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Linearized Oscillation of First-Order Nonlinear Neutral Delay Differential Equations[1]

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In this paper we establish a equivalence of oscillation for the first-order nonlinear neutral delay differential equation,

$$[x(t) - px(t-\tau)]' + \sum_{i=1}^{m} q_i f_i(x(t-\sigma_i)) = 0,$$

and its corresponding linear equation,

$$[x(t) - px(t-\tau)]' + \sum_{i=1}^{m} q_i x(t-\sigma_i) = 0,$$

under a weaker condition on f_i. © 2001 Academic Press

Key Words: neutral delay differential equation; linearized; oscillation; equivalence; critical state.

1. INTRODUCTION

Consider the nonlinear neutral delay differential equation

$$[x(t) - px(t-\tau)]' + \sum_{i=1}^{m} q_i f_i(x(t-\sigma_i)) = 0, \quad t \geq t_0, \tag{1.1}$$

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and its corresponding linear equation

$$[x(t) - px(t-\tau)]' + \sum_{i=1}^{m} q_i x(t-\sigma_i) = 0, \qquad t \geq t_0, \tag{1.2}$$

where $p \in [0, 1)$, $\tau \in (0, \infty)$, $f_i \in C(R, R)$, $q_i \in (0, \infty)$, and $\sigma_i \in [0, \infty)$, $i = 1, \ldots, m$.

It is a known fact (see [3]), that every solution of (1.2) oscillates if and only if its characteristic equation

$$F(\lambda) \equiv -\lambda + \lambda p e^{\lambda\tau} + \sum_{i=1}^{m} q_i e^{\lambda\sigma_i} = 0 \tag{1.3}$$

has no real roots.

In [3], Györi and Ladas related the oscillation of the nonlinear equation (1.1) to the oscillation of the linear equation (1.2). Indeed, they proved that if the following conditions hold:

(H_1) $uf_i(u) > 0$, $u \neq 0$ and $\lim_{u\to 0} \frac{f_i(u)}{u} = 1$, $i = 1, \ldots, m$

(H_2) There exists a $\delta > 0$ such that either

$$f_i(u) \leq u, \ u \in [0, \delta), \quad i = 1, \ldots, m$$

or

$$f_i(u) \geq u, \ u \in (-\delta, 0], \quad i = 1, \ldots, m,$$

then every solution of (1.1) oscillates if and only if every solution of (1.2) oscillates.

Here (H_1) and (H_2) are the slightly modified version of [3, Corollary 6.6.1]. For related work, see also [4].

We remark that (H_2) is an additional condition to the linearized condition (H_1), which restricts by u the tendency for functions $f_i(u)$ to vary in a neighborhood of the origin, and so many known equations may fail to satisfy it. For example, the delay logistic equation

$$x'(t) + \sum_{i=1}^{m} p_i\left(e^{\alpha_i x(t-\sigma_i)} - 1\right) = 0 \tag{1.4}$$

is such an equation, where $p_i, \alpha_i, \sigma_i \in (0, \infty)$ and $f_i(u) = \alpha_i^{-1}(e^{\alpha_i u} - 1)$, $i = 1, \ldots, m$. For this reason, (H_2) reduces the scope of applicability of the foregoing equivalence theorem. On the other hand, the example given in [7] shows that (H_2) cannot be removed in general. Therefore, it is valuable and necessary to relax the condition (H_2).

In this paper we further investigate the equivalence of oscillation of (1.1) and (1.2) in two cases: a noncritical case and a critical case. Consequently,

we show that (H_2) can be replaced by the following condition:

(H_3) There exists $r > 0, \delta > 0$, and $K > 0$ such that for each $i = 1, \ldots, m$, $f_i(u)$ is nondecreasing in $(-\delta, \delta)$ and either

$$f_i(u) \le u + K|u|^{1+r}, \qquad u \in [0, \delta),\ i = 1, \ldots, m$$

or

$$f_i(u) \ge u - K|u|^{1+r}, \qquad u \in (-\delta, 0], \quad i = 1, \ldots, m.$$

It is easy to see that (H_1) implies that $f_i'(0) = 1$, $i = 1, \ldots, m$. Therefore, under (H_1), is a harmfulless condition that f_i is nondecreasing in some neighborhood of the origin. Moreover, (H_3) is also easily satisfied. For example, the function $f_i(u) = \alpha_i^{-1}(e^{\alpha_i u} - 1)$ in (1.4) $(i = 1, \ldots, m)$ satisfies (H_3) but not (H_2). Indeed, function f_i $(i = 1, \ldots, m)$ always satisfies (H_3) whenever f_i is twice differentiable continuously in some neighborhood of the origin and $f_i'(0) = 1$ for $i = 1, \ldots, m$.

DEFINITION. If there exists a $\lambda_0 \in (-\infty, \infty)$ such that

$$F(\lambda_0) = 0 \quad \text{and} \quad F(\lambda) > 0 \quad \text{for} \quad \lambda \in (-\infty, \lambda_0) \cup (\lambda_0, \infty), \tag{1.5}$$

then we say that (1.1) or (1.2) is in a critical state. Otherwise, (1.1) or (1.2) is considered to be in a noncritical state.

In a noncritical state, we have either

$$F(\lambda) > 0 \ \text{ for } \ \lambda \in (-\infty, \infty) \tag{1.6}$$

or

$$F(\lambda^*) < 0 \ \text{ for some } \ \lambda^* \in (-\infty, \infty). \tag{1.7}$$

As is customary, a solution is called oscillatory if it has arbitrary large 0's. Otherwise, it is called nonoscillatory.

2. THE NONCRITICAL CASE

LEMMA 2.1 [3]. *Every solution of* (1.2) *oscillates if and only if the inequality*

$$[x(t) - px(t - \tau)]' + \sum_{i=1}^{m} q_i x(t - \sigma_i) \le 0, \quad t \ge t_0 \tag{2.1}$$

has no eventually positive solution.

LEMMA 2.2 [3]. *Every solution of* (1.2) *oscillates if and only if the characteristic equation* (1.3) *has no real roots*.

LEMMA 2.3 [3]. *Assume that* (H_1) *holds. Then the fact that every solution of* (1.2) *oscillates implies that every solution of* (1.1) *oscillates*.

THEOREM 2.1. *Assume that* (H_1) *and either* (1.6) *or* (1.7) *hold. Then every solution* (1.1) *oscillates if and only if every solution of* (1.2) *oscillates*.

Proof. First, assume that (1.6) holds. In view of Lemmas 2.2 and 2.3, every solution of both (1.1) and (1.2) oscillates. Next, assume that (1.7) holds. Then there exists a $\lambda^* \in (-\infty, +\infty)$ such that $F(\lambda^*) < 0$. Because $\lim_{t\to-\infty} F(\lambda) = \lim_{t\to\infty} F(\lambda) = \infty$. It follows that (1.3) has a real root at least. Hence Lemma 2.2 implies that (1.2) has an eventually positive solution. The proof will be complete if we show that (1.1) also has an eventually positive solution. To this end, let $\varepsilon_0 \in (0, 1)$ such that $\varepsilon_0 \sum_{i=1}^m q_i \times e^{\lambda^*\sigma_i} < -F(\lambda^*)$. Set

$$F^+_{\varepsilon_0}(\lambda) = -\lambda + \lambda p e^{\lambda\tau} + (1+\varepsilon_0)\sum_{i=1}^m q_i e^{\lambda\sigma_i}.$$

Then

$$\begin{aligned} F^+_{\varepsilon_0}(\lambda^*) &= -\lambda^* + \lambda^* p e^{\lambda^*\tau} + (1+\varepsilon_0)\sum_{i=1}^m q_i e^{\lambda^*\sigma_i} \\ &= F(\lambda^*) + \varepsilon_0 \sum_{i=1}^m q_i e^{\lambda^*\sigma_i} < 0. \end{aligned}$$

It follows that there exists a $\lambda_1 \in (-\infty, +\infty)$ such that $F^+_{\varepsilon_0}(\lambda_1) = 0$. By (H_1), we may choose $\delta_1 > 0$ such that

$$f_i(u) \le (1+\varepsilon_0)u, \qquad u \in [0, \delta_1), \quad i = 1, \ldots, m. \tag{2.2}$$

Hence it follows in a manner similar to that in the proof of [3, Theorem 6.6.2] that (1.1) has an eventually positive solution. The proof is complete. ∎

3. THE CRITICAL CASE

Consider the nonlinear nonautonomous delay differential equation

$$\begin{aligned} &[x(t) - px(t-\tau)]' + \sum_{i=1}^m q_i x(t-\sigma_i) \\ &\quad + f(t, x(t-\delta_1(t)), \ldots, x(t-\delta_n(t))) = 0, \quad t \ge t_0, \end{aligned} \tag{3.1}$$

where $p \in [0, 1)$, $\tau \in (0, \infty)$, $q_i \in (0, \infty)$, $\sigma_i \in [0, \infty)$, $i = 1, \ldots, m$; $\delta_j \in C([t_0, \infty), [0, \delta])$, $j = 1, \ldots, n$, $\delta \in (0, \infty)$; and $f \in C([t_0, \infty) \times R^n, R)$.

In this section we first establish a equivalence of oscillation for (3.1) and the second-order ordinary differential equation

$$y''(t) + 2\left(2\tau p e^{\lambda_0 \tau} + \lambda_0 \tau^2 p e^{\lambda_0 \tau} + \sum_{i=1}^{m} \sigma_i^2 q_i e^{\lambda_0 \sigma_i}\right)^{-1} e^{\lambda_0 t} \times f\left(t, y(t)e^{-\lambda_0(t-\delta_1(t))}, \ldots, y(t)e^{-\lambda_0(t-\delta_n(t))}\right) = 0, \quad t \geq t_0 \tag{3.2}$$

under the critical state (1.5). Then we apply this equivalence theorem to (1.1) to conclude the condition that guarantees that (1.1) and (1.2) have the same oscillatory behavior.

THEOREM 3.1. *Assume that* (1.5) *and the following condition hold*:

(H_4) For any $t \in [t_0, \infty)$

(i) $f(t, u_1, \ldots, u_n) \geq 0$, $u_1, \ldots, u_n > 0$ and $f(t, u_1, \ldots, u_n) \leq 0$, $u_1, \ldots, u_n < 0$

(ii) $f(t, u_1, \ldots, u_n)$ is nondecreasing in $u_1, \ldots, u_n$.

Then every solution of (3.1) *oscillates if and only if every solution of* (3.2) *oscillates*.

Proof. If $f(t, u_1, \ldots, u_n) \equiv 0$ for large t and $u_1, \ldots, u_n > 0$ or $u_1, \ldots, u_n < 0$, then it is easy to see that both (3.1) and (3.2) have nonoscillatory solutions. In the sequel, we assume that $f(t, u_1, \ldots, u_n) \not\equiv 0$ for large t and $u_1 \cdot u_j > 0$, $j = 1, \ldots, n$. Set $\sigma = \max\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$, $\rho = \max\{\tau, \sigma, \delta\}$, and

$$\alpha = 2\left(2\tau p e^{\lambda_0 \tau} + \lambda_0 \tau^2 p e^{\lambda_0 \tau} + \sum_{i=1}^{m} \sigma_i^2 q_i e^{\lambda_0 \sigma_i}\right)^{-1}.$$

It follows from (1.5) that

$$F(\lambda_0) = 0 \qquad \text{and} \qquad F'(\lambda_0) = 0$$

or

$$\lambda_0 = \lambda_0 p e^{\lambda_0 \tau} + \sum_{i=1}^{m} q_i e^{\lambda_0 \sigma_i} \qquad \text{and}$$
$$p e^{\lambda_0 \tau} + \lambda_0 \tau p e^{\lambda_0 \tau} + \sum_{i=1}^{m} \sigma_i q_i e^{\lambda_0 \sigma_i} = 1. \tag{3.3}$$

It is easy to see that $\lambda_0 > 0$.

Sufficiency. Let $x(t)$ be an eventually positive solution of (3.1). Then there exists a $t_1 \geq t_0$ such that $x(t) > 0$ for $t \geq t_1$. Set $v(t) = x(t)e^{\lambda_0 t}$ for $t \geq t_1$. Then $v(t) > 0$ for $t \geq t_1$. From (3.1) and (3.3), we have

$$\left[v(t) - pe^{\lambda_0\tau}v(t-\tau) - \lambda_0 pe^{\lambda_0\tau}\int_{t-\tau}^{t} v(s)ds - \sum_{i=1}^{m} q_i e^{\lambda_0\sigma_i}\int_{t-\sigma_i}^{t} v(s)ds\right]'$$
$$+ e^{\lambda_0 t} f\left(t, e^{-\lambda_0(t-\delta_1(t))}v(t-\delta_1(t)), \ldots, e^{-\lambda_0(t-\delta_n(t))}v(t-\delta_n(t))\right) = 0,$$
$$t \geq t_1 + \rho. \tag{3.4}$$

Let

$$u(t) = v(t) - pe^{\lambda_0\tau}v(t-\tau) - \lambda_0 pe^{\lambda_0\tau}\int_{t-\tau}^{t} v(s)ds$$
$$- \sum_{i=1}^{m} q_i e^{\lambda_0\sigma_i}\int_{t-\sigma_i}^{t} v(s)ds. \tag{3.5}$$

Similar to the proof of [9, Lemma 1], one can easily show that there exists a $t_2 > t_1 + \rho$ such that

$$u(t) > 0, \qquad u'(t) \leq 0, \qquad t \geq t_2. \tag{3.6}$$

Set $M = 2^{-1}\min\{v(t) | t_2 - \rho \leq t \leq t_2\}$. Then it follows from (3.3), (3.5), and (3.6) that

$$v(t) > M, \qquad t \geq t_2 - \rho. \tag{3.7}$$

Let $\lim_{t\to\infty} u(t) = l$. There are two possible cases:

Case 1. $l = 0$. Let $t_3 > t_2$ such that $u(t) < M/2\rho\alpha$ for $t \geq t_3$. Then for any $\bar{t} \geq t_3$, we have

$$v(t) > \alpha\int_{\bar{t}}^{t+\rho} u(s)ds, \quad t \in [\bar{t}, \bar{t} + \rho].$$

Case 2. $l > 0$. Noting that $u'(t) \leq 0$ for $t \geq t_2$, we have $u(t) \geq l$ for $t \geq t_2$. From (3.3), (3.5), and (3.7), we get

$$v(t) \geq l + pe^{\lambda_0\tau}v(t-\tau) + \lambda_0 pe^{\lambda_0\tau}\int_{t-\tau}^{t} v(s)ds + \sum_{i=1}^{m} q_i e^{\lambda_0\sigma_i}\int_{t-\sigma_i}^{t} v(s)ds$$
$$\geq l + M, \quad t \geq t_2.$$

By induction, one can easily show that

$$v(t) \geq nl + M, \quad t \geq t_2 + (n-1)\rho, \quad n = 1, 2, \ldots,$$

and so $\lim_{t\to\infty} v(t) = \infty$, which implies that there exists a $t_4 \geq t_3$ such that

$$v(t) > \alpha \int_{t_4}^{t+\rho} u(s)ds, \quad t \in [t_4, t_4 + \rho].$$

Combining Cases 1 and 2, we see that there exists a $T \geq t_3$ such that

$$v(t) > \alpha \int_{T}^{t+\rho} u(s)ds, \quad t \in [T, T + \rho].$$

Next we prove that

$$v(t) > \alpha \int_{T}^{t+\rho} u(s)ds, \quad t \geq T. \tag{3.8}$$

If (3.8) does not hold, then there exists a $T^* \geq T + \rho$ such that

$$v(t) > \alpha \int_{T}^{t+\rho} u(s)ds \text{ for } t \in [T, T^*) \qquad \text{and} \qquad v(T^*) = \alpha \int_{T}^{T^*+\rho} u(s)ds.$$

By (3.3), (3.5), and (3.6), and using the fact that $u(t)$ is nonincreasing in $[t_2, \infty)$, we have

$$\begin{aligned}
&\alpha \int_{T}^{T^*+\rho} u(s)ds \\
&\quad = u(T^*) + pe^{\lambda_0\tau} v(T^* - \tau) + \lambda_0 pe^{\lambda_0\tau} \int_{T^*-\tau}^{T^*} v(s)ds \\
&\qquad + \sum_{i=1}^{m} q_i e^{\lambda_0\sigma_i} \int_{T^*-\sigma_i}^{T^*} v(s)ds \\
&\quad > u(T^*) + \alpha pe^{\lambda_0\tau} \int_{T}^{T^*+\rho-\tau} u(s)ds + \alpha\lambda_0 pe^{\lambda_0\tau} \int_{T^*-\tau}^{T^*} ds \\
&\qquad \times \int_{T}^{s+\rho} u(\xi)d\xi + \alpha \sum_{i=1}^{m} q_i e^{\lambda_0\sigma_i} \int_{T^*-\sigma_i}^{T^*} ds \int_{T}^{s+\rho} u(\xi)d\xi \\
&\quad = u(T^*) + \alpha pe^{\lambda_0\tau} \left[\int_{T}^{T^*+\rho} u(s)ds - \int_{T^*+\rho-\tau}^{T^*+\rho} u(s)ds \right] \\
&\qquad + \alpha\lambda_0 pe^{\lambda_0\tau} \left[\tau \int_{T}^{T^*+\rho} u(s)ds - \int_{T^*+\rho-\tau}^{T^*+\rho} (s + \tau - \rho - T^*)u(s)ds \right] \\
&\qquad + \alpha \sum_{i=1}^{m} q_i e^{\lambda_0\sigma_i} \left[\sigma_i \int_{T}^{T^*+\rho} u(s)ds - \int_{T^*+\rho-\sigma_i}^{T^*+\rho} (s + \sigma_i - \rho - T^*)u(s)ds \right] \\
&\quad = \alpha \int_{T}^{T^*+\rho} u(s)ds + u(T^*) \\
&\qquad - \alpha \left[pe^{\lambda_0\tau} \int_{T^*+\rho-\tau}^{T^*+\rho} u(s)ds + \lambda_0 pe^{\lambda_0\tau} \int_{T^*+\rho-\tau}^{T^*+\rho} (s + \tau - \rho - T^*)u(s)ds \right.
\end{aligned}$$

$$+\sum_{i=1}^{m} q_i e^{\lambda_0\sigma_i}\int_{T^*+\rho-\sigma_i}^{T^*+\rho}(s+\sigma_i-\rho-T^*)u(s)ds\Bigg]$$

$$\geq \alpha\int_{T}^{T^*+\rho}u(s)ds+u(T^*)$$

$$-\alpha u(T^*)\Bigg[pe^{\lambda_0\tau}\int_{T^*+\rho-\tau}^{T^*+\rho}ds+\lambda_0 pe^{\lambda_0\tau}\int_{T^*+\rho-\tau}^{T^*+\rho}(s+\tau-\rho-T^*)ds$$

$$+\sum_{i=1}^{m} q_i e^{\lambda_0\sigma_i}\int_{T^*+\rho-\sigma_i}^{T^*+\rho}(s+\sigma_i-\rho-T^*)ds\Bigg]$$

$$=\alpha\int_{T}^{T^*+\rho}u(s)ds.$$

This contradiction implies that (3.8) holds. From (3.8), we obtain

$$v(t-\sigma_j(t))\geq\alpha\int_{T}^{t}u(s)ds,\quad t\geq T+\rho,\quad j=1,\ldots,n. \tag{3.9}$$

Let $y(t)=\alpha\int_T^t u(s)ds$. Then $y'(t)=\alpha u(t)$ and $y''(t)=\alpha u'(t)$ for $t\geq T$. From (H_4), (3.4), (3.5), and (3.9), we have

$$y''(t)+\alpha e^{\lambda_0 t}f\left(t, y(t)e^{-\lambda_0(t-\delta_1(t))},\ldots,y(t)e^{-\lambda_0(t-\delta_n(t))}\right)\leq 0,$$
$$t\geq T+\rho. \tag{3.10}$$

This shows that inequality (3.10) has an eventually positive solution. Similar to the proof of [8, Lemma 2.4], it is easy to show that the corresponding equation (3.2) also has an eventually positive solution. This leads to a contradiction.

Necessity. Without loss of generality, let $y(t)$ be an eventually positive solution of (3.2). Then there exists a $T\geq t_0$ such that

$$y(t)>0,\qquad y'(t)>0,\qquad \text{and}\qquad y''(t)\leq 0,\qquad t\geq T. \tag{3.11}$$

Set $u(t)=y'(t)$ for $t\geq T$. Then $u(t)>0$ and $u'(t)\leq 0$ for $t\geq T$, and

$$y(t)=\int_{T}^{t}u(s)ds+y(T),\quad t\geq T. \tag{3.12}$$

Let

$$\omega(t)=\alpha\left(\int_{T}^{t}u(s)ds+y(T)\right),\quad t\geq T. \tag{3.13}$$

Then for $t \geq T + \rho$,

$$\begin{aligned}
&pe^{\lambda_0\tau}\omega(t-\tau) + \lambda_0 pe^{\lambda_0\tau}\int_{t-\tau}^{t}\omega(s)ds + \sum_{i=1}^{m} q_i e^{\lambda_0\sigma_i}\int_{t-\sigma_i}^{t}\omega(s)ds \\
&= \alpha pe^{\lambda_0\tau}\int_{T}^{t-\tau}u(s)ds + \alpha\lambda_0 pe^{\lambda_0\tau}\int_{t-\tau}^{t}ds\int_{T}^{s}u(\xi)d\xi \\
&\quad + \alpha\sum_{i=1}^{m} q_i e^{\lambda_0\sigma_i}\int_{t-\sigma_i}^{t}ds\int_{T}^{s}u(\xi)d\xi + \alpha y(T) \\
&= \alpha pe^{\lambda_0\tau}\left(\int_{T}^{t}u(s)ds - \int_{t-\tau}^{t}u(s)ds\right) \\
&\quad + \alpha\lambda_0 pe^{\lambda_0\tau}\left(\tau\int_{T}^{t}u(s)ds - \int_{t-\tau}^{t}(s+\tau-t)u(s)ds\right) \\
&\quad + \alpha\sum_{i=1}^{m} q_i e^{\lambda_0\sigma_i}\left(\sigma_i\int_{T}^{t}u(s)ds - \int_{t-\sigma_i}^{t}(s+\sigma_i-t)u(s)ds\right) + \alpha y(T) \\
&\leq \alpha\left(\int_{T}^{t}u(s)ds + y(T)\right) - \alpha u(t)\Bigg[pe^{\lambda_0\tau}\int_{t-\tau}^{t}ds \\
&\quad + \lambda_0 pe^{\lambda_0\tau}\int_{t-\tau}^{t}(s+\tau-t)ds + \sum_{i=1}^{m} q_i e^{\lambda_0\sigma_i}\int_{t-\sigma_i}^{t}(s+\sigma_i-t)ds\Bigg] \\
&= \omega(t) - u(t)
\end{aligned}$$

or

$$\begin{aligned}
\omega(t) \geq u(t) &+ pe^{\lambda_0\tau}\omega(t-\tau) + \lambda_0 pe^{\lambda_0\tau}\int_{t-\tau}^{t}\omega(s)ds \\
&+ \sum_{i=1}^{m} q_i e^{\lambda_0\sigma_i}\int_{t-\sigma_i}^{t}\omega(s)ds, \quad t \geq T + \rho. \qquad (3.14)
\end{aligned}$$

Define a sequence $\{v_n(t)\}_{n=0}^{\infty}$ of functions as follows:

$$v_0(t) = \omega(t), \quad t \geq T$$

and

$$v_n(t) = \begin{cases} u(t) + pe^{\lambda_0\tau}v_{n-1}(t-\tau) + \lambda_0 pe^{\lambda_0\tau}\int_{t-\tau}^{t}v_{n-1}(s)ds \\ \quad + \sum_{i=1}^{m} q_i e^{\lambda_0\sigma_i}\int_{t-\sigma_i}^{t}v_{n-1}(s)ds, \quad t \geq T+\rho, \quad n = 1, 2, \ldots \\ u(t) + \frac{v_n(T+\rho)-u(T+\rho)}{\omega(T+\rho)-u(T+\rho)}[\omega(t)-u(t)], \quad T \leq t < T+\rho, \end{cases}$$

From (3.14), by induction we have

$$u(t) < v_n(t) \leq v_{n-1}(t) \leq \omega(t), \quad t \geq T, \quad n = 1, 2 \ldots .$$

Then for $t \geq T$, $v(t) = \lim_{n\to\infty} v_n(t)$ exists and

$$u(t) \leq v(t) \leq \omega(t), \quad t \geq T \tag{3.15}$$

and

$$v(t) = u(t) + pe^{\lambda_0\tau}v(t-\tau) + \lambda_0 pe^{\lambda_0\tau}\int_{t-\tau}^{t} v(s)ds + \sum_{i=1}^{m} q_i e^{\lambda_0\sigma_i}\int_{t-\sigma_i}^{t} v(s)ds, \quad t \geq T+\rho. \tag{3.16}$$

By (3.12), (3.13), and (3.15), we have

$$v(t-\delta_j(t)) \leq \omega(t-\delta_j(t)) < \omega(t) = \alpha y(t), \quad t \geq T+\rho, \quad j = 1, \ldots, n. \tag{3.17}$$

It follows from (H_4), (3.2), (3.16), and (3.17) that

$$\left[v(t) - pe^{\lambda_0\tau}v(t-\tau) - \lambda_0 pe^{\lambda_0\tau}\int_{t-\tau}^{t} v(s)ds - \sum_{i=1}^{m} q_i e^{\lambda_0\sigma_i}\int_{t-\sigma_i}^{t} v(s)ds\right]' + \alpha e^{\lambda_0 t} f\Big(t, \alpha^{-1}e^{-\lambda_0(t-\delta_1(t))}v(t-\delta_1(t)), \ldots, \alpha^{-1}e^{-\lambda_0(t-\delta_n(t))} \times v(t-\delta_n(t))\Big) \leq 0, \quad t \geq T+\rho. \tag{3.18}$$

Set $x(t) = \alpha^{-1}v(t)e^{-\lambda_0 t}$ for $t \geq T$. Then $x(t) > 0, t \geq T$, and from (3.3) and (3.18), we have

$$[x(t) - px(t-\tau)]' + \sum_{i=1}^{m} q_i x(t-\sigma_i) + f(t, x(t-\delta_1(t)), \ldots, x(t-\delta_n(t))) \leq 0, \quad t \geq T+\rho. \tag{3.19}$$

This shows that inequality (3.19) has an eventually positive solution. Similar to the proof of [2, Theorem 5.1.1], one can show that the corresponding equation (3.1) has also an eventually positive solution, leading to a contradiction. The proof is complete.

For the sake of simplicity, we now let

$$f^*(t,u) = 2\left(2\tau pe^{\lambda_0\tau} + \lambda_0\tau^2 pe^{\lambda_0\tau} + \sum_{i=1}^{m}\sigma_i^2 q_i e^{\lambda_0\sigma_i}\right)^{-1} \times e^{\lambda_0 t} f\Big(t, e^{-\lambda_0(t-\delta_1(t))}u, \ldots, e^{-\lambda_0(t-\delta_n(t))}u\Big). \tag{3.20}$$

Clearly, if (H_4) holds, then $uf^*(t,u) \geq 0$, and $f^*(t,u)$ is nondecreasing in u for $t \geq t_0$.

THEOREM 3.2. *Assume that* (1.5) *and* (H_4) *hold. Let* $\phi \in C(R, R)$ *be nondecreasing and satisfy that* $u\phi(u) > 0$ *for* $u \neq 0$ *and*

$$\int_{\beta}^{\infty} \frac{du}{\phi(u)} < \infty \qquad \text{and} \qquad \left| \int_{-\beta}^{-\infty} \frac{du}{\phi(u)} \right| < \infty. \tag{3.21}$$

If there exists $c_0 > 0$ *and* $K > 0$ *such that*

$$\liminf_{|u| \to \infty} \frac{f^*(t, u)}{\phi(u)} \geq K |f^*(t, c_0)|, \tag{3.22}$$

then every solution of (3.1) *oscillates if and only if*

$$\left| \int^{\infty} t f^*(t, c) dt \right| = \infty \quad \text{for } c \in [-c_0, 0) \cup (0, c_0]. \tag{3.23}$$

Proof of Sufficiency. Suppose to that (3.1) has a nonoscillatory solution. By Theorem 3.1, (3.2) has also a nonoscillatory solution. We can assume, without loss of generality, that (3.2) has an eventually positive solution $y(t)$. Then there exists a $T \geq t_0$ such that

$$y(t) > 0, \quad y'(t) > 0, \quad \text{and} \quad y''(t) \leq 0, \quad t \geq T. \tag{3.24}$$

There are two possible cases to consider.

Case 1. $\lim_{t \to \infty} y(t) = \beta < \infty$. It follows from (3.24) that $y(T) \leq y(t) < \beta$ for $t \geq T$. Integrating (3.2) from $t \geq T$ to ∞ and using the fact that $f^*(t, u)$ is nondecreasing in u, we obtain

$$y'(t) \geq \int_t^{\infty} f^*(s, y(s)) ds \geq \int_t^{\infty} f^*(s, y(T)) ds, \quad t \geq T.$$

Integrating again the both sides of the foregoing from T to ∞, we obtain

$$\beta - y(T) \geq \int_T^{\infty} \int_t^{\infty} f^*(s, y(T)) ds dt = \int_T^{\infty} (t - T) f^*(t, y(T)) dt.$$

It follows that $\int_T^{\infty} t f^*(t, y(T)) dt < \infty$, which contradicts (3.23).

Case 2. $\lim_{t \to \infty} y(t) = \infty$. Integrating (3.2) from $t \geq T$ to ∞ yields

$$y'(t) \geq \int_t^{\infty} f^*(s, y(s)) ds, \quad t \geq T,$$

and so

$$\frac{y'(t)}{\phi(y(t))} \geq \frac{1}{\phi(y(t))} \int_t^{\infty} f^*(s, y(s)) ds \geq \int_t^{\infty} \frac{f^*(s, y(s))}{\phi(y(s))} ds, \quad t \geq T. \tag{3.25}$$

In view of (3.22) and the fact that $\lim_{t\to\infty} y(t) = \infty$, we may choose $T_1 \geq T$ such that

$$\frac{f^*(t, y(t))}{\phi(y(t))} \geq \frac{1}{2}K|f^*(t, c_0)|, \quad t \geq T_1. \tag{3.26}$$

Hence, integrating (3.25) from T_1 to ∞ and using (3.26), we have

$$\begin{aligned}\int_{T_1}^{\infty} \frac{y'(t)}{\phi(y(t))}dt &\geq \frac{K}{2}\int_{T_1}^{\infty}\int_{t}^{\infty} |f^*(s, c_0)|dsdt \\ &= \frac{K}{2}\int_{T_1}^{\infty}(t - T_1)|f^*(t, c_0)|dt. \end{aligned}\tag{3.27}$$

On the other hand, from (3.21) it follows that

$$\int_{T_1}^{\infty} \frac{y'(t)}{\phi(y(t))}dt = \int_{y(t_1)}^{\infty} \frac{du}{\phi(u)} < \infty. \tag{3.28}$$

(3.28), together with (3.27), implies that $\int_{T_1}^{\infty} t|f^*(t, c_0)|dt < \infty$, which contradicts (3.23). ∎

Proof of Necessity. Suppose, to the contrary, that (3.23) does not hold. Without loss of generality, assume that there exists $c \in (0, c_0)$ such that

$$\int_{t_0}^{\infty} tf^*(t, c)dt < \infty.$$

It follows that

$$\int_{t_0}^{\infty}\int_{t}^{\infty} f^*(s, c)dsdt < \infty. \tag{3.29}$$

Let T be large so that $\int_T^{\infty}\int_t^{\infty} f^*(s, c)dsdt < c/2$. Define a function $z(t)$ as follows:

$$z(t) = \frac{c}{2} + \int_T^t\int_s^{\infty} f^*(\xi, c)d\xi ds, \quad t \geq T. \tag{3.30}$$

Clearly, $z(t)$ is continuous on $[T, \infty)$ and satisfies

$$c/2 \leq z(t) \leq c, \quad t \geq T.$$

Substituting this into (3.30) and using the fact that $f^*(t, u)$ is nondecreasing in u, we obtain

$$z(t) \geq \frac{c}{2} + \int_T^t\int_s^{\infty} f^*(\xi, z(\xi))d\xi ds, \quad t \geq T. \tag{3.31}$$

From (3.31), it is not difficult to show that the corresponding equation,

$$y(t) = \frac{c}{2} + \int_T^t\int_s^{\infty} f^*(\xi, y(\xi))d\xi ds, \quad t \geq T, \tag{3.32}$$

has a solution $y(t)$ that satisfies $c/2 \leq y(t) \leq z(t)$. Clearly, $y(t)$ is also an eventually positive solution of (3.2). In view of Theorem 3.1, (3.1) has an eventually positive solution. The proof is complete. ∎

THEOREM 3.3. *Assume that* (1.5), (H_1), *and* (H_3) *hold. Then every solution of* (1.1) *oscillates if and only if every solution of* (1.2) *oscillates*.

Proof. Clearly, (1.2) has a nonoscillatory solution $x(t) = e^{-\lambda_0 t}$. To complete our proof, we show that (1.1) has also a nonoscillatory solution. Let $n = m$, $\delta_j(t) = \sigma_j$, $j = 1, \ldots, m$, $f(t, u_1, \ldots, u_m) = \sum_{j=1}^{m} q_j |u_j|^{1+r} \operatorname{sign} u_j$. Then (H_4) holds. Moreover, taking $\phi(u) = |u|^{1+r} \operatorname{sign} u$, $\phi(u)$ satisfies the conditions in Theorem 3.2, and

$$\begin{aligned} f^*(t, u) &= 2\left[\left(2\tau p e^{\lambda_0 \tau} + \lambda_0 \tau^2 p e^{\lambda_0 \tau} + \sum_{i=1}^{m} \sigma_i^2 q_i e^{\lambda_0 \sigma_i}\right)^{-1}\right. \\ &\quad \left.\times \left(\sum_{j=1}^{m} q_j e^{(1+r)\lambda_0 \sigma_j}\right)\right] e^{-\lambda_0 r t} |u|^{1+r} \operatorname{sign} u \\ &\triangleq p(t)\phi(u). \end{aligned}$$

It is easy to see that $\left|\int^{\infty} tp(t)dt\right| < \infty$. In view of Theorem 3.2, the first-order neutral delay differential equation

$$\begin{aligned} &[x(t) - px(t-\tau)]' + \sum_{i=1}^{m} q_i x(t-\sigma_i) \\ &\quad + K \sum_{i=1}^{m} q_i |x(t-\sigma_i)|^{1+r} \operatorname{sign} x(t-\sigma_i) = 0 \end{aligned} \tag{3.32}$$

has a nonoscillatory solution. It follows from (H_3) that either the inequality

$$[x(t) - px(t-\tau)]' + \sum_{i=1}^{m} q_i f_i(x(t-\sigma_i)) \le 0, \quad t \ge t_0 \tag{3.33}$$

has an eventually positive solution, or the inequality

$$[x(t) - px(t-\tau)]' + \sum_{i=1}^{m} q_i f(x(t-\sigma_i)) \ge 0, \quad t \ge t_0 \tag{3.34}$$

has an eventually negative solution. Similar to the proof of [2, Theorem 5.1.1]. It is easy to show that (1.1) also has a nonoscillatory solution. The proof is complete. ∎

THEOREM 3.4. *Assume that* (1.5) *and* (H_4) *hold. If for any* $M \neq 0$,

$$\int^{\infty} \left| e^{\lambda_0 t} f\left(t, Me^{-\lambda_0(t-\delta_1(t))}, \ldots, Me^{-\lambda_0(t-\delta_n(t))}\right)\right| dt = \infty, \tag{3.35}$$

then every solution of (3.1) *oscillates*.

Proof. Suppose, to the contrary, that (3.1) has a nonoscillatory solution. By Theorem 3.1, (3.2) also has a nonoscillatory solution. We can assume, without loss of generality, that (3.2) has an eventually positive solution $y(t)$. Then there exists a $T \geq t_0$ such that (3.24) holds and $y(t) \geq y(T)$ for $t \geq T$. It follows from (3.2) and (H_4) that

$$\int_T^\infty f^*(t, y(T))dt \leq y'(T) < \infty,$$

which yields

$$\int_T^\infty \left| e^{\lambda_0 t} f\left(t, y(T)e^{-\lambda_0(t-\delta_n(t))}, \ldots, y(T)e^{-\lambda_0(t-\delta_n(t))}\right)\right| dt < \infty.$$

This contradicts (3.35), and so the proof is complete. ∎

4. SOME REMARKS

From Theorems 2.1 and 3.3, we have the following immediately.

THEOREM 4.1. *Assume that* (H_1) *and* (H_3) *hold. Then every solution of* (1.1) *oscillates if and only if every solution of* (1.2) *oscillates*.

The following example shows that (H_3) is an essential condition that guarantees that (1.1) and (1.2) have the same oscillatory behavior in the critical case.

EXAMPLE 4.1. Consider the nonlinear equation

$$\left[x(t) - \frac{e^{-1/3}}{2}x(t-1)\right]' + \frac{e^{-2/3}}{6}[x(t-2) + g(x(t-2))] = 0, \quad t \geq 0 \tag{4.1}$$

and its corresponding linear equation

$$\left[x(t) - \frac{e^{-1/3}}{2}x(t-1)\right]' + \frac{e^{-2/3}}{6}x(t-2) = 0, \quad t \geq 0, \tag{4.2}$$

where $p = 1/2e^{-1/3}$, $q = 1/6e^{-2/3}$, $\tau = 1$, $\sigma = 2$, and

$$g(u) = \begin{cases} 0, & u = 0 \\ \frac{u}{1-\ln|u|}, & 0 < |u| \leq 1 \\ u, & |u| > 1. \end{cases} \tag{4.3}$$

It is easy to see that (4.1) and (4.2) are in a critical state, because the characteristic equation

$$F(\lambda) = -\lambda + \frac{\lambda}{2}e^{-1/3+\lambda} + \frac{1}{6}e^{-2/3+2\lambda} = 0$$

has a unique real root $\lambda_0 = 1/3$, and $F(\lambda) > 0$ for $\lambda \neq 1/3$.

For any $M \neq 0$,

$$\int_{2+3\ln|M|}^{\infty} \left| e^{\lambda_0 t} g(Me^{-\lambda_0(t-2)}) \right| dt = \int_{2+3\ln|M|}^{\infty} \frac{|M| e^{2\lambda_0}}{1 + \lambda_0(t-2) - \ln|M|} dt = \infty.$$

This shows that (3.35) holds. By Theorem 3.4, every solution of (4.1) oscillates. However, (4.2) has an eventually positive solution $x(t) = e^{-t/3}$. Consequently, (4.1) and (4.2) have different oscillatory behavior.

Trace something to its source, because (4.1) does not satisfy (H_3). In fact, for $r > 0$,

$$\lim_{u \to 0} \frac{|g(u)|}{|u|^{1+r}} = \lim_{u \to 0} \frac{1}{|u|^r (1 - \ln|u|)} = \infty.$$

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