

# Comparison Theorems for Controllability of Nonlinear Volterra Integrodifferential Systems

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Sufficient conditions are derived for controllability of nonlinear Volterra integrodifferential systems. Here the controllability problem is transformed into a fixed point problem for some operator with the help of the notion of the comparison principle and the results are obtained by using the Schauder fixed point theorem.

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*Key Words:* controllability, Volterra systems, fixed point theorem.

## 1. INTRODUCTION

The problem of controllability of nonlinear systems has been studied by several authors by means of fixed point principles [2]. However, Khanh [6] and Yamamoto [7] studied the controllability of general nonlinear systems with the help of the notion of the comparison principle. Balachandran [1] established a set of sufficient conditions for the controllability of nonlinear Volterra integrodifferential systems. In this paper we shall prove some comparison theorems for the controllability of nonlinear perturbations of Volterra integrodifferential systems by utilizing the nonlinear variation of parameters formula developed in [5]. The results generalize the results of [3, 4].

Consider the nonlinear perturbations of the Volterra integrodifferential system of the form

$$\begin{aligned} \dot{x}(t) &= g(t, x(t)) + \int_{t_0}^t h(t, s, x(s)) ds + B(t, x(t))u(t) \\ &\quad + f(t, x(t), S(x)(t), u(t)), \quad t \in [t_0, t_1] = J, \\ x(t_0) &= x_0, \end{aligned} \quad (1)$$

where the operator  $S$  is defined by

$$(Sx)(t) = \int_{t_0}^t k(t, s, x(s)) ds.$$

Here  $x \in R^n$ ,  $u \in R^m$ , and the functions  $g$ ,  $h$ ,  $f$ ,  $B$ , and  $k$  satisfy the following conditions:

(i)  $g: J \times R^n \rightarrow R^n$  is continuous and continuously differentiable with respect to  $x$ .

(ii)  $h: J \times J \times R^n \rightarrow R^n$  is continuous and continuously differentiable with respect to  $x$  and there exists a positive constant  $a_1 > 0$  such that  $|h(t, s, x)| \leq a_1$ , for  $x \in B_r = \{x \in R^n : |x| < r\}$ .

(iii)  $B: J \times R^n \rightarrow R^n \times R^m$  is continuous and there exists a positive constant  $a_2 > 0$  such that  $\|B(t, x)\| \leq a_2$ , for  $x \in B_r$ .

(iv)  $k: J \times J \times R^n \rightarrow R^n$  is continuous and

$$|k(t, s, x(s))| \leq a(t, s)|x|$$

where  $a(t, s)$  is a scalar-valued continuous function on  $J \times J$  such that

$$\max \int_{t_0}^t a(t, s) ds < 1.$$

(v)  $f: J \times R^n \times R^n \times R^m \rightarrow R^n$  is continuous and

$$|f(t, x, Sx, u)| \leq p(t, |x|, |x|, |u|),$$

where  $p(t, \alpha, \alpha, \beta)$  is a continuous function with respect to its arguments and nondecreasing for any  $\alpha > 0$ ,  $\beta > 0$ .

Here the norm of a matrix is taken as the usual matrix norm. Let  $x(t, t_0, x_0)$  be the unique solution of the equation

$$\dot{x}(t) = g(t, x) + \int_{t_0}^t h(t, s, x(s)) ds. \quad (2)$$

Define  $H(t, t_0, x_0) = (\partial g / \partial x)(t, x(t, t_0, x_0))$  and  $G(t, s, t_0, x_0) = (\partial h / \partial x)(t, s, x(s, t_0, x_0))$ . Then  $X(t, t_0, x_0) = (\partial x / \partial x_0)(t, t_0, x_0)$  exists and is the solution of

$$\dot{y}(t) = H(t, t_0, x_0)y(t) + \int_{t_0}^t G(t, s, t_0, x_0)y(s) ds$$

such that  $X(t_0, t_0, x_0) = I$ , the identity matrix.

DEFINITION 1.1. The system (1) is controllable from  $(x_0, t_0)$  to  $(x_1, t_1)$  if, for some control  $u(t)$ ,  $t_0 \leq t \leq t_1$ , the solution of (1) with  $x(t_0) = x_0$  is such that  $x(t_1) = x_1$  where  $x_1$  and  $t_1$  are a preassigned terminal state and time, respectively. If the system is controllable for all  $x_0$  at  $t = t_0$  and for all  $x_1$  at  $t = t_1$ , it will be called controllable on  $J = [t_0, t_1]$ .

## 2. PRELIMINARY RESULTS

Choose  $x_0, x_1 \in R^n$  and consider how to determine an appropriate control  $u$  which steers the solutions of the system (1) with  $x(t_0) = x_0$  to  $x(t_1) = x_1$ . The solution  $y(t)$  of the system (1) with  $y(t_0) = x_0$  is given by [5]

$$\begin{aligned} y(t) &= x(t, t_0, x_0) + \int_{t_0}^t X(t, s, x(s)) \\ &\quad \times [B(s, x(s))u(s) + f(s, x(s), S(x)(s), u(s))] ds \\ &\quad + \int_{t_0}^t \int_s^t [X(t, \tau, x(\tau)) - R(t, \tau, s, x(s))] h(\tau, s, x(s)) d\tau ds, \end{aligned} \tag{3}$$

where  $R(t, s, t_0, x_0)$  is the solution of the equation

$$\begin{aligned} \frac{\partial R}{\partial s}(t, s, t_0, x_0) + R(t, s, t_0, x_0)H(s, t_0, x_0) \\ + \int_s^t R(t, \tau, t_0, x_0)G(\tau, s, t_0, x_0) d\tau = 0 \end{aligned}$$

such that  $R(t, t, t_0, x_0) = I$  on the interval  $t_0 \leq s \leq t$  and  $R(t, t_0, t_0, x_0) = X(t, t_0, x_0)$ . Define the matrix  $W$  by

$$\begin{aligned} W(t_1, t_0, x) \\ = \int_{t_0}^{t_1} X(t_1, s, x(s))B(s, x(s))(X(t_1, s, x(s))B(s, x(s)))^* ds, \end{aligned}$$

where the asterisk denotes the matrix transpose. Assume the following hypothesis:

(H) The matrix  $W(t_1, t_0, x)$  has an inverse for all  $x \in C_n(J)$ , a Banach space of  $n$ -dimensional continuous functions defined on  $J$ .

In order to establish our results the following state with  $v \in C_m(J)$ :

$$\begin{aligned} y(t) = & x(t, t_0, x_0) + \int_{x_0}^t X(t, s, x(s)) \\ & \times [B(s, x(s))v(s) + f(s, x(s), S(x)(s), u(s))] ds \\ & + \int_{t_0}^t \int_s^t [X(t, \tau, x(\tau)) - R(t, \tau, s, x(s))] h(\tau, s, x(s)) d\tau ds. \end{aligned} \quad (4)$$

If the system (1) satisfies the condition (H) then one of the controls which steers the state (4) to a given  $x_1$  at time  $t_1$  is given by

$$\begin{aligned} v(t) = & B^*(t, x) X^*(t_1, t, x) W^{-1}(t_1, t_0, x) \\ & \times \left[ x_1 - x(t_1, t_0, x_0) - q(t_1, t_0, x) \right. \\ & \left. - \int_{t_0}^{t_1} X(t_1, s, x) (f(s, x(s), (Sx)(s), u(s)) ds \right], \end{aligned} \quad (5)$$

where

$$q(t, t_0, x) = \int_{t_0}^t \int_s^t [X(t, \tau, x(\tau)) - R(t, \tau, s, x(s))] h(\tau, s, x(s)) d\tau ds.$$

Substituting Eq. (5) into (4), we see that  $y(t_1) = x_1$ . If the vectors  $y, v$  agree with  $x, u$ , which result from Eqs. (4) and (5), respectively, then these vectors are also solutions of the original problem for system (1), and the controllability of system (1) is guaranteed. Thus, the controllability problem for system (1) becomes an existence problem of fixed points for Eqs. (4) and (5). Equations (5) and (4) are also considered as nonlinear operator equations which assign  $(x, u) \in C_{n+m}(J)$  to  $(y, v) \in C_{n+m}(J)$ , where  $C_{n+m}(J)$  is a Banach space of  $(n + m)$ -dimensional continuous functions defined on  $J$ . Thus, the original controllability problem becomes a fixed point problem for the operator  $\Phi$  defined by  $(y, v) = \Phi(x, u)$ . The nonlinear operator  $\Phi$  is obviously continuous on  $C_{n+m}(J)$ . Furthermore, if there exists a closed bounded convex subset  $Q$  of  $C_{n+m}(J)$  such that the operator  $\Phi$  is invariant for  $Q$ , that is,

$$(y, v) = \Phi(x, u) \in Q \quad \text{for any } (x, u) \in Q, \quad (6)$$

then as derived from Eqs. (5) and (4) the operator  $\Phi$  is bounded and equicontinuous, hence by Schauder's fixed point theorem there exists at least one fixed point for  $\Phi$ . Thus, we obtain the following lemma.

**LEMMA 2.1.** *If there exists a closed bounded convex subset  $Q$  of  $C_{n+m}(J)$  such that the operator  $\Phi$  is invariant for  $Q$ , then the system (1) which satisfies the hypothesis (H) is completely controllable on  $J$ .*

### 3. COMPARISON THEOREMS FOR CONTROLLABILITY

Now we examine the conditions such that there exists a subset  $Q$  which satisfies the lemma. Define the set  $Q$  by

$$Q = \{(x, u) \in C_{n+m}(J) : |x(t)| \leq \alpha(t), |u(t)| \leq \beta(t)\}. \quad (7)$$

Then we have

$$\begin{aligned} |y(t)| &\leq |x(t, t_0, x_0)| + \int_{t_0}^t \|X(t, s, x(s))\| \\ &\quad \times [\|B(s, x(s))\| \|v(s)\| + |f(s, x(s), S(x)(s), u(s))|] ds \\ &\quad + \int_{t_0}^t \int_s^t [\|X(t, \tau, x(\tau))\| + \|R(t, \tau, s, x(s))\|] |h(\tau, s, x(s))| d\tau ds \\ &\leq a_0 + a_4 a_2 \int_{t_0}^t \beta(s) ds + a_4 \int_{t_0}^t p(s, |x(s)|, |Sx(s)|, |u(s)|) ds \\ &\quad + a_1(a_3 + a_4) \int_{t_0}^t (t - s) ds, \end{aligned}$$

and so

$$\begin{aligned} |y(t)| &\leq a_0 + a_2 a_4 \int_{t_0}^t \beta(s) ds + a_4 \int_{t_0}^t p(s, \alpha(s), \alpha(s), \beta(s)) ds \\ &\quad + a_1(a_3 + a_4) \frac{(t_1 - t_0)^2}{2}, \end{aligned} \quad (8)$$

where  $a_0 = \max|x(t, t_0, x_0)|$ ,  $a_3 = \sup\|R(t, \tau, s, x(s))\|$ ,  $a_4 = \sup\|X(t, s, x(s))\|$ . In a way similar to Eq. (5), we obtain

$$\begin{aligned} |v(t)| &\leq \|B^*(t, x)X^*(t_1, t, x)W^{-1}(t_1, t_0, x_0) \\ &\quad \times [x_1 - x(t_1, t_0, x_0) - q(t_1, t_0, x)]\| \\ &\quad + \|B^*(t, x)X^*(t_1, t, x)W^{-1}(t_1, t_0, x_0)\| \\ &\quad \int_{t_0}^{t_1} \|X(t_1, s, x)\| |(f(s, x(s)), (Sx)(s), u(s))| ds \\ &\leq c_0 + a_4 c_1 \int_{t_0}^{t_1} p(s, \alpha(s), \alpha(s), \beta(s)) ds, \end{aligned} \quad (9)$$

where

$$\begin{aligned} c_0 &= \sup\|B^*(t, x)X^*(t_1, t, x)W^{-1}(t_1, t_0, x_0) \\ &\quad \times [x_1 - x(t_1, t_0, x_0) - q(t_1, t_0, x)]\|, \\ c_1 &= \sup\|B^*(t, x)X^*(t_1, t, x)W^{-1}(t_1, t_0, x_0)\|. \end{aligned}$$

Here  $a_0$  depends on the initial value  $x_0$  and  $c_0$  depends on both the initial and terminal values  $x_0, x_1$ . But  $a_1, a_2, a_3, a_4, c_1$  are constants defined only by the system parameter and the control interval  $J$ . Therefore, in order that a subset  $Q$  from Eq. (7) will satisfy the lemma, it is sufficient that the right-hand sides of Eqs. (8) and (9) are smaller than  $\alpha(t)$  and  $\beta(t)$ .

**THEOREM 3.1.** *For the system (1) which satisfies the assumptions (i)–(v) and Hypothesis (H) to be completely controllable on  $J$ , it is sufficient that the inequality relations*

$$\begin{aligned} a_0 + a_2 a_4 \int_{t_0}^t \beta(s) ds + a_4 \int_{t_0}^t p(s, \alpha(s), \alpha(s), \beta(s)) ds \\ + a_1(a_3 + a_4) \frac{(t_1 - t_0)^2}{2} \leq \alpha(t), \end{aligned} \quad (10)$$

$$c_0 + a_4 c_1 \int_{t_0}^{t_1} p(s, \alpha(s), \alpha(s), \beta(s)) ds \leq \beta(t) \quad (11)$$

have at least one nonnegative solution  $(\alpha(t), \beta(t))$  for any  $a_0, c_0 > 0$  and for some  $a_1, a_2, a_3, a_4, c_1$  defined by the system equations.

This theorem can be simplified when the nonlinear functions do not depend on  $x$  or  $u$ . In Eq. (1), if  $f$  does not contain  $u$  (that is,  $f(\cdot) =$

$f(t, x(t), (Sx)(t))$  and the assumption (v) changes to

$$|f(t, x(t), (Sx)(t))| \leq p(t, |x|, |(Sx)|), \tag{12}$$

then (10) and (11) are respectively converted to

$$\begin{aligned} a_0 + a_2 a_4 \int_{t_0}^t \beta(s) ds + a_4 \int_{t_0}^t p(s, \alpha(s), \alpha(s)) ds \\ + a_1(a_3 + a_4) \frac{(t_1 - t_0)^2}{2} \leq \alpha(t) \\ c_0 + a_4 c_1 \int_{t_0}^{t_1} p(s, \alpha(s), \alpha(s)) ds \leq \beta(t). \end{aligned}$$

Then from these inequalities we have

$$\begin{aligned} \dot{\alpha}(t) &\geq a_2 a_4 \beta(t) + a_4 p(t, \alpha(t), \alpha(t)) \\ &\geq a_2 a_4 \left[ c_0 + a_4 c_1 \int_{t_0}^{t_1} p(s, \alpha(s), \alpha(s)) ds \right] + a_4 p(t, \alpha(t), \alpha(t)) \\ &= d_1 + d_2 \int_{t_0}^{t_1} p(s, \alpha(s), \alpha(s)) ds + a_4 p(t, \alpha(t), \alpha(t)) \end{aligned}$$

and

$$\alpha(t_0) \geq a_0 + d_0,$$

where  $d_0 = a_1(a_3 + a_4)((t_1 - t_0)^2/2)$ ,  $d_1 = a_2 a_4 c_0$ , and  $d_2 = a_4^2 a_2 c_1$ .

**THEOREM 3.2.** *Consider the nonlinear integrodifferential control system*

$$\begin{aligned} \dot{x}(t) &= g(t, x(t)) + \int_{t_0}^t h(t, s, x(s)) ds + B(t, x(t))u(t) \\ &\quad + f(t, x(t), S(x)(t)) \end{aligned} \tag{13}$$

which satisfies (iii) and (12) and Hypothesis (H). If there exists at least one nonnegative solution  $\alpha(t)$  of the inequalities

$$\begin{aligned} \dot{\alpha}(t) &\geq d_1 + d_2 \int_{t_0}^{t_1} p(s, \alpha(s), \alpha(s)) ds + a_4 p(t, \alpha(t), \alpha(t)), \\ \alpha(t_0) &\geq a_0 + d_0 \end{aligned}$$

for any  $a_0, d_0 > 0$  and for some positive constants  $a_1, a_4, d_1$  then the system (13) is completely controllable on  $J$ .

In the case that  $f$  in (1) does not contain  $x$  (that is,  $f(\cdot) = f(t, u(t))$ ) and if assumption (v) changes to

$$|f(t, u(t))| \leq p(t, |u|), \quad (14)$$

then (10) and (11) are respectively altered to

$$a_0 + a_2 a_4 \int_{t_0}^t \beta(s) ds + a_4 \int_{t_0}^t p(s, \beta(s)) ds + d_0 \leq \alpha(t),$$

$$c_0 + a_4 c_1 \int_{t_0}^{t_1} p(s, \beta(s)) ds \leq \beta(t).$$

But if the second inequality has a solution  $\beta(t)$ , then the first inequality always has a solution sufficiently large. Therefore, we have the following theorem.

**THEOREM 3.3.** *Consider the nonlinear integrodifferential control system*

$$\dot{x}(t) = g(t, x(t)) + \int_{t_0}^t h(t, s, x(s)) ds + B(t, x(t))u(t) + f(t, u(t)) \quad (15)$$

which satisfies (iii) and (14) and Hypothesis (H). If there exists at least one nonnegative solution  $\beta(t)$  of the inequality

$$\beta(t) \geq c_0 + a_4 c_1 \int_{t_0}^{t_1} p(s, \beta(s)) ds$$

for any  $c_0 > 0$  and for some positive constants  $a_4, c_1$ , then the system (15) is completely controllable on  $J$ .

*Remark.* The theorems established above are based on the notion of the comparison principle; so we call these theorems comparison theorems for nonlinear Volterra integrodifferential systems.

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