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Gaussian Estimates for Second-Order Operators with Unbounded Coefficients

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We study second-order differential operators A with lower-order coefficients in some $\mathbf{L}_q + \mathbf{L}_\infty$. We prove the generation of positive, quasi-contractive C_0 semigroups on $\mathbf{L}_p$ for all $p \in (1, \infty)$. If the second-order coefficients are in some $\mathbf{L}_q + \mathbf{L}_\infty$, we get upper pseudo-Gaussian bounds of the heat kernel. Maximal regularity, spectral independence on $\mathbf{L}_p$, and analyticity of the generated semigroup on $\mathbf{L}_1$ are studied for these operators. © 2001 Academic Press

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0. INTRODUCTION

There is now an extensive literature on Gaussian bounds for heat kernels associated with strongly elliptic operators (see Davies [8], Robinson [18] and references therein). The best upper bounds have been derived by a technique introduced by Davies in [7]. We use this technique and ideas developed in Stampacchia [19], Daners [6], and Arendt and ter Elst [2] to analyze the Cauchy problem,

$$\left.\begin{aligned} d_t u(t) + Au(t) &= 0 && \text{on } \Omega, \text{for all } t \in (0,\infty) \\ u(0) &= u_0 && \text{on } \Omega \\ u(t) &= 0 && \text{on } \partial\Omega, \text{for all } t \in (0,\infty) \end{aligned}\right\}, \qquad \text{(CP)}$$

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where Ω is an open subset of $\mathbf{R}^n$ and the operator A is associated with the closure of the second-order form $\mathscr{E}$ given by

$$\begin{aligned}\mathscr{E}(u,v) = &\sum_{i,j=1}^{n} \int_\Omega a_{i,j}(x)\,\partial_i u(x)\,\partial_j v(x)\,dx \\ &+ \sum_{j=1}^{n} \int_\Omega b_{1,j}(x)\,\partial_i u(x)\,v(x)\,dx \\ &+ \sum_{j=1}^{n} \int_\Omega b_{2,j}(x)u(x)\,\partial_j v(x)\,dx + \int_\Omega c(x)u(x)v(x)\,dx. \quad (1)\end{aligned}$$

This form is well defined for all $u, v \in \mathbf{W}^1_{\infty,c}$ if the coefficients $a_{i,j}$, $b_{l,j}$, and c are in $\mathbf{L}_{1,\mathrm{loc}}$ and are real valued.

Suppose that $c_- \in Ł_q := \mathbf{L}_q + \mathbf{L}_\infty$ and $b_{l,j} \in Ł_{2q}$ for some $q \geq 1$. Then we show that the operator $-A$ generates a semigroup T on $\mathbf{L}_2$. We prove that in this case T can be extended to consistent, quasi-contractive, positive C_0 semigroups on $\mathbf{L}_p$ for all $p \in (1,\infty)$. If $b_{l,j} \in Ł_p$ and $c_- \in Ł_q$ for some $p > 2$, $q > 1$, then $T(t)$ is in $\mathscr{L}(\mathbf{L}_1, \mathbf{L}_\infty)$ and is quasi-ultracontractive, in particular, $\|T(t)\|_{\mathscr{L}(\mathbf{L}_1,\mathbf{L}_\infty)} \leq Ct^{-n/2}e^{\omega t}$. Of course, in this situation T is associated with a kernel k such that

$$T(t)f(x) = \int_\Omega k(t,x,y)f(y)\,dy \quad \text{for all } f \in \mathbf{L}_1. \qquad (2)$$

Assume in addition that $a_{i,j} \in Ł_p$ for some $p > 2$. We then prove a pseudo-Gaussian bound of order $m \geq 2$, explicitly

$$|k(t,x,y)| \leq Ct^{-n/2}\exp\Big(\omega_1 t - \omega_2\big(|x-y|^m/t\big)^{1/m-1}\Big) \qquad (3)$$

for almost all $x, y \in \Omega$, $t \in (0,\infty)$ and some $C, \omega_i > 0$. If $a_{i,j} \in \mathbf{L}_\infty$, then we show $m = 2$ and obtain a classical Gaussian kernel as an upper bound. This extends the result of Aronson [4], Arendt and ter Elst [2], and Daners [6]. Using estimate (3), we obtain some results about spectral independence, maximal regularity, and analyticity.

The plan of this article is as follows. In Section 1 we state the exact assumptions and the main results. We give some estimates for the coefficients in Section 2. In Section 3 we derive the key estimate (6) and use it to show that the form $\mathscr{E}$ generates a C_0 semigroup on $\mathbf{L}_2$. In Section 4, we prove that $\mathscr{E}$ generates quasi-dissipative C_0 semigroups on $\mathbf{L}_p$ $(1 < p < \infty)$ and also give an $\mathbf{L}_1$–$\mathbf{L}_\infty$ estimate. We derive the upper Gaussian estimates in Section 5 and apply these results in Section 6.

In this article we fix the open set $\Omega \subset \mathbf{R}^n$ and use the following notation. By $\mathbf{N}$ and $\mathbf{R}$ we denote the set of all natural numbers (without 0) and the set of all real numbers, respectively. We use $\mathbf{R}^+ = [0, \infty)$ for short. We write $\mathbf{L}_p$ to denote the real Banach space of all functions from Ω into $\mathbf{R}$ with norm $\|f\|_p = (\int_\Omega |f(x)|^p \, dx)^{1/p}$, if $1 \le p < \infty$ and $\|f\|_\infty = \operatorname{ess\,sup}_{x \in \Omega} |f(x)|$. We use $\text{Ł}_p = \mathbf{L}_p + \mathbf{L}_\infty$ for short. If $p = 2$, then $\mathbf{L}_2$ is a Hilbert space with scalar product $\langle u, v \rangle = \int_\Omega u(x) v(x) \, dx$. By $\mathbf{L}_{1,\text{loc}}$ we denote the set of all real measurable functions that are integrable on every compact subset of Ω. The space $\mathbf{W}^1_{\infty, c}$ is the subset of $\mathbf{L}_\infty$ whose functions have compact support and (weak) derivatives in $\mathbf{L}_\infty$, while $\mathscr{D}$ is the set of all infinitely differentiable functions on Ω with compact support. By $\mathbf{H}^1$ we denote the Hilbert space of all functions in $\mathbf{L}_2$ whose first (weak) derivatives are also in $\mathbf{L}_2$, with scalar product $\langle u, v \rangle_{\mathbf{H}^1} = \langle u, v \rangle + \sum_{k=1}^n \langle \partial_k u, \partial_k v \rangle$. Finally, $\mathbf{H}^1_0$ denotes the closure of $\mathscr{D}$ with respect to $\mathbf{H}^1$.

For functions with two arguments like $f(t, x)$, we write $f(t)$ for the mapping $x \mapsto f(t, x)$ and $f(x)$ for the mapping $t \mapsto f(t, x)$ if there is no ambiguity. By $f_\pm$ we denote $\max\{0, \pm f\}$ for short, while $\operatorname{sgn} f = \mathbf{1}_{\{x;\, f(x) \neq 0\}} f/|f|$, where $\mathbf{1}_A$ is the characteristic function of the set A. Given a form $\mathscr{E}$ on a Hilbert space H, we write $\mathscr{E}_\alpha(u, v) = \mathscr{E}(u, v) + \alpha \langle u, v \rangle$, where $\alpha \in \mathbf{R}$.

A family $(A_p)_{p \in \Pi}$ with $\Pi \subset [1, \infty]$ and $A_p \colon \mathbf{L}_p \to \mathbf{L}_p$ for all $p \in \Pi$ is called consistent if for all $p, q \in \Pi$, $f \in \mathbf{L}_p \cap \mathbf{L}_q$ holds $A_p f = A_q f$. A continuous form $\mathscr{E}$ on some Hilbert space H is called coercive if there exists a constant $c > 0$ such that $\mathscr{E}(u, u) \ge c \|u\|_H^2$ for all $u \in D(\mathscr{E})$.

1. CONDITIONS AND MAIN RESULTS

We distinguish three different conditions; each one enables us to state interesting theorems.

CONDITION 1.1. The coefficients of the form $\mathscr{E}$ satisfy

(a) The growth condition. We have $a_{i,j}, c \in \mathbf{L}_{1,\text{loc}}$, $b_{l,j} \in \text{Ł}_{2q}$ for $i, j = 1, \dots, n$, $l = 1, 2$, and $c_- \in \text{Ł}_q$, where

$$q \in \begin{cases} \{1\} & \text{if } n = 1 \\ (1, \infty) & \text{if } n = 2 \\ [n/2, \infty) & \text{if } n > 2. \end{cases}$$

(b) The strong ellipticity condition. There exists $\nu \in (0, \infty)$ such that

$$\nu |\xi|^2 \le \sum_{i,j=1}^n a_{i,j}(x) \xi_i \xi_j \tag{4}$$

for all $\xi \in \mathbf{R}^n$ and almost all $x \in \Omega$.

(c) The antisymmetric boundedness condition. For $i, j = 1, \dots, n$, we have $a_{i,j} - a_{j,i} \in \mathbf{L}_\infty$.

These requirements are quite weak. The second-order coefficients are as arbitrary as possible, and the lower-order ones may still be unbounded. The boundedness of $a_{i,j} - a_{j,i}$ is only needed in Lemma 3.5 to prove the continuity of $\mathscr{E}$. The closure of $\mathscr{E}$ is associated with a generator A and its generated C_0 semigroup T. These weak assumptions are strong enough to prove the following.

THEOREM 1.2. *If Condition* 1.1 *is satisfied, then there is a consistent family* $(T_p)_{p \in (1,\infty)}$ *of quasi-contractive, positive* C_0 *semigroups* T_p *on* $\mathbf{L}_p$ *with generator* $-A_p$ *and* $A_2 = A$.

We cannot expect to always get a quasi-contractive semigroup on $\mathbf{L}_1$. A counterexample has been given by Ouhabaz [17, Remark 4.3.a].

CONDITION 1.3. In addition to Condition 1.1, we assume that the coefficients of the form $\mathscr{E}$ satisfy $b_{l,j} \in Ł_{p_1}$ for all $j = 1, \dots, n$, $l = 1, 2$ and $c_- \in Ł_{p_0}$, where $p_1 > 2q$ and $p_0 > q$; i.e., any choice of

$$p_1 \in \begin{cases} (2, \infty] & \text{if } n = 1 \\ (n, \infty] & \text{if } n \geq 2 \end{cases} \qquad p_0 \in \begin{cases} (1, \infty] & \text{if } n = 1 \\ (n/2, \infty] & \text{if } n \geq 2 \end{cases}$$

is possible if the coefficients are appropriate.

This condition is strong enough to prove the following kernel bound.

THEOREM 1.4. *If Condition* 1.3 *is satisfied, then* T *is ultracontractive. Moreover,*

$$\|T(t)\|_{\mathscr{L}(\mathbf{L}_1, \mathbf{L}_\infty)} \leq C t^{-n/2} e^{\beta t} \quad \textit{for all } t \in (0, \infty)$$

and some $C, \beta > 0$ (*see Theorem* 3.3 *and Remark* 4.7).

CONDITION 1.5. The coefficients of the form $\mathscr{E}$ satisfy Condition 1.3 and $a_{i,j} \in Ł_{p_2}$ for all $i, j = 1, \dots, n$ and some $p_2 > 2q$.

THEOREM 1.6. *If Condition* 1.5 *is satisfied, then the associated semigroup* T *is given by* $(T(t)f)(x) = \int_\Omega k(t, x, y) f(y)\, dy$, *and the kernel* $k\colon (0, \infty) \times \Omega \times \Omega \to \mathbf{R}$ *satisfies* $0 \leq k(t, x, y)$ *and a pseudo-Gaussian estimate of order* m, *i.e.*,

$$k(t, x, y) \leq C t^{-n/2} \exp\Big(\omega_1 t - \omega_2 (|x - y|^m / t)^{1/m - 1} \Big) \tag{5}$$

for all $x, y \in \Omega$, $t \in (0, \infty)$, *some* $C, \omega_i > 0$, *and* $m = 2 + \frac{4q}{p_2 - 2q}$. *If* $m = 2$, *then we have the classical Gaussian estimate* (*of order* 2).

THEOREM 1.7. *If Condition* 1.5 *is satisfied, then* $\sigma(A_p) = \sigma(A_2)$ *for all* $p \in (1, \infty)$. *If the semigroup extends to a* C_0 *semigroup on* $\mathbf{L}_1$ *with generator* A_1 (*see Remark* 6.4), *then* $\sigma(A_1) = \sigma(A_2)$ *as well.*

THEOREM 1.8. *If Condition* 1.5 *is satisfied and* $a_{i,j} \in \mathbf{L}_\infty$ *for all* $i, j = 1, \dots, n$, *then* A *has maximal* $\mathbf{L}_p$–$\mathbf{L}_q$ *regularity for all* $1 < p, q < \infty$; *i.e., for every* $f \in \mathbf{L}_p(\mathbf{R}^+, \mathbf{L}_q(\Omega))$ *there exists an unique solution* $u \in \mathbf{W}_p^1(\mathbf{R}^+, \mathbf{L}_q(\Omega)) \cap \mathbf{L}_p(\mathbf{R}^+, D(A_q))$ *of* $u'(t) = Au(t) + f(t)$ *for* $t \in \mathbf{R}^+$ *and* $u(0) = 0$.

THEOREM 1.9. *If Condition* 1.5 *is satisfied and the associated semigroup* $T\colon \Sigma_\theta \to \mathscr{L}(\mathbf{L}_2)$ *is analytic with* $\Sigma_\theta = \{re^{i\varphi};\ r > 0,\ |\varphi| < \theta\}$, *where* $\theta \in (0, \pi/2]$, *then for all* $0 < \theta_1 < \theta$ *there is an analytic mapping* $T_1\colon \Sigma_{\theta_1} \to \mathscr{L}(\mathbf{L}_1)$ *such that* $T(z)$ *and* $T_1(z)$ *are consistent for all* $z \in \Sigma_{\theta_1}$.

If we have a classical Gaussian estimate of the kernel, then we obtain an analytic semigroup on $\mathbf{L}_1$, *in particular at* 0.

2. COEFFICIENTS

As we are discussing quite general forms $\mathscr{E}$, we introduce Ł_p, a larger space than $\mathbf{L}_p$, as the space from which the coefficients are taken.

DEFINITION 2.1. For $1 \le p \le \infty$, let $\text{Ł}_p = \mathbf{L}_p + \mathbf{L}_\infty$. Moreover, for $0 \le \epsilon < \infty$ and $f \in \text{Ł}_p$, let

$$[f]_{\epsilon,p} = \inf\bigl\{k \in [0,\infty];\ \exists f_i \in \text{Ł}_p : f = f_1 + f_2,\ \|f_1\|_p \le \epsilon,\ \|f_2\|_\infty \le k\bigr\}$$

and

$$\|f\|_{\text{Ł}_p} = \inf\bigl\{\epsilon + [f]_{\epsilon,p};\ 0 \le \epsilon < \infty\bigr\}.$$

Remark 2.2. Clearly, $(\text{Ł}_p, \|\cdot\|_{\text{Ł}_p})$ is a Banach space and $[f]_{\epsilon,p}$ is nonincreasing in ϵ. For all $f, g \in \text{Ł}_p$ with $0 \le f \le g$, we have $[f]_{\epsilon,p} \le [g]_{\epsilon,p}$.

Some other properties of the spaces Ł_p are collected in the following.

LEMMA 2.3. *Let* $1 \le p \le \infty$, $f_i \in \text{Ł}_p$, $g \in \mathbf{L}_\infty$ *and* $0 \le \epsilon < \infty$. *Then*

(i) $[g]_{\epsilon,p} \le \|g\|_\infty$

(ii) *for all* $\theta \in [0,1]$, *we have* $[f_1 + f_2]_{\epsilon,p} \le [f_1]_{\theta\epsilon,p} + [f_2]_{(1-\theta)\epsilon,p}$

(iii) *for all* $\rho \in \mathbf{R}$, *we have* $[\rho f_1]_{\epsilon|\rho|,p} = |\rho|[f_1]_{\epsilon,p}$

(iv) *If* $p < q < \infty$, $\epsilon \neq 0$ *and* $f \in \mathbf{L}_q$, *then* $f \in \text{Ł}_p$ *and* $[f]_{\epsilon,p} \le \|f\|_q^{q/(q-p)}\epsilon^{-(p/(q-p))}$.

Proof. Parts (i) and (ii) are clear by the definition of $[f]_\epsilon$. We show part (iii):

$$
\begin{aligned}
&[\rho f_1]_{\epsilon|\rho|,p}\\
&\quad= \inf\left\{k \in [0,\infty];\ \exists g_i \in \mathbf{Ł}_p : f = g_1 + g_2,\ \|\rho g_1\|_p \le \epsilon|\rho|,\ \|\rho g_2\|_\infty \le k\right\}\\
&\quad= |\rho| \inf\left\{k \in [0,\infty];\ \exists g_i \in \mathbf{Ł}_p : f = g_1 + g_2,\ \|g_1\|_p \le \epsilon,\ \|g_2\|_\infty \le k\right\}\\
&\quad= |\rho| [f_1]_{\epsilon,p}.
\end{aligned}
$$

To prove part (iv), let $k = \|f\|_1^{q/(q-p)} \epsilon^{-(p/(q-p))}$ and set $f_k = \min\{k, \max\{-k, f\}\}$, $g_k = f - f_k$. Then we have $\|f_k\|_\infty \le k$ and

$$
\begin{aligned}
\|g_k\|_p &\le \|f \mathbf{1}_{\{|f|>k\}}\|_p \le \|f\|_q \|\mathbf{1}_{\{|f|>k\}}\|_1^{1/p-1/q} \le \|f\|_q \left\| \frac{|f|}{k} \right\|_q^{q(1/p-1/q)}\\
&\le \|f\|_q^{q/p} k^{-((q/p)-1)} = \epsilon.
\end{aligned}
$$

Thus, $[f]_{\epsilon,p} \le k$. ∎

To estimate terms in which the coefficients appear, we need the following well-known result.

LEMMA 2.4 (Sobolev embedding). *Let $n \in \mathbf{N}$ and*

$$
q \in \begin{cases} \{\infty\} & \text{if } n = 1 \\ [2,\infty) & \text{if } n = 2 \\ \left[2, \frac{2n}{n-2}\right] & \text{if } n > 2 \end{cases}.
$$

Then there exists a constant $C \in \mathbf{R}^+$ such that

$$
\|u\|_q \le C(\|\nabla u\|_2 + \|u\|_2)
$$

for all functions $u \in \mathscr{D}$. We call the infimum of such constants C the Sobolev embedding constant for n and q.

See, for example, Adams [1, Theorem 5.4].

LEMMA 2.5. *Let*

$$
q \in \begin{cases} \{2\} & \text{if } n = 1 \\ (2,\infty] & \text{if } n = 2 \\ [n,\infty] & \text{if } n > 2 \end{cases}.
$$

For all $a_i \in \mathbf{Ł}_q$, $v, u \in \mathbf{W}^1_{\infty, c}$, $\epsilon \in (0, \infty)$, *we have*

$$\sum_{i=1}^{n} \|a_i u \partial_i v\|_1$$
$$\leq \epsilon(\|\nabla u\|_2 + \|u\|_2)\|\nabla v\|_2 + \left(\sum_{i=1}^{n} [a_i]^2_{\epsilon/\sqrt{n}\, C_1, q}\right)^{1/2} \|u\|_2 \|\nabla v\|_2$$
$$\leq \epsilon\|\nabla u\|_2\|\nabla v\|_2 + \epsilon\|\nabla v\|_2^2 + \frac{1}{2\epsilon}\left(\epsilon^2 + \sum_{i=1}^{n} [a_i]_{2\epsilon/\sqrt{n}\, C_1, q}\right)\|u\|_2^2,$$

with C_1 *the Sobolev embedding constant for* n *and* q.

Proof. For all $\eta > 0$, there are $b_i, c_i \in \mathbf{Ł}_q$ such that $a_i = b_i + c_i$, $\|b_i\|_q \leq \epsilon/(\sqrt{n}\, C_1)$ and $\|c_i\|_\infty \leq [a_i]_{\epsilon/(\sqrt{n}\, C_1), q} + \eta$. Let $1/2 = 1/p + 1/q$. Then, by Lemma 2.4,

$$\|a_i u \partial_i v\|_1 \leq \|b_i u \partial_i v\|_1 + \|c_i u \partial_i v\|_1 \leq \|b_i u\|_2 \|\partial_i v\|_2 + \|c_i\|_\infty \|u \partial_i v\|_1$$
$$\leq \|b_i\|_q \|u\|_p \|\partial_i v\|_2 + \left([a_i]_{\epsilon/(\sqrt{n}\, C_1), q} + \eta\right)\|u \partial_i v\|_1$$
$$\leq \frac{\epsilon}{\sqrt{n}}(\|\nabla u\|_2 + \|u\|_2)\|\partial_i v\|_2 + \left([a_i]_{\epsilon/(\sqrt{n}\, C_1), q} + \eta\right)\|u \partial_i v\|_1.$$

Letting $\eta \searrow 0$, we obtain

$$\|a_i u \partial_i v\|_1 \leq \frac{\epsilon}{\sqrt{n}}(\|\nabla u\|_2 + \|u\|_2)\|\partial_i v\|_2 + [a_i]_{\epsilon/(\sqrt{n}\, C_1), q}\|u \partial_i v\|_1.$$

Summing up and using that $\sum_{i=1}^{n} \|\partial_i f\|_2 \leq \sqrt{n}\, \|\nabla f\|_2$, we obtain

$$\sum_{i=1}^{n} \|a_i u \partial_i v\|_1$$
$$\leq \frac{\epsilon}{\sqrt{n}}(\|\nabla u\|_2 + \|u\|_2) \sum_{i=1}^{n} \|\partial_i v\|_2 + \sum_{i=1}^{n} [a_i]_{\epsilon/(\sqrt{n}\, C_1), q}\|u|\partial_i v|\,\|_1$$
$$\leq \epsilon(\|\nabla u\|_2 + \|u\|_2)\|\nabla v\|_2 + \left(\sum_{i=1}^{n} [a_i]^2_{\epsilon/(\sqrt{n}\, C_1), q}\right)^{1/2} \|u\|_2\|\nabla v\|_2$$
$$\leq \epsilon\|\nabla u\|_2\|\nabla v\|_2 + \epsilon\|\nabla v\|_2^2 + (2\epsilon)^{-1}\left(\epsilon^2 + \sum_{i=1}^{n} [a_i]^2_{\epsilon/(\sqrt{n}\, C_1), q}\right)\|u\|_2^2$$

by $xy \leq \epsilon x^2 + (4\epsilon)^{-1} y^2$ for all $\epsilon > 0$. ∎

LEMMA 2.6. *Let*

$$q \in \begin{cases} \{1\} & \text{if } n = 1 \\ (1, \infty] & \text{if } n = 2 \\ [n/2, \infty] & \text{if } n > 2 \end{cases}.$$

For all $f \in Ł_q, v, u \in \mathbf{W}^1_{\infty, c}$, $\epsilon \in \mathbf{R}^+$, *we have*

$$\|fuv\|_1 \le \epsilon\big(\|\nabla u\|_2 \|\nabla v\|_2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \|u\|_2 \|v\|_2\big) + \frac{\epsilon}{4}\big(\|u\|_2^2 + \|v\|_2^2\big) + [f]_{\epsilon/C_2^2, q} \|uv\|_1,$$

with C_2 *the Sobolev embedding constant for* n *and* q.

Proof. For all $\eta > 0$, there are $g, h \in Ł_q$ such that $f = g + h$, $\|g\|_q \le \epsilon/C_2^2$ and $\|h\|_\infty \le [f]_{\epsilon/C_2^2, q} + \eta$. Let $1 = 2/p + 1/q$. Then, by Lemma 2.4,

$$\begin{aligned} \|fuv\|_1 &\le \|guv\|_1 + \|huv\|_1 \le \|gu\|_{p/(p-1)} \|v\|_p + \|h\|_\infty \|uv\|_1 \\ &\le \|g\|_q \|u\|_p \|v\|_p + \big([f]_{\epsilon/C_2^2, q} + \eta\big)\|uv\|_1 \\ &\le \epsilon(\|\nabla u\|_2 + \|u\|_2)(\|\nabla v\|_2 + \|v\|_2) + \big([f]_{\epsilon/C_2^2, q} + \eta\big)\|uv\|_1. \end{aligned}$$

Letting $\eta \searrow 0$, we obtain

$$\begin{aligned} \|fuv\|_1 &\le \epsilon(\|\nabla u\|_2 + \|u\|_2)(\|\nabla v\|_2 + \|v\|_2) + [f]_{\epsilon/C_2^2, q} \|uv\|_1 \\ &\le \epsilon\big(\|\nabla u\|_2 \|\nabla v\|_2 + \|\nabla u\|_2^2 + \|\nabla v\|_2^2 + \|u\|_2 \|v\|_2\big) \\ &\quad + \frac{\epsilon}{4}\big(\|u\|_2^2 + \|v\|_2^2\big) + [f]_{\epsilon/C_2^2, q} \|uv\|_1 \end{aligned}$$

by $xy \le x^2 + y^2/4$. ∎

3. APPROXIMATION

In this section we prove the key to this article, Theorem 3.3. We use this to prove that $\mathscr{E}$ is a closable, coercive, continuous form. Since we need estimates about $\mathbf{L}_p$ in the next section, we use a substitution and the $\mathbf{L}_2$ norm. This enables us to approximate the $\mathbf{L}_p$ norm using a technique of Daners [6].

DEFINITION 3.1. For any $x \in \Omega$, $u\colon \Omega \to \mathbf{R}$, $k \in [1,\infty)$, $p \in [2,\infty)$, let

$$f_{k,p}(x) = \begin{cases} |x|^{p/2} & \text{if } |x| \le k \\ k^{p/2} + \frac{p}{2}k^{p/2-1}(|x|-k) & \text{if } |x| > k \end{cases},$$

$$g_{k,p}(x) = f_{k,p}(x)\,\partial f_{k,p}(x),$$

$$w_{k,p,u}(x) = f_{k,p}(u(x)),$$

$$v_{k,p,u}(x) = g_{k,p}(u(x)).$$

We put together some properties of these functions.

LEMMA 3.2. *Let $u \in \mathbf{W}^1_{\mathrm{loc}}$, $k \in [1,\infty)$, $p \in [2,\infty)$; then*

$$f_{k,p}(x) = \begin{cases} |x|^{p/2} & \textit{if } |x| \le k \\ \frac{p}{2}k^{(p/2)-1}\left(|x| - \frac{p-2}{p}k\right) & \textit{if } |x| > k \end{cases},$$

$$f'_{k,p}(x) = \begin{cases} \frac{p}{2}|x|^{(p/2)-2}x & \textit{if } |x| \le k \\ \frac{p}{2}k^{(p/2)-1}|x|^{-1}x & \textit{if } |x| > k \end{cases},$$

$$f''_{k,p}(x) = \begin{cases} \frac{p}{2}\frac{p-2}{2}|x|^{(p/2)-2} & \textit{if } |x| \le k \\ 0 & \textit{if } |x| > k \end{cases},$$

$$f'_{k,p},\, g'_{k,p} \in \mathbf{L}_\infty(\mathbf{R}),$$

$$w^2_{k,p,u} \le uv_{k,p,u} \le \frac{p}{2}w^2_{k,p,u},$$

$$\partial_i uv_{k,p,u} = w_{k,p,u}\,\partial_i w_{k,p,u} \quad a.e.,$$

$$|u\,\partial_i v_{k,p,u}| \le (p-1)w_{k,p,u}|\partial_i w_{k,p,u}| \quad a.e.$$

Proof. We obtain the derivatives of $f_{k,p}$ by simple differentiation and see that $f'_{k,p}, g'_{k,p} \in \mathbf{L}_\infty(\mathbf{R})$. Now we fix k, p, and u and write $w = w_{k,p,u}$ and $v = v_{k,p,u}$ for short. Then we have $w^2 = |u|^p$ and $uv = \frac{p}{2}|u|^2$ if $|u(x)| \le k$, while $\frac{w^2}{uv} = 1 - \frac{p-2}{p}\frac{k}{|u|} \in [2/p, 1]$ if $|u(x)| \ge k$. This leads to

$$w^2 \le uv \le \frac{p}{2}w^2.$$

The next identity is shown using $\partial_i uv = \partial_i uwf'_{k,p}(u) = w\partial_i w$ a.e. by Gilbarg and Trudinger [10, Theorem 7.8]. Again by [10], we have for $|u(x)| \le k$,

$$u\,\partial_i v = u\,\partial_i u \frac{p}{2}|u|^{p-2}\left(\frac{p}{2} + \frac{p-2}{2}\right) = (p-1)w\partial_i w \quad \text{a.e.}$$

and

$$u\,\partial_i v = u\,\partial_i u\left(\frac{p}{2}\right)^2 k^{p-2} = \frac{p}{2}k^{p/2-1}|u|\,\partial_i w \quad \text{a.e.}$$

if $|u(x)| \ge k$. As $k^{p/2-1}|u| = k^{p/2} + k^{p/2-1}(|u| - k) \le w$ and $\frac{p}{2} \le p - 1$, the last inequality of the Lemma is proved. ▮

In Section 5 we need to control δ. Hence we provide its value in the following theorem.

THEOREM 3.3. *Let* $k \ge 1, \eta, \gamma \in [0,1)$, $2 \le p < \infty$ *and* $u \in \mathbf{W}^1_{\infty,c}$. *Assume that Condition* 1.1 *is satisfied. Then*

$$\mathscr{E}(u, v_{k,p,u}) + \delta(p,\eta,\gamma)\|w_{k,p,u}\|^2_{\mathbf{L}_2} \ge \gamma\mathscr{E}'(u, v_{k,p,u}) + \nu\eta(1-\gamma)\|\nabla w_{k,p,u}\|^2_{\mathbf{L}_2} \tag{6}$$

with

$$\delta(p,\eta,\gamma) = \frac{3(1-\gamma)(1-\eta)\nu}{8} + \frac{2(p-1)^2}{(1-\gamma)(1-\eta)\nu} \times \sum_{\substack{j=1\\ l=1,2}}^{n} [b_{l,j}]^2_{(\nu(1-\eta)(1-\gamma))/(4\sqrt{n}C_1(p-1)),2q} + \frac{p}{2}[c_-]_{(\nu(1-\eta)(1-\gamma))/(3C_2^2 p),q}, \tag{7}$$

$$\mathscr{E}'(u,v) = \sum_{i,j=1}^{n} \int_\Omega a_{i,j}(x)\,\partial_i u(x)\,\partial_j v(x)\,dx,$$

and C_1 (*resp.* C_2) *the Sobolev constant for* n *and* q (*resp.* $2q$). *Let* $\delta(p) = \delta(p, 1/2, 0)$ *for short.*

Proof. Let $E_k = \{x \in \Omega;\ |u(x)| \le k\}$, $F_k = \Omega \setminus E_k$. Fix the constants k and p and the functions $u, w = w_{k,p,u}$ and $v = v_{k,p,u}$ in the rest of the proof. Note that $w, v \in \mathbf{W}^1_{\infty,c}$ (cf. Gilbarg and Trudinger [10, Theorem 7.8]). This is the reason why we do not use $\mathscr{D}$ to define the form $\mathscr{E}$, although the closure would be the same.

On E_k, we have

$$\partial_j w \partial_l w = \frac{p}{2(p-1)} \partial_j u \partial_l v \quad \text{a.e.}$$

Applying this and Lemma 3.2 to (4), we obtain a.e.

$$\begin{aligned}
&(1-\gamma)\nu|\nabla w|^2 \\
&\quad \leq (1-\gamma) \sum_{i,j=1}^{n} a_{i,j} \partial_i w \partial_j w \leq \frac{(1-\gamma)p}{2(p-1)} \sum_{i,j=1}^{n} a_{i,j} \partial_i u \partial_j v \\
&\quad \leq -\gamma a'(\cdot, u, v) + a(\cdot, u, v) + \sum_{j=1}^{n} \left(|b_{1,j} \partial_j uv| + |b_{2,j} u \partial_j v| \right) + c_{-}|uv| \\
&\quad \leq -\gamma a'(\cdot, u, v) + a(\cdot, u, v) + \sum_{j=1}^{n} \left(|b_{1,j} w \partial_j w| + (p-1)|b_{2,j} w \partial_j w| \right) \\
&\qquad + \frac{p}{2} c_{-} w^2,
\end{aligned}$$

where $a'(x, u, v) = \sum_{i,j=1}^{n} a_{i,j}(x) \partial_i u(x) \partial_j v(x)$ and $a(\cdot, u, v)$ is the integrand of (1).

On F_k, we have $\partial_j w \partial_l w = \partial_j u \partial_l v$ a.e. Applying this and Lemma 3.2 to (4), we obtain a.e.

$$\begin{aligned}
(1-\gamma)\nu|\nabla w|^2 &\leq (1-\gamma) \sum_{i,j=1}^{n} a_{i,j} \partial_i w \partial_j w \\
&\leq (1-\gamma) \sum_{i,j=1}^{n} a_{i,j} \partial_i u \partial_j v \\
&\leq -\gamma a'(\cdot, u, v) + a(\cdot, u, v) \\
&\qquad + \sum_{j=1}^{n} \left(|b_{1,j} \partial_j uv| + |b_{2,j} u \partial_j v| \right) + c_{-}|uv| \\
&\leq -\gamma a'(\cdot, u, v) + a(\cdot, u, v) \\
&\qquad + \sum_{j=1}^{n} \left(|b_{1,j} w \partial_j w| + \frac{p}{2} |b_{2,j} w \partial_j w| \right) + \frac{p}{2} c_{-} w^2.
\end{aligned}$$

As $2 \leq p$ and $E_k \cup F_k = \Omega$, the inequality for E_k holds on Ω. Integration over Ω gives

$$\begin{aligned}
(1-\gamma)\nu\|\nabla w\|_{\mathbf{L}_2}^2 \\
&\leq -\gamma \mathscr{E}'(u,v) + \mathscr{E}(u,v) + (p-1)\sum_{\substack{j=1\\ l=1,2}}^{n} \|b_{l,j} w \partial_j w\|_{\mathbf{L}_1} + \frac{p}{2}\|c_- w^2\|_{\mathbf{L}_1} \\
&\leq -\gamma \mathscr{E}'(u,v) + \mathscr{E}(u,v) \\
&\quad + \frac{p}{2}\left(3\epsilon_2 \|\nabla w\|_{\mathbf{L}_2}^2 + \frac{3\epsilon_2}{2}\|w\|_{\mathbf{L}_2}^2 + [c_-]_{\epsilon_2/C_2^2}\|w\|_2^2\right) \\
&\quad + (p-1)2\epsilon_1 \|\nabla w\|_{\mathbf{L}_2}^2 + \frac{p-1}{2\epsilon_1}\|w\|_{\mathbf{L}_2}^2\left(\epsilon_1^2 + \sum_{\substack{j=1\\ l=1,2}}^{n} [b_{l,j}]^2_{\epsilon_1/(\sqrt{n}C_1)}\right),
\end{aligned}$$

where we used Lemma 2.5, 2.6, and 2.3. By choosing $\epsilon_1 = \frac{\nu(1-\eta)(1-\gamma)}{4(p-1)}$ and $\epsilon_2 = \frac{\nu(1-\eta)(1-\gamma)}{3p}$, the Theorem is proved. ∎

We deduce from this the following corollary.

COROLLARY 3.4. *The form* $\mathscr{E}_{\delta(2,\eta,\gamma)}$ *is positive for all* $\eta, \gamma \in [0,1)$ *if Condition* 1.1 *is satisfied.*

Proof. Choose $p = 2$ in (6) and use $u = v_{k,2,u} = w_{k,2,u} \operatorname{sgn} u$ to see the positivity. ∎

LEMMA 3.5. *If Condition* 1.1 *is satisfied, then the form* $\mathscr{E}_{\delta(2,\eta,\gamma)}$ *is continuous with respect to its graph norm* $\|u\|^2_{\mathscr{E}_{\delta(2,\eta,\gamma)}} = (1+\delta(2,\eta,\gamma))\|u\|_2^2 + \mathscr{E}(u,u)$ *for all* $\eta \in [0,1)$ *and* $\gamma \in (0,1)$.

Proof. By Lemma 3.3 and Eq. (4) we know that $0 \leq \gamma\nu\|\nabla u\|_2^2 \leq \gamma\mathscr{E}'(u,u) \leq \mathscr{E}(u,u) + \delta(2,\eta,\gamma)\|u\|_2^2$. Fix $\eta, \gamma \in [0,1)$ and let $\delta = \delta(2,\eta,\gamma)$ in the following.

Let $\check{\mathscr{E}}(u,v) = \frac{1}{2}(\mathscr{E}(u,v) - \mathscr{E}(v,u))$ and $\check{\mathscr{E}}'(u,v) = \frac{1}{2}(\mathscr{E}'(u,v) - \mathscr{E}'(v,u))$ be the antisymmetric parts of $\mathscr{E}$ (resp. $\mathscr{E}'$). Choose $M > 0$ such that $|a_{i,j} - a_{j,i}| \leq M\nu^2$ for all $i, j = 1, \ldots, n$. Then the Hölder inequality and (4) lead to

$$\begin{aligned}
4|\check{\mathscr{E}}'(u,v)|^2 &\leq \left(\sum_{i,j=1}^{n}\int_\Omega M\nu^2|\partial_i u\, \partial_i v|\right)^2 \leq M^2\left(\int_\Omega \nu^2|\nabla u|\,|\nabla v|\right)^2 \\
&\leq M^2\|\nu\nabla u\|_2^2\,\|\nu\nabla v\|_2^2 \leq M^2\mathscr{E}'(u,u)\mathscr{E}'(v,v) \\
&\leq M^2\gamma^{-2}\mathscr{E}_{\delta+1}(u,u)\mathscr{E}_{\delta+1}(v,v)
\end{aligned}$$

for any $u, v \in \mathbf{W}^1_{\infty, c}$. Now we can estimate, by Lemma 2.5,

$$\begin{aligned}
2|\check{\mathscr{E}}(u,v) - \check{\mathscr{E}}'(u,v)| \\
&= \left| \sum_{j=1}^{n} (b_{1,j} - b_{2,l})(v\partial_j u - u\partial_j v) \right| \\
&\leq \sum_{j=1}^{n} \left(\|(b_{1,j} - b_{2,j})v\partial_j u\|_1 + \|(b_{1,j} - b_{2,j})u\partial_j v\|_1 \right) \\
&\leq \epsilon(\|\nabla u\|_2 + \|u\|_2)\|\nabla v\|_2 \\
&\quad + \left(\sum_{j=1}^{n} [b_{1,j} - b_{2,j}]_{\epsilon/(\sqrt{n}C_1), 2q} \right)^{1/2} \|u\|_2 \|\nabla v\|_2 \\
&\quad + \epsilon(\|\nabla v\|_2 + \|v\|_2)\|\nabla u\|_2 \\
&\quad + \left(\sum_{j=1}^{n} [b_{1,j} - b_{2,j}]_{\epsilon/(\sqrt{n}C_1), 2q} \right)^{1/2} \|v\|_2 \|\nabla u\|_2 .
\end{aligned}$$

As $\gamma\nu\|\nabla u\|_2^2 \leq \gamma\mathscr{E}'(u,u) \leq \mathscr{E}_{\delta+1}(u,u)$ and $\gamma\|u\|_2^2 \leq \gamma\mathscr{E}'_1(u,u) \leq \mathscr{E}_{\delta+1}(u,u)$ by (4) and Lemma 3.3, we can continue the estimate by

$$\begin{aligned}
2|\check{\mathscr{E}}(u,v) - \check{\mathscr{E}}'(u,v)| \\
&\leq 2\mathscr{E}_{\delta+1}(u,u)^{1/2}\mathscr{E}_{\delta+1}(v,v)^{1/2} \\
&\qquad \cdot \gamma^{-1}\left(\frac{\epsilon}{\nu} + \frac{\epsilon}{\sqrt{\nu}} + \frac{1}{\sqrt{\nu}} \left(\sum_{j=1}^{n} [b_{1,j} - b_{2,j}]_{\epsilon/(\sqrt{n}C_1), 2q} \right)^{1/2} \right).
\end{aligned}$$

Using these inequalities, we obtain

$$\begin{aligned}
|\check{\mathscr{E}}(u,v)| \\
&\leq |\check{\mathscr{E}}'(u,v)| + |\check{\mathscr{E}}(u,v) - \check{\mathscr{E}}'(u,v)| \\
&\leq \mathscr{E}_{\delta+1}(u,u)^{1/2}\mathscr{E}_{\delta+1}(v,v)^{1/2} \\
&\qquad \cdot \gamma^{-1}\left(\frac{\epsilon}{\nu} + \frac{\epsilon}{\sqrt{\nu}} + \frac{1}{\sqrt{\nu}} \left(\sum_{j=1}^{n} [b_{1,j} - b_{2,j}]_{\epsilon/(\sqrt{n}C_1), 2q} \right)^{1/2} + M/2 \right).
\end{aligned}$$

Together with the Cauchy–Schwarz inequality for the symmetric part of $\mathscr{E}_\delta$, this implies

$$|\mathscr{E}_\delta(u,v)| \leq K\mathscr{E}_{\delta+1}(u,u)^{1/2}\mathscr{E}_{\delta+1}(v,v)^{1/2}$$

with

$$K = 1 + \delta + \gamma^{-1}\left(\frac{\epsilon}{\nu} + \frac{\epsilon}{\sqrt{\nu}} + \frac{1}{\sqrt{\nu}}\left(\sum_{j=1}^{n}\left[b_{1,j} - b_{2,j}\right]_{\epsilon/(\sqrt{n}C_1),2q}\right)^{1/2} + \frac{M}{2}\right)$$

for all $\epsilon > 0$. This completes the proof. ∎

LEMMA 3.6. *If Condition 1.1 is satisfied, then the form $\mathscr{E}_{\delta(2,\eta,\gamma)}$ is closable for all $\eta \in [0,1)$ and $\gamma \in (0,1)$. The closure is a coercive form. The forms $\mathscr{E}_{\delta(2,\eta,\gamma)}$ and $\mathscr{E}'$ define equivalent norms.*

Proof. Let $\hat{\mathscr{E}}'(u,v) = \frac{1}{2}(\mathscr{E}'(u,v) + \mathscr{E}'(v,u))$ be the symmetric part of $\mathscr{E}'$. Fix $\eta, \gamma \in [0,1)$ and let $\hat{\delta} = \delta(2,\eta,\gamma)$ in the following. Using Lemmas 2.5 and 2.6, we see that

$$\begin{aligned}
&|\mathscr{E}(u,u)| \\
&\quad\le |\hat{\mathscr{E}}'(u,u)| + \sum_{j=1}^{n}\int_\Omega |b_{1,j} + b_{2,j}|\,|u\,\partial_j u| + \int_\Omega |c|u^2 \\
&\quad\le \hat{\mathscr{E}}'(u,u) + 2\epsilon\|\nabla u\|_2^2 + \frac{1}{2\epsilon}\left(\epsilon^2 + \sum_{j=1}^{n}\left[b_{1,j} + b_{2,j}\right]_{\epsilon/(\sqrt{n}C_1),2q}\right)\|u\|_2^2 \\
&\qquad + 3\epsilon\|\nabla u\|_2^2 + \left(\epsilon + \frac{1}{2\epsilon} + [c]_{\epsilon/C_2^2,q}\right)\|u\|_2^2 \\
&\quad\le \mathscr{E}_1'(u,u)\left(1 + \frac{5\epsilon}{\nu} + \frac{3\epsilon}{2} + \frac{1}{2\epsilon}\right. \\
&\qquad\qquad \left. + \frac{1}{2\epsilon}\sum_{j=1}^{n}\left[b_{1,j} + b_{2,j}\right]_{\epsilon/(\sqrt{n}C_1),2q} + [c]_{\epsilon/C_2^2,q}\right),
\end{aligned}$$

by (4) and the inequality $\mathscr{E}(u,u) + \delta\|u\|_2^2 \ge \gamma\hat{\mathscr{E}}'(u,u) \ge \gamma\nu\|\nabla u\|_2^2 \ge 0$ derived from Theorem 3.3. Using these inequalities, we obtain

$$c^{-1}\mathscr{E}_1'(u,u) \le \mathscr{E}_{\hat{\delta}+\gamma}(u,u) \le \mathscr{E}_{\hat{\delta}+1}(u,u) \le c\mathscr{E}_1'(u,u),$$

with

$$\begin{aligned}
c = \gamma^{-1} + \hat{\delta} + 2 + \frac{5\epsilon}{\nu} + \frac{3\epsilon}{2} + \frac{1}{2\epsilon} \\
+ \frac{1}{2\epsilon}\sum_{j=1}^{n}\left[b_{1,j} + b_{2,j}\right]_{\epsilon/(\sqrt{n}C_1),2q} + [c]_{\epsilon/C_2^2,q}.
\end{aligned}$$

Thus the associated norms are equivalent. By Ma and Röckner [14, I.3, Proposition 3.5] the form $\mathscr{E}_{\hat{\delta}+\gamma}$ is closable since $\mathscr{E}'$ is closable (see [14, II.2d, Case 1]).

Since $\mathscr{E}_\delta$ is continuous, we have $|\mathscr{E}_\delta(u,v)|^2 \le K^2\mathscr{E}_{\delta+1}(u,u)\mathscr{E}_{\delta+1}(v,v)$ for some $K > 0$. Thus

$$\begin{aligned}|\mathscr{E}_{\delta+1}(u,v)|^2 &\le (K+1)^2\hat{\mathscr{E}}_{\hat{\delta}+1}(u,u)\hat{\mathscr{E}}_{\hat{\delta}+1}(v,v)\\ &\le (K+1)^2c^2\mathscr{E}'_1(u,u)\mathscr{E}'_1(v,v).\end{aligned}$$

Together with Ma and Röckner [14, I.3, Proposition 3.5], this shows that the closure of $\mathscr{E}_\delta$ is a closed coercive form. ▮

DEFINITION 3.7. If Condition 1.1 is satisfied, then $\mathscr{E}_{\delta(2,\eta,\gamma)}$ has a unique, positive, closed, coercive, continuous extension to the real Hilbert space

$$V = \overline{\mathbf{W}^1_{\infty,c}} \text{ with respect to } \langle u,v\rangle_V = \langle u,v\rangle + \frac{1}{2}(\mathscr{E}'(u,v) + \mathscr{E}'(v,u))$$

for all $\eta \in [0,1)$, $\gamma \in (0,1)$. This extension is the well-known Friedrichs extension and is denoted by $\mathscr{E}_{\delta(2,\nu,\gamma)}$. Using this extension, we can extend $\mathscr{E}$ to V by setting $\mathscr{E}(u,v) = \mathscr{E}_{\delta(2,\nu,\gamma)}(u,v) - \delta(2,\nu,\gamma)\langle u,v\rangle$.

It is well known that in this situation $\mathscr{E}$ is associated with an unique operator A with $D(A) = \{u \in V;\ \exists w \in \mathbf{L}_2 \forall v \in V : \langle w,v\rangle = \mathscr{E}(u,v)\}$ and $Au = w$ for all $u \in D(A)$, where $\langle w,v\rangle = \mathscr{E}(u,v)$ for all $v \in V$. The operator $-A$ is the generator of a C_0 semigroup T on $\mathbf{L}_2$.

LEMMA 3.8. *Suppose that $f \in \mathscr{C}(\mathbf{R})$ is piecewise in $\mathscr{C}^1$ with $f' \in \mathbf{L}_\infty(\mathbf{R})$ and $f(0) = 0$. The substitution $u \mapsto f \circ u$ is continuous on V and on $\mathbf{L}_2$.*

Proof. First, we must show that the range of the substitution is in V. Let $u \in V$ and choose $u_n \in \mathbf{W}^1_{\infty,c}$ such that $u_n \to u$ in V. Then, by Eq. (4),

$$\begin{aligned}0 \le \sup_{n\in\mathbf{N}} \mathscr{E}'(f\circ u_n, f\circ u_n) &= \sup_{n\in\mathbf{N}} \int_\Omega \sum_{i,j=1}^n a_{i,j}\partial_i u_n(x)\partial_j u_n(x)(f'\circ u_n)^2\\ &\le \sup_{n\in\mathbf{N}} \|f'\|_\infty^2 \mathscr{E}'(u_n,u_n) < \infty,\end{aligned}$$

since $\mathscr{E}'(u_n,u_n) \to \mathscr{E}'(u,u)$ for $n \to \infty$ and $\partial_i f\circ u_n = f'\circ u_n \partial_i u_n$ (see Gilbarg and Trudinger [10, Theorem 7.8]). By the Banach–Alaoglu theorem, $f\circ u_{n_k} \to v$ weakly in V for some subsequence u_{n_k} of u_n and some $v \in V$. Since $f\circ u_n \to f\circ u$ in $\mathbf{L}_2$, we obtain $f\circ u = v \in V$.

Second, let $u_n \in V$ such that $u_n \to u \in V$. Then, again by Eq. (4) and Gilbarg and Trudinger [10, Theorem 7.8], we have

$$0 \le \mathscr{E}'(f \circ u_n, f \circ u_n) = \int_\Omega \sum_{i,j=1}^{n} a_{i,j} \partial_i u_n(x) \partial_j u_n(x) (f' \circ u_n)^2$$
$$\le \|f'\|_\infty^2 \mathscr{E}'(u_n, u_n) \to \|f'\|_\infty^2 \mathscr{E}'(u, u)$$

for $n \to \infty$. Thus there exists a subsequence u_{n_k} and a function $g \in \mathbf{L}_1$ such that

$$0 \le \left(f' \circ u_{n_k}\right)^2 \sum_{i,j=1}^{n} a_{i,j} \partial_i u_{n_k}(x) \partial_j u_{n_k}(x) \le g$$

a.e. By the dominated convergence theorem, $\mathscr{E}'(f \circ u_{n_k}, f \circ u_{n_k})$ converges to $\mathscr{E}'(f \circ u, f \circ u)$ for $n \to \infty$. This implies continuity, since there is such a dominated subsequence for all subsequences of u_n.

Third, $f \circ u \in \mathbf{L}_2$ whenever $u \in \mathbf{L}_2$ as $|f \circ u| \le \|f'\|_\infty |u|$ a.e. Finally, $|f \circ u - f \circ v| \le \|f'\|_\infty |u - v|$ a.e. for all $u, v \in \mathbf{L}_2$. Thus we have $\|f \circ u - f \circ v\|_2 \le \|f'\|_\infty \|u - v\|_2$. ∎

COROLLARY 3.9. *If Condition* 1.1 *is satisfied, then the semigroup T associated with $\mathscr{E}$ is positive. The mapping $u \mapsto |u|$ is continuous on V.*

Proof. Fix $\eta, \gamma \in [0, 1)$ and $\delta = \delta(2, \eta, \gamma)$ in the proof. Lemma 3.8 implies that V is a lattice and $u \mapsto |u|$ is continuous on V, since $x \mapsto |x|$ is smooth on $\mathbf{R} \setminus \{0\}$ with essentially bounded derivative. Thus we can apply the Beurling–Deny criteria for nonsymmetric forms (see Ouhabaz [17, Theorem 2.4]). Since $\partial_i u_+ = \partial_i u \mathbf{1}_{\{x \in \Omega;\ u(x) > 0\}}$, we have $\mathscr{E}_\delta(u_+, u_-) = 0$ for any $u \in V$, and we obtain the positivity of T immediately. ∎

4. UPPER BOUNDS

From the previous section, we know that there are unique solutions of (CP). In this section we derive upper bounds of these solutions. To do this, we need a simple generalization of Theorem 3.3.

Remark 4.1. Since $f_{k,p}$ and $g_{k,p}$ are piecewise smooth with bounded derivative, (see Lemma 3.2), $w_{k,p,u}, v_{k,p,u} \in V$ for all $u \in V$, by Lemma 3.8.

Obviously Theorem 3.3 holds for all $u \in V$. Since all terms of (6) are defined and the first part of the proof is a pointwise calculation, while the second part only uses that $w_{k,p,u} \in \mathbf{H}^1 \supset V$, the proof carries over to this more general situation.

LEMMA 4.2. *If Condition* 1.1 *is satisfied and u is an orbit with initial value* $u(0) \in D(A)$, *then*

$$d_t \|w_{k,p,u}(t)\|_2^2 \leq -\nu \|\nabla w_{k,p,u}(t)\|_2^2 + 2\delta(p,\eta,\gamma)\|w_{k,p,u}(t)\|_2^2$$

for all $t > 0$, $\eta, \gamma \in [0,1)$.

Proof. Since $u(0) \in D(A)$, we have $u \in \mathscr{C}(\mathbf{R}^+, D(A)) \cap \mathscr{C}^1(\mathbf{R}^+, \mathbf{L}_2)$. Thus the mapping $t \mapsto f_{k,p} \circ u(t)$: $\mathbf{R}^+ \to \mathbf{L}_2$ has the derivative $f'_{k,p} \circ u(t)\partial_t u(t)$ a.e. by Gilbarg and Trudinger [10, Theorem 7.8]. Now we see that

$$\begin{aligned} d_t \|w_{k,p,u}(t)\|_2^2 &= 2\left\langle f'_{k,p} \circ u(t)\, \partial_t u(t), w_{k,p,u}(t)\right\rangle = 2\left\langle \partial_t u(t), v_{k,p,u}(t)\right\rangle \\ &= -2\left\langle Au(t), v_{k,p,u}(t)\right\rangle = -2\mathscr{E}\big(u(t), v_{k,p,u}(t)\big) \\ &\leq -\nu \|\nabla w_{k,p,u}(t)\|_2^2 + 2\delta(p,\eta,\gamma)\|w_{k,p,u}(t)\|_2^2 \end{aligned}$$

as in Remark 4.1. ∎

Proof (Proof of Theorem 1.2). By Corollary 3.9, we have the positivity of the semigroup. Thus we need only show the quasi-contractivity on $\mathbf{L}_p$.

If u is an orbit with initial value $u(0) \in D(A)$, then by Lemma 4.2, we have

$$\begin{aligned} d_t \|w_{k,p,u}(t)\|_2^2 &\leq -\nu \|\nabla w_{k,p,u}(t)\|_2^2 + 2\delta(p)\|w_{k,p,u}(t)\|_2^2 \\ &\leq 2\delta(p)\|w_{k,p,u}(t)\|_2^2. \end{aligned}$$

From this differential inequality, it follows that

$$\|w_{k,p,u}(t)\|_2^2 \leq \|w_{k,p,u}(0)\|_2^2 e^{2\delta(p)t}.$$

Since $D(A)$ is dense in $\mathbf{L}_2$, we can choose any initial value in $\mathbf{L}_2(\Omega) \cap \mathbf{L}_p(\Omega)$ by approximation; see Lemma 3.8. Since $w_{k,p,u}(t,x) \nearrow |u(t,x)|^{p/2}$ as $k \to \infty$, we have

$$\|\, |u(t)|^{p/2}\|_2^2 \leq \|\, |u(0)|^{p/2}\|_2^2 e^{2\delta(p)t}$$

and

$$\|u(t)\|_p \leq \|u(0)\|_p e^{2\delta(p)t/p}.$$

The proposition is finished by a limiting argument. ∎

The subsequent $\mathbf{L}_p$ estimates of semigroup kernels are based on an inequality first derived by Nash. See [16] for a discussion of parabolic partial differential equations on $\mathbf{L}_2(\mathbf{R}^n)$.

PROPOSITION 4.3 (Nash inequality). *There exists a constant* $M_1 > 0$ *such that*

$$\|u\|_2^{2+4/n} \le M_1 \|\nabla u\|_2^2 \|u\|_1^{4/n}$$

for all $u \in \mathbf{H}_0^1 \cap \mathbf{L}_1$. *We call* $M_1 = \frac{4\pi n}{n+2}\left(\frac{\Gamma(1+n/2)}{1+n/2}\right)^{2/n}$ *the Nash constant.*

Proof. Extend the function u by 0 to $\mathbf{R}^n$ and use Nash's result [16, p. 936]; or see Davies [8, Theorem 2.4.6]. ∎

LEMMA 4.4. *If Condition* 1.1 *is satisfied and u is an orbit with initial value* $u(0) \in D(A)$, *then we have*

$$\begin{aligned}&\left(\|w_{k,2p,u}(t)\|_2^{1/p} e^{-(\delta(2p)/p)t}\right)^{-4p/n}\\ &\quad \ge \frac{2\nu}{M_1 n}\int_0^t \left(\|w_{k,p,u}(s)\|_2^{2/p} e^{-(\delta(2p)/p)s}\right)^{-4p/n} ds\end{aligned}$$

for all $t \in J := \{t \in \mathbf{R}^+;\ \inf_{0 \le s \le t}\|u(s)\|_2 > 0\}$, $p \ge 2$, $k > 1$ *and* M_1 *the Nash constant.*

Proof. By Lemma 4.2 and Proposition 4.3, we have

$$\begin{aligned}&d_t\|w_{k,2p,u}(t)\|_2^2\\ &\quad \le -\nu\|\nabla w_{k,2p,u}(t)\|_2^2 + 2\delta(2p)\|w_{k,2p,u}(t)\|_2^2\\ &\quad \le -\frac{\nu}{M_1}\|w_{k,2p,u}(t)\|_2^{2+4/n}\|w_{k,2p,u}(t)\|_1^{-4/n} + 2\delta(2p)\|w_{k,2p,u}(t)\|_2^2\\ &\quad \le -\frac{\nu}{M_1}\|w_{k,2p,u}(t)\|_2^{2+4/n}\|w_{k,p,u}(t)\|_2^{-8/n} + 2\delta(2p)\|w_{k,2p,u}(t)\|_2^2.\end{aligned}$$

The last inequality holds by the definition of $w_{k,p,u}$.

For positive functions f with $f' \le g + cf$ on $\mathbf{R}^+$, we have

$$f^{-M}(t)e^{cMt} \ge -M\int_0^t f^{-M-1}(s)g(s)e^{cMs}\,ds$$

for all $M > 0$. Choosing $M = 2/n$, we obtain

$$\|w_{k,2p,u}(t)\|_2^{-4/n}e^{4(\delta(2p)/n)t} \ge \frac{2\nu}{M_1 n}\int_0^t \|w_{k,p,u}(s)\|_2^{-8/n}e^{4(\delta(2p)/n)s}\,ds,$$

which implies the claim. ∎

LEMMA 4.5. *If Condition* 1.3 *is satisfied and* $r \geq \max\{2 + \frac{4q}{p_1 - 2q}, 1 + \frac{q}{p_0 - q}\}$, *then there exists* $\delta_0 > 0$ *such that* $\delta(p) \leq f(p)\delta_0$ *for all* $p \geq 2$ *with* $f(p) = (p-1)^r$. *Further, we have* $\sum_{j=1}^{\infty} 2^{-j} \log f(2^{j+1}) \leq 3r \log 2$.

Proof. Choose $\hat{b}_{l,j}$, $\hat{c} \in \mathbf{L}_\infty$, $\check{b}_{l,j} \in \mathbf{L}_{p_1}$, and $\check{c} \in \mathbf{L}_{p_0}$ such that $c_- = \check{c} + \hat{c}$ and $b_{l,j} = \hat{b}_{l,j} + \hat{b}_{l,j}$. By direct estimate using Theorem 3.3 and Lemma 2.3(iv), we obtain

$$\delta_0 = \frac{3}{16}\nu + \frac{4}{\nu}\sum_{\substack{j=1\\ l=1,2}}^{n}\left(\|\hat{b}_{l,j}\|_\infty + \|\check{b}_{l,j}\|_{p_1}^{p_1/(p_1-2q)}\left(\frac{8\sqrt{n}\,C_1}{\nu}\right)^{2q/(p_1-2q)}\right)^2$$

$$+ \frac{6}{\nu} + \|\hat{c}\|_\infty + \|\check{c}\|_{p_0}^{p_0/(p_0-q)}\left(\frac{6C_2^2}{\nu}\right)^{q/(p_0-q)},$$

which satisfies the inequality.

The last inequality of this lemma is quite easy, since

$$\sum_{j=1}^{\infty} 2^{-j}\log f(2^{j+1}) = \sum_{j=1}^{\infty} 2^{-j}\log(2^{j+1})^r \leq r\sum_{j=1}^{\infty} 2^{-j}(j+1)\log 2$$

$$= r\log 2\sum_{j=1}^{\infty} 2^{-j}(j+1) = 3r\log 2,$$

using the derivative of the geometric series at $1/2$. ∎

LEMMA 4.6. *Let* $k > 1$, $l \in \mathbf{N}$, $p = 2^l$ *and let* u *be an orbit. Then there exists* $C, \gamma > 0$ *such that*

$$\|w_{k,p,u}(t)\|_2^{2/p} \leq 2^{-n/4}\sqrt{C}\,t^{-(n/2)(\frac{1}{2}-\frac{1}{p})}e^{\beta t}\|u(0)\|_2 \tag{8}$$

holds for all $t \in (0,\infty)$ *if Condition* 1.3 *is satisfied.*

Remark 4.7. Taking δ_0, f, r as in Lemma 4.5 and choosing $\epsilon \in (0,1)$, the constants in Lemma 4.6 are given by

$$\beta = \epsilon\delta_0\delta(2) \qquad \text{and} \qquad C = \left(\frac{M_1 n 12^r}{\nu\epsilon(3^r - \epsilon)}\right)^{n/2}.$$

Proof. Let u be an orbit with initial value $u(0) \in D(A)$, $\epsilon \in (0,1)$, the function $h_2(t) = e^{\delta(2)t}$, and

$$h_{2p}(t) = h_p(t) e^{\delta(2p)\epsilon/(pf(2p))t} \left(\frac{M_1 n}{2\nu 3^r \epsilon (3^r - \epsilon)} \right)^{n/(4p)} (f(2p) - \epsilon)^{n/(4p)}.$$

Then $h_p(t) \le 2^{-n/4} \sqrt{C} e^{\beta t}$ holds, by Lemma 4.5, and h_p is nondecreasing. To prove the claim, we do an induction over $l = \log_2 p$. We see that $\|w_{k,2}(s)\|_2 = \|u(t)\|_2 \le h_2(t)\|u(0)\|_2$. By Lemma 4.4 and the uniqueness of the zero solution, we obtain

$$\begin{aligned}
&\|w_{k,2p,u}(t)\|_2^{1/p} \\
&\le e^{(\delta(2p)/p)t} \left(\frac{M_1 n}{2\nu} \right)^{n/(4p)} \left(\int_0^t \left(\|w_{k,p,u}(s)\|_2^{2/p} e^{-(\delta(2p)/p)s} \right)^{-(4p/n)} ds \right)^{-n/(4p)} \\
&\le e^{(\delta(2p)/p)t} \left(\frac{M_1 n}{2\nu} \right)^{n/(4p)} \\
&\quad \times \left(\int_0^t \left(s^{-(n/2)\frac{1}{2} - \frac{1}{p}} h_p(s) e^{-(\delta(2p)/p)s} \right)^{-(4p/n)} ds \right)^{-n/(4p)} \|u(0)\|_2 \\
&\le e^{(\delta(2p)/p)t} \left(\frac{M_1 n}{2\nu} \right)^{n/(4p)} h_p(t) \left(\int_{t(1-\epsilon/f(2p))}^t s^{p-2}\, ds \right)^{-n/(4p)} \\
&\quad \times e^{-(\delta(2p)/p)t(1-\epsilon/f(2p))} \|u(0)\|_2 \\
&\le e^{(\delta(2p)\epsilon/(pf(2p)))t} \left(\frac{M_1 n}{2\nu} \right)^{n/(4p)} h_p(t) \\
&\quad \times \left(\frac{1}{p-1} t^{p-1} \left(1 - \left(1 - \frac{\epsilon}{f(2p)} \right)^{p-1} \right) \right)^{-n/(4p)} \|u(0)\|_2 \\
&\le h_{2p}(t) t^{-(n/2)(\frac{1}{2} - \frac{1}{2p})} \|u(0)\|_2,
\end{aligned}$$

since $1 - (1 - \epsilon/f(2p))^{p-1} \ge \frac{(f(4)-\epsilon)(p-1)}{f(2p)-\epsilon}(\epsilon/f(4))$, which can be seen by the following elementary calculus. First, $g(x) = \frac{e^x - 1}{x}$ is increasing on $\mathbf{R}$. Second, the function $\gamma(p) = -(p-1)\log(1 - \epsilon/f(2p)) = \epsilon\frac{p-1}{f(2p)} \sum_{j=0}^{\infty} \frac{\epsilon^j}{(j+1)(f(2p))^j}$ is decreasing, since $f(p) = (p-1)^r$ and $r \ge 2$. Third, $\tilde{\gamma}(x) = -(x - \epsilon)\log(1 - \epsilon/x)$ is increasing for $x > \epsilon$, as its derivative is $\tilde{\gamma}'(x) = -\log(1 - \epsilon/x) - \epsilon/x \ge 0$. Finally, $\tilde{\gamma}(f(2p)) g(-\gamma(p)) = \frac{f(2p)-\epsilon}{p-1}(1 - (1 - \epsilon/f(2p))^{p-1})$ is nondecreasing.

Lemma 3.8, with regard to $f_{k,p}$, shows that we can approximate any initial value in $\mathbf{L}_2$. ∎

COROLLARY 4.8. *If Condition* 1.3 *is satisfied, then there exists* $C, \beta > 0$ *such that*

$$\|T(t)\|_{\mathscr{L}(\mathbf{L}_2,\mathbf{L}_\infty)} \leq 2^{-n/4}\sqrt{C}\,t^{-(n/4)}e^{\beta t} \quad \textit{for all } t > 0.$$

The constants are given in Remark 4.7.

Proof. Let $h(t) = 2^{-n/4}\sqrt{C}\,e^{\beta t}$. By definition, we have

$$\begin{aligned}
\|T(t)\|_{\mathscr{L}(\mathbf{L}_2,\mathbf{L}_\infty)} &= \sup_{\|u_0\|_2=1} \|T(t)u_0\|_\infty \leq \sup_{\|u_0\|_2=1} \limsup_{j\to\infty} \|T(t)u_0\|_{2^j} \\
&= \sup_{\|u_0\|_2=1} \limsup_{j\to\infty} \lim_{k\to\infty} \|f_{k,2^j} \circ T(t)u_0\|_2^{2/2^j} \\
&\leq \sup_{\|u_0\|_2=1} \limsup_{j\to\infty} \lim_{k\to\infty} t^{-(n/2)(\frac{1}{2}-2^{-j})} h(t)\|u_0\|_2 \\
&= t^{-(n/4)}h(t).
\end{aligned}$$

∎

Proof (Proof of Theorem 1.4). By duality, we obtain

$$\|T(t)\|_{\mathscr{L}(\mathbf{L}_1,\mathbf{L}_\infty)} \leq \|T(t/2)\|_{\mathscr{L}(\mathbf{L}_2,\mathbf{L}_\infty)}\|T(t/2)\|_{\mathscr{L}(\mathbf{L}_1,\mathbf{L}_2)} \leq Ct^{-n/2}e^{(\delta(2)+\gamma)t}.$$

∎

5. GAUSSIAN ESTIMATES

In this section we use a technique developed by Davies [7] to obtain pseudo-Gaussian upper bounds of the kernel of T. We assume Condition 1.5 in this section.

DEFINITION 5.1. Let $\Psi = \{\psi \in \mathscr{D}(\mathbf{R}^n);\ |\nabla\psi| \leq 1\}$. For all real numbers ρ and $\psi \in \Psi$, let

$$\begin{aligned}
(B_{\rho,\psi}f)(x) &= e^{\rho\psi(x)}f(x) \\
T_{\rho,\psi}(t) &= B_{\rho,\psi}T(t)B_{-\rho,\psi}.
\end{aligned}$$

LEMMA 5.2. *For any real ρ and $\psi \in \Psi$, the semigroup $T_{\rho,\psi}$ is associated with the form $\mathscr{E}_{\rho,\psi}$ given by*

$$\mathscr{E}_{\rho,\psi}(u,v) = \sum_{i,j=1}^{n} \int_{\Omega} a_{i,j} \partial_i u \partial_j v + \sum_{j=1}^{n} \int_{\Omega} \left(b_{1,j,\rho,\psi} \partial_j u v + b_{2,j,\rho,\psi} v \partial_j v \right)$$
$$+ \int_{\Omega} c_{\rho_1 \psi} u v,$$

where

$$b_{1,j,\rho,\psi} = b_{1,j} + \rho \sum_{k=1}^{n} a_{j,k} \partial_k \psi, \qquad b_{2,j,\rho,\psi} = b_{2,j} - \rho \sum_{k=1}^{n} a_{k,j} \partial_k \psi,$$

$$c_{\rho,\psi} = c - \rho^2 \sum_{i,j=1}^{n} a_{i,j} \partial_j \psi \partial_i \psi - \rho \sum_{j=1}^{n} \left(b_{1,j} - b_{2,j} \right) \partial_j \psi$$

for all $u, v \in V$.

Proof. For every $u \in D(A)$, $v \in \mathbf{W}^1_{\infty,c}$, we have

$$\begin{aligned} \partial_t \langle T_{\rho,\psi}(t) u, v \rangle &= \partial_t \langle T(t) e^{-\rho\psi} u, e^{\rho\psi} v \rangle = \langle AT(t) e^{-\rho\psi} u, e^{\rho\psi} v \rangle \\ &= \mathscr{E}\left(T(t) e^{-\rho\psi} u, e^{\rho\psi} v \right) = \mathscr{E}\left(e^{-\rho\psi} T_{\rho,\psi}(t) u, e^{\rho\psi} v \right) \\ &= \mathscr{E}_{\rho,\psi}\left(T_{\rho,\psi}(t) u, v \right). \end{aligned}$$

By approximation, this holds actually for all $u \in V$, from which the lemma follows by a direct calculation. ∎

THEOREM 5.3. *If $\rho \in \mathbf{R}$, $\psi \in \Psi$, $t \in (0,\infty)$, then there exists Cd, $\delta_i > 0$ such that*

$$\|T_{\rho,\psi}(t)\|_{\mathscr{L}(\mathbf{L}_1,\mathbf{L}_\infty)} \leq C t^{-n/2} \exp\left(\left(\delta_2 |\rho|^m + \delta_1 \right) t \right), \tag{9}$$

where

$$m = 2 + \frac{4q}{p_2 - 2q} \in [2, \infty). \tag{10}$$

Remark 5.4. The exact values of the constants are not needed in the following, but are given here for completeness. Choosing $\epsilon \in (0,1)$ and analyzing the proof provides

$$C = \left(\frac{M_1 n 12^r}{\nu (3^r - \epsilon) \epsilon} \right)^{n/2}, \ \delta_i = (1 + \epsilon) \hat{\delta}_i,$$

$$f(p) = (p-1)^r,\ r = \max\left\{2 + \frac{4q}{\min\{p_2, p_1\} - 2q},\ 1 + \frac{q}{p_0 - q}\right\},$$

$$\hat{\delta}_1 = \hat{\delta}_2 + \frac{12}{\nu} \sum_{\substack{j=1\\ l=1,2}}^{n} \left(\|\hat{b}_{l,j}\|_\infty + \|\check{b}_{l,j}\|_{p_1}^{p_1/(p_1-2q)} \left(\frac{16\sqrt{n}\, C_1}{\nu} \right)^{2q/(p_1-2q)} \right)^2$$

$$+ \|\hat{c}\|_\infty + \|\check{c}\|_{p_0}^{p_0/(p_0-q)} \left(\frac{18 C_2^2}{\nu} \right)^{q/(p_0-q)} + \frac{3}{16}\nu,$$

$$\hat{\delta}_2 = \frac{12}{\nu} \sum_{j=1}^{n} \left(\left(\sum_{i=1}^{n} \|\hat{a}_{i,j}\|_\infty \right)^2 + \left(\sum_{i=1}^{n} \|\check{a}_{i,j}\|_{p_2}^{p_2/(p_2-2q)} \left(\frac{32 n\sqrt{n}\, C_1}{\nu} \right)^{2q/(p_2-2q)} \right)^2 \right.$$

$$\left. + \left(\sum_{i=1}^{n} \|\hat{a}_{j,i}\|_\infty \right)^2 + \left(\sum_{i=1}^{n} \|\check{a}_{j,i}\|_{p_2}^{p_2/(p_2-2q)} \left(\frac{32 n\sqrt{n}\, C_1}{\nu} \right)^{2q/(p_2-2q)} \right)^2 \right)$$

$$+ \sum_{i,j=1}^{n} \left(\|\hat{a}_{i,j}\|_\infty + \|\check{a}_{i,j}\|_{p_2}^{p_2/(p_2-q)} \left(\frac{18 n^2 C_2^2}{\nu} \right)^{q/(p_2-q)} \right)$$

$$+ \sum_{j=1}^{n} \left(\|\hat{b}_{1,j} - \hat{b}_{2,j}\|_\infty + \|\check{b}_{1,j} - \check{b}_{2,j}\|_{p_1}^{2p_1/(p_1-q)} \left(\frac{18 n C_2^2}{\nu} \right)^{q/(2p_1-q)} \right)$$

for any choice of $\hat{a}_{i,j}, \hat{b}_{l,j}, \hat{c} \in \mathbf{L}_\infty$, $\check{a}_{i,j} \in \mathbf{L}_{p_2}$, $\check{b}_{l,j} \in \mathbf{L}_{p_1}$, and $\check{c} \in \mathbf{L}_{p_0}$ such that $p_2 > 2q$ or $p_1 = \infty$ and $a_{i,j} = \check{a}_{i,j} + \hat{a}_{i,j}$, $b_{l,j} = \check{b}_{l,j} + \hat{b}_{l,j}$ for all $i, j = 1, \dots, n$, $l = 1, 2$, and $c_- = \check{c} + \hat{c}$. If $a_{i,j} \in \mathbf{L}_\infty$ for all $i, j = 1, \dots, n$, then choose $\check{a}_{i,j} = 0$, $p_2 = \infty$, and $m = 2$.

Proof. Since the form $\mathscr{E}_{p,\psi}$ satisfies Condition 1.3, Theorem 1.4 holds with $\delta_{\rho,\psi}$ corresponding to (7). To estimate $\delta_{\rho,\psi}$, we choose functions $\hat{a}_{i,j}, \hat{b}_{l,j}, \hat{c} \in \mathbf{L}_\infty$, $\check{a}_{i,j} \in \mathbf{L}_{p_2}$, $\check{b}_{l,j} \in \mathbf{L}_{p_1}$, and $\check{c} \in \mathbf{L}_{p_0}$ such that $a_{i,j} = \check{a}_{i,j} + \hat{a}_{i,j}$, $b_{l,j} = \check{b}_{l,j} + \hat{b}_{l,j}$, and $c_- = \check{c} + \hat{c}$. If $a_{i,j} \in \mathbf{L}_\infty$, then choose $\check{a}_{i,j} = 0$, $p_2 = \infty$, and $m = 2$. With the help of these decompositions and Lemma 2.3(ii),(iv), we obtain

$$[b_{1,j,\rho,\psi}]_\theta \le [b_{1,j}]_{\theta/2} + |\rho| \sum_{k=1}^{n} \|\hat{a}_{j,k}\|_\infty$$

$$+ |\rho| \sum_{k=1}^{n} \|\check{a}_{j,k}\|_{p_2}^{p_2/(p_2-2q)} \left(\frac{2n|\rho|}{\theta} \right)^{2q/(p_2-2q)}$$

$$\leq \left[b_{1,j} \right]_{\theta/2} + |\rho| \sum_{k=1}^{n} \|\hat{a}_{j,k}\|_\infty + |\rho| \sum_{k=1}^{n} \|\check{a}_{j,k}\|_{p_2}^{p_2/(p_2-2q)} \left(\frac{2n|\rho|}{\theta} \right)^{2q/(p_2-2q)}$$

$$\left[b_{1,j,\rho,\psi} \right]_\theta^2 \leq 3\left[b_{1,j} \right]_\theta^2 + 3(1+|\rho|^m) \left(\sum_{k=1}^{n} \|\hat{a}_{j,k}\|_\infty \right)^2 + 3(1+|\rho|^m) \left(\sum_{k=1}^{n} \|\check{a}_{j,k}\|_{p_2}^{p_2/(p_2-2q)} \left(\frac{2n}{\theta} \right)^{2q/(p_2-2q)} \right)^2$$

$$\left[b_{2,j,\rho,\psi} \right]_\theta^2 \leq 3\left[b_{2,j} \right]_\theta^2 + 3(1+|\rho|^m) \left(\sum_{k=1}^{n} \|\hat{a}_{k,j}\|_\infty \right)^2 + 3(1+|\rho|^m) \left(\sum_{k=1}^{n} \|\check{a}_{k,j}\|_{p_2}^{p_2/(p_2-2q)} \left(\frac{2n}{\theta} \right)^{2q/(p_2-2q)} \right)^2$$

$$\begin{aligned}
[c_{\rho,\psi -}]_\theta &\leq [c_-]_{\theta/3} + \rho^2 \sum_{i,j=1}^{n} \|\hat{a}_{i,j}\|_\infty + \rho^2 \sum_{i,j=1}^{n} \|\check{a}_{i,j}\|_{p_2}^{p_2/(p_2-q)} \left(\frac{3n^2\rho^2}{\theta} \right)^{q/(p_2-q)} \\
&\quad + |\rho| \sum_{j=1}^{n} \|\hat{b}_{1,j} - \hat{b}_{2,j}\|_\infty + |\rho| \sum_{j=1}^{n} \|\check{b}_{1,j} - \check{b}_{2,j}\|_{p_1}^{p_1/(p_1-q)} \left(\frac{3n|\rho|}{\theta} \right)^{q/(p_1-q)} \\
&\leq [c_-]_{\theta/3} + (1+|\rho|^m) \\
&\quad \times \left(\sum_{i,j=1}^{n} \left(\|\hat{a}_{i,j}\|_\infty + \|\check{a}_{i,j}\|_{p_2}^{p_2/(p_2-q)} \left(\frac{3n^2}{\theta} \right)^{q/(p_2-q)} \right) \right. \\
&\quad \left. + \sum_{j=1}^{n} \left(\|\hat{b}_{1,j} - \hat{b}_{2,j}\|_\infty + \|\check{b}_{1,j} - \check{b}_{2,j}\|_{p1}^{p_1/(p_1-q)} \left(\frac{3n}{\theta} \right)^{q/(p_1-q)} \right) \right).
\end{aligned}$$

Together with (7), we obtain the estimate $\delta_{\rho,\psi}(p) \leq (\hat{\delta}_1 + |\rho|^m \hat{\delta}_2) f(p)$. Then, by Theorem 1.4 and the values of the constants given in Remark 4.7,

$$\|T_{\rho,\psi}(t)\|_{\mathscr{L}(\mathbf{L}_1,\mathbf{L}_\infty)} \leq Ct^{-n/2} \exp\big((\delta_2|\rho|^m + \delta_1) t \big)$$

∎

Proof (Proof of Theorem 1.6). Let $\rho \in \mathbf{R}$, $\psi \in \Psi$, k and $k_{\rho,\psi}$ be the kernel associated with T and $T_{\rho,\psi}$, respectively. Then $k(t,x,y) = e^{-\rho(\psi(x)-\psi(y))}k_{\rho,\psi}(x,y,t) \geq 0$ for all $t > 0$ and almost all $x, y \in \Omega$, since the semigroup T is positive. By Theorem 5.3, we obtain

$$k(t,x,y) \leq Ct^{-n/2}\exp\big((\delta_2|\rho|^m + \delta_1)t - \rho(\psi(x) - \psi(y))\big).$$

Replacing ρ by $-\rho$, it follows that

$$k(t,x,y) \leq Ct^{-n/2}\exp\big((\delta_2|\rho|^m + \delta_1)t - \rho|\psi(x) - \psi(y)|\big).$$

Since $\sup_{\psi \in \Psi}|\psi(x) - \psi(y)| = |x-y|$, we have

$$k(t,x,y) \leq Ct^{-n/2}\exp\big((\delta_2|\rho|^m + \delta_1)t - \rho|x-y|\big).$$

Choosing $\rho^{m-1} = (\delta_2 mt)^{-1}|x-y|$ proves the claim with $\omega_1 = \delta_1$ and $\omega_2 = (\delta_2 m)^{-(m/(m-1))}(1 - \frac{1}{m})$. ∎

6. APPLICATIONS

Here we can use the results in the literature about kernel estimates. First, we get p-independence of the spectrum $\sigma(A)$ using the following.

THEOREM 6.1 (Kunstmann [13, Theorem 1.1]). *Let Q be an integral operator on $\mathbf{L}_2(\Omega)$ given by a kernel k which satisfies*

$$|k(x,y)| \leq g(x-y) \ \forall x, y \in \Omega,$$

where g is a function satisfying

$$x \mapsto \exp(\rho(|x|))g(x) \in \mathbf{L}_1(\mathbf{R}^n) \cap \mathbf{L}_{p_0}(\mathbf{R}^n)$$

for some $p_0 \in (1,\infty]$ and some Ω-admissible function ρ (i.e., ρ: $\mathbf{R}^+ \to \mathbf{R}^+$ is nondecreasing and subadditive, $\lim_{t\to 0}\rho(t) = 0$, and $x \mapsto \exp(-\rho(|x|)) \in \mathbf{L}_1(\tilde{\Omega})$, where $\tilde{\Omega} = \{x - y;\ x, y \in \Omega\}$). Then Q extends to consistent operators Q_p on all $\mathbf{L}_p(\Omega)$, $p \in [1,\infty]$, and the spectrum $\sigma(Q_p)$ does not depend on p.

COROLLARY 6.2. *Let $m > 1$ and $d \in \mathbf{R}$ such that $n > m(d-1)$. Let $R_2 \in \mathscr{L}(\mathbf{L}_2)$ be given by the formula*

$$R_2 f(x) = \int_\Omega k(x,y)f(y)\,dx \ \forall f \in \mathbf{L}_2(\Omega)$$

with some measurable kernel k: $\Omega \times \Omega \to \mathbf{C}$. Assume

$$|k(x,y)| \leq g_{\mu,b}(x-y) = C\int_0^\infty t^{-d}e^{-\mu t - b(|x-y|^m/t)^{1/(m-1)}}\,dt$$

for some $\mu, b, C > 0$. *Then there is a consistent family* $(R_p \in \mathscr{L}(\mathbf{L}_p))_{p \in [1,\infty]}$, *and the spectrum* $\sigma(R_p) = \sigma(R_2)$ *is independent of* $p \in [1, \infty]$.

Proof. Using the continuous Minkowski inequality and the substitution $x = yt^{1/m}$, we obtain $\|g_{\mu,b}\|_p < \infty$ if $p^{-1} > \frac{m}{n}(d-1)$. Observe that the range of p does not depend on b. For any $c \in (0, b)$ and $a > 0$, we have

$$\sup_{s \geq 0} as - c(s^m/t)^{1/(m-1)} = t\left(\frac{a}{m}\right)^m \left(\frac{m-1}{c}\right)^{m-1} =: t\gamma(a, m, c).$$

Thus, choosing $a > 0$ and $c \in (0, b)$ such that $\gamma(a, m, c) < \mu$, we have

$$|e^{a|x|} g_{\mu_1 b}(x)| \leq g_{\mu - \gamma(a,m,c), b-c}(x)$$

for all $x \in \mathbf{R}^n$. Now choose $\rho(r) = ar$ and $p_0 > 1$ such that $p_0^{-1} > \frac{m}{n}(d-1)$. Then the assumptions of Theorem 6.1 are given. Therefore, R_2 extends to a consistent family $(R_p)_{p \in [1,\infty]}$, and the spectrum of R_p is independent of $p \in [1, \infty]$. ∎

LEMMA 6.3. *Let* $n \in \mathbf{N}$, $M_i \subset \{z \in \mathbf{C};\ \operatorname{Re} z < w\}$ *for some* $w \in \mathbf{R}$ *and* $N_{i,\lambda} = \{(\lambda - z)^n;\ z \in M_i\}$, *where* $i = 1, 2$. *If* $N_{1,\lambda} = N_{2,\lambda}$ *for all* $\lambda > w$, *then* $M_1 = M_2$.

Proof. First, let $B(x, r) = \{y \in \mathbf{C};\ |x - y| < r\}$, $S_{i,\lambda} = M_i \cap B(\lambda, \frac{\lambda - w}{\cos(\pi/n)})$, and $T_{i,\lambda} = \{(\lambda - z)^n;\ z \in S_{i,\lambda}\}$. Then $T_{i,\lambda} = N_{i,\lambda} \cap B(0, (\frac{\lambda - w}{\cos(\pi/n)})^n)$, $\lambda - S_{i,\lambda} = (\lambda - M_i) \cap B(0, \frac{\lambda - w}{\cos(\pi/n)})$ is a subset of $\Sigma_{\pi/n} := \{re^{\mathbf{i}\varphi};\ r > 0, |\varphi| < \pi/n\}$ and $T_{1,\lambda} = T_{2,\lambda}$. Of course the mapping $z \mapsto z^n\colon \Sigma_{\pi/n} \to \mathbf{C}$ is injective. Thus $\lambda - S_{1,\lambda} = \lambda - S_{2,\lambda}$ and then also $S_{1,\lambda} = S_{2,\lambda}$, for all $\lambda > w$.

Now we finish the proof by the equations

$$\begin{aligned} M_1 &= M_1 \cap \bigcup_{\lambda > w} B\left(\lambda, \frac{\lambda - w}{\cos(\pi/n)}\right) = \bigcup_{\lambda > w} M_1 \cap B\left(\lambda, \frac{\lambda - w}{\cos(\pi/n)}\right) \\ &= \bigcup_{\lambda > w} S_{1,\lambda} = \bigcup_{\lambda > w} S_{2,\lambda} = \bigcup_{\lambda > w} M_2 \cap B\left(\lambda, \frac{\lambda - w}{\cos(\pi/n)}\right) \\ &= M_2 \cap \bigcup_{\lambda > w} B\left(\lambda, \frac{\lambda - w}{\cos(\pi/n)}\right) = M_2, \end{aligned}$$

since $\mathbf{C} = \bigcup_{\lambda > w} B(\lambda, \frac{\lambda - w}{\cos(\pi/n)})$. ∎

Proof (Proof of Theorem 1.7). If T generates a C_0 semigroup on $\mathbf{L}_1$, then let $P = [1, \infty)$, otherwise $P = (1, \infty)$.

Using the resolvent identity and Theorem 1.6, we have

$$\begin{aligned}\left(R(\lambda, A_p)\right)^{n+1} f(x) &= \frac{(-1)^n}{n!} \frac{d^n}{d\lambda^n} R(\lambda, A_p) f(x) \\ &= \int_\Omega \int_0^\infty \frac{1}{n!} t^n e^{-\lambda t} k(t, x, y)\, dt f(y)\, dy\end{aligned}$$

for all $f \in \mathbf{L}_p$ and $\lambda > \omega_1$. Applying Corollary 6.2, we obtain $\sigma((R(\lambda, A_2))^{n+1}) = \sigma((R(\lambda, A_p))^{n+1})$ for all $\lambda > \omega_1$ and $p \in P$.

Using the spectral mapping theorem for the resolvent, i.e.,

$$\sigma(R(\lambda, A)) \setminus \{0\} = \frac{1}{\lambda - \sigma(A)} = \left\{ \frac{1}{\lambda - \mu};\ \mu \in \sigma(A) \right\},$$

and for bounded operators S and polynomials $g(\sigma(g(S)) = g(\sigma(S)))$, we obtain

$$\left\{ 1/z;\ z \in \sigma\left(\left(R(\lambda, A_p)\right)^{n+1} \right) \setminus \{0\} \right\} = \left\{ (\lambda - \mu)^{n+1};\ \mu \in \sigma(A_p) \right\}$$

for all $p \in P$ and $\lambda > \omega_1$. Applying Lemma 6.3 to $N_{p,\lambda} = (\lambda - \sigma(A_p))^{n+1}$ finishes the proof. ∎

Remark 6.4. If $a_{i,j} \in \mathbf{L}_\infty$, then, by Theorem 1.6, the kernel associated with the semigroup satisfies a classical Gaussian estimate. Thus the semigroup extends to a C_0 semigroup on $\mathbf{L}_1$.

Another way to show that we have a C_0 semigroup on $\mathbf{L}_1$ uses the $\mathbf{L}_1$ contractivity of $S(t) = e^{-kt}T(t)$. The semigroup S is $\mathbf{L}_1$ contractive if $k + c - \sum_{j=1}^n \partial_j b_{1,j} \geq 0$ as a distribution. In fact, by Lemma 3.8, $\min\{u_+, 1\} \in V$ whenever $u \in V$. Now direct calculation shows that $\mathscr{E}(u - \min\{u_+, 1\}, \min\{u_+, 1\}) \geq 0$ for all $u \in \mathscr{D}$. This remains true for all $u \in V$, as $\mathscr{D}$ is dense in V. By Ma and Röckner [14, Proposition I.4.3, Theorem I.4.4], T is $\mathbf{L}_1$ contractive.

The same arguments show that T is quasi-$\mathbf{L}_\infty$ contractive if there is an $k \in \mathbf{R}$ such that $k + c - \sum_{j=1}^n \partial_j b_{2,j} \geq 0$ as a distribution. In general, $\mathscr{E}$ is not quasi-$\mathbf{L}_\infty$ contractive (see Ouhabaz [17, Remark 4.3.a]), even though Condition 1.5 is satisfied.

Second, we have maximal $\mathbf{L}_p$–$\mathbf{L}_q$ regularity if Condition 1.5 is satisfied and $a_{i,j} \in \mathbf{L}_\infty$ for all $i, j = 1, \ldots, n$. Since we get classical Gaussian estimates by Theorem 1.6, we can use the result of Hieber and Pruess [11, Theorem 3.1] to prove Theorem 1.8. Moreover, there exists a constant $M > 0$ such that

$$\int_0^\infty \|u(t)\|_q^p\, dt + \int_0^\infty \|u'(t)\|_q^p\, dt + \int_0^\infty \|A_q u(t)\|_q^p\, dt \leq M \int_0^\infty \|f(t)\|_q^p\, dt$$

for all $f \in \mathbf{L}_p(\mathbf{R}^+, \mathbf{L}_q(\Omega))$.

Third, we can prove Theorem 1.9, getting an analytic mapping on $\mathbf{L}_1$ in an open sector.

Proof (Proof of Theorem 1.9). This proof uses Theorem 1.6 and a technique of Arendt [3]. Let $0 < (m-1)\theta_1 < (m-1)\theta_2 < \theta_3 < \theta_4 < \theta$. Replacing T by $(e^{-wt}T(t))_{t \geq 0}$ for some $w > \omega_1$, we can assume that

$$\|T(z)\|_{\mathscr{L}(\mathbf{L}_2)} \leq M_1 \quad \text{for all } z \in \Sigma_{\theta_4},$$

$$0 \leq k(t,x,y) \leq Ct^{-n/2}e^{-b(|x-y|^m/t)^{1/m-1}} \quad \text{for all } t > 0, x, y \in \Omega.$$

By Corollary 4.8 and duality, we know that if w is sufficiently large, then

$$\|T(t)\|_{\mathscr{L}(\mathbf{L}_2,\mathbf{L}_\infty)} \leq M_2 t^{-n/4}$$

and

$$\|T(t)\|_{\mathscr{L}(\mathbf{L}_1,\mathbf{L}_2)} \leq M_3 t^{-n/4} \quad \text{for all } t > 0.$$

Now choose $\delta \in (0,1)$ such that $\delta t + \mathbf{i}s \in \Sigma_{\theta_4}$ whenever $t + \mathbf{i}s \in \Sigma_{\theta_3}$. For any $z = t + \mathbf{i}s \in \Sigma_{\theta_3}$, we have

$$\begin{aligned}
&\|T(z)\|_{\mathscr{L}(\mathbf{L}_1,\mathbf{L}_\infty)} \\
&\quad \leq \|T((1-\delta)t/2)\|_{\mathscr{L}(\mathbf{L}_1,\mathbf{L}_2)} \|T(\delta t + \mathbf{i}s)\|_{\mathscr{L}(\mathbf{L}_2)} \|T((1-\delta)t/2)\|_{\mathscr{L}(\mathbf{L}_2,\mathbf{L}_\infty)} \\
&\quad \leq M_1 M_2 M_3 t^{-n/2} \left(\frac{1-\delta}{2}\right)^{-n/2} =: M_4 (\operatorname{Re} z)^{-n/2}.
\end{aligned}$$

Applying [3, Theorem 4.2], obtain the analyticity of $T\colon \Sigma_{\theta_3} \to \mathscr{L}(\mathbf{L}_1, \mathbf{L}_\infty) \sim \mathbf{L}_\infty(\Omega \times \Omega)$. By [3, Lemma 4.1], there exists $K\colon \Sigma_{\theta_3} \times \Omega \times \Omega \to \mathbf{C}$ such that $K(\cdot, x, y)$ is analytic for all $x, y \in \Omega$, $K(z, \cdot, \cdot) \in \mathbf{L}_\infty(\Omega \times \Omega)$ for all $z \in \Sigma_{\theta_3}$ and $T(z)f(x) = \int_\Omega K(z,x,y)f(y)\,dy$ for all $f \in \mathbf{L}_1 \cap \mathbf{L}_2$. As $\mathbf{L}_\infty(\Omega \times \Omega)$ is an isometric isomorph to $\mathscr{L}(\mathbf{L}_1, \mathbf{L}_\infty)$ by $k \mapsto (f \mapsto \int_\Omega k(\cdot, y) f(y)\,dy)$, we obtain $|K(z,x,y)| \leq M_4 (\operatorname{Re} z)^{-n/2}$ for all $z \in \Sigma_{\theta_3}$ and almost all $x, y \in \Omega$ and $K(t,x,y) = k(t,x,y)$ for all $t > 0$ and almost all $x, y \in \Omega$.

Applying [3, Lemma 4.4] to $H(z,x,y) = K(z^{m-1}, x, y) z^{(m-1)n/2}$, we obtain the estimate $|K(z,x,y)| \leq M_0 |z|^{-n/2} e^{-\beta(|x-y|^m/|z|)^{1/(m-1)}}$ for all $z \in \Sigma_{\theta_1}$ and almost all $x, y \in \Omega$, where $M_0 = \sup\{M_4, C\}$ and $\beta = \frac{\sin(\gamma - \theta_1)}{\sin \gamma} b$.

Thus $\sup_{z \in \Sigma_{\theta_1}, |z| > r} \|T_p(z)\|_p < \infty$ for all $r > 0$, $p \in [1, \infty)$. Since the operators T_p are consistent, we can finish the proof by applying [3, Theorem 4.2] again. ∎

We cannot prove the boundedness of the semigroup T on $\mathbf{L}_1$ near 0 using only a pseudo-Gaussian estimate of order $m \neq 2$.

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