

## On a Model Connected with the Kirchhoff Equation

Fabio Catalano<sup>1</sup>

*Département de Mathématiques Appliquées, Université Bordeaux I,  
351, Cours de la Libération, 33405 Talence Cedex, France; and  
Dipartimento di Matematica Pura ed Applicata, Università degli Studi L'Aquila,  
Via Vetoio, Loc. Coppito, 67100 L'Aquila, Italy  
E-mail: catalano@math.u-bordeaux.fr*

*Submitted by H. A. Levine*

Received July 5, 2000

We prove a global existence result in suitable Sobolev spaces for the solution of the nonlinear Cauchy problem

$$\partial_{tt}u(t, x) - \left(1 + \int_{\mathbf{R}^n} dy K(x - y) |\nabla_y u(t, y)|^2\right) \Delta_x u(t, x) = 0,$$

$$u(0, x) = \epsilon u_0(x), \quad (\partial_t u)(0, x) = \epsilon u_1(x), \quad \epsilon > 0,$$

where  $n > 3$  is the space dimension, the initial data are  $C^\infty$  with compact support in  $\mathbf{R}^n$ , and  $K(z)$  is a positive smooth function rapidly decreasing as  $|z| \rightarrow \infty$ . Our approach is based on the energy estimates combined with the classical Von Wahl inequalities. © 2002 Elsevier Science (USA)

*Key Words:* energy inequalities; Von Wahl estimates, contraction method.

### INTRODUCTION

In this paper we study the nonlinear hyperbolic Cauchy problem

$$\partial_{tt}u(t, x) - \left(1 + \int_{\mathbf{R}^n} dy K(x - y) |\nabla_y u(t, y)|^2\right) \Delta_x u(t, x) = 0, \quad (1)$$

$$u(0, x) = \epsilon u_0(x), \quad (\partial_t u)(0, x) = \epsilon u_1(x), \quad \epsilon > 0,$$

where  $t \geq 0$  and  $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ ,  $\Delta_x = \sum_{i=1, \dots, n} \partial_{x_i}^2$  is the Laplace operator in  $\mathbf{R}^n$ ,  $(u_0, u_1) \in C_c^\infty(\mathbf{R}^n) \times C_c^\infty(\mathbf{R}^n)$  and  $\epsilon > 0$  is a small parameter. In the paper we suppose that  $n > 3$  and we assume that  $K(z)$  is a

<sup>1</sup>The author was supported by I.N.d.A.M. Istituto Nazionale di Alta Matematica “F. Severi.”

positive function satisfying the property

$$\max_{0 \leq |\gamma| \leq s} |\nabla_z^\gamma K(z)| \leq \frac{c}{(1 + |z|)^N}, \quad (2)$$

where  $c > 0$  is a constant,  $N > n$  is a suitable integer, and  $s \geq 2\left[\frac{n}{2}\right] + 6$  (see Theorem 0.2).

In Eq. (1), we may set

$$v(t, x, K, \nabla u) = 1 + \int_{R^n} dy K(x - y) |\nabla_y u(t, y)|^2$$

and write

$$\partial_{tt} u(t, x) - v(t, x, K, \nabla u) \Delta_x u(t, x) = 0, \quad (3)$$

$$u(0, x) = \epsilon u_0(x), \quad (\partial_t u)(0, x) = \epsilon u_1(x), \quad \epsilon > 0.$$

Then we may interpret  $v(t, x, K, \nabla u)$  as a variable speed of propagation, depending on the first-order derivative of  $u$ .

Problem (1) generalizes the classical Kirchhoff equation

$$\partial_{tt} u(t, x) - (1 + \|\nabla_x u(t, \cdot)\|_{L^2(R^n)}^2) \Delta u(t, x) = 0, \quad (4)$$

studied in many papers (see [2–7, 15, 16]). Here

$$\|\nabla_x u(t, \cdot)\|_{L^2(R^n)} = \left( \int_{R^n} dx |\nabla_x u(t, x)|^2 \right)^{1/2}$$

denotes the  $L^2$  norm with respect to the space variables. On the other hand, the integrodifferential term  $\|\nabla_x u(t, \cdot)\|_{L^2(R^n)}^2$  may be considered as a connection between the speed of propagation and the classical energy associated to the solution.

One of the first results concerning the Kirchhoff equation (4) is due to Bernstein [2] in the one-dimensional case. He proved a local existence result for initial data in suitable Sobolev spaces as well as a global existence result for real analytic data. Next, Carrier [4] studied an approximate nonlinear equation for the transversal motion of a string. Dickey and Nishida [5–7, 16] investigated some models for strings of finite and infinite length. Recently, D'Ancona and Spagnolo [8] examined the nonlinear Cauchy problem

$$\begin{aligned} \partial_{tt} u(t, x) - \left( 1 + \int_{R^n} dx |\nabla_x u(t, x)|^2 \right) \Delta u(t, x) \\ = F(u(t, x), \partial_t u(t, x), \nabla_x u(t, x)) \\ u(0, x) = \epsilon u_0(x) \quad (\partial_t u)(0, x) = \epsilon u_1(x) \end{aligned} \quad (5)$$

and they proved the following.

**THEOREM 0.1 ([8]).** *Let  $(u_0, u_1) \in C_c^\infty(\mathbf{R}^n) \times C_c^\infty(\mathbf{R}^n)$  and put  $s = (u, \partial_t u, \nabla_x u)$ . Assume that  $F(s) = O(|s|^{\lambda+1})$  near  $s = 0$ . Then for  $n \geq 2$  there exists  $\epsilon_0 = \epsilon_0(u_0, u_1, F)$  sufficiently small and  $\lambda_0 = \lambda_0(n)$  so that problem (5) admits a unique global solution  $u \in C^\infty(\mathbf{R}_+ \times \mathbf{R}^n)$ , provided that  $0 < \epsilon < \epsilon_0(u_0, u_1, F)$  and  $\lambda > \lambda_0(n)$ .*

Going back to Eq. (1), we can write it in the following form

$$\partial_{tt}u(t, x) - (1 + (K \star |\nabla_x u|^2)(t, x))\Delta_x u(t, x) = 0, \quad (6)$$

where  $\star$  denotes the convolution product with respect to the space variables. In the case  $K(z) = 1$ , Eq. (6) becomes the Kirchhoff equation (4). On the other hand, if  $\delta$  denotes the Dirac distribution, setting  $K = \delta$ , we obtain from (6) the quasi-linear wave equation

$$\partial_{tt}u(t, x) - (1 + |\nabla_x u(t, x)|^2)\Delta u(t, x) = 0. \quad (7)$$

The last observation shows that problem (1) can be considered as an interpolation between Eqs. (4) and (7).

Our main result is the following.

**THEOREM 0.2.** *Let  $s \geq 2\left[\frac{n}{2}\right] + 6$ . Then there exists  $\epsilon_0 = \epsilon_0(\|u_0(\cdot)\|_{H^s(\mathbf{R}^n)}, \|u_1(\cdot)\|_{H^{s-1}(\mathbf{R}^n)})$  sufficiently small such that for any  $0 < \epsilon < \epsilon_0$  problem (1) admits a unique global solution*

$$u \in \bigcap_{j=0, \dots, s} C^j([0, +\infty[; H^{s-j}(\mathbf{R}^n)).$$

The plan of the paper is the following. Sections 1 and 2 are devoted to the proof of suitable energy estimates for problem (1). In Section 3, we introduce  $L^\infty \rightarrow L^2$ -type inequalities. In Section 4 the contraction method and the continuation principle for differential equations are applied to complete the proof of Theorem 0.2. In the Appendix we recall two lemmas used in the paper.

In the exposition below we use the following notations. By  $H^s$  and  $C_c^\infty$  we denote, respectively, the usual Sobolev space and the set of  $C^\infty$  functions compactly supported in  $\mathbf{R}^n$ . Also we use the notations  $\partial_t u = u_t$  and  $\partial_x^\alpha u = u^{(\alpha)}$ .

## 1. PRELIMINARY ESTIMATES

Let

$$B(0, R) = \{x \in \mathbf{R}^n : |x| \leq R, 0 < R < +\infty\}$$

denote the ball with center at the origin and radius  $R$ . Without loss of generality, we can assume that

$$\text{supp}(u_0) \cup \text{supp}(u_1) \subseteq B(0, R).$$

The regularity of the initial data implies the existence of a local solution to problem (1) for  $t \in [0, T]$  with sufficiently small  $T > 0$ .

DEFINITION 1.1. Set

$$D(T) = \max \left\{ \sqrt{\left(1 + \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2\right)} : t \in [0, T], |y| \leq R \right\}. \quad (8)$$

Define the characteristic cone by

$$\Omega(T) = \{(t, x) \in [0, T] \times \mathbf{R}^n : |x| \leq R - D(T)t\}. \quad (9)$$

DEFINITION 1.2. Let  $s \geq 2\left[\frac{n}{2}\right] + 6$  and suppose that  $u(t, x)$  is a local solution to (1). We denote by

$$E(t) = \int_{|x| \leq R-D(T)t} dx \left( \frac{|u_t(t, x)|^2}{2} + \left(1 + \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2\right) \times \frac{|\nabla_x u(t, x)|^2}{2} \right) \quad (10)$$

and

$$E_{(\alpha)}(t) = \int_{|x| \leq R-D(T)t} dx \left( \frac{|u_t^{(\alpha)}(t, x)|^2}{2} + \left(1 + \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2\right) \times \frac{|\nabla_x u^{(\alpha)}(t, x)|^2}{2} \right), \quad (11)$$

respectively, the energy associated to  $u$  and  $u^{(\alpha)}$  for every fixed  $\alpha$ ,  $0 \leq |\alpha| \leq s$ .

Our purpose is to prove the following.

PROPOSITION 1.3. Let  $u(t, x)$  be a local solution to (1). Then

$$E(T) \leq E(0) - \int_{\Omega(T)} dt dx \left( \nabla_x \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2 \right) u_t(t, x) \nabla_x u(t, x) + \int_{\Omega(T)} dt dx \left( \partial_t \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2 \right) \frac{|\nabla_x u(t, x)|^2}{2}. \quad (12)$$

*Proof.* We will divide the proof into four steps.

Step 1:

DEFINITION 1.4. Let  $\gamma : [0, T] \times \mathbf{R}^n \rightarrow \mathbf{R}_+$  be a smooth function with respect to the time and space variables. Define the domain

$$\Omega_\gamma(T) = \{(t, x) \in [0, T] \times \mathbf{R}^n : |x| \leq R - \gamma(t, x)\}, \quad (13)$$

where the function  $\gamma(t, x)$  will be given in (22), Step 3.

The proof of Proposition 1.3 is based on a suitable application of the divergence theorem in  $\Omega_\gamma(T)$ . The choice of this domain is closely related to Definition 1.1. In fact, setting  $\gamma(t, x) = D(T)t$  (see Step 3), we just find  $\Omega(T)$  and the required estimate (12) follows.

Multiplying Eq. (1) by  $u_t(t, x)$  and integrating in  $\Omega_\gamma(T)$ , we have

$$\int_{\Omega_\gamma(T)} dt dx \left( u_t(t, x) u_{tt}(t, x) - \left( 1 + \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2 \right) \times u_t(t, x) \Delta u(t, x) \right) = 0.$$

After some calculations, the last identity takes the form

$$\begin{aligned} & \int_{\Omega_\gamma(T)} dt dx \left[ \partial_t \left( \frac{|u_t(t, x)|^2}{2} + \left( 1 + \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2 \right) \frac{|\nabla_x u(t, x)|^2}{2} \right) \right] \\ & - \int_{\Omega_\gamma(T)} dt dx \operatorname{div} \left[ \left( 1 + \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2 \right) u_t(t, x) \nabla_x u(t, x) \right] \\ & + \int_{\Omega_\gamma(T)} dt dx \left( \nabla_x \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2 \right) u_t(t, x) \nabla_x u(t, x) \\ & - \int_{\Omega_\gamma(T)} dt dx \left( \partial_t \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2 \right) \frac{|\nabla_x u(t, x)|^2}{2} = 0. \end{aligned} \quad (14)$$

Step 2:

DEFINITION 1.5. Let

$$\Gamma_\gamma(T) = \{(t, x) \in ]0, T[ \times \mathbf{R}^n : |x| = R - \gamma(t, x)\}$$

be the lateral surface of  $\Omega_\gamma(T)$ . For  $(t, x) \in \Gamma_\gamma(T)$  let

$$\begin{aligned} \nu(t, x) &= (\nu_t(t, x), \nu_x(t, x)) \\ &= \frac{1}{\sqrt{|\gamma_t(t, x)|^2 + \left| \frac{x}{|x|} + \nabla_x \gamma(t, x) \right|^2}} \left( \gamma_t(t, x), \frac{x}{|x|} + \nabla_x \gamma(t, x) \right) \end{aligned} \quad (15)$$

denote the unit normal vector coming out from  $\Gamma_\gamma(T)$ .

Notice that, for  $t = 0$  and  $t = T$ , the base sides of  $\Omega_\gamma(T)$  have the form

$$B_{inf} = \{x \in \mathbf{R}^n : |x| \leq R\}$$

and

$$B_{sup} = \{x \in \mathbf{R}^n : |x| \leq R - D(T)T\}.$$

The unit vectors coming out from  $B_{inf}$  and  $B_{sup}$  are respectively given by  $(-1, 0, \dots, 0)$  and  $(1, 0, \dots, 0)$ . Recalling (14), we apply the divergence theorem to the term

$$\begin{aligned} J = \int_{\Omega_\gamma(T)} dt dx & \left[ \partial_t \left( \frac{|u_t(t, x)|^2}{2} + \left( 1 + \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2 \right) \right. \right. \\ & \left. \left. \times \frac{|\nabla_x u(t, x)|^2}{2} \right) \right] - \int_{\Omega_\gamma(T)} dt dx \\ & \times \operatorname{div} \left[ \left( 1 + \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2 \right) u_t(t, x) \nabla_x u(t, x) \right]. \end{aligned}$$

Thus we are going to study

$$J = J_1 + J_2 + J_3,$$

where

$$\begin{aligned} J_1 &= - \int_{|x| \leq R} dx \left( \frac{|u_t(0, x)|^2}{2} + \left( 1 + \int_{R^n} dy K(x-y) |\nabla_y u(0, y)|^2 \right) \right. \\ & \quad \left. \times \frac{|\nabla_x u(0, x)|^2}{2} \right), \\ J_2 &= \int_{|x| \leq R - \gamma(T, x)} dx \left( \frac{|u_t(T, x)|^2}{2} + \left( 1 + \int_{R^n} dy K(x-y) |\nabla_y u(T, y)|^2 \right) \right. \\ & \quad \left. \times \frac{|\nabla_x u(T, x)|^2}{2} \right) \end{aligned}$$

and

$$\begin{aligned} J_3 &= \int_{\Gamma_\gamma(T)} d\sigma_{t,x} \left( \frac{|u_t(t, x)|^2}{2} + \left( 1 + \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2 \right) \right. \\ & \quad \left. \times \frac{|\nabla_x u(t, x)|^2}{2} \right) \cdot \nu_t(t, x) \\ & \quad - \int_{\Gamma_\gamma(T)} d\sigma_{t,x} \left( 1 + \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2 \right) u_t(t, x) \nabla_x u(t, x) \cdot \nu_x(t, x). \end{aligned}$$

Here,  $J_1$  and  $J_2$  denote respectively the boundary contributions for  $t = 0$  and  $t = T$ , while  $J_3$  is the remaining term evaluated on the surface  $\Gamma_\gamma(T)$ .

Step 3: To get inequality of Proposition 1.3, it remains to show that  $J_3$  is positive. For  $(t, x) \in \Gamma_\gamma(T)$  set

$$\begin{aligned} J_{31} &= \left( \frac{|u_t(t, x)|^2}{2} + \left( 1 + \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2 \right) \frac{|\nabla_x u(t, x)|^2}{2} \right) \cdot \nu_t(t, x) \\ & \quad - \left( 1 + \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2 \right) u_t(t, x) \nabla_x u(t, x) \cdot \nu_x(t, x). \end{aligned}$$

We will show that  $J_{31}$  is non-negative. Taking into account (15), for simplicity, we put

$$\phi(t, x, \gamma) = \frac{1}{\sqrt{|\gamma_t(t, x)|^2 + \left|\frac{x}{|x|} + \nabla_x \gamma(t, x)\right|^2}} > 0.$$

Setting

$$\begin{aligned} A(t, x, \gamma) &= \frac{\gamma_t(t, x)}{2}, \\ B(t, x, \gamma, K, \nabla u) &= \frac{\gamma_t(t, x)(1 + \int_{\mathbb{R}^n} dy K(x - y) |\nabla_y u(t, y)|^2)}{2} \end{aligned} \quad (16)$$

and

$$C(t, x, \gamma, K, \nabla u) = \frac{(1 + \int_{\mathbb{R}^n} dy K(x - y) |\nabla_y u(t, y)|^2) \left(\frac{x}{|x|} + \nabla_x \gamma(t, x)\right)}{2},$$

we deduce

$$\begin{aligned} J_{31} &= \phi(t, x, \gamma) \left( A(t, x, \gamma) |u_t(t, x)|^2 + B(t, x, \gamma, K, \nabla u) |\nabla_x u(t, x)|^2 \right. \\ &\quad \left. - 2C(t, x, \gamma, K, \nabla u) u_t(t, x) \nabla_x u(t, x) \right). \end{aligned}$$

**ASSUMPTION 1.6.** Assume that  $\gamma_t(t, x) > 0$ ,  $\forall (t, x) \in \Gamma_\gamma(T)$ . Then, we have  $A(t, x, \gamma) > 0$  and  $B(t, x, \gamma, K, \nabla u) > 0$ ,  $\forall (t, x) \in \Gamma_\gamma(T)$ .

We refer to Remark 1.8 below for more details concerning Assumption 1.6. To complete the proof of Proposition 1.3, observe that, whenever

$$C^2(t, x, \gamma, K, \nabla u) \leq A(t, x, \gamma) B(t, x, \gamma, K, \nabla u), \quad (17)$$

the following estimate

$$\begin{aligned} &A(t, x, \gamma) |u_t(t, x)|^2 + B(t, x, \gamma, K, \nabla u) |\nabla_x u(t, x)|^2 \\ &\quad - 2C(t, x, \gamma, K, \nabla u) u_t(t, x) \nabla_x u(t, x) \\ &\quad \geq \left( \sqrt{A(t, x, \gamma) u_t(t, x)} - \sqrt{B(t, x, \gamma, K, \nabla u) \nabla_x u(t, x)} \right)^2 \geq 0 \end{aligned}$$

holds. This allows us to conclude that  $J_3 \geq 0$ .

**LEMMA 1.7.** Inequality (17) holds if we take  $\gamma(t, x) = D(T)t$  for  $(t, x) \in [0, T] \times \mathbb{R}^n$ .

*Proof.* It is convenient to write (17) as follows

$$\left| \left( \frac{x}{|x|} + \nabla_x \gamma(t, x) \right) \right| \leq \frac{|\gamma_t(t, x)|}{\sqrt{(1 + \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2)}}. \quad (18)$$

Taking

$$\beta(t, x) = |x| + \gamma(t, x), \quad (19)$$

we have

$$\beta_t(t, x) = \gamma_t(t, x) \quad \text{and} \quad \nabla_x \beta(t, x) = \frac{x}{|x|} + \nabla_x \gamma(t, x);$$

hence (18) yields

$$|\nabla_x \beta(t, x)| \leq \frac{\beta_t(t, x)}{\sqrt{(1 + \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2)}}.$$

Without loss of generality, we can assume that

$$|\nabla_x \beta(t, x)| = \frac{\beta_t(t, x)}{\sqrt{(1 + \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2)}} \quad (20)$$

holds. We call this equality *eicoidal equation*. Recall that by Definition 1.1 the solution to (20) is given by

$$\beta(t, x) = |x| + D(T)t. \quad (21)$$

Consequently, using (19), we conclude that

$$\gamma(t, x) = D(T)t, \quad (22)$$

so we may omit the index  $\gamma$  using the notations

$$\Omega(T) = \{(t, x) \in [0, T] \times \mathbf{R}^n : |x| \leq R - D(T)t\},$$

$$\Gamma(T) = \{(t, x) \in ]0, T[ \times \mathbf{R}^n : |x| = R - D(T)t\}$$

instead of  $\Omega_\gamma(T)$  and  $\Gamma_\gamma(T)$ .

*Remark 1.8.* Notice that from (22) we get  $\gamma_t(t, x) = D(T) > 0$ .

Step 4: Using (10), we see that  $J_1 = -E(0)$  and  $J_2 = E(T)$ . Since  $J_3 \geq 0$ , from (14) we conclude that

$$\begin{aligned} E(T) - E(0) + \int_{\Omega(T)} dt dx \left( \nabla_x \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2 \right) u_t(t, x) \nabla_x u(t, x) \\ - \int_{\Omega(T)} dt dx \left( \partial_t \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2 \right) \frac{|\nabla_x u(t, x)|^2}{2} \leq 0. \end{aligned}$$

This completes the proof of Proposition 1.3.



## 2. GENERALIZED ENERGY ESTIMATES

In this section we extend the previous results to higher order derivatives of the solution.

**PROPOSITION 2.1.** *Let  $s \geq 2[\frac{n}{2}] + 6$  and let  $\alpha$  be fixed so that  $0 \leq |\alpha| \leq s$ . Assume that  $u(t, x)$  is a local solution to (1). Then we have*

$$\begin{aligned} E_{(\alpha)}(T) &\leq E_{(\alpha)}(0) + \int_{\Omega(T)} dt dx \left( \partial_t \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2 \right) \\ &\quad \times \frac{|\nabla_x u^{(\alpha)}(t, x)|^2}{2} - \int_{\Omega(T)} dt dx \left( \nabla_x \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2 \right) \\ &\quad \times u_t^{(\alpha)}(t, x) \nabla_x u^{(\alpha)}(t, x) + \int_{\Omega(T)} dt dx \sum_{\alpha_1 + \alpha_2 = \alpha} C_{\alpha_1, \alpha_2}^\alpha u_t^{(\alpha)}(t, x) \\ &\quad \times \left( \partial_x^{\alpha_1} \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2 \right) \Delta u^{(\alpha_2)}(t, x), \end{aligned} \quad (23)$$

where  $1 \leq |\alpha_1| \leq |\alpha|$ ,  $0 \leq |\alpha_2| \leq |\alpha| - 1$ , and  $C_{\alpha_1, \alpha_2}^\alpha$  are suitable constants.

*Proof.* Our approach follows that used in Section 1. We divide the proof into two steps.

Step 1: Fix  $\alpha$  so that  $0 \leq |\alpha| \leq s$  and consider the identity

$$\partial_x^\alpha \left( u_{tt}(t, x) - \left[ 1 + \left( \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2 \right) \right] \Delta u(t, x) \right) = 0.$$

A simple calculation yields

$$\begin{aligned} u_{tt}^{(\alpha)}(t, x) - \left[ 1 + \left( \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2 \right) \right] \Delta u^{(\alpha)}(t, x) \\ = \sum_{\alpha_1 + \alpha_2 = \alpha} C_{\alpha_1, \alpha_2}^\alpha \left( \partial_x^{\alpha_1} \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2 \right) \Delta u^{(\alpha_2)}(t, x), \end{aligned} \quad (24)$$

where  $1 \leq |\alpha_1| \leq |\alpha|$  and  $0 \leq |\alpha_2| \leq |\alpha| - 1$ . Multiplying (24) by  $u_t^{(\alpha)}(t, x)$  and integrating in  $\Omega(T)$ , we obtain

$$\begin{aligned} \int_{\Omega(T)} dt dx u_t^{(\alpha)}(t, x) u_{tt}^{(\alpha)}(t, x) - \int_{\Omega(T)} dt dx u_t^{(\alpha)}(t, x) \\ \times \left[ 1 + \left( \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2 \right) \right] \Delta u^{(\alpha)}(t, x) \\ = \int_{\Omega(T)} dt dx \sum_{\alpha_1 + \alpha_2 = \alpha} C_{\alpha_1, \alpha_2}^\alpha u_t^{(\alpha)}(t, x) \\ \times \left( \partial_x^{\alpha_1} \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2 \right) \Delta u^{(\alpha_2)}(t, x). \end{aligned} \quad (25)$$

Step 2: Write (25) as follows

$$\begin{aligned}
& \int_{\Omega(T)} dt dx \left[ \partial_t \left( \frac{|u_t^{(\alpha)}(t, x)|^2}{2} + \left( 1 + \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2 \right) \right. \right. \\
& \quad \left. \left. \times \frac{|\nabla_x u^{(\alpha)}(t, x)|^2}{2} \right) \right] \\
& - \int_{\Omega(T)} dt dx \operatorname{div} \left[ \left( 1 + \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2 \right) u_t^{(\alpha)}(t, x) \nabla_x u^{(\alpha)}(t, x) \right] \\
& + \int_{\Omega(T)} dt dx \left( \nabla_x \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2 \right) u_t^{(\alpha)}(t, x) \nabla_x u^{(\alpha)}(t, x) \\
& - \int_{\Omega(T)} dt dx \left( \partial_t \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2 \right) \frac{|\nabla_x u^{(\alpha)}(t, x)|^2}{2} \\
& = \int_{\Omega(T)} dt dx \sum_{\alpha_1 + \alpha_2 = \alpha} C_{\alpha_1, \alpha_2}^\alpha u_t^{(\alpha)}(t, x) \left( \partial_x^{\alpha_1} \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2 \right) \\
& \quad \times \Delta u^{(\alpha_2)}(t, x).
\end{aligned}$$

Proceeding in the same way as in Section 1, we apply the divergence theorem to the term

$$\begin{aligned}
J_1^{(\alpha)} &= \int_{\Omega(T)} dt dx \left[ \partial_t \left( \frac{|u_t^{(\alpha)}(t, x)|^2}{2} + \left( 1 + \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2 \right) \right. \right. \\
& \quad \left. \left. \times \frac{|\nabla_x u^{(\alpha)}(t, x)|^2}{2} \right) \right] - \int_{\Omega(T)} dt dx \\
& \quad \operatorname{div} \left[ \left( 1 + \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2 \right) u_t^{(\alpha)}(t, x) \nabla_x u^{(\alpha)}(t, x) \right].
\end{aligned}$$

The statement of Proposition 2.1 follows combining (11) with Steps 3 and 4 of Section 1.

### 3. VON WAHL INEQUALITY

Introduce the uniform norm

$$\|f(t, \cdot)\|_{L^\infty(\mathbf{R}^n)} = \sup_{x \in \mathbf{R}^n} |f(t, x)|.$$

DEFINITION 3.1. Let  $s \geq 2\left[\frac{n}{2}\right] + 6$  and let  $u(t, x)$  be a local solution to (1). We define the weighted- $L^\infty$  norm

$$H_{[s/2]+1}(T) = \sup_{(t, x) \in \Omega(T), 0 \leq |\beta| \leq [s/2]+1} \left\{ (1+t)^{(n-1)/2} |\partial_x^\beta u(t, x)|, (1+t)^{(n-1)/2} \times |\partial_t \partial_x^\beta u(t, x)| \right\}, \quad (26)$$

where  $[a]$  denotes the greater integer less than  $a \in \mathbf{R}_+$ .

Our goal is to prove the following.

PROPOSITION 3.2. Let  $u(t, x)$  be a local solution to (1). Then

$$H_{[s/2]+1}(T) \leq c_0 \epsilon + c_1 H_{[s/2]+1}^2(T) \sup_{t \in [0, T]} \{ \|u(t, \cdot)\|_{H^{[s/2]+[n/2]+3}(R^n)} \},$$

where  $c_0$  and  $c_1$  denote suitable positive constants.

*Proof.* Our approach is based on the classical  $L^\infty(\mathbf{R}^n) \rightarrow L^2(\mathbf{R}^n)$  inequalities established by Von Wahl [19]. Consider the non-homogeneous Cauchy problem

$$\begin{aligned} w_{tt}(t, x) - \Delta_x w(t, x) &= F(t, x, w(t, x), \nabla_{t,x} w(t, x), \nabla_{t,x}^2 w(t, x)) \quad (27) \\ w(0, x) &= \epsilon w_0(x), \quad (\partial_t w)(0, x) = \epsilon w_1(x), \quad \epsilon > 0, \end{aligned}$$

where  $x \in \mathbf{R}^n$ ,  $(w_0, w_1) \in C_c^\infty(\mathbf{R}^n) \times C_c^\infty(\mathbf{R}^n)$ , and  $\epsilon > 0$  is a small parameter. Suppose that problem (27) admits a local solution for  $t$  in some interval  $[0, T]$ ,  $0 < T < +\infty$ . Then the Von Wahl inequality ([19]) implies the estimate

$$\begin{aligned} \|w(T, \cdot)\|_{L^\infty(R^n)} &\leq c_0 \frac{\|\epsilon w_0(\cdot)\|_{H^{[n/2]+1}(R^n)} + \|\epsilon w_1(\cdot)\|_{H^{[n/2]}(R^n)}}{(1+T)^{(n-1)/2}} \\ &\quad + c_1 \int_0^T dt \frac{\|F(t, \cdot, w(t, \cdot), \nabla_{t,x} w(t, \cdot), \nabla_{t,x}^2 w(t, \cdot))\|_{H^{[n/2]}(R^n)}}{(1+T-t)^{(n-1)/2}}, \end{aligned}$$

where  $c_0$  and  $c_1$  are positive constants.

We write the problem (1) as

$$\begin{aligned} u_{tt}(t, x) - \Delta_x u(t, x) &= \left( \int_{R^n} dy K(x-y) |\nabla_y u(t, y)|^2 \right) \Delta_x u(t, x), \quad (28) \\ u(0, x) &= \epsilon u_0(x), \quad (\partial_t u)(0, x) = \epsilon u_1(x), \quad \epsilon > 0 \end{aligned}$$

and assume that  $u(t, x)$  is a local solution to (1). Therefore, the Von Wahl inequality yields

$$\begin{aligned} & \|u(T, \cdot)\|_{L^\infty(R^n)} \\ & \leq c_0 \frac{\|\epsilon u_0(\cdot)\|_{H^{[n/2]+1}(R^n)} + \|\epsilon u_1(\cdot)\|_{H^{[n/2]}(R^n)}}{(1+T)^{(n-1)/2}} \\ & \quad + c_1 \int_0^T dt \frac{\left\| \left( \int_{R^n} dy K(\cdot - y) |\nabla_y u(t, y)|^2 \right) \Delta_x u(t, \cdot) \right\|_{H^{[n/2]}(R^n)}}{(1+T-t)^{(n-1)/2}}. \end{aligned} \quad (29)$$

Let  $\beta$  be an index such that  $0 \leq |\beta| \leq [\frac{s}{2}] + 1$ . Applying the operator  $\partial_x^\beta$  to (28), we get

$$\begin{aligned} u_{tt}^{(\beta)}(t, x) - \Delta_x u^{(\beta)}(t, x) &= \partial_x^\beta \left[ \left( \int_{R^n} dy K(x - y) |\nabla_y u(t, y)|^2 \right) \Delta_x u(t, x) \right], \\ \partial_x^\beta u(0, x) &= \epsilon \partial_x^\beta u_0(x), \quad (\partial_x^\beta \partial_t u)(0, x) = \epsilon \partial_x^\beta u_1(x). \end{aligned}$$

A simple calculation gives

$$\begin{aligned} & \partial_x^\beta \left[ \left( \int_{R^n} dy K(x - y) |\nabla_y u(t, y)|^2 \right) \Delta_x u(t, x) \right] \\ &= \sum_{\beta_1 + \beta_2 = \beta} C_{\beta_1, \beta_2}^\beta \left( \partial_x^{\beta_1} \int_{R^n} dy K(x - y) |\nabla_y u(t, y)|^2 \right) \Delta_x u^{(\beta_2)}(t, x), \end{aligned}$$

where  $C_{\beta_1, \beta_2}^\beta$  are constants,  $0 \leq |\beta_1| \leq |\beta|$ , and  $0 \leq |\beta_2| \leq |\beta|$ . From the above estimates we deduce

$$\begin{aligned} & \|u^{(\beta)}(T, \cdot)\|_{L^\infty(R^n)} \\ & \leq c_{0, \beta} \frac{\|\epsilon u_0(\cdot)\|_{H^{[n/2]+1+|\beta|}(R^n)} + \|\epsilon u_1(\cdot)\|_{H^{[n/2]+|\beta|}(R^n)}}{(1+T)^{(n-1)/2}} + c_{1, \beta} \int_0^T dt \\ & \quad \times \frac{\left\| \sum_{\beta_1 + \beta_2 = \beta} (\partial_x^{\beta_1} \int_{R^n} dy K(\cdot - y) |\nabla_y u(t, y)|^2) \Delta_x u^{(\beta_2)}(t, \cdot) \right\|_{H^{[n/2]}(R^n)}}{(1+T-t)^{(n-1)/2}}, \end{aligned} \quad (30)$$

with suitable positive constants  $c_{0, \beta}$ ,  $c_{1, \beta}$ . Next applying Lemma 5.1, we get

$$\partial_x^{\beta_1} \int_{R^n} dy K(x - y) |\nabla_y u(t, y)|^2 = \left( \int_{R^n} dy (\nabla_x^{\beta_1} K(x - y)) |\nabla_y u(t, y)|^2 \right). \quad (31)$$

Exploiting Assumption (2) on  $K(z)$ , we conclude that

$$\|\nabla_z^{\beta_1} K(\cdot)\|_{L_z^1(\mathbf{R}^n)} \leq c_{\beta_1} < +\infty, \quad (32)$$

for every  $0 \leq |\beta_1| \leq |\beta|$ , with some constants  $c_{\beta_1}$ . Using Lemma 5.1 once more, we have

$$\begin{aligned}
 & \left\| \left( \partial_x^{\beta_1} \int_{R^n} dy K(\cdot - y) |\nabla_y u(t, y)|^2 \right) \Delta_x u^{(\beta_2)}(t, \cdot) \right\|_{H^{[n/2]}(R^n)} \\
 & \leq \left\| \partial_x^{\beta_1} \int_{R^n} dy K(\cdot - y) |\nabla_y u(t, y)|^2 \right\|_{L^\infty(R^n)} \left\| \Delta_x u^{(\beta_2)}(t, \cdot) \right\|_{H^{[n/2]}(R^n)} \\
 & \leq \left\| \nabla_x^{\beta_1} K(\cdot) \right\|_{L^1(R^n)} \left\| \nabla_x u(t, \cdot) \right\|_{L^\infty(R^n)}^2 \left\| \Delta_x u^{(\beta_2)}(t, \cdot) \right\|_{H^{[n/2]}(R^n)} \\
 & \leq c_{\beta_1} \frac{H_{[s/2]+1}^2(T)}{(1+t)^{n-1}} \|u(t, \cdot)\|_{H^{[n/2]+|\beta_2|+2}(R^n)}.
 \end{aligned}$$

Combining the previous estimates, we obtain

$$\begin{aligned}
 \|u^{(\beta)}(T, \cdot)\|_{L^\infty(R^n)} & \leq c_{0,\beta} \frac{\|\epsilon u_0(\cdot)\|_{H^{[n/2]+1+|\beta|}(R^n)} + \|\epsilon u_1(\cdot)\|_{H^{[n/2]+|\beta|}(R^n)}}{(1+T)^{(n-1)/2}} \\
 & \quad + c_{1,\beta} H_{[\frac{s}{2}]+1}^2(T) \\
 & \quad \times \int_0^T dt \frac{\sum_{0 \leq |\beta_2| \leq |\beta|} \|u(t, \cdot)\|_{H^{[n/2]+|\beta_2|+2}(R^n)}}{(1+T-t)^{(n-1)/2} (1+t)^{n-1}}.
 \end{aligned}$$

Finally, summing over  $0 \leq |\beta| \leq [\frac{s}{2}] + 1$  and recalling that

$$\|u_0(\cdot)\|_{H^{[s/2]+[n/2]+2}(R^n)} < +\infty \quad \text{and} \quad \|u_1(\cdot)\|_{H^{[s/2]+[n/2]+1}(R^n)} < +\infty,$$

we conclude that

$$\begin{aligned}
 & \sum_{0 \leq |\beta| \leq [\frac{s}{2}]+1} \|u^{(\beta)}(T, \cdot)\|_{L^\infty(R^n)} \\
 & \leq \frac{c_0 \epsilon}{(1+T)^{(n-1)/2}} + c_1 H_{[s/2]+1}^2(T) \\
 & \quad \times \int_0^T dt \sum_{0 \leq |\beta_2| \leq [\frac{s}{2}]+1} \frac{\|u(t, \cdot)\|_{H^{[n/2]+|\beta_2|+2}(R^n)}}{(1+t)^{n-1} (1+T-t)^{(n-1)/2}} \\
 & \leq \frac{c_0 \epsilon}{(1+T)^{(n-1)/2}} + c_1 H_{[s/2]+1}^2(T) \left( \sup_{t \in [0, T]} \|u(t, \cdot)\|_{H^{[n/2]+[s/2]+3}(R^n)} \right) \\
 & \quad \times \int_0^T dt \frac{1}{(1+t)^{n-1} (1+T-t)^{(n-1)/2}}. \tag{33}
 \end{aligned}$$

Now from Lemma 5.2 we have

$$\int_0^T dt \frac{1}{(1+t)^{n-1} (1+T-t)^{(n-1)/2}} \leq \frac{c}{(1+T)^{(n-1)/2}},$$

with a suitable constant  $c$  depending on  $n$ . Thus, (33) takes the form

$$\sum_{0 \leq |\beta| \leq [\frac{s}{2}] + 1} \|u^{(\beta)}(T, \cdot)\|_{L^\infty(R^n)} \leq \frac{c_0 \epsilon}{(1+T)^{(n-1)/2}} + c_1 \frac{H_{[s/2]+1}^2(T)}{(1+T)^{(n-1)/2}} \times \left( \sup_{t \in [0, T]} \|u(t, \cdot)\|_{H^{[s/2]+[n/2]+3}(R^n)} \right). \quad (34)$$

Multiplying inequality (34) by  $(1+T)^{(n-1)/2}$ , we complete the proof of Proposition 3.2.

#### 4. PROOF OF THEOREM 0.2.

Going back to (11), the estimate (34) takes the following form

$$H_{[s/2]+1}(T) \leq c_0 \epsilon + c_1(n, s) H_{[s/2]+1}^2(T) \sup_{t \in [0, T]} \left\{ \sqrt{E_{([s/2]+[n/2]+3)}(t)} \right\}. \quad (35)$$

On the other hand, by (23), for every fixed  $0 \leq |\alpha| \leq s$ , we have

$$\begin{aligned} E_{(\alpha)}(T) &\leq E_{(\alpha)}(0) + \int_0^T dt \left\| \int_{R^n} dy (\nabla_x K(\cdot - y)) |\nabla_y u(t, y)|^2 \right\|_{L^\infty} \|u_t^{(\alpha)}(t, \cdot)\|_{L^2} \\ &\quad \times \|\nabla_x u^{(\alpha)}(t, \cdot)\|_{L^2} + \int_0^T dt \left\| \int_{R^n} dy K(\cdot - y) \partial_t (|\nabla_y u(t, y)|^2) \right\|_{L^\infty} \\ &\quad \times \|\nabla_x u^{(\alpha)}(t, \cdot)\|_{L^2} \frac{\|\nabla_x u^{(\alpha)}(t, \cdot)\|_{L^2}}{2} \\ &\quad + \sum_{\alpha_1 + \alpha_2 = \alpha} C_{\alpha_1, \alpha_2}^\alpha \int_0^T dt \left\| \int_{R^n} dy (\partial_x^{\alpha_1} K(\cdot - y)) |\nabla_y u(t, y)|^2 \right\|_{L^\infty} \\ &\quad \times \|u_t^\alpha(t, \cdot)\|_{L^2} \frac{\|\Delta u^{\alpha_2}(t, \cdot)\|_{L^2}}{2}, \end{aligned} \quad (36)$$

where  $1 \leq |\alpha_1| \leq |\alpha|$ ,  $0 \leq |\alpha_2| \leq |\alpha| - 1$  and  $C_{\alpha_1, \alpha_2}^\alpha$  are suitable constants. Since

$$|\partial_t |\nabla_y u(t, y)|^2| \leq 2 |\nabla_y u(t, y)| |\partial_t \nabla_y u(t, y)| \leq 2 \frac{H_{[s/2]+1}^2(T)}{(1+t)^{n-1}},$$

applying (2), we get

$$\begin{aligned} &\left\| \int_{R^n} dy (\nabla_x K(\cdot - y)) |\nabla_y u(t, y)|^2 \right\|_{L^\infty} + \left\| \int_{R^n} dy K(\cdot - y) \partial_t (|\nabla_y u(t, y)|^2) \right\|_{L^\infty} \\ &\leq c \frac{H_{[s/2]+1}^2(T)}{(1+t)^{n-1}} \end{aligned}$$

and

$$\left\| \int_{\mathbb{R}^n} dy (\partial_x^{\alpha_1} K(\cdot - y)) |\nabla_y u(t, y)|^2 \right\|_{L^\infty} \leq c_{\alpha_1} \frac{H_{[s/2]+1}^2(T)}{(1+t)^{n-1}},$$

for every  $1 \leq |\alpha_1| \leq |\alpha|$ . Here and below  $c$  and  $c_{\alpha_1}$  denote suitable constants. Thus from (36) we have

$$E_{(\alpha)}(T) \leq E_{(\alpha)}(0) + c_{\alpha} H_{[s/2]+1}^2(T) \int_0^T dt \frac{E_{(\alpha)}(t)}{(1+t)^{n-1}}. \quad (37)$$

Without loss of generality, we may suppose that  $E_{(\alpha)}(t)$  is an increasing function in  $t$  for every fixed  $\alpha$  such that  $0 \leq |\alpha| \leq s$ . Therefore, by (37) we deduce

$$E_{(\alpha)}(T) \leq E_{(\alpha)}(0) + c_{\alpha} E_{(\alpha)}(T) H_{[s/2]+1}^2(T) \int_0^T dt \frac{1}{(1+t)^{n-1}}.$$

Summing over  $0 \leq |\alpha| \leq s$  and observing that the condition  $n > 3$  implies the convergence of the integral as  $T \rightarrow \infty$ , we conclude that

$$E_{(s)}(T) \leq E_{(s)}(0) + c_2(s, n) E_{(s)}(T) H_{[s/2]+1}^2(T).$$

Taking into account (35), we finally obtain the following system of inequalities

$$\begin{aligned} H_{[s/2]+1}(T) &\leq \epsilon + c_1(n, s) H_{[s/2]+1}^2(T) \sqrt{E_{([s/2]+[n/2]+3)}(T)} \\ E_{(s)}(T) &\leq E_{(s)}(0) + c_2(n, s) E_{(s)}(T) H_{[s/2]+1}^2(T). \end{aligned} \quad (38)$$

Here we have  $s \geq [\frac{s}{2}] + [\frac{n}{2}] + 3$  since  $s \geq 2[\frac{n}{2}] + 6$ . On the other hand, the fact that the initial data are small leads to  $E_{(s)}(0) \simeq c\epsilon^2$ . Thus we conclude that

$$\begin{aligned} H_{[s/2]+1}(T) &\leq \epsilon + c_1(n, s) H_{[s/2]+1}^2(T) \sqrt{E_{([s/2]+[n/2]+3)}(T)} \\ E_{(s)}(T) &\leq \epsilon^2 + c_2(s, n) E_{(s)}(T) H_{[s/2]+1}^2(T). \end{aligned} \quad (39)$$

Putting

$$\theta(u; T; s) = E_{(s)}(T) + H_{[s/2]+1}(T),$$

and using (39), we obtain

$$\theta(u; T; s) \leq c(s, n) (\epsilon + \epsilon^2 + \theta(u; T; s)^{5/2} + \theta(u; T; s)^3). \quad (40)$$

**LEMMA 4.1.** *Assume (40) fulfilled. Then there exists a constant  $C_1$  and a number  $\epsilon_0 = \epsilon_0(\|u_0(\cdot)\|_{H^s(\mathbb{R}^n)}, \|u_1(\cdot)\|_{H^{s-1}(\mathbb{R}^n)})$  sufficiently small such that, for all  $\epsilon \in ]0, \epsilon_0[$  and for all  $t \in [0, T]$  we have*

$$\theta(u; t; s) < C_1 \epsilon.$$

*Proof.* We will proceed by contradiction. Put  $C_1 = 2C_0$  and suppose that there exists  $t_1 \in ]0, T[$  so that

$$\theta(u; t; s) < 2C_0\epsilon \quad \forall 0 \leq t < t_1, \quad \text{while} \quad \theta(u; t_1; s) = 2C_0\epsilon. \quad (41)$$

Using (40), for all  $0 \leq t < t_1$ , we have

$$\theta(u; t; s) < c(s, n)(\epsilon + \epsilon^2 + (2C_0\epsilon)^{5/2} + (2C_0\epsilon)^3). \quad (42)$$

We take  $\epsilon_0 > 0$  sufficiently small, so that for all  $0 < \epsilon < \epsilon_0$ , the right-hand side of (42) becomes smaller than  $\frac{3}{2}C_0\epsilon$ . Hence, by (42), for all  $0 \leq t < t_1$  and for all  $0 < \epsilon < \epsilon_0$  we have

$$\theta(u; t; s) \leq \frac{3}{2}C_0\epsilon. \quad (43)$$

The continuity of  $\theta(u; t; s)$  yields the estimate

$$\theta(u; t_1; s) \leq \frac{3}{2}C_0\epsilon. \quad (44)$$

Since  $t_1$  is arbitrary, we obtain a contradiction. The continuation principle for differential equations shows that we have no restrictions on the time variable and we can therefore extend the result to the case  $T = +\infty$ . The uniqueness of the solution follows from the standard contraction argument [14, 18]. This completes the proof of Theorem 0.2.

## 5. APPENDIX

The following lemma concerns some properties related to the convolution product. For more details and for a complete proof we refer to [3].

LEMMA 5.1. *Let  $r \geq 0$  be an integer and let  $C_c^r(\mathbf{R}^n) = \{h \in C^r(\mathbf{R}^n) : \text{supp}(h) \text{ is compact in } \mathbf{R}^n\}$ . Assume that  $1 \leq p \leq +\infty$ ,  $f \in C_c^r(\mathbf{R}^n)$ , and  $g \in L_{\text{loc}}^1(\mathbf{R}^n) \cap L^p(\mathbf{R}^n)$ . Then*

$$\|f \star g\|_{L^p(\mathbf{R}^n)} \leq c(p, n)\|f\|_{L^1(\mathbf{R}^n)}\|g\|_{L^p(\mathbf{R}^n)},$$

where  $c(p, n)$  is a suitable constant and  $\star$  denotes the convolution product with respect to the space variables. Moreover,

$$(f \star g) \in C^r(\mathbf{R}^n) \quad \text{and} \quad (\partial_x^j(f \star g))(x) = ((\partial_x^j f) \star g)(x),$$

for every index  $0 \leq |j| \leq r$ .

We will prove the following estimate used in Section 3.



LEMMA 5.2. Assume  $n > 3$  and  $0 < T \leq +\infty$ . Then we have

$$\int_0^T dt \frac{1}{(1+T-t)^{(n-1)/2}(1+t)^{n-1}} \leq \frac{c}{(1+T)^{(n-1)/2}}. \quad (45)$$

*Proof.* Writing  $[0, T] = [0, T/2] \cup [T/2, T]$ , we get

$$\begin{aligned} & \int_0^{T/2} dt \frac{1}{(1+T-t)^{(n-1)/2}(1+t)^{n-1}} \\ & \leq \left\{ \max_{t \in [0, T/2]} \frac{1}{(1+T-t)^{(n-1)/2}} \right\} \int_0^{T/2} dt \frac{1}{(1+t)^{n-1}} \\ & = \frac{1}{(1+\frac{T}{2})^{(n-1)/2}} \left( \frac{1}{n-2} - \frac{1}{(n-2)(1+\frac{T}{2})^{n-2}} \right) \\ & \leq \frac{c(n)}{(1+\frac{T}{2})^{(n-1)/2}} \leq \frac{\tilde{c}(n)}{(1+T)^{(n-1)/2}} \end{aligned}$$

and

$$\begin{aligned} & \int_{T/2}^T dt \frac{1}{(1+T-t)^{(n-1)/2}(1+t)^{n-1}} \\ & \leq \left\{ \max_{t \in [T/2, T]} \frac{1}{(1+t)^{n-1}} \right\} \int_{T/2}^T dt \frac{1}{(1+T-t)^{(n-1)/2}} \\ & = \frac{1}{(1+\frac{T}{2})^{n-1}} \left( \frac{2}{n-3} - \frac{2}{(n-3)(1+\frac{T}{2})^{(n-3)/2}} \right) \\ & \leq \frac{c(n)}{(1+\frac{T}{2})^{n-1}} \leq \frac{\tilde{c}(n)}{(1+T)^{(n-1)/2}}. \end{aligned}$$

## ACKNOWLEDGMENTS

The author expresses his sincere gratitude to Professors V. Georgiev and V. Petkov for their helpful suggestions and constant encouragement.

## REFERENCES

1. A. Arosio and S. Spagnolo, Global solution to the Cauchy problem for a nonlinear equation, in "Nonlinear PDE's and Their Applications," Collège de France Seminar (H. Brézis and J. L. Lions, Eds.), Vol. VI; Pitman, Boston; *Research Notes Math.* **109** (1984), 1–26.
2. S. Bernstein, Sur une classe d'équations fonctionnelles aux dérivées partielles, *Izv. Akad. Nauk SSSR, Ser. Math.* **4** (1940), 17–26.

3. H. Brezis, "Analyse Fonctionnelle: Théorie et Applications," Masson, Paris, 1983.
4. G. F. Carrier, On the nonlinear vibration problem of the elastic string, *Quart Appl. Math.* **3** (1945), 157–165.
5. R. W. Dickey, Infinite systems of nonlinear oscillation equations related to the string, *Proc. Am. Math. Soc.* **23** (1969), 459–468.
6. R. W. Dickey, Infinite systems of nonlinear oscillation equations with linear damping, *SIAM J. Appl. Math.* **19** (1970), 208–214.
7. R. W. Dickey, The initial value problem for a nonlinear semi-infinite string, in workshop, University of Texas, Austin, March 1977.
8. P. D'Ancona and S. Spagnolo, Nonlinear perturbations of the Kirchhoff equation, *Commun. Pure Appl. Math.* **47** (1994), 1005–1029.
9. P. D'Ancona and S. Spagnolo, Kirchhoff type equation depending on small parameter, *Chinese Ann. Math.* **16B**, No. 4 (1995), 413–430.
10. P. D'Ancona and S. Spagnolo, Global solvability for the degenerate Kirchhoff equation with real analytic data, *Invent. Math.* **108** (1992), 247–262.
11. P. D'Ancona and S. Spagnolo, Global existence for the generalized Kirchhoff equation with small data, *Arch. Rat. Mech. Anal.* **124** (1993), 201–219.
12. J. M. Greenberg and S. C. Hu, The initial value problem for a stretched string, *Quart. Appl. Math.* (1980), 289–331.
13. S. Klainerman and M. Machedon, Smoothing estimates for null forms and applications, *Duke Math. J.* **81** (1995–1996), 99–133.
14. S. Klainerman and G. Ponce, Global small amplitude solutions to nonlinear evolution equations, *Comm. Pure Appl. Math.* **36** (1983), 133–141.
15. R. Narisimha, Nonlinear vibrations of an elastic string, *J. Sound Vibration* **8** (1968), 134–146.
16. T. Nishida, A note on the nonlinear vibrations of the elastic string, *Mem. Fac. Engrg. Kyoto Univ.* **33** (1971), 329–341.
17. G. Perla Menzala, On a classical solution of a quasilinear hyperbolic equation, *Nonlinear Anal.* **3** (1979), 613–627.
18. J. Shatah, Global existence of small solutions to nonlinear evolution equations, *J. Differential Equations* **46** (1982), 409–425.
19. W. Von Wahl,  $L^p$ -decay rates for homogeneous wave equation, *Math. Zeit.* **120** (1971), 93–106.