

Remarks on global existence for the supercritical nonlinear Schrödinger equation with a harmonic potential [☆]

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Abstract

This paper is concerned with the nonlinear Schrödinger equation with a harmonic potential which describes the attractive Bose–Einstein condensate under the magnetic trap. By combining the best constant of Gagliardo–Nirenberg’s inequality with the characteristic of this equation, we derive out a global existence condition for the supercritical equation which coincides with the critical case.

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1. Introduction

In this paper we consider the nonlinear Schrödinger equation with a harmonic potential

$$i\varphi_t = -\Delta\varphi + |x|^2\varphi - |\varphi|^{p-1}\varphi, \quad x \in \mathbf{R}^N, \quad t \geq 0, \quad (1.1)$$

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where $1 < p < \frac{N+2}{N-2}$ when $N \geq 3$ and $1 < p < \infty$ when $N = 1, 2$. Equation (1.1) may model the Bose–Einstein condensate with attractive inter-particle interactions under the magnetic trap [2,6,11,12].

For the above nonlinear Schrödinger equation with a general real-valued potential function $V(x)$, when $|D^\alpha V(x)|$ is bounded for all $\alpha \geq 2$, Fujiwara [7] provided that the smoothness of Schrödinger kernel for potentials of quadratic growth. And Yajima [20] showed that for super-quadratic potentials, the Schrödinger kernel is nowhere C^1 . From Oh [16] it is known that quadratic potentials are the highest order potentials for local well-posedness of the equation. Therefore $V(x) = |x|^2$ is the critical potential for the local existence of the Cauchy problem.

For Eq. (1.1), we impose the initial datum as follows:

$$\varphi(x, 0) = \varphi_0(x), \quad x \in \mathbf{R}^N. \quad (1.2)$$

In the course of nature, we set

$$H(\mathbf{R}^N) := \left\{ u \in H^1(\mathbf{R}^N) : \int |x|^2 |u|^2 dx < \infty \right\}. \quad (1.3)$$

Here and hereafter, for simplicity, we denote $\int_{\mathbf{R}^N} dx$ by $\int dx$. $H(\mathbf{R}^N)$ becomes a Hilbert space, continuously embedded in $H^1(\mathbf{R}^N)$, when endowed with the inner product

$$\langle \varphi, \psi \rangle_{H(\mathbf{R}^N)} = \int \nabla \varphi \nabla \bar{\psi} + \varphi \bar{\psi} + |x|^2 \varphi \bar{\psi} dx, \quad (1.4)$$

whose associated norm we denote by $\|\cdot\|_{H(\mathbf{R}^N)}$.

In the case of Eq. (1.1), Oh [16] and Cazenave [5] established the local existence of the Cauchy problem in the energy space. When $1 < p < 1 + 4/N$, Zhang [22] proved that global solutions of the Cauchy problem of Eqs. (1.1)–(1.2) exist for any initial data in the energy space. When $p = 1 + 4/N$, Zhang [21] showed that there exists a sharp condition of the global existence for the Cauchy problem of Eqs. (1.1)–(1.2). When $p > 1 + 4/N$, Cazenave [5], Carles [3,4] and Tsurumi and Wadati [18] showed that the solutions of the Cauchy problem of Eqs. (1.1)–(1.2) blow up in a finite time for some initial data, especially for a class of sufficiently large initial data; but the solutions of the Cauchy problem of Eqs. (1.1)–(1.2) globally exist for other initial data, especially for a class of sufficiently small initial data [3,4,18]. So the problem of finding the sharp threshold for the global existence of the Cauchy problem of Eqs. (1.1)–(1.2) arises for $p > 1 + 4/N$. From view point of physics, this problem is also pursued strongly [2,6,11–13].

In the following we recall some relevant known results of Eq. (1.1) without any potential. Ginibre and Velo [8] established the local existence of the Cauchy problem in the energy space. When $1 < p < 1 + 4/N$, Ginibre and Velo [9] proved the solutions exist globally. When $p = 1 + 4/N$, Weinstein [19] obtained the sharp condition of global solutions. When $p > 1 + 4/N$, Glassey [10], Ogawa and Tsutsumi [14,15] proved the solutions blow up in a finite time for some initial data, especially for a class of sufficiently large initial data; but Ginibre and Velo [9], Cazenave [5] and Strauss [17] also got the solutions globally exist for other initial data, especially for a class of sufficiently small initial data. Moreover, Bégout [1] attempted to search the sharp condition of the global solutions for this case.

So in this paper we shall fix $1 + 4/N < p < \frac{N+2}{N-2}$ when $N \geq 3$ and $1 + 4/N < p < \infty$ when $N = 1, 2$. Originating in Weinstein [19], Bégout [1] and Zhang [21,23], combining the best constant of Gagliardo–Nirenberg’s inequality with the characteristic of Eq. (1.1), we obtain a similar results with Zhang [21]. As we will see, we establish a set $\mathcal{A} := \{u \in H(\mathbf{R}^N) : \|u\|_2 \leq h(\|\nabla u\|_2^2 + \|xu\|_2^2)\}$, where h is defined by (3.1). For any initial value $\varphi_0 \in \mathcal{A}$, the corresponding solution $\varphi(t)$ of the Cauchy problem (1.1)–(1.2) globally exists. Especially, it is interesting to note that \mathcal{A} is an unbounded subset of $H(\mathbf{R}^N)$. Hence we can arbitrarily take initial values with the $H(\mathbf{R}^N)$ norm large as we want. Moreover, when $p \searrow 1 + 4/N$, $h(\|\nabla \varphi_0\|_2^2 + \|x\varphi_0\|_2^2) \rightarrow \|Q\|_2$. Here Q is the solution of Eq. (2.3). Then it yields the sharp condition of global existence, $\|\varphi_0\|_2 < \|Q\|_2$, for the case of $p = 1 + 4/N$, which coincides with the results of Zhang [21]. However, it is regret that we do not know this condition is sharp for the case of $p > 1 + 4/N$.

This paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we show the main result and the proof.

2. Preliminaries

Firstly we give the local well-posedness of the Cauchy problem (1.1)–(1.2).

Proposition 2.1. [5,8,9,16] Assume that $1 < p < \frac{N+2}{N-2}$ when $N \geq 3$ and $1 < p < \infty$ when $N = 1, 2$, and $\varphi_0 \in H(\mathbf{R}^N)$. Then there exists a unique solution $\varphi(t, x)$ of the Cauchy problem (1.1)–(1.2) in $C([0, T]; H(\mathbf{R}^N))$ for some $T \in [0, \infty)$ (maximal existence time), and $\varphi(t, \cdot)$ satisfies the following two conservation laws of the mass:

$$\int |\varphi(t)|^2 dx = \int |\varphi_0|^2 dx \quad (2.1)$$

and energy

$$E(\varphi(t)) = \int \frac{1}{2} |\nabla \varphi(t)|^2 + \frac{1}{2} |x|^2 |\varphi(t)|^2 - \frac{1}{p+1} |\varphi(t)|^{p+1} dx = E(\varphi_0) \quad (2.2)$$

for all $t \in [0, T)$. Furthermore, we have the following alternative: $T = \infty$ or else $T < \infty$ and $\lim_{t \rightarrow T} \|\varphi\|_{H(\mathbf{R}^N)} = \infty$ (collapse).

Proposition 2.2. [5] Let $\varphi_0 \in H(\mathbf{R}^N)$. Then for $1 < p < 1 + \frac{4}{N}$, the Cauchy problem (1.1)–(1.2) has a unique bounded global solution $\varphi(t, x)$ on $t \in [0, \infty)$ in $H(\mathbf{R}^N)$. For $1 + \frac{4}{N} \leq p < \frac{N+2}{N-2}$ when $N \geq 3$ and $1 + \frac{4}{N} \leq p < \infty$ when $N = 1, 2$, when $\|\varphi_0\|_{H(\mathbf{R}^N)}$ is sufficiently small, the Cauchy problem (1.1)–(1.2) has a unique bounded global solution on $t \in [0, \infty)$ in $H(\mathbf{R}^N)$.

Proposition 2.3. [5] For $1 + \frac{4}{N} \leq p < \frac{N+2}{N-2}$ when $N \geq 3$ and $1 + \frac{4}{N} \leq p < \infty$ when $N = 1, 2$, when $E(\varphi_0) < 0$, the solution $\varphi(t, x)$ of the Cauchy problem (1.1)–(1.2) blows up at a finite time in $H(\mathbf{R}^N)$.

Lemma 2.1. [19] Let $1 < p < \frac{N+2}{N-2}$ when $N \geq 3$ and $1 < p < \infty$ when $N = 1, 2$ and Q be the ground state solution of the following nonlinear elliptic equation:

$$-\Delta u + u = |u|^{p-1}u \quad \text{in } \mathbf{R}^N. \quad (2.3)$$

Then the best constant $C_* > 0$ of the Gagliardo–Nirenberg’s inequality,

$$\|f\|_{L^{p+1}}^{p+1} \leq C_* \|f\|_{L^2}^{p+1 - \frac{N(p-1)}{2}} \|\nabla f\|_{L^2}^{\frac{N(p-1)}{2}}, \quad (2.4)$$

is given by

$$C_* = \frac{2(p+1)}{N(p-1)} \left(\frac{4 - (p-1)(N-2)}{N(p-1)} \right)^{\frac{N(p-1)-4}{4}} \|Q\|_{L^2}^{-(p-1)}. \quad (2.5)$$

3. The condition of global existence

In this section, we show the main result and the proof.

Firstly, we make some preliminaries. We set a real value function $h(\lambda)$ in \mathbf{R} as follows:

$$h(\lambda) = \left(\frac{N(p-1)-4}{4 - (p-1)(N-2)} \right)^{\frac{N(p-1)-4}{2((N+2)-(N-2)p)}} \|Q\|_2^{\frac{2(p-1)}{(N+2)-(N-2)p}} \lambda^{-\frac{N(p-1)-4}{2((N+2)-(N-2)p)}} \quad (3.1)$$

for any $\lambda > 0$,

where $1 + \frac{4}{N} < p < \frac{N+2}{N-2}$ when $N \geq 3$, $1 + \frac{4}{N} < p < \infty$ when $N = 1, 2$, and Q is the ground state solution of Eq. (2.3).

Proposition 3.1. [1] Let $I \subseteq \mathbf{R}$ be an open interval, $t_0 \in I$, $\theta > 1$, $a > 0$, $b > 0$ and $\Phi(t) \in C(I; \mathbf{R}^+)$. Set $f(x) = a - x + bx^\theta$ for any $x \geq 0$, $x_* = (b\theta)^{-\frac{1}{\theta-1}}$ and $b_* = \frac{\theta-1}{\theta} x_*$. Assume that $\Phi(t_0) < x_*$, $a \leq b_*$ and $f \circ \Phi > 0$. Then $\Phi(t) < x_*$ for any $t \in I$.

Proof. Since $\Phi(t_0) < x_*$ and Φ is a continuous function, there exists a $\delta > 0$ such that $\Phi(t) < x_*$ for any $t \in (t_0 - \delta, t_0 + \delta) \subseteq I$. If $\Phi(t) < x_*$ were not true for any $t \in I$, by continuity, there would exist a $t_* \in I$ satisfying $\Phi(t_*) = x_*$. Then $f \circ \Phi(t_*) = f(x_*) = a - b_* \leq 0$. However, this is impossible from $f \circ \Phi > 0$. Therefore $\Phi(t) < x_*$ for any $t \in I$. The proof is complete. \square

Then we give the main result of this paper.

Theorem 3.1. Let $1 + \frac{4}{N} < p < \frac{N+2}{N-2}$ when $N \geq 3$ and $1 + \frac{4}{N} < p < \infty$ when $N = 1, 2$ and Q be the ground state solution of Eq. (2.3). If $\|\varphi_0\|_2 \leq h(\|\nabla \varphi_0\|_2^2 + \|x\varphi_0\|_2^2)$, then the corresponding solution $\varphi(t)$ of the Cauchy problem (1.1)–(1.2) exists globally in $H(\mathbf{R}^N)$.

Moreover,

$$\int |\nabla \varphi|^2 + |x|^2 |\varphi|^2 dx < \frac{2N(p-1)}{N(p-1)-4} E(\varphi_0).$$

Proof. It follows from (2.2), (2.1) and Lemma 2.1 that

$$\begin{aligned}
 & \int |\nabla \varphi|^2 + |x|^2 |\varphi|^2 dx \\
 &= 2E(\varphi) + \frac{2}{p+1} \int |\varphi|^{p+1} dx \\
 &= 2E(\varphi_0) + \frac{2}{p+1} \int |\varphi|^{p+1} dx \\
 &< \int |\nabla \varphi_0|^2 + |x|^2 |\varphi_0|^2 dx + \frac{2}{p+1} \int |\varphi|^{p+1} dx \\
 &\leq \int |\nabla \varphi_0|^2 + |x|^2 |\varphi_0|^2 dx + \frac{2}{p+1} C_* \|\varphi\|_2^{p+1-\frac{N(p-1)}{2}} \|\nabla \varphi\|_2^{\frac{N(p-1)}{2}} \\
 &= \int |\nabla \varphi_0|^2 + |x|^2 |\varphi_0|^2 dx + \frac{2}{p+1} C_* \|\varphi_0\|_2^{p+1-\frac{N(p-1)}{2}} \|\nabla \varphi\|_2^{\frac{N(p-1)}{2}} \\
 &\leq \int |\nabla \varphi_0|^2 + |x|^2 |\varphi_0|^2 dx + \frac{2}{p+1} C_* \|\varphi_0\|_2^{p+1-\frac{N(p-1)}{2}} \\
 &\quad \times \left(\int |\nabla \varphi|^2 + |x|^2 |\varphi|^2 dx \right)^{\frac{N(p-1)}{4}} \\
 &:= a + b(\Phi(t))^\theta.
 \end{aligned} \tag{3.2}$$

Since $1 + \frac{4}{N} < p < \frac{N+2}{N-2}$ when $N \geq 3$ and $1 + \frac{4}{N} < p < \infty$ when $N = 1, 2$, here $a := \int |\nabla \varphi_0|^2 + |x|^2 |\varphi_0|^2 dx > 0$, $b := \frac{2}{p+1} C_* \|\varphi_0\|_2^{p+1-\frac{N(p-1)}{2}} > 0$, $\theta := \frac{N(p-1)}{4} > 1$ and $\Phi(t) := \int |\nabla \varphi|^2 + |x|^2 |\varphi|^2 dx$. Obviously, $\Phi(0) = a$. At the same time, we define $f(x) = a - x + bx^\theta$. Then (3.2) becomes that

$$f \circ \Phi(t) = a - \Phi(t) + b(\Phi(t))^\theta > 0. \tag{3.3}$$

On the other hand, since $\|\varphi_0\|_2 \leq h(\|\nabla \varphi_0\|_2^2 + \|x\varphi_0\|_2^2) = h(a)$, we have

$$\|\varphi_0\|_2 \leq \left(\frac{N(p-1)-4}{4-(p-1)(N-2)} \right)^{\frac{N(p-1)-4}{2((N+2)-(N-2)p)}} \|\varphi_0\|_2^{\frac{2(p-1)}{(N+2)-(N-2)p}} a^{-\frac{N(p-1)-4}{2((N+2)-(N-2)p)}}, \tag{3.4}$$

which yields that

$$a \leq \frac{N(p-1)-4}{4-(p-1)(N-2)} \|\varphi_0\|_2^{\frac{4(p-1)}{N(p-1)-4}} \|\varphi_0\|_2^{-\frac{2((N+2)-(N-2)p)}{N(p-1)-4}}, \tag{3.5}$$

because $1 + \frac{4}{N} < p < \frac{N+2}{N-2}$ when $N \geq 3$ and $1 + \frac{4}{N} < p < \infty$ when $N = 1, 2$.

Therefore, let $x_* = (b\theta)^{-\frac{1}{\theta-1}}$ and $b_* = \frac{\theta-1}{\theta} x_*$. Then $b_* < x_*$. At the same time, noting (3.5), we have $\Phi(0) = a \leq b_* < x_*$. From this point, (3.3) and Proposition 3.1, we can obtain that $\Phi(t) < x_*$. Then it follows from (2.1) that $\int |\nabla \varphi|^2 + |x|^2 |\varphi|^2 + |\varphi|^2 dx < x_* + \int |\varphi_0|^2 dx = C$. Here and hereafter, we use C to denote various positive constants. In other words, we get that $\varphi(t)$ is bounded in $H(\mathbf{R}^N)$. Therefore, the solution $\varphi(t)$ of Eq. (1.1) corresponding the initial data φ_0 exists globally in $H(\mathbf{R}^N)$ by Proposition 2.1.

Furthermore,

$$\begin{aligned}
 E(\varphi_0) &= E(\varphi) = \frac{1}{2} \int |\nabla \varphi|^2 + |x|^2 |\varphi|^2 dx - \frac{1}{p+1} \int |\varphi|^{p+1} dx \\
 &\geq \frac{1}{2} \int |\nabla \varphi|^2 + |x|^2 |\varphi|^2 dx - \frac{1}{p+1} C_* \|\varphi\|_2^{p+1-\frac{N(p-1)}{2}} \|\nabla \varphi\|_2^{\frac{N(p-1)}{2}} \\
 &= \frac{1}{2} \int |\nabla \varphi|^2 + |x|^2 |\varphi|^2 dx - \frac{1}{p+1} C_* \|\varphi_0\|_2^{p+1-\frac{N(p-1)}{2}} \|\nabla \varphi\|_2^{\frac{N(p-1)}{2}} \\
 &\geq \frac{1}{2} \int |\nabla \varphi|^2 + |x|^2 |\varphi|^2 dx - \frac{1}{p+1} C_* \|\varphi_0\|_2^{p+1-\frac{N(p-1)}{2}} \\
 &\quad \times \left(\int |\nabla \varphi|^2 + |x|^2 |\varphi|^2 dx \right)^{\frac{N(p-1)}{4}} \\
 &= \frac{1}{2} \int |\nabla \varphi|^2 + |x|^2 |\varphi|^2 dx \\
 &\quad \times \left[1 - \frac{4}{N(p-1)} \left[\left(\frac{N(p-1)}{2(p+1)} C_* \|\varphi_0\|_2^{p+1-\frac{N(p-1)}{2}} \right)^{-\frac{4}{N(p-1)-4}} \right. \right. \\
 &\quad \left. \left. \times \left(\int |\nabla \varphi|^2 + |x|^2 |\varphi|^2 dx \right)^{-1} \right]^{-\frac{N(p-1)-4}{4}} \right]. \tag{3.6}
 \end{aligned}$$

At the meantime, from above we have obtained $\Phi(t) < x_*$ which is that

$$\int |\nabla \varphi|^2 + |x|^2 |\varphi|^2 dx < \left(\frac{N(p-1)}{2(p+1)} C_* \|\varphi_0\|_2^{p+1-\frac{N(p-1)}{2}} \right)^{-\frac{4}{N(p-1)-4}}. \tag{3.7}$$

Then we have

$$\left(\frac{N(p-1)}{2(p+1)} C_* \|\varphi_0\|_2^{p+1-\frac{N(p-1)}{2}} \right)^{-\frac{4}{N(p-1)-4}} \left(\int |\nabla \varphi|^2 + |x|^2 |\varphi|^2 dx \right)^{-1} > 1. \tag{3.8}$$

Since $1 + \frac{4}{N} < p < \frac{N+2}{N-2}$ when $N \geq 3$ and $1 + \frac{4}{N} < p < \infty$ when $N = 1, 2$, we have $\frac{N(p-1)-4}{4} > 0$. Noting this point and (3.8), we obtain that

$$\begin{aligned}
 0 &< \left[\left(\frac{N(p-1)}{2(p+1)} C_* \|\varphi_0\|_2^{p+1-\frac{N(p-1)}{2}} \right)^{-\frac{4}{N(p-1)-4}} \right. \\
 &\quad \left. \times \left(\int |\nabla \varphi|^2 + |x|^2 |\varphi|^2 dx \right)^{-1} \right]^{-\frac{N(p-1)-4}{4}} < 1. \tag{3.9}
 \end{aligned}$$

Therefore, it follows from (3.6) and (3.9) that

$$\begin{aligned}
 E(\varphi_0) &> \frac{1}{2} \int |\nabla \varphi|^2 + |x|^2 |\varphi|^2 dx \left[1 - \frac{4}{N(p-1)} \right] \\
 &= \frac{N(p-1)-4}{2N(p-1)} \int |\nabla \varphi|^2 + |x|^2 |\varphi|^2 dx. \tag{3.10}
 \end{aligned}$$

The proof is complete. \square

Remark 3.1. Set $\mathcal{A} := \{u \in H(\mathbf{R}^N) : \|u\|_2 \leq h(\|\nabla u\|_2^2 + \|xu\|_2^2)\}$. For any initial value $\varphi_0 \in \mathcal{A}$, the corresponding solution $\varphi(t)$ of Cauchy problem (1.1)–(1.2) is global and uniformly bounded in $H(\mathbf{R}^N)$. It is interesting to note that \mathcal{A} is an unbounded subset of $H(\mathbf{R}^N)$. Hence Theorem 3.1 gives a general result for global existence for which we can arbitrarily take initial values with the $H(\mathbf{R}^N)$ norm large as we want.

Remark 3.2. When $p \searrow 1 + \frac{4}{N}$, $h(\lambda) \rightarrow \|Q\|_2$. Hence it yields the sharp condition of global existence for the case of $p = 1 + \frac{4}{N}$, $\|\varphi_0\|_2 < \|Q\|_2$, which coincides with the results of Zhang [21]. However, we do not know the function h is optimum for the case of $p > 1 + \frac{4}{N}$.

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