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Local completeness, drop theorem and Ekeland's variational principle

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Abstract

By using a very general drop theorem in locally convex spaces we obtain some extended versions of Ekeland's variational principle, which only need assume local completeness of some related sets and improve Hamel's recent results. From this, we derive some new versions of Caristi's fixed points theorems. In the framework of locally convex spaces, we prove that Daneš' drop theorem, Ekeland's variational principle, Caristi's fixed points theorem and Phelps lemma are equivalent to each other.

Keywords: Locally convex spaces; Local completeness; Drop theorem; Ekeland's variational principle

1. Introduction

Let $(X, \| \|)$ be a Banach space and $B(X)$ be its closed unit ball $\{x \in X\colon \|x\| \leqslant 1\}$. For any $x_0 \notin B(X)$, the convex hull of the set $\{x_0\} \cup B(X)$ is called a drop determined by the point x_0 and $B(X)$ and it is denoted by $D(x_0, B(X))$. If a nonempty closed subset A of X at a positive distance from the closed unit ball $B(X)$ is given, then there exists $a \in A$ such that $D(a, B(X)) \cap A = \{a\}$, which is the so-called Danes' drop theorem; see [1]. The drop theorem was used in various situations (see, for instance, [2–6]) and it is equivalent to Ekeland's variational principle (see [7]). In the framework of locally convex spaces (here and

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in the following, a locally convex space always means a Hausdorff locally convex topological vector space), Cheng, Zhou, and Zhang [8], Mizoguchi [9] and Zheng [10] obtained various kinds of drop theorem and deduced the corresponding versions of Ekeland's variational principle. Recently Hamel [11] proved a drop theorem in locally convex spaces as follows.

Theorem 1.1 [11, Theorem 7]. *Let X be a sequentially complete locally convex space. Let $A \subset X$ be a nonempty sequentially closed set and $B \subset X$ a nonempty sequentially closed bounded convex set. Let $\{p_\lambda\}_{\lambda \in \Lambda}$ be a family of seminorms defining the topology on X (see, for instance,* [12, Chapter 2] *or* [13, I, pp. 203–204]) *and assume that there exist $\mu \in \Lambda$, $\delta > 0$ such that $p_\mu(a - b) \geqslant \delta$, $\forall a \in A$, $\forall b \in B$. Then for each $x_0 \in A$, there exists a point $a \in D(x_0, B) \cap A$ such that $D(a, B) \cap A = \{a\}$.*

Here a seminorm family $\{p_\lambda\}_{\lambda \in \Lambda}$ defining the topology on X means that the system $\{\bigcap_{i=1}^{n}(p_{\lambda_i} < \epsilon)\colon n \in N, \lambda_i \in \Lambda, \epsilon > 0\}$ forms a base of 0-neighborhoods in X. Obviously, the condition that there exists $\mu \in \Lambda$, $\delta > 0$ such that $p_\mu(a - b) \geqslant \delta$, $\forall a \in A$, $\forall b \in B$, is equivalent to one that $0 \notin \mathrm{cl}(A - B)$. Hamel also gave the following two versions of Ekeland's variational principle in locally convex spaces.

Theorem 1.2 [11, Theorem 3]. *Let X be a sequentially complete locally convex space. Let $f : X \to (-\infty, +\infty]$ be a sequentially lower semicontinuous proper function, bounded from below. Let $\{p_\lambda\}_{\lambda \in \Lambda}$ be a family of seminorms defining the topology on X and $\{\alpha_\lambda\}_{\lambda \in \Lambda}$ a family of positive real numbers. Then for each $x_0 \in \operatorname{dom} f$ there exists $z \in X$ such that*

(i) $f(z) + \alpha_\lambda p_\lambda(z - x_0) \leqslant f(x_0)$ *for all* $\lambda \in \Lambda$;
(ii) *for any $x \neq z$, there exists $\mu \in \Lambda$ such that*

$$f(z) < f(x) + \alpha_\mu p_\mu(x - z).$$

Theorem 1.3 [11, Theorem 2]. *Let X be a sequentially complete locally convex space. Let $f : X \to (-\infty, +\infty]$ be a sequentially lower semicontinuous proper function, bounded from below. Let $S \subset X$ be a sequentially closed bounded convex set such that $0 \in S$. Then, for each $\alpha > 0$, $x_0 \in \operatorname{dom} f$, there exists $z \in X$ such that*

(i) $f(z) + \alpha p_S(z - x_0) \leqslant f(x_0)$;
(ii) *for any $x \neq z$, $f(z) < f(x) + \alpha p_S(x - z)$.*

Here p_S denotes the Minkowski functional of S.

In Theorems 1.1–1.3, the assumption that X is sequentially complete cannot be omitted. As is well known, for locally convex spaces there are various kinds of completeness, for example, completeness, quasicompleteness, sequential completeness, Σ-completeness, l^∞-completeness, local completeness and so on; for details, please refer to [14, Chapter 5] and [15]. Up to now, we know that local completeness is the weakest kind of completeness. In [16] we proved a very general version of the drop theorem in locally convex spaces, which only needs the assumption on local completeness of some related sets.

Theorem 1.4 [16, Theorem 3.1]. *Let A be a locally closed subset of a locally convex space X and B a locally closed, bounded convex subset of X. Moreover, assume that there exists a locally convex topology τ on X such that $0 \notin \mathrm{cl}_\tau(A - B)$, where $\mathrm{cl}_\tau(A - B)$ denotes the τ-closure of $A - B$. Then for each $x_0 \in A$, there exists $a \in D(x_0, B) \cap A$ such that $D(a, B) \cap A = \{a\}$ provided that either of the following conditions is satisfied*:

(i) *the local closure of $B \cap L(A)$ is locally complete, where $L(A)$ denotes the linear manifold generated by A*;
(ii) *A is locally complete.*

From Theorem 1.4, we have the following deductions.

Theorem 1.5 (Refer to [16, Corollary 3.1]). *Let A be a locally closed subset of a locally convex space X and B a locally closed, bounded convex subset of X with $0 \notin \mathrm{cl}(A - B)$. If either A or B is locally complete, then for each $x_0 \in A$, there exists $a \in D(x_0, B) \cap A$ such that $D(a, B) \cap A = \{a\}$.*

Theorem 1.6 (Refer to [16, Corollary 3.2]). *Let X be a locally complete locally convex space, A be a locally closed subset of X and B be a locally closed, bounded convex subset of X. If $0 \notin \mathrm{cl}(A - B)$, then for each $x_0 \in A$, there exists $a \in D(x_0, B) \cap A$ such that $D(a, B) \cap A = \{a\}$.*

In Section 2, we review the notions of locally complete sets and locally closed sets. We shall see that a sequentially complete locally convex space is locally complete and a sequentially closed set is locally closed; but neither of the two converses is true. Hence the assumption in Theorem 1.6 (respectively, in Theorems 1.4 and 1.5) is strictly weaker than the assumption in Theorem 1.1. In Section 3, following the way of [7], we use Theorem 1.5 to deduce two new versions of Ekeland's variational principle, which improve Theorems 1.2 and 1.3. By using the improved Ekeland's variational principles, we obtain two extended versions of Caristi's fixed theorem. In Section 4, we point out that the two versions of Ekeland's variational principle, the two versions of Caristi's fixed theorem and the drop Theorem 1.5 are equivalent to each other. In Section 5, we give a direct proof of a general Phelps lemma in locally convex spaces. Moreover, we prove the equivalence between the Phelps lemma and the Ekeland's variational principle.

2. Sequential completeness and local completeness

In this section, we recall some basic facts concerning sequential completeness and local completeness (for example, see [14, Chapter 5]). Let X be a locally convex space and X^* be its topological dual. A locally convex space is said to be sequentially complete if every Cauchy sequence in X is convergent. For brevity, we call a bounded absolutely convex set B a disc. Denote $\mathrm{sp}[B]$ the vector subspace spanned by B and denote p_B the Minkowski functional of B, then $E_B := (\mathrm{sp}[B], p_B)$ is a normed space. If E_B is a Banach space, then B is called a Banach disc. A sequence $\{x_n\}$ in X is said to be locally convergent to an element

x if there is a disc B in X such that the sequence $\{x_n\}$ is convergent to x in E_B and $\{x_n\}$ is said to be locally Cauchy if there is a disc B in X such that $\{x_n\}$ is a Cauchy sequence in E_B. In [12, pp. 225–226], a locally convergent sequence is called a convergent sequence in Mackey sense and some properties of locally convergent sequences were investigated.

Definition 2.1 [14, Chapter 5]. A locally convex space X is locally complete if every locally Cauchy sequence is locally convergent. This is equivalent to that each bounded subset of X is contained in a certain Banach disc. Let A be a nonempty subset of X, then A is said to be locally complete if every locally Cauchy sequence in A is locally convergent to a point in A. And A is said to be locally closed if for any locally convergent sequence in A, its local limit point belongs to A.

It is easy to prove that every sequentially complete disc is a Banach disc (see [14, Corollary 3.2.5], [17, pp. 91–92], or [18, Theorem 6-1-17]). From this we know:

Theorem 2.1 (See [13, II, p. 135] or [14, Corollary 5.1.8]). *Every sequentially complete locally convex space is locally complete.*

As shown by [14, Example 5.1.12], the converse of Theorem 2.1 is not true. In fact, since local completeness is invariant for all compatible locally convex topologies of the dual pair (X, X^*) (see [14, Corollary 5.1.7]), we can easily construct a locally complete locally convex space which is not sequentially complete. For example, see [14, Example 5.1.12], [15, Example 1], [16, Example 3.1], and [18, Problem 10-2-119].

Similarly we see that every sequentially closed set is locally closed, but the converse is not true (see [16, Example 3.1]). A proper function $f : X \to (-\infty, +\infty]$ is called a locally lower semicontinuous if for each $r \in R$, the set $\{x \in X\colon\ f(x) \leqslant r\}$ is locally closed in X. Clearly every sequentially lower semicontinuous function is locally lower semicontinuous and the converse is not true.

3. Ekeland's variational principle in locally convex spaces

In this section, motivated by the paper of Penot [7], we use Theorem 1.5 to deduce two versions of Ekeland's principle in locally convex spaces, which improve Theorems 1.2 and 1.3, respectively.

Theorem 3.1. *Let X be a locally convex space, $\{p_\lambda\}_{\lambda\in\Lambda}$ be a family of seminorms defining the topology on X and $\{\alpha_\lambda\}_{\lambda\in\Lambda}$ be a family of positive real numbers. Let $f : X \to (-\infty, +\infty]$ be a locally lower semicontinuous, bounded from below, proper function and let $x_0 \in \operatorname{dom} f$. Assume that the set $\bigcap_{\lambda\in\Lambda}\{x \in X\colon\ \alpha_\lambda p_\lambda(x) \leqslant 1\}$ or the set $\{x \in X\colon\ f(x) \leqslant f(x_0)\}$ is locally complete, then there exists $z \in X$ such that*

(i) $f(z) + \alpha_\lambda p_\lambda(z - x_0) \leqslant f(x_0)$ *for all* $\lambda \in \Lambda$,
(ii) *for any $x \neq z$, there exists $\mu \in \Lambda$ such that $f(z) < f(x) + \alpha_\mu p_\mu(x - z)$.*

Proof. Without loss of generality, we may assume that $x_0 = 0$ and $f(x_0) = 0$. Put $E := X \times R$ with the product topology, then the topology can be generated by a family $\{q_\lambda\}_{\lambda \in \Lambda}$ of seminorms, where $q_\lambda(x,t) = p_\lambda(x) + |t|$, $\forall (x,t) \in E = X \times R$. Let A be the set $\{(x,t) \in E\colon\ f(x) \leqslant t \leqslant 0\}$ and let $m := \inf\{t\colon\ (x,t) \in A\}$, then $-\infty < m \leqslant 0$. Take any fixed real number $r < m$ and put

$$B := \left\{(x,r) \in E\colon\ p_\lambda(x) \leqslant \frac{-r}{\alpha_\lambda},\ \forall \lambda \in \Lambda\right\},$$

then $K := \operatorname{cone}(B)$ is exactly the set $\{(y,t) \in E\colon\ -t \geqslant \alpha_\lambda p_\lambda(y),\ \forall \lambda \in \Lambda\}$, where $\operatorname{cone}(B)$ denotes the cone generated by B, i.e.

$$\operatorname{cone}(B) := \{\alpha(x,r)\colon\ \alpha \geqslant 0,\ (x,r) \in B\}.$$

By the assumption we see that either A or B is locally complete, B is a bounded closed convex subset of E and $q_\lambda(A - B) \geqslant m - r > 0$. By Theorem 1.5, there exists

$$(z,s) \in A \cap D\big((0,0), B\big) \subset A \cap K \tag{1}$$

such that

$$A \cap D\big((z,s), B\big) = \big\{(z,s)\big\}. \tag{2}$$

From (1), $(z,s) \in A \cap K$, hence

$$f(z) \leqslant s \leqslant 0 \tag{3}$$

and for all $\lambda \in \Lambda$,

$$-s \geqslant \alpha_\lambda p_\lambda(z). \tag{4}$$

Combining (3) and (4), we have

$$-f(z) \geqslant -s \geqslant \alpha_\lambda p_\lambda(z), \quad \forall \lambda \in \Lambda. \tag{5}$$

Remarking the assumption that $x_0 = 0$ and $f(x_0) = 0$, we can write (5) as

$$f(x_0) - f(z) \geqslant \alpha_\lambda p_\lambda(z - x_0), \quad \forall \lambda \in \Lambda.$$

That is, the result (i) holds. By (3) and the meanings of r and m, we have

$$r < m \leqslant f(z) \leqslant s \leqslant 0.$$

Put

$$\delta := \frac{f(z) - r}{s - r}, \quad \text{then } 0 < \delta \leqslant 1.$$

It is easy to verify that

$$\delta s + (1 - \delta) r = \frac{f(z) - r}{s - r} s + \frac{s - f(z)}{s - r} r = f(z).$$

Hence

$$\big(z, f(z)\big) = \big(z, \delta s + (1 - \delta) r\big) = \delta(z,s) + (1 - \delta)(z,r). \tag{6}$$

By (5),

$$p_\lambda(z) \leqslant \frac{-f(z)}{\alpha_\lambda} < \frac{-r}{\alpha_\lambda}, \quad \forall \lambda \in \Lambda,$$

which means that

$$(z, r) \in B. \tag{7}$$

Combining (6) and (7), we have $(z, f(z)) \in D((z, s), B)$. Also, clearly $(z, f(z)) \in A$. Hence we have

$$\big(z, f(z)\big) \in A \cap D\big((z, s), B\big).$$

On the other hand, by (2), we have

$$A \cap D\big((z, s), B\big) = \big\{(z, s)\big\}.$$

Thus we have shown that $(z, f(z)) = (z, s)$ and $s = f(z)$.

For any $x \in X$, $x \neq z$, we consider the following two cases:

Case 1. Let $(x, f(x)) \notin A$, then $f(x) > 0$. Thus for all $\lambda \in \Lambda$,

$$f(x) + \alpha_\lambda p_\lambda(z - x) \geqslant f(x) > 0 \geqslant f(z).$$

Case 2. Let $(x, f(x)) \in A$, we shall show that $(x, f(x)) \notin (z, s) + K$. If not, we assume that $(x - z, f(x) - s) \in K$, i.e.

$$s - f(x) \geqslant \alpha_\lambda p_\lambda(x - z) \quad \text{for all } \lambda \in \Lambda.$$

Since $x \neq z$ and $\{p_\lambda\}_{\lambda\in\Lambda}$ separates points in X, we conclude that $s - f(x) > 0$. Put

$$\eta := \frac{s - f(x)}{s - r}, \quad \text{then } 0 < \eta < 1.$$

Since K is a cone,

$$\left(\frac{x - z}{\eta}, \frac{f(x) - s}{\eta}\right) \in K,$$

that is,

$$\left(\frac{x - z}{\eta}, r - s\right) \in K. \tag{8}$$

By (1),

$$(z, s) \in K. \tag{9}$$

Since K is a convex cone, by (8) and (9) we have

$$(z, s) + \left(\frac{x - z}{\eta}, r - s\right) \in K, \quad \text{i.e.}$$

$$\left(z + \frac{x - z}{\eta}, r\right) \in K \cap \big(X \times \{r\}\big) = B.$$

It is easy to verify that

$$(1 - \eta)s + \eta r = \frac{f(x) - r}{s - r}s + \frac{s - f(x)}{s - r}r = f(x),$$

hence

$$\big(x, f(x)\big) = (1-\eta)(z,s) + \eta\left(z + \frac{x-z}{\eta}, r\right) \in D\big((z,s), B\big).$$

Thus we have

$$\big(x, f(x)\big) \in D\big((z,s), B\big) \cap A.$$

By (2),

$$A \cap D\big((z,s), B\big) = \big\{(z,s)\big\},$$

which leads to that $(x, f(x)) = (z,s)$ and hence $x = z$, a contradiction. This shows that $(x, f(x)) \notin (z,s) + K = (z, f(z)) + K$, i.e. there exists $\mu \in \Lambda$ such that

$$f(z) - f(x) < \alpha_\mu p_\mu(x - z).$$

That is to say, the result (ii) holds. $\square$

If X is locally complete, then both $\bigcap_{\lambda\in\Lambda}\{x \in X\colon \alpha_\lambda p_\lambda(x) \leqslant 1\}$ and $\{x \in X\colon f(x) \leqslant f(x_0)\}$ are locally complete. Hence the following corollary is direct.

Corollary 3.1. *Let X be a locally complete locally convex space, $\{p_\lambda\}_{\lambda\in\Lambda}$ be a family of seminorms defining the topology on X and $\{\alpha_\lambda\}_{\lambda\in\Lambda}$ be a family of positive real numbers. Let $f : X \to (-\infty, +\infty]$ be a locally lower semicontinuous, bounded from below, proper function and let $x_0 \in \operatorname{dom} f$. Then there exists $z \in X$ such that*

(i) $f(z) + \alpha_\lambda p_\lambda(z - x_0) \leqslant f(x_0)$ *for all* $\lambda \in \Lambda$;
(ii) *for any $x \neq z$, there exists $\mu \in \Lambda$ such that*

$$f(z) < f(x) + \alpha_\mu p_\mu(x - z).$$

Let $S \subset X$ be a convex set containing 0. As usual, we define the Minkowski functional of S to be

$$p_S(x) := \begin{cases} \inf\{\alpha > 0\colon x \in \alpha A\}, & \text{if there exists } \alpha > 0 \text{ such that } x \in \alpha A;\\ +\infty, & \text{if } x \notin \alpha A \text{ for all } \alpha > 0.\end{cases}$$

When the perturbation function is the Minkowski functional of a bounded set, we can also use Theorem 1.5 to deduce the following Theorem 3.2, which improves Theorem 1.3.

Theorem 3.2. *Let X be a locally convex space, $S \subset X$ be a locally closed, bounded convex set containing 0, α be a positive real number, $f : X \to (-\infty, +\infty]$ be a locally lower semicontinuous, bounded from below, proper function and $x_0 \in \operatorname{dom} f$. If the set $\{x \in X\colon f(x) \leqslant f(x_0)\}$ or S is locally complete, then there exists $z \in \operatorname{dom} f$ such that*

(i) $f(z) + \alpha p_S(z - x_0) \leqslant f(x_0)$;
(ii) *for any $x \neq z$, $f(z) < f(x) + \alpha p_S(x - z)$.*

The proof of Theorem 3.2 is similar to that of Theorem 3.1 and we omit it. From Theorem 3.2 we immediately obtain the following:

Corollary 3.2. *Let X be a locally complete locally convex space, $S \subset X$ be a locally closed, bounded convex set containing* 0, *α be a positive real number, and $f : X \to (-\infty, +\infty]$ be a locally lower semicontinuous, bounded from below, proper function. Then for each $x_0 \in \operatorname{dom} f$, there exists $z \in \operatorname{dom} f$ such that*

(i) $f(z) + \alpha p_S(z - x_0) \leqslant f(x_0)$;
(ii) *for any $x \neq z$, $f(z) < f(x) + \alpha p_S(x - z)$.*

Obviously Corollaries 3.1 and 3.2 improve Theorems 1.2 and 1.3 (see Section 2), respectively. Mizoguchi [9] and Fang [19] considered the extended versions of Caristi's fixed point theorem [20] in complete uniform spaces and in sequentially complete topological vector spaces, respectively. Here, from Theorems 3.1 and 3.2 we obtain the following two versions of Caristi's fixed point theorem in locally convex spaces.

Corollary 3.3. *Let X be a locally convex space, $\{p_\lambda\}_{\lambda \in \Lambda}$ be a family of seminorms defining the topology on X, $\{\alpha_\lambda\}_{\lambda \in \Lambda}$ be a family of positive real numbers and $f : X \to (-\infty, +\infty]$ be a locally lower semicontinuous, bounded from below, proper function. Moreover, assume that the set $\{x \in X\colon\ \alpha_\lambda p_\lambda(x) \leqslant 1,\ \forall \lambda \in \Lambda\}$ is locally complete or assume that there exists $x_0 \in \operatorname{dom} f$ such that $\{x \in X\colon\ f(x) \leqslant f(x_0)\}$ is locally complete* (*particularly we may assume that X is locally complete*). *If $T : X \to 2^X$ has the property that for each $x \in X$ and $y \in Tx$,*

$$\alpha_\lambda p_\lambda(x - y) + f(y) \leqslant f(x), \quad \forall \lambda \in \Lambda;$$

then there exists $z \in Tx_0$ such that $Tz = \{z\}$.

Corollary 3.4. *Let X be a locally convex space, $S \subset X$ be a locally closed, bounded convex set containing* 0, *α be a positive real number and $f : X \to (-\infty, +\infty]$ be a locally lower semicontinuous, bounded from below, proper function. Moreover, assume that S is locally complete or assume that there exists $x_0 \in \operatorname{dom} f$ such that $\{x \in X\colon\ f(x) \leqslant f(x_0)\}$ is locally complete* (*particularly we may assume that X is locally complete*). *If $T : X \to 2^X$ has the property that for each $x \in X$ and $y \in Tx$,*

$$\alpha p_S(y - x) + f(y) \leqslant f(x);$$

then there exists $z \in Tx_0$ such that $Tz = \{z\}$.

4. Equivalences between drop theorem, Ekeland's variational principle and Caristi's fixed point theorem

In Section 3 by using Theorem 1.5 we obtained Theorems 3.1 and 3.2, the two different versions of Ekeland's variational principle in locally convex spaces. In fact they are equivalent.

Theorem 4.1. *Theorems* 3.1 *and* 3.2 *are mutually equivalent.*

Proof. First we show that Theorem 3.2 implies Theorem 3.1. Put

$$S = \bigcap_{\lambda\in\Lambda} \bigl\{x \in X\colon\ \alpha_\lambda p_\lambda(x) \leqslant 1\bigr\},$$

then $S \subset X$ is a bounded, closed absolutely convex set. Let p_S be the Minkowski functional of S, then

$$p_S(x) = \sup_{\lambda\in\Lambda} \alpha_\lambda p_\lambda(x), \quad \forall x \in X. \tag{10}$$

By the assumption that S or $\{x \in X\colon\ f(x) \leqslant f(x_0)\}$ is locally complete, then by Theorem 3.2 (taking $\alpha = 1$) we have $z \in X$ such that

(i) $f(z) + p_S(z - x_0) \leqslant f(x_0)$;
(ii) for any $x \neq z$, $f(z) < f(x) + p_S(x - z)$.

Remarking (10), we know that (i) and (ii) in Theorem 3.1 hold.

Conversely we can prove that Theorem 3.1 implies Theorem 3.2. From Theorem 3.1 we easily deduce the following proposition $(*)$:

Let $(X, \|\ \|)$ be a normed space and $f : (X, \|\ \|) \to (-\infty, +\infty]$ be a lower semicontinuous, bounded from below, proper function and $x_0 \in \operatorname{dom} f$. If $(X, \|\ \|)$ is complete or the set $\{x \in X\colon\ f(x) \leqslant f(x_0)\}$ is complete, then for any $\alpha > 0$, there exists $z \in X$ such that

(i) $f(z) + \alpha\|z - x_0\| \leqslant f(x_0)$;
(ii) for any $x \neq z$, $f(z) < f(x) + \alpha\|x - z\|$.

Let $T = \Gamma(x_0, S)$ be the absolutely convex hull of $S \cup \{x_0\}$. Then $(X_T, \|\ \|_T)$ is a normed space. Put

$$C = \bigl\{x \in X_T\colon\ f(x) + \alpha p_S(x - x_0) \leqslant f(x_0)\bigr\},$$

then C is closed in $(X_T, \|\ \|_T)$ since f and p_S are locally lower semicontinuous. Define a function g on X_T as following:

$$g(x) = \begin{cases} f(x), & \text{if } x \in C, \\ +\infty, & \text{if } x \in X_T \setminus C. \end{cases} \tag{11}$$

Then g is a bounded from below, lower semicontinuous proper function and $x_0 \in \operatorname{dom} g$. If S is locally complete, then T is a Banach disk and $(X_T, \|\ \|_T)$ is a Banach space. If the set $\{x \in X\colon\ f(x) \leqslant f(x_0)\}$ is locally complete, then $\{x \in X_T\colon\ g(x) \leqslant g(x_0)\} = C \cap \{x \in X\colon\ f(x) \leqslant f(x_0)\}$ is a complete set in $(X_T, \|\ \|_T)$. By proposition $(*)$, there exists $z \in X_T$ such that

(i) $g(z) + \alpha\|z - x_0\|_T \leqslant g(x_0) = f(x_0)$;
(ii) for any $x \in X_T$ and $x \neq z$,

$$g(z) < g(x) + \alpha\|x - z\|_T. \tag{12}$$

From (i) we know that $g(z) < \infty$, and hence $z \in C$, that is,

$$f(z) + \alpha p_S(z - x_0) \leqslant f(x_0). \tag{13}$$

Thus the result (i) in Theorem 3.2 holds. Next we show that the result (ii) in Theorem 3.2 according to the following three cases:

Case 1. Let $x \neq z$ and $x \in C$, then (12) becomes

$$f(z) < f(x) + \alpha \|x - z\|_T \leqslant f(x) + \alpha p_S(x - z).$$

Case 2. Let $x \neq z$ and $x \in X_T \setminus C$, then by the definition of C we have

$$f(x) + \alpha_S(x - x_0) > f(x_0).$$

Combining this with (13), we have

$$\begin{aligned} f(z) + \alpha p_S(z - x_0) &\leqslant f(x_0) < f(x) + \alpha p_S(x - x_0) \\ &\leqslant f(x) + \alpha p_S(x - z) + \alpha p_S(z - x_0). \end{aligned} \tag{14}$$

From (13) we know $\alpha p_S(z - x_0) < \infty$. By subtracting $\alpha p_S(z - x_0)$ from the two sides of (14), we have $f(z) < f(x) + \alpha p_S(z - x_0)$.

Case 3. Let $x \neq z$ and $x \notin X_T$, then $p_S(z - x_0) = +\infty$ and certainly $f(z) < f(x) + \alpha p_S(z - x_0)$. □

As shown in Section 3, we see that the drop theorem (Theorem 1.5) implies the two versions of Ekeland's variational principle (i.e. Theorems 3.1 and 3.2). Now Theorem 4.1 points out that the two versions are mutually equivalent. Moreover, we shall see that the two versions of Ekeland's variational principle and the drop theorem are equivalent to each other.

Theorem 4.2. *Theorems* 3.1, 3.2, *and* 1.5 *are equivalent to each other.*

Proof. It is sufficient to prove that Theorem 3.2 implies Theorem 1.5. The proof is similar to one of Theorem 2 in [21]. Here for the sake of completeness we sketch out the main points. Without loss of generality we may assume that $0 \in B$. Since $0 \notin \mathrm{cl}(A - B)$, there exists a closed absolutely convex 0-neighborhood W such that

$$(A - B) \cap (3W) = \emptyset \quad \text{or} \quad (B + 3W) \cap A = \emptyset. \tag{15}$$

Denote $\Gamma(x_0, B)$ the absolutely convex hull of the set $\{x_0\} \cup B$, then there is α, $0 < \alpha < 1$, such that $\alpha \Gamma(x_0, B) \subset W$. Let G be the local closure of the set $B + \alpha \Gamma(x_0, B)$ and p be the Minkowski functional of G. Clearly

$$\frac{\alpha}{2}\big(B + \alpha \Gamma(x_0, B)\big) \subset \frac{1}{2}W + \frac{1}{2}W = W \quad \text{and} \quad \frac{\alpha}{2}G \subset W.$$

Thus

$$G + \frac{\alpha}{2}G \subset B + \alpha \Gamma(x_0, B) + W + W \subset B + W + W + W = B + 3W.$$

Combining this with (15), we have

$$\left(G + \frac{\alpha}{2}G\right) \cap A = \emptyset.$$

This yields that

$$p(x) \geqslant 1 + \frac{\alpha}{2}, \quad \forall x \in A. \tag{16}$$

Define f as follows: $f(x) = p(x)$ for any $x \in D(x_0, B) \cap A$; or else $f(x) = +\infty$. Then f is locally lower semicontinuous and bounded from below. Since

$$x_0 \in \frac{1}{\alpha}\big(\alpha\Gamma(x_0, B)\big) \subset \frac{1}{\alpha}G \quad \text{and} \quad x_0 \in D(x_0, B) \cap A,$$

we have $f(x_0) = p(x_0) \leqslant 1/\alpha$. Thus

$$\big\{x \in X\colon\ f(x) \leqslant f(x_0)\big\} = p(x_0)G \cap D(x_0, B) \cap A.$$

If A or B is locally complete, then $\{x \in X\colon\ f(x) \leqslant f(x_0)\}$ is locally complete. By using Theorem 3.2 (α is replaced by $\alpha^2/4$ and S is replaced by G), we know that there exists a point $z \in \operatorname{dom} f = D(x_0, B) \cap A$ such that

$$\frac{\alpha^2}{4} p(x - z) + f(x) > f(z), \quad \forall x \in X \text{ and } x \neq z. \tag{17}$$

For any $x \in D(z, B) \cap A$, we may write $x = tz + (1-t)b$, where $b \in B \subset G$ and $0 \leqslant t \leqslant 1$. Clearly $\alpha(b/2 - z/2) \in \alpha\Gamma(x_0, B) \subset G$. Thus

$$p\left(\alpha\left(\frac{1}{2}b - \frac{1}{2}z\right)\right) \leqslant 1 \quad \text{and} \quad p(b) \leqslant 1.$$

Now we have

$$\begin{aligned}
f(x) + \frac{\alpha^2}{4} p(x - z) &= p\big(tz + (1-t)b\big) + \frac{\alpha^2}{4} p\big((1-t)(b - z)\big) \\
&\leqslant tp(z) + (1-t)p(b) + \frac{\alpha}{2}(1-t)p\left(\frac{\alpha}{2}(b - z)\right) \\
&\leqslant tp(z) + (1-t) + \frac{\alpha}{2}(1-t) \\
&= tp(z) + (1-t)\left(1 + \frac{\alpha}{2}\right) \\
&\leqslant tp(z) + (1-t)p(z) \\
&= p(z) = f(z).
\end{aligned}$$

Combining this with (17), we conclude that $x = z$. This completes the proof. □

From Theorems 3.1 and 3.2 we deduce respectively Corollaries 3.3 and 3.4, the two versions of Caristi's fixed point theorem in locally convex spaces. Conversely we shall prove that Corollary 3.3/Corollary 3.4 implies Theorem 3.1/Theorem 3.2, respectively. Combining this with Theorems 4.1 and 4.2 we know that Theorems 1.5, 3.1, 3.2, Corollaries 3.3 and 3.4 are equivalent to each other.

Theorem 4.3. *Theorem* 3.1 *and Corollary* 3.3 *are mutually equivalent.*

Proof. It is sufficient to prove that Corollary 3.3 implies Theorem 3.1. Define $T: X \to 2^X$ as follows:

$$Tx = \{y \in X\colon\ \alpha_\lambda p_\lambda(y-x) + f(y) \leqslant f(x),\ \forall \lambda \in \Lambda\}.$$

Obviously, for any $x \in X$, $Tx \neq \emptyset$. And for each $x \in X$ and $y \in Tx$,

$$\alpha_\lambda p_\lambda(x-y) + f(y) \leqslant f(x), \quad \forall \lambda \in \Lambda.$$

By Corollary 3.3, there exists $z \in Tx_0$ such that $Tz = \{z\}$. Since $z \in Tx_0$, we have

$$\alpha_\lambda p_\lambda(z-x_0) + f(z) \leqslant f(x_0), \quad \forall \lambda \in \Lambda.$$

That is, the result (i) in Theorem 3.1 holds. Since $Tz = \{z\}$, for any $x \in X$, $x \neq z$, we have $x \notin Tz$. That is, there exists $\mu \in \Lambda$, such that $\alpha_\mu p_\mu(x-z) + f(x) > f(z)$. Hence the result (ii) in Theorem 3.1 holds. □

Similarly we can prove the following:

Theorem 4.4. *Theorem* 3.2 *and Corollary* 3.4 *are mutually equivalent.*

5. The equivalence between Phelps lemma and Ekeland's variational principle

Phelps obtained a lemma known as his name in complete locally convex spaces [22]. Hamel [11, Theorem 1] gave a generalization of Phelps lemma to sequentially complete locally convex spaces and proved the equivalence between the Phelps lemma and the Ekeland's variational principle. For the case of complete metric spaces, the equivalence can be found in [23]. In this section we shall give an improved version of Hamel's result and prove that the version is equivalent to Theorem 3.2. First we give some lemmas.

Lemma 5.1. *Let $(X, \|\,\|)$ be a normed space and $B \subset X$ be a bounded closed convex set with $0 \notin B$. Let $K = \mathrm{cone}(B) := \{x \in X\colon\ \exists \alpha \geqslant 0,\ b \in B$ such that $x = \alpha b\}$, then K is a closed convex cone. Moreover, if B is complete, then K is complete.*

Proof. On the proof of K being a closed convex cone, see [24, p. 121]. Now assume that B is complete, we show below that K is complete. Let $\{x_n\} \subset K$ be a Cauchy sequence. We may assume that $x_n = \lambda_n b_n$, $\lambda_n \geqslant 0$, $b_n \in B$, $\forall n \in N$. If there exists a subsequence $\{\lambda_{n_i}\}$ of $\{\lambda_n\}$ such that $\lambda_{n_i} \to 0$, as $i \to \infty$, then $x_{n_i} = \lambda_{n_i} b_{n_i} \to 0$, as $i \to \infty$. Thus $x_n \to 0$, as $n \to \infty$ and $0 \in K$. Or else, we may assume that there is $m \in N$ such that $\inf\{\lambda_n\colon\ n \geqslant m\} = \eta > 0$. For convenience, we assume that $\lambda_n \geqslant \eta > 0$, $\forall n \in N$. Since the Cauchy sequence $\{x_n\}$ is bounded, there exists $\beta > 0$ such that $\|x_n\| \leqslant \beta$, $\forall n \in N$. And since $0 \notin B$ and B is closed, there exists $\delta > 0$ such that $\|b\| \geqslant \delta > 0$, $\forall b \in B$. Thus we have:

$$\lambda_n \delta \leqslant \lambda_n \|b_n\| = \|\lambda_n b_n\| = \|x_n\| \leqslant \beta.$$

From this,

$$\lambda_n \leqslant \frac{\beta}{\delta} \quad \text{and hence} \quad \{\lambda_n\} \subset \left[\eta, \frac{\beta}{\delta}\right].$$

By the compactness of $[\eta, \beta/\delta]$, there exists a subsequence $\{\lambda_{n_i}\}$ of $\{\lambda_n\}$ such that $\lambda_{n_i} \to \lambda_0 \in [\eta, \beta/\delta]$. Observe that $\{\lambda_{n_i} b_{n_i}\}_{i \in N}$ is a Cauchy sequence and $(\lambda_{n_i} - \lambda_0) b_{n_i} \to 0$, as $i \to \infty$. We conclude that $\{\lambda_0 b_{n_i}\} = \{\lambda_{n_i} b_{n_i}\} + \{(\lambda_0 - \lambda_{n_i}) b_{n_i}\}$ is still a Cauchy sequence. Since $\lambda_0 \geqslant \eta > 0$, we know that $\{b_{n_i}\}$ is a Cauchy sequence too. By the hypothesis that B is complete, there exists $b_0 \in B$ such that $b_{n_i} \to b_0$. Thus the subsequence $\{x_{n_i}\} = \{\lambda_{n_i} b_{n_i}\}$ is convergent to $\lambda_0 b_0 \in K$, which implies that the Cauchy sequence $\{x_n\}$ is convergent to $x_0 = \lambda_0 b_0 \in K$. □

Lemma 5.2 (See [25, Lemmas 1.1 and 1.2]). *Let $(X, \|\ \|)$ be a normed space, $B \subset X$ be a bounded closed convex set with $0 \notin B$ and $A \subset X$ be closed. Assume that A or B is complete, then for each $x_0 \in A$ such that $A \cap (x_0 + K)$ is bounded, there exists $z \in A \cap (x_0 + K)$ such that $A \cap (z + K) = \{z\}$, where $K = \operatorname{cone}(B)$.*

Proof. By the assumption that A or B is complete and by Lemma 5.1 we know that $A \cap (x_0 + K)$ is complete, where K denotes $\operatorname{cone}(B)$. By modifying the proof of [25, Lemmas 1.1 and 1.2], we can deduce the result. □

Lemma 5.3 (Refer to [11, Proposition 4]). *Let X be a locally convex space, $T \subset X$ be a bounded absolutely convex set and $B \subset T$ with $0 \notin \operatorname{cl}(B)$. If $M \subset K$ is bounded in X, then M is bounded in $(X_T, \|\ \|_T)$. Here K denotes $\operatorname{cone}(B)$ and X_T denotes $\operatorname{span} T$.*

Proof. Since $0 \notin \operatorname{cl}(B)$, there exists a continuous seminorm p_μ on X and $\delta > 0$ such that $p_\mu(b) \geqslant \delta$, $\forall b \in B$. Since M is bounded in X, there exists $\beta > 0$ such that $p_\mu(y) \leqslant \beta$, $\forall y \in M$. For any $y \in M \subset K$, we may assume that $y = \lambda b$, $\lambda \geqslant 0$, $b \in B$. Thus

$$\lambda\delta \leqslant \lambda p_\mu(b) = p_\mu(\lambda b) = p_\mu(y) \leqslant \beta.$$

From this, $\lambda \leqslant \beta/\delta$ and hence

$$\|y\|_T = \|\lambda b\|_T = \lambda\|b\|_T \leqslant \lambda \leqslant \frac{\beta}{\delta}.$$

That is, M is bounded in $(X_T, \|\ \|_T)$. □

Now we can give the following Phelps lemma in locally convex spaces, which only need assume local completeness of some related sets (particularly, which only need assume that the locally convex space is locally complete).

Theorem 5.1. *Let X be a locally convex space, $A \subset X$ be a locally closed set and $B \subset X$ be a locally closed bounded convex set with $0 \notin \operatorname{cl}(B)$. Assume that A or B is locally complete, then for each $x_0 \in A$ such that $A \cap (x_0 + K)$ is bounded, there exists $z \in A \cap (x_0 + K)$ such that $\{z\} = A \cap (z + K)$. Here K denotes $\operatorname{cone}(B)$.*

Proof. Let T be the local closure of $\Gamma(x_0, B)$, then $(X_T, \|\ \|_T)$ is a normed space. Since A is locally closed, $A \cap X_T$ is closed in $(X_T, \|\ \|_T)$. Since B is locally closed and $B \subset T$, then B is a bounded closed convex set in $(X_T, \|\ \|_T)$ with $0 \notin B$. It is easy to prove that if A is locally complete then $A \cap X_T$ is complete in $(X_T, \|\ \|_T)$. And if B is locally complete, then B is complete in $(X_T, \|\ \|_T)$. Now $x_0 \in A$ such that $A \cap (x_0 + K)$ is bounded in X, which implies that $(A \cap X_T) \cap (x_0 + K)$ is bounded in X. By Lemma 5.3, we know that $(A \cap X_T) \cap (x_0 + K)$ is bounded in $(X_T, \|\ \|_T)$. By Lemma 5.2, there exists $z \in (A \cap X_T) \cap (x_0 + K)$ such that

$$\{z\} = (A \cap X_T) \cap (z + K).$$

Since $z + K \subset x_0 + K + K = x_0 + K \subset X_T$, we have

$$A \cap (z + K) = (A \cap X_T) \cap (z + K) = \{z\}.$$

This completes the proof. $\square$

We shall see that the above Phelps lemma turns out to be equivalent to the Ekeland's variational principle, Theorem 3.2. By modifying the proof of Lemma 5.1 we can show the following:

Lemma 5.4. *Let B be a locally closed bounded convex set and $0 \notin B$, then $K = \mathrm{cone}(B)$ is locally closed. If B is locally complete, then K is also locally complete.*

Theorem 5.2. *Theorem* 3.2 *implies Theorem* 5.1.

Proof. Let $S = \mathrm{co}(\{0\} \cup B)$ be the convex hull of $\{0\} \cup B$. Since B is a locally closed convex set, S is also a locally closed convex set (see [16, Lemma 2.1]). Let p_S be the Minkowski functional of S. Since $0 \notin \mathrm{cl}(B)$, there exists $l \in X^*$ and $\alpha > 0$ such that $l(b) \geqslant \alpha > 0$, $\forall b \in B$. For any $x \in K := \mathrm{cone}(B)$, we may assume that $x = \lambda b$ for some $\lambda \geqslant 0$ and some $b \in B$. Remarking that $p_S(b) \leqslant 1$, we have

$$\alpha p_S(x) = \alpha p_S(\lambda b) = \lambda \alpha p_S(b) \leqslant \lambda \alpha \leqslant \lambda l(b) = l(\lambda b) = l(x).$$

Therefore

$$K \subset K_\alpha := \bigl\{x \in X\colon\ \alpha p_S(x) \leqslant l(x)\bigr\}.$$

Since B is locally closed bounded convex set, by Lemma 5.4, $K = \mathrm{cone}(B)$ is locally closed and $x_0 + K$ is locally closed. Put $A_0 = A \cap (x_0 + K)$, then A_0, as the intersection of the two locally closed sets, is still locally closed (see [14, Proposition 5.1.17]). Define

$$f(x) = \begin{cases} -l(x), & \text{if } x \in A_0, \\ +\infty, & \text{if } x \notin A_0. \end{cases} \tag{18}$$

Then f is a locally lower semicontinuous, bounded from below, proper function. It is easy to see that $x_0 \in \mathrm{dom}\, f$ and that $\{x \in X\colon\ f(x) \leqslant f(x_0)\} = \{x \in A_0\colon\ l(x_0 - x) \leqslant 0\} = A \cap (x_0 + K) \cap \{x \in X\colon\ l(x) \geqslant l(x_0)\}$ is a locally closed subset of A. If A is locally complete, then $\{x \in X\colon\ f(x) \leqslant f(x_0)\}$, as a locally closed subset of A, is still locally complete. If B is locally complete, then $S = \mathrm{co}(\{0\} \cup B)$ is still locally complete (see [16, Lemma 2.1]). Now applying Theorem 3.2, we know that there exists $z \in X$ such that

(i) $\alpha p_S(z - x_0) + f(z) \leqslant f(x_0)$;
(ii) for any $x \neq z$, $f(z) < f(x) + \alpha p_S(x - z)$.

By (i), $f(z) \leqslant f(x_0) < +\infty$, so $z \in A_0 = A \cap (x_0 + K)$.

Next we show that (ii) implies that $\{z\} = A \cap (z + K)$. Assume that $x \neq z$ and $x \in A \cap (z + K)$. We consider the following two cases:

Case 1. Let $x \notin A_0$, i.e. $x \notin A \cap (x_0 + K)$. Since $x \in A$, we conclude that $x \notin x_0 + K$. On the other hand, $x \in z + K$ and $z \in A_0 \subset x_0 + K$. Thus $x \in z + k \subset x_0 + K + K = x_0 + K$, a contradiction.

Case 2. Let $x \in A_0$, then (ii): $f(z) < f(x) + \alpha p_S(x - z)$ becomes

$$-l(z) < -l(x) + \alpha p_S(x - z), \quad \text{that is,} \quad l(x - z) < \alpha p_S(x - z).$$

Thus $x - z \notin K_\alpha$ and since $K \subset K_\alpha$, we have $x \notin z + K$. This contradicts the assumption that $x \in A \cap (z + K) \subset z + K$. □

Theorem 5.3. *Theorem* 5.1 *implies Theorem* 3.2.

First we prove that Theorem 5.1 implies the following Bishop–Phelps lemma.

Lemma 5.5 (Refer to [21, Lemma 2]). *Let X be a locally convex space and $p : X \to R^+ \cup \{+\infty\}$ a locally lower semicontinuous, positive-homogeneous, sub-additive function such that $B := \{x \in X\colon\ p(x) \leqslant 1\}$ is bounded. Suppose that A is locally closed nonempty subset of $X \times R$ and that* $\inf\{r\colon\ (x, r) \in A\} = 0$. *If A or B is locally complete, then for any $\alpha > 0$ and any $(x_0, r_0) \in A$, there exists $(\bar{x}, \bar{r}) \in A \cap (K_\alpha + (x_0, r_0))$ such that $\{(\bar{x}, \bar{r})\} = A \cap (K_\alpha + (\bar{x}, \bar{r}))$, where $K_\alpha := \{(x, r) \in X \times R\colon\ \alpha p(x) \leqslant -r\}$.*

Proof. Put $\hat{B} := \{(x, -1) \in X \times R\colon\ \alpha p(x) \leqslant 1\}$, then $\hat{B}$ is a locally closed bounded convex set in $X \times R$, $(0, 0) \notin \mathrm{cl}(\hat{B})$ and $K_\alpha = \mathrm{cone}(\hat{B})$. Obviously the condition that A or B is locally complete means that A or $\hat{B}$ is locally complete. If we can prove that $A \cap ((x_0, r_0) + K_\alpha)$ is bounded in $X \times R$, then the result follows from Theorem 5.1. Take any $(x, r) \in A \cap ((x_0, r_0) + K_\alpha)$. Then $(x, r) \in A$ and since $\inf\{r\colon\ (x, r) \in A\} = 0$, we have

$$0 \leqslant r < +\infty. \tag{19}$$

On the other hand, $(x, r) \in (x_0, r_0) + K_\alpha$, hence

$$\alpha p(x - x_0) \leqslant r_0 - r. \tag{20}$$

By (19) and (20), we know that $\alpha p(x - x_0) \leqslant r_0$. Take $\epsilon = 1$, then $x - x_0 \in (\epsilon + r_0/\alpha)B = (1 + r_0/\alpha)B$ and hence $x \in x_0 + (1 + r_0/\alpha)B$, the right side is a bounded set in X. Again by (19) and (20), we know that $0 \leqslant r \leqslant r_0 - \alpha p(x - x_0) \leqslant r_0$. Thus we have shown that

$$A \cap \big((x_0, r_0) + K_\alpha\big) \subset \big(x_0 + (1 + r_0/\alpha)B\big) \times [0, r_0],$$

which is bounded in $X \times R$. □

Proof of Theorem 5.3. Now we have already shown that Theorem 5.1 implies Lemma 5.5. Just as we did in the proof of [21, Theorem 1], we can prove that Lemma 5.5 implies Theorem 3.2. □

Remark 5.1. Summing up the main points in Sections 4 and 5 we conclude that the two versions of Ekeland's variational principle (Theorems 3.1 and 3.2), the two versions of Caristi's fixed point theorem (Corollaries 3.3 and 3.4), the drop theorem (Theorem 1.5), the Phelps lemma (Theorem 5.1) and the Bishop–Phelps lemma (Lemma 5.5) are equivalent to each other.

Remark 5.2. Just like Corollaries 3.1 and 3.2, if we assume that X is a locally complete locally convex space, then the condition on local completeness of some related subsets is automatically satisfied. Hence in the case, we can omit the condition and all the results remain true.

Remark 5.3. The referee(s) pointed out that a direct proof of Theorem 3.1 using the induction argument is also possible. Here we use Theorem 1.5 to prove Theorem 3.1 and stress the connection between them.

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