387072x^7} \right. \\
& \quad \left. - \frac{5011}{17280x^9} + \frac{4861}{12096x^{11}} \right) \Bigg] + O(\mu^5), \\
z_3(x; \mu) &= 4\kappa \left[\frac{1}{x^3} + \mu \left(\frac{1}{2x^3} - \frac{3}{4x^5} \right) + \mu^2 \left(\frac{1}{36x^3} - \frac{5}{8x^5} + \frac{27}{32x^7} \right) \right. \\
& \quad + \mu^3 \left(-\frac{1}{72x^3} - \frac{1}{48x^5} + \frac{47}{64x^7} - \frac{119}{128x^9} \right) \\
& \quad + \mu^4 \left(\frac{1}{108x^3} + \frac{1}{96x^5} + \frac{2131}{120960x^7} - \frac{5767}{6912x^9} + \frac{1962979}{1935360x^{11}} \right) \Bigg] \\
& \quad + O(\mu^5), \\
z_4(x; \mu) &= 1, \\
z_5(x; \mu) &= \kappa \left[1 + \frac{3}{x^4} + \mu \left(-\frac{1}{2} + \frac{3}{2x^4} - \frac{3}{x^6} \right) \right. \\
& \quad + \mu^2 \left(\frac{1}{3} + \frac{1}{12x^4} - \frac{7}{3x^6} + \frac{43}{12x^8} \right) \\
& \quad + \mu^3 \left(-\frac{1}{4} - \frac{1}{24x^4} - \frac{1}{12x^6} + \frac{73}{24x^8} - \frac{25}{6x^{10}} \right) \\
& \quad + \mu^4 \left(\frac{1}{5} + \frac{1}{36x^4} + \frac{1}{24x^6} + \frac{19}{240x^8} - \frac{1327}{360x^{10}} + \frac{3407}{720x^{12}} \right) \Bigg] \\
& \quad + O(\mu^5), \\
z_6(x; \mu) &= -2 \left[x + \frac{\mu}{4x} - \frac{13\mu^2}{288x^3} + \frac{7\mu^3}{384x^5} - \frac{3097\mu^4}{387072x^7} \right] + O(\mu^5), \\
z_7(x; \mu) &= - \left[1 + \frac{\mu^2}{12x^4} - \frac{\mu^3}{12x^6} + \frac{19\mu^4}{240x^8} \right] + O(\mu^5), \\
z_8(x; \mu) &= - \left[x + \frac{\mu}{4x} - \frac{7\mu^2}{96x^3} + \frac{5\mu^3}{128x^5} - \frac{787\mu^4}{30720x^7} \right] + O(\mu^5).
\end{aligned}$$

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