Available online at www.sciencedirect.com

SCIENCE DIRECT

J. Math. Anal. Appl. 306 (2005) 730–739

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

A family of the Cauchy type mean-value theorems

Josip E. Pečarić [a], Ivan Perić [b], H.M. Srivastava [c,*]

[a] *Faculty of Textile Technology, University of Zagreb, Pierottijeva 6, HR-10000 Zagreb, Croatia*
[b] *Faculty of Chemical Engineering and Technology, University of Zagreb, Marulićev trg. 19, HR-10000 Zagreb, Croatia*
[c] *Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3P4, Canada*

Received 6 October 2004

Available online 4 February 2005

Submitted by William F. Ames

Abstract

The Cauchy type mean-value theorems for the Riemann–Liouville fractional derivative are deduced here from known mean-value theorems of the Lagrange type. A general method for deducing these Cauchy type formulas is extracted. Two Cauchy type formulas are then deduced without a priori knowledge about the Lagrange type mean-value theorems.
© 2004 Elsevier Inc. All rights reserved.

Keywords: Cauchy mean-value theorem; Lagrange mean-value theorem; Riemann–Liouville fractional integral and fractional derivative; Newton–Cotes quadrature formulas; Jensen's inequality; Trapezoidal rule

1. Introduction

Mean-value theorems are of great importance in mathematical analysis. In particular, the Lagrange type and the Cauchy type mean-value theorems are most frequently used. The usual approach is to prove first the Lagrange type mean-value theorems and then deduce from them the Cauchy type mean-value theorems. As a typical example of this method,

* Corresponding author.
E-mail addresses: pecaric@hazu.hr (J.E. Pečarić), iperic@pbf.hr (I. Perić), hmsri@uvvm.uvic.ca, harimsri@math.uvic.ca (H.M. Srivastava).

0022-247X/$ – see front matter © 2004 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2004.10.018

in Section 2, we first show how this approach works for the Riemann–Liouville fractional derivative. Then, in Section 3, we extract a general abstract method which contains the crucial step in this procedure. Finally, in Section 4, we make use of the perfect symmetry of the Cauchy type mean-value theorems in order to show that, in many cases, one can easily guess the form of the Cauchy type mean-value theorem and then deduce from it the exact form of the Lagrange type mean-value theorem.

2. Generalized Cauchy type formulas for the Riemann–Liouville fractional derivative

Let us first consider the *Riemann–Liouville fractional integral of order* $-\alpha$, that is,

$$D_a^\alpha f(x) = I_a^{-\alpha} f(x).$$

Here the *Riemann–Liouville fractional integral operator* I_a^β is defined as follows:

$$I_a^\beta f(x) := \frac{1}{\Gamma(\beta)} \int_a^x (x-t)^{\beta-1} f(t)\,dt \qquad (x > a;\ a \in \mathbb{R};\ \beta \in \mathbb{R}^+) \tag{1}$$

and

$$D_a^\alpha f(x) = D^n I_a^{n-\alpha} f(x) \qquad \left(D := \frac{d}{dx};\ n-1 \leqq \alpha < n;\ n \in \mathbb{N} := \{1,2,3,\ldots\}\right) \tag{2}$$

with, of course,

$$I_a^0 f(x) = f(x).$$

The *sequential fractional derivative* is denoted by (see, for example, [5, p. 86 et seq.])

$$D_a^{n\alpha} := D_a^\alpha D_a^{(n-1)\alpha} \qquad \big(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\big). \tag{3}$$

Let Ω be a real interval and $\alpha \in [0,1]$. Let $F(\Omega)$ denote the space of Lebesgue measurable functions with domain in Ω and suppose that $x_0 \in \Omega$. Then a function f is called α-continuous at x_0 if there exists $\lambda \in [0, 1-\alpha)$ for which the function g given by

$$g(x) = |x - x_0|^\lambda f(x)$$

is continuous at x_0. Thus, in the present terminology, the function f is called 1-continuous at x_0 if it is continuous at x_0. Moreover, the function f is called α-continuous on Ω if it is α-continuous for every $x \in \Omega$. We now denote, for convenience, the class of α-continuous functions on Ω by

$$C_\alpha(\Omega) := \big\{f\colon\ f \in F(\Omega) \text{ and } f \text{ is } \alpha\text{-continuous on } \Omega\big\},$$

so that

$$C_1(\Omega) = C(\Omega).$$

For $a \in \Omega$, a function f is called a-singular of order α if

$$\lim_{x \to a} \frac{f(x)}{|x-a|^{\alpha-1}} = k \quad (0 \neq k < \infty).$$

Let $\alpha \in \mathbb{R}^+$, $a \in \Omega$, and $E \subset \Omega$ such that $a \leqq x$ for every $x \in E$. Then we write

$$_aI_\alpha(E) := \big\{f\colon\ f \in F(\Omega) \text{ and } I_a^\alpha f(x) < \infty\ (\forall x \in E)\big\}, \tag{4}$$

where, as before, $F(\Omega)$ denotes the space of Lebesgue measurable functions with domain in Ω.

Recently, Trujillo et al. [6] proved the following results.

Theorem 1 (A generalized mean-value theorem). *Let $\alpha \in [0,1]$ and $f \in C(a,b]$ such that $D_a^\alpha f \in C[a,b]$. Then*

$$f(x) = \big[(x-a)^{1-\alpha} f(x)\big](a+)(x-a)^{\alpha-1} + D_a^\alpha f(\xi)\left(\frac{(x-a)^\alpha}{\Gamma(\alpha+1)}\right), \tag{5}$$

for every $x \in [a,b]$ with $a \leqq \xi \leqq x$.

Theorem 2 (A generalized Taylor's formula). *Let $\alpha \in [0,1]$ and $n \in \mathbb{N}$. Let f be a continuous function on $(a,b]$ satisfying each of the following conditions*:

(i) *$D_a^{j\alpha} f \in C(a,b]$ and $D_a^{j\alpha} f \in {}_aI_\alpha[a,b]$ for $j = 1,\dots,n$.*
(ii) *$D_a^{(n+1)\alpha} f$ is continuous on $[a,b]$.*
(iii) *If $\alpha < 1/2$, then, for each $j \in \{1,\dots,n\}$ such that $(j+1)\alpha \leqq 1$, $D_a^{(j+1)\alpha} f(x)$ is γ-continuous at $x = a$ for some γ $(1-(j+1)\alpha \leqq \gamma \leqq 1)$ or a-singular of order α.*

Then, for every $x \in (a,b]$,

$$R_n(f;x,a) = \frac{D_a^{(n+1)\alpha} f(\xi)}{\Gamma\big((n+1)\alpha+1\big)}(x-a)^{(n+1)\alpha} \quad (a \leqq \xi \leqq x), \tag{6}$$

where

$$R_n(f;x,a) := f(x) - \sum_{j=0}^{n} \frac{c_j (x-a)^{(j+1)\alpha-1}}{\Gamma\big((j+1)\alpha\big)} \tag{7}$$

and

$$c_j = \Gamma(\alpha)\big[(x-a)^{1-\alpha} D_a^{j\alpha} f(x)\big](a+) = I_a^{1-\alpha} D_a^{j\alpha} f(a+) \quad (j \in \mathbb{N}_0;\ 0 \leqq j \leqq n). \tag{8}$$

In our present investigation, we propose to give some related results by using the methodology given in [4]. We begin by stating our first result as follows.

Theorem 3. *Let $\alpha \in [0,1]$ and let $f, g \in C(a,b]$ be such that*

$$D_a^\alpha f,\ D_a^\alpha g \in C[a,b],$$

where

$$D_a^\alpha g(x) \neq 0 \quad \textit{for every } x \in [a,b].$$

Then, for every $x \in (a,b]$, *there is a* ξ $(a \leqq \xi \leqq x)$ *such that*

$$\frac{f(x) - [(x-a)^{1-\alpha} f(x)](a+)(x-a)^{\alpha-1}}{g(x) - [(x-a)^{1-\alpha} g(x)](a+)(x-a)^{\alpha-1}} = \frac{D_a^\alpha f(\xi)}{D_a^\alpha g(\xi)}. \tag{9}$$

Proof. Let $x \in [a,b]$ be fixed. Denote by $\mathcal{K}_1$ and $\mathcal{K}_2$ the following functions:

$$\mathcal{K}_1 = f(x) - \big[(x-a)^{1-\alpha} f(x)\big](a+)(x-a)^{\alpha-1}$$

and

$$\mathcal{K}_2 = g(x) - \big[(x-a)^{1-\alpha} g(x)\big](a+)(x-a)^{\alpha-1}.$$

We consider the function $F(t)$ given by

$$F(t) = \mathcal{K}_2 f(t) - \mathcal{K}_1 g(t) \quad \big(t \in [a,b]\big).$$

Since f and g satisfy the conditions in Theorem 1, the same is valid for F, so we have

$$F(x) - \big[(x-a)^{1-\alpha} F(x)\big](a+)(x-a)^{\alpha-1} = D_a^\alpha F(\xi)\left(\frac{(x-a)^\alpha}{\Gamma(\alpha+1)}\right) \tag{10}$$

for some ξ $(a \leqq \xi \leqq x)$. This gives us

$$\begin{aligned} 0 &= \mathcal{K}_2\big[f(x) - \big[(x-a)^{1-\alpha} f(x)\big](a+)(x-a)^{\alpha-1}\big] \\ &\quad - \mathcal{K}_1\big[g(x) - \big[(x-a)^{1-\alpha} g(x)\big](a+)(x-a)^{\alpha-1}\big] \\ &= \frac{(x-a)^\alpha}{\Gamma(\alpha+1)}\big(\mathcal{K}_2 D_a^\alpha f(\xi) - \mathcal{K}_1 D_a^\alpha g(\xi)\big), \end{aligned} \tag{11}$$

from which the assertion (9) of Theorem 3 follows easily. □

Corollary 1. *Let* $\alpha \in [0,1]$ *and let* $f, g \in C(a,b]$ *be such that*

$$D_a^\alpha\big[(x-a)^{\alpha-1} f(x)\big], D_a^\alpha\big[(x-a)^{\alpha-1} g(x)\big] \in C[a,b],$$

where

$$D_a^\alpha\big[(x-a)^{\alpha-1} g(x)\big] \neq 0 \quad \textit{for every } x \in [a,b].$$

Then, for every $x \in (a,b]$, *there is a* ξ $(a \leqq \xi \leqq x)$ *such that*

$$\frac{f(x) - f(a+)}{g(x) - g(a+)} = \frac{\big[D_a^\alpha\big((x-a)^{\alpha-1} f(x)\big)\big](\xi)}{\big[D_a^\alpha\big((x-a)^{\alpha-1} g(x)\big)\big](\xi)}. \tag{12}$$

Proof. Upon replacing $f(x)$ by $(x-a)^{\alpha-1} f(x)$, and $g(x)$ by $(x-a)^{\alpha-1} g(x)$, Theorem 3 readily yields Corollary 1. □

Theorem 4. *Suppose that the functions f and g satisfy the conditions in Theorem* 2*, where*

$$D_a^{(n+1)\alpha} g(x) \neq 0 \quad \textit{for every } x \in [a,b].$$

Then, for every $x \in (a,b]$, there is a ξ $(a \leqq \xi \leqq z)$ such that

$$\frac{R_n(f;x,a)}{R_n(g;x,a)} = \frac{D_a^{(n+1)\alpha} f(\xi)}{D_a^{(n+1)\alpha} g(\xi)}, \tag{13}$$

where R_n is defined by (7).

Proof. Let $x \in (a,b]$ be fixed. In terms of R_n defined by (7), we denote by $\mathcal{K}_1$ and $\mathcal{K}_2$ the following functions:

$$\mathcal{K}_1 = R_n(g;x,a) \quad \text{and} \quad \mathcal{K}_2 = R_n(f;x,a),$$

and consider the function F defined by

$$F(t) = \mathcal{K}_1 f(t) - \mathcal{K}_2 g(t) \quad \big(t \in [a,b]\big).$$

Using the linearity property of R_n defined by (7), the rest of the proof of Theorem 4 is as in the proof of Theorem 3. □

A simple consequence of Theorem 4 is given by the following corollary.

Corollary 2. *Let F and G be functions defined on $(a,b]$ such that the functions*

$$f(x) = (x-a)^{\alpha-1} F(x) \quad \textit{and} \quad g(x) = (x-a)^{\alpha-1} G(x)$$

satisfy the conditions of Theorem 4*. Then, for every $x \in (a,b]$, there is a ξ $(a \leqq \xi \leqq x)$ such that*

$$\frac{\tilde{R}_n(F;x,a)}{\tilde{R}_n(G;x,a)} = \frac{[D_a^{(n+1)\alpha}(x-a)^{\alpha-1}F(x)](\xi)}{[D_a^{(n+1)\alpha}(x-a)^{\alpha-1}G(x)](\xi)}, \tag{14}$$

where

$$\tilde{R}_n(F;x,a) = F(x) - \sum_{j=0}^{n} \frac{c_j (x-a)^{j\alpha}}{\Gamma\big((j+1)\alpha\big)} \tag{15}$$

and

$$c_j = \big[I_a^{1-\alpha} D_a^{j\alpha} (x-a)^{\alpha-1} F(x)\big](a+) \quad (j \in \mathbb{N}_0;\ 0 \leqq j \leqq n).$$

3. An outline of the general method

Let Ξ be a set and let $\mathcal{F}$ be some appropriately chosen vector space of real-valued functions defined on Ξ. Let Φ be a functional on $\mathcal{F}$ and let $A : \mathcal{F} \to \mathcal{R}$ be a linear operator, where $\mathcal{R}$ is the vector space of *all* real-valued functions defined on Ξ.

Suppose that, for each $f \in \mathcal{F}$, there is a $\xi \in \Xi$ such that

$$\Phi(f) = A(f)(\xi). \tag{16}$$

Theorem 5. *For every $f, g \in \mathcal{F}$, there is a $\xi \in \Xi$ such that*

$$A(g)(\xi)\Phi(f) = A(f)(\xi)\Phi(g), \tag{17}$$

where Φ is given by (16) *in terms of the linear operator $A : \mathcal{F} \to \mathcal{R}$.*

Proof. Consider the following linear combination:

$$h = \Phi(g)f - \Phi(f)g.$$

Obviously, we have

$$\Phi(h) = 0.$$

On the other hand, there is a $\xi \in \Xi$ such that

$$A(h)(\xi) = \Phi(h) = 0,$$

which completes the proof of Theorem 5 by using the linearity property of the operator A. □

In order to obtain Theorem 3 from Theorem 5 (using Theorem 1), we set

$$\Xi = [a, x], \qquad \Phi(f) = f(x) - \big[(x-a)^{1-\alpha} f(x)\big](a+)(x-a)^{\alpha-1},$$

and

$$A(f) = D_a^\alpha f,$$

and let $\mathcal{F}$ contain continuous functions f on $(a, b]$ for which $D_a^\alpha f$ is continuous on $[a, b]$.

In exactly the same way, we can obtain results of this kind from any *Newton–Cotes quadrature formula*, provided that we know its error term. For example, in the case of Simpson's rule, we set

$$\Xi = [a, b], \qquad \Phi(f) = \frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right],$$

and

$$A(f) = -\frac{(b-a)^4}{2880} f^{(4)},$$

and then let $\mathcal{F}$ contain functions with continuous fourth derivative on $[a, b]$.

4. Derivation of the reversed results

In the preceding sections, the mean-value theorems of the Lagrange type were known and the mean-value theorems of the Cauchy type were deduced from them. In this section, we prove some new Cauchy type mean-value theorems and deduce therefrom some (possibly new and useful) Lagrange type mean-value theorems.

The following result is connected with Jensen's inequality and its discrete form was proved by Mercer [2] (see also [3]).

Theorem 6. *Let $\mathcal{I}$ be a compact real interval and let $\phi, \psi \in C^2(\mathcal{I})$. Also let h be an integrable function with respect to a normalized weight ω on $[a,b] \subset \mathbb{R}$ such that the range of h is a subset of $\mathcal{I}$. Then, for some $\xi \in \mathcal{I}$,*

$$\frac{\int_a^b \phi\big(h(x)\big)\omega(x)\,dx - \phi\big(\int_a^b h(x)\omega(x)\,dx\big)}{\int_a^b \psi\big(h(x)\big)\omega(x)\,dx - \psi\big(\int_a^b h(x)\omega(x)\,dx\big)} = \frac{\phi''(\xi)}{\psi''(\xi)}, \tag{18}$$

provided that the denominator on the left-hand side of (18) *is non-zero.*

Proof. Define $A := \int_a^b h(x)\omega(x)\,dx$ and (analogously as in [2]) write

$$(Q\phi)(t) = \int_a^b \phi\big(th(x) + (1-t)A\big)\omega(x)\,dx - \phi(A),$$

so that

$$(Q\phi)'(t) = \int_a^b \big(h(x) - A\big)\phi'\big(th(x) + (1-t)A\big)\omega(x)\,dx$$

and

$$(Q\phi)''(t) = \int_a^b \big(h(x) - A\big)^2\phi''\big(th(x) + (1-t)A\big)\omega(x)\,dx.$$

We now consider the function $W(t)$ given by

$$W(t) = (Q\psi)(1)(Q\phi)(t) - (Q\phi)(1)(Q\psi)(t),$$

which immediately yields

$$W(0) = W'(0) = W(1) = 0,$$

so that two applications of the mean-value theorem give us

$$W''(\mu) = 0 \quad \text{for some } \mu \in (0,1).$$

This implies that

$$\int_a^b \big(h(x) - A\big)^2\big[(Q\psi)(1)\phi''\big(\mu h(x) + (1-\mu)A\big) - (Q\phi)(1)\psi''\big(\mu h(x) + (1-\mu)A\big)\big]\omega(x)\,dx = 0.$$

For any fixed μ, the expression in the square brackets is a continuous function of x and hence it vanishes for some value of x in (a,b). Corresponding to this value of $x \in (a,b)$, we get a number $\xi \in \mathcal{I}$ such that

$$(Q\psi)(1)\phi''(\xi) - (Q\phi)(1)\psi''(\xi) = 0,$$

which completes the proof of Theorem 6. $\square$

As our first application of Theorem 6, let us consider the integral power means $M_r(h)$ $(h \geqq 0)$, which are defined as follows:

$$M_r(h) = \left(\int_a^b \left[h(x)\right]^r \omega(x)\, dx \right)^{1/r} \quad (r \neq 0)$$

and

$$M_0(h) = \exp\left(\int_a^b \omega(x) \log\bigl(h(x)\bigr)\, dx \right).$$

Choose

$$\phi(x) = x^{r/s} \quad \text{and} \quad \psi(x) = x^{l/s}$$

in (18) and then put $h(x) = [u(x)]^s$. We thus find that

$$\left| \frac{r(r-s)}{l(l-s)} \right| m \leqq \left| \frac{M_r^r(u) - M_s^r(u)}{M_l^l(u) - M_s^l(u)} \right| \leqq \left| \frac{r(r-s)}{l(l-s)} \right| M, \tag{19}$$

where M and m denote the maximum and minimum of x^{r-l} over the range of $u(x)$. The estimation in (19) is also meaningful in their limiting cases when $s \to 0$, $s \to \infty$, and $s \to -\infty$.

As our second application of Theorem 6, we put $\psi(x) = x^2$ in Theorem 6. Upon rearranging (18), we get

$$\int_a^b \phi\bigl(h(x)\bigr)\omega(x)\, dx - \phi\left(\int_a^b h(x)\omega(x)\, dx \right)$$
$$= \frac{1}{2}\phi''(\xi) \int_a^b \left[h(x) - \int_a^b h(x)\omega(x)\, dx \right]^2 \omega(x)\, dx,$$

which immediately gives Jensen's inequality for a convex (concave) function ψ.

Finally, we give a Cauchy type mean-value theorem which is a generalization of the classical *trapezoid rule* (see, for details, [1]).

Theorem 7. *Let $f, g : [a, b] \to \mathbb{R}$ be two functions, each of which possesses a continuous derivative of order $n \geqq 2$. If*

$$f^{(k)}(a) = g^{(k)}(a) = 0 \quad (k = 2, \ldots, n-2),$$

then

$$\frac{(b-a)\frac{f(a)+f(b)}{2} - \int_a^b f(x)\, dx}{(b-a)\frac{g(a)+g(b)}{2} - \int_a^b g(x)\, dx} = \frac{f^{(n)}(\xi)(\xi - a) + (n-2) f^{(n-1)}(\xi)}{g^{(n)}(\xi)(\xi - a) + (n-2) g^{(n-1)}(\xi)} \tag{20}$$

for some $\xi \in (a, b)$.

Proof. Consider the function

$$(Qf)(t) = \frac{f(t) + f(a)}{2}(t-a) - \int_a^t f(s)\,ds,$$

so that

$$(Qf)'(t) = \frac{1}{2}f'(t)(t-a) - \frac{1}{2}\big(f(t) - f(a)\big)$$

and

$$(Qf)^{(k)}(t) = \frac{1}{2}\big[f^{(k)}(t)(t-a) + (k-2)f^{(k-1)}(t)\big] \quad (2 \leqq k \leqq n).$$

We note that

$$(Qf)(a) = (Qf)'(a) = (Qf)''(a) = 0.$$

We now consider the function $W(t)$ given by

$$W(t) = (Qg)(b)(Qf)(t) - (Qf)(b)(Qg)(t),$$

which readily yields

$$W(a) = W'(a) = \cdots = W^{(n-1)}(a) = W(b) = 0,$$

so that n successive applications of the mean-value theorem give us

$$W^{(n)}(\xi) = 0 \quad \text{for some } \xi \in (a,b).$$

This evidently completes the proof of Theorem 7. □

By setting $g(x) = x^2$ and $n = 2$ (in which case we do not have any boundary conditions), our assertion (20) of Theorem 7 reduces to the classical *trapezoidal rule*. Furthermore, by letting

$$g(x) = (x-a)^n$$

and assuming that

$$f^{(n-1)}(a) = 0$$

in Theorem 7, the following estimation would result from the assertion (20) of Theorem 7:

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \leqq \frac{(n-1)(b-a)^n}{2 \cdot (n+1)!} M_n,$$

where

$$M_n = \max_{x \in [a,b]} f^{(n)}(x).$$

Acknowledgments

The present investigation was supported, in part, by the *Natural Sciences and Engineering Research Council of Canada* under Grant OGP0007353.

References

[1] P.J. Davis, P. Rabinowitz, Methods of Numerical Integration, second ed., Series on Computer Science and Applied Mathematics, Academic Press, New York, 1984.
[2] A.McD. Mercer, Some new inequalities involving elementary mean values, J. Math. Anal. Appl. 229 (1999) 677–681.
[3] A.McD. Mercer, An "error term" for the Ky Fan inequality, J. Math. Anal. Appl. 220 (1998) 774–777.
[4] J. Pečarić, P.J.Y. Wong, Polynomial interpolation and generalizations of mean value theorem, Nonlinear Funct. Anal. Appl. 6 (2001) 329–340.
[5] I. Podlubny, Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications, Mathematics in Science and Engineering, vol. 198, Academic Press, New York, 1999.
[6] J.J. Trujillo, M. Rivero, B. Bonilla, On a Riemann–Liouville generalized Taylor's formula, J. Math. Anal. Appl. 231 (1999) 255–265.