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Hausdorff dimension of fiber-coding sub-Sierpinski carpets

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Abstract

In this paper we study a class of subset of Sierpinski carpets for which the allowed digits in the expansions fall into each fiber set with a prescribed frequency. We calculate the Hausdorff and packing dimensions of these subsets and give necessary and sufficient conditions for the corresponding Hausdorff and packing measures to be finite.

Keywords: Sierpinski carpets; Hausdorff dimension; Hausdorff measure; Packing dimension; Packing measure

1. Introduction

Let T be the expanding endomorphism of the 2-torus $\mathbf{T}^2 = \mathbf{R}^2/\mathbf{Z}^2$ given by the matrix $\operatorname{diag}(n, m)$ where $2 \leqslant m < n$ are integers. The simplest invariant sets for T have the form

$$K(T, D) = \left\{ \sum_{k=1}^{\infty} \begin{pmatrix} n^{-1} & 0 \\ 0 & m^{-1} \end{pmatrix}^k d_k \colon d_k \in D \text{ for all } k \geqslant 1 \right\},$$

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where $D \subseteq I \times J$ is a set of digits with $I = \{0, 1, \dots, n-1\}$ and $J = \{0, 1, \dots, m-1\}$. Throughout this paper, we always assume that D contains at least two elements. Alternatively, define a map $K_T : (I \times J)^{\mathbf{N}} \to \mathbf{T}^2$ by

$$K_T(d) = \sum_{k=1}^{\infty} \begin{pmatrix} n^{-1} & 0 \\ 0 & m^{-1} \end{pmatrix}^k d_k, \quad d = (d_k)_{k=1}^{\infty} \in (I \times J)^{\mathbf{N}}.$$

Then $K(T, D) = K_T(D^{\mathbf{N}})$. So each element of $K(T, D)$ can be represented as an expansion in base $\operatorname{diag}(n^{-1}, m^{-1})$ with digits in D. The set $K(T, D)$, called as the Sierpinski carpet, was first studied by C. McMullen [6] and T. Bedford [1], independently, to determine its Hausdorff and box-counting dimensions. From then on, some further problems related to the Sierpinski carpet $K(T, D)$ are proposed and considered by lots of authors. Y. Peres [8,9] studied its packing and Hausdorff measures. R. Kenyon and Y. Peres [4,5] extended the results of McMullen [6] and Bedford [1] to the compact subsets of the 2-torus corresponding to shifts of finite type or sofic shifts and to the Sierpinski sponges. O.A. Nielsen [7] studied a certain subset of $K(T, D)$ by insisting that the allowed digits in the expansions occur with prescribed frequencies.

Let σ denote the projection of $\mathbf{R}^2$ onto its second coordinate. Let $B = \sigma(D) = \{b_1, b_2, \dots, b_s\}$. Set $\Gamma_j = \{d \in D\colon \sigma(d) = b_j\}$, $j = 1, 2, \dots, s$. Then Γ_js are the horizontal fibers of D and form a partition of D, i.e., $D = \bigcup_{j=1}^{s} \Gamma_j$ with disjoint union. Let $(c_1, c_2, \dots, c_s)$ be a probability vector, i.e., $\sum_{j=1}^{s} c_j = 1$ with $c_j > 0$. Let

$$\Omega = \left\{ d = (d_i)_{i=1}^{\infty} \in D^{\mathbf{N}}\colon \lim_{k\to\infty} \frac{\#\{1 \leqslant i \leqslant k\colon d_i \in \Gamma_j\}}{k} = c_j,\ j = 1, 2, \dots, s \right\}, \tag{1}$$

where and throughout this paper we use $\#A$ to denote the cardinality of a finite set A. Then Ω is such a subset of $D^{\mathbf{N}}$: for each element of Ω its entry falls into each fiber set Γ_j with a prescribed (group) frequency c_j. And so $K_T(\Omega)$ is the subset of the $K(T, D)$ whose elements have their codings falling into Γ_j with prescribed (group) frequencies c_j, $j = 1, 2, \dots, s$. We call $K_T(\Omega)$ to be a fiber-coding sub-Sierpinski carpet. Clearly, $K_T(\Omega)$ is T-invariant, dense in $K(T, D)$ but not compact in general.

For any Borel subset E of $\mathbf{R}^2$, let $\dim_H E$ and $\dim_P E$, respectively, denote its Hausdorff and packing dimensions, and $\mathcal{H}^\gamma(E)$ and $\mathcal{P}^\gamma(E)$, respectively, denote its γ-dimensional Hausdorff and packing measures. Let $\alpha = \log_n m$ and $n_j = \#\Gamma_j$, $j = 1, 2, \dots, s$. In this paper, we obtain the following results.

Theorem 1.1.

(I) $\dim_H K_T(\Omega) = \dim_P K_T(\Omega) = \sum_{j=1}^{s} c_j(\log_m n_j^\alpha - \log_m c_j)$;
(II) $\dim_H K_T(\Omega) = \dim_H K(T, D)$ *if and only if* $c_j = \frac{n_j^\alpha}{\sum_{i=1}^{s} n_i^\alpha}$, $j = 1, 2, \dots, s$;
(III) $\dim_P K_T(\Omega) = \dim_P K(T, D)$ *if and only if* $c_i = c_j$ *and* $n_i = n_j$ *for all* $1 \leqslant i, j \leqslant s$.

Theorem 1.2. *Let* $\gamma = \dim_H K_T(\Omega) = \dim_P K_T(\Omega) = \sum_{j=1}^{s} c_j(\log_m n_j^\alpha - \log_m c_j)$. *Then*

(I) $0 < \mathcal{H}^\gamma(K_T(\Omega)) \leqslant \mathcal{P}^\gamma(K_T(\Omega)) < \infty$ *if and only if* $c_i = c_j$ *and* $n_i = n_j$ *for all* $1 \leqslant i, j \leqslant s$;
(II) *if there exist some* $1 \leqslant i \neq j \leqslant s$ *such that* $c_i \neq c_j$ *or* $n_i \neq n_j$, *then* $\mathcal{H}^\gamma(K_T(\Omega)) = \mathcal{P}^\gamma(K_T(\Omega)) = +\infty$.

Our Theorems 1.1 and 1.2 can be considered as a generalization of the results given by O.A. Nielsen in [7] where the subset $K_T(\Omega^*)$ of the Sierpinski carpet was studied. From the definition of Ω^* defined by (6), it follows that $K_T(\Omega^*) \subseteq K_T(\Omega)$ if taking $c_j = \sum_{d \in \Gamma_j} p_d$ for $1 \leqslant j \leqslant s$. Each digit $d \in D$ is required to occur as an entry of elements of Ω^* with a prescribed frequency p_d, while its occurrence as an entry of elements of Ω is relatively more freely. However, the method for the proofs of our Theorems 1.1 and 1.2 comes from that given by O.A. Nielsen in [7], mainly by making use of Lemmas 2.1 and 2.2 which appear as Lemmas 4 and 5 in [7].

The rest of this paper is organized as follows. In Section 2, some basic facts and known results needed in the proof of our theorems are described. Proofs of Theorems 1.1 and 1.2 are arranged in Section 3.

2. Preliminaries

As in [6–9], a class of approximate squares are used to calculate the various dimensions of the Sierpinski carpets and its subsets. For each $\omega = (\omega_j)_{j=1}^{\infty} \in (I \times J)^{\mathbf{N}}$ and each positive integer k, let

$$Q_k(\omega) = \bigl\{K_T(y)\colon\ y = (y_j)_{j=1}^{\infty} \in (I \times J)^{\mathbf{N}},\ y_j = \omega_j \text{ for } 1 \leqslant j \leqslant [\alpha k] \text{ and}$$
$$\sigma(y_j) = \sigma(\omega_j) \text{ for } [\alpha k] + 1 \leqslant j \leqslant k\bigr\},$$

where, as usual, $[x]$ with $x \in \mathbf{R}$ denotes the greatest integer function. The sets $Q_k(\omega)$ are approximate squares in $[0, 1]^2$, whose sizes have length $n^{-[\alpha k]}$ and m^{-k}. Note that the radio of the sizes of $Q_k(\omega)$ is at most n, and their diameters diam $Q_k(\omega)$ satisfy

$$\sqrt{2}m^{-k} \leqslant \operatorname{diam} Q_k(\omega) \leqslant \sqrt{2}nm^{-k}.$$

So in the definition of Hausdorff measure, we can restrict attention to covers by such approximate squares since any set of diameter less than m^{-k} can be covered by a bounded number of approximate squares $Q_k(\omega)$. The following two lemmas appear in [7] in which the approximate square $Q_k(\omega)$ behaves as an analogue as the ball does in the classical density theorems. The first of them, involved in Hausdorff measure, is just a reformulation of the Rogers–Taylor density theorem as stated by Peres in Section 2 of [9]. The proof for the second of them, involved in packing measure, was given by Nielsen [7] as Lemma 5 in Section 2.

Lemma 2.1. (Nielsen [7, Lemma 4]) *Suppose that δ is a positive number, that μ is a finite Borel measure in* $[0, 1]^2$*, and that E is a subset of* $(I \times J)^{\mathbf{N}}$ *such that $K_T(E)$ is a Borel subset of* $[0, 1]^2$*, and* $\mu(K_T(E)) > 0$*, put*

$$A(\omega) = \limsup_{k \to \infty} \bigl(k\delta + \log_m \mu\bigl(Q_k(\omega)\bigr)\bigr)$$

for each point $\omega \in E$.

(1) *If $A(\omega) = -\infty$ for all $\omega \in E$, then $\mathcal{H}^{\delta}(K_T(E)) = +\infty$.*
(2) *If $A(\omega) = +\infty$ for all $\omega \in E$, then $\mathcal{H}^{\delta}(K_T(E)) = 0$.*
(3) *If there are numbers a and b such that $a \leqslant A(\omega) \leqslant b$ for all $\omega \in E$, then* $0 < \mathcal{H}^{\delta}(K_T(E)) < +\infty$.

Lemma 2.2. (Nielsen [7, Lemma 5]) *Suppose that δ, μ and E are as in Lemma* 2.1 *and put*

$$B(\omega) = \liminf_{k \to \infty} \bigl(k\delta + \log_m \mu\bigl(Q_k(\omega)\bigr)\bigr)$$

for each point $\omega \in E$.

(1) *If* $B(\omega) = -\infty$ *for all* $\omega \in E$, *then* $\mathcal{P}^{\delta}(K_T(E)) = +\infty$.
(2) *If* $B(\omega) = +\infty$ *for all* $\omega \in E$, *then* $\mathcal{P}^{\delta}(K_T(E)) = 0$.
(3) *If there are numbers* a *and* b *such that* $a \leqslant B(\omega) \leqslant b$ *for all* $\omega \in E$, *then* $0 < \mathcal{P}^{\delta}(K_T(E)) < +\infty$.

The Borel measures on $[0,1]^2$ to which the above lemmas will be applied are constructed as follows. Let $\mathbf{p} = (p_d)_{d\in D}$ be a probability vector on D, i.e., $\sum_{d\in D} p_d = 1$ with each $p_d > 0$. Then $\mathbf{p}$ determines a unique infinite product Borel probability measure, denoted by $\mu_{\mathbf{p}}$, on $D^{\mathbf{N}}$. For any finite sequence $(\omega_1, \omega_2, \ldots, \omega_k) \in D^k$,

$$\mu_{\mathbf{p}}\bigl(\mathbf{C}(\omega_1, \omega_2, \ldots, \omega_k)\bigr) = \prod_{j=1}^{k} p_{\omega_j},$$

where $\mathbf{C}(\omega_1, \omega_2, \ldots, \omega_k) := \{d = (d_j)_{j=1}^{\infty} \in D^{\mathbf{N}}\colon\ d_j = \omega_j \text{ for } 1 \leqslant j \leqslant k\}$ is a cylinder set of $D^{\mathbf{N}}$ with base $(\omega_1, \omega_2, \ldots, \omega_k)$. Let $\tilde{\mu}_{\mathbf{p}}$ be the Borel probability measure on $K_T(D^{\mathbf{N}})$ which is the image measure of $\mu_{\mathbf{p}}$ under K_T, i.e., $\tilde{\mu}_{\mathbf{p}}(A) = \mu_{\mathbf{p}}(K_T^{-1}A)$ for Borel set $A \subseteq \mathbf{R}^2$. From the fact that the approximate square $Q_k(\omega)$ is a finite union of cylinder sets, it follows that for any $\omega \in D^{\mathbf{N}}$ (cf. formula (4) in [7], also formula (4.4) in [3]),

$$\tilde{\mu}_{\mathbf{p}}\bigl(Q_k(\omega)\bigr) = \prod_{j=1}^{[\alpha k]} p_{\omega_j} \cdot \prod_{j=[\alpha k]+1}^{k} q_{\sigma(\omega_j)}, \tag{2}$$

where, also below, $q_{b_i} = \sum_{d\in\Gamma_i} p_d$ for $b_i \in B$ (recall $B = \sigma(D) = \{b_1, b_2, \ldots, b_s\}$ and $\Gamma_i = \{d \in D\colon\ \sigma(d) = b_i\}$, $i = 1, 2, \ldots, s$).

The following lemma shows that $K_T(\Omega)$ is of full $\tilde{\mu}_{\mathbf{p}}$-measure for some properly selected $\mathbf{p}$.

Lemma 2.3. *Let* Ω *be defined as* (1). *If the probability vector* $\mathbf{p} = (p_d)_{d\in D}$ *satisfies* $\sum_{d\in\Gamma_i} p_d = c_i$, $i = 1, 2, \ldots, s$, *then* $\tilde{\mu}_{\mathbf{p}}(K_T(\Omega)) = 1$.

Proof. Let

$$\Omega_i = \left\{\omega \in D^{\mathbf{N}}\colon\ \lim_{k\to\infty} \frac{\#\{1 \leqslant j \leqslant k\colon\ \omega_j \in \Gamma_i\}}{k} = c_i\right\}, \quad i = 1, 2, \ldots, s.$$

Then $\Omega = \bigcap_{i=1}^{s} \Omega_i$. So it suffices to show that $\mu_{\mathbf{p}}(\Omega_i) = 1$, $i = 1, 2, \ldots, s$. Fix $i \in \{1, 2, \ldots, s\}$ and define a sequence of random variables $\{X_j\}_{j=1}^{\infty}$ on the probability space $(D^{\mathbf{N}}, \mathcal{F}, \mu_{\mathbf{p}})$ ($\mathcal{F}$ is the Borel σ-algebra) by letting

$$X_j(\omega) = 1 \quad \text{if } \omega_j \in \Gamma_i, \qquad \text{otherwise } = 0,$$

for $j \in \mathbf{N}$ and $\omega = (\omega_j)_{j=1}^{\infty} \in D^{\mathbf{N}}$. Then $X_1, X_2, \ldots$ are independent and identically distributed random variables with $\mu_{\mathbf{p}}(X_1 = 1) = \mu_{\mathbf{p}}(\bigcup_{d\in\Gamma_i} \mathbf{C}(d)) = \sum_{d\in\Gamma_i} p_d = c_i$ and $\mu_{\mathbf{p}}(X_1 = 0) = 1 - c_i$. By Kolmogorov strong law of large numbers, we have that for $\mu_{\mathbf{p}}$-a.e. $\omega \in D^{\mathbf{N}}$,

$$\lim_{k\to\infty} \frac{\#\{1 \leqslant j \leqslant k\colon\ \omega_j \in \Gamma_i\}}{k} = \lim_{k\to\infty} \frac{1}{k} \sum_{j=1}^{k} X_j(\omega) = E(X_1) = c_i,$$

implying $\mu_{\mathbf{p}}(\Omega_i) = 1$. $\square$

Finally, we will use the following lemma in next section which proof can be found in [2, Corollary 1.5]. In addition, we always use the notation log to denote the natural logarithm, i.e., $\log = \log_e$.

Lemma 2.4. *For a probability vector $(u_1, u_2, \ldots, u_t)$ and any $(\gamma_j)_{j=1}^t \in \mathbf{R}^t$ we have*

$$\sum_{j=1}^{t} u_j(-\log u_j + \gamma_j) \leqslant \log \sum_{j=1}^{t} e^{\gamma_j},$$

with equality if and only if $u_j = \frac{e^{\gamma_j}}{\sum_{i=1}^t e^{\gamma_i}}$, $j = 1, 2, \ldots, t$.

3. Proofs

Proof of Theorem 1.1. (I) We will make use of Lemmas 2.1 and 2.2. To do this, we take $E = \Omega$ defined by (1), $\mu = \tilde{\mu}_{\mathbf{p}}$ with $p_d = \frac{c_j}{n_j}$ for $d \in \Gamma_j$, $1 \leqslant j \leqslant s$. Then $\tilde{\mu}_{\mathbf{p}}(K_T(\Omega)) = 1 > 0$ by Lemma 2.3. For each $\omega = (\omega_j)_{j=1}^{\infty} \in D^{\mathbf{N}}$, $d \in D$ and $k \in \mathbf{N}$ let

$$N_k(\omega, d) = \#\{1 \leqslant j \leqslant k\colon\ \omega_j = d\}. \tag{3}$$

Then

$$\#\{1 \leqslant j \leqslant k\colon\ \omega_j \in \Gamma_i\} = \sum_{d \in \Gamma_i} N_k(\omega, d) \quad \text{for } i = 1, 2, \ldots, s. \tag{4}$$

By (2) we have

$$\begin{aligned}
\log_m \tilde{\mu}_{\mathbf{p}}\big(Q_k(\omega)\big) &= \sum_{j=1}^{[\alpha k]} \log_m p_{\omega_j} + \sum_{j=[\alpha k]+1}^{k} \log_m q_{\sigma(\omega_j)} \\
&= \sum_{j=1}^{[\alpha k]} \log_m p_{\omega_j} + \sum_{j=1}^{k} \log_m q_{\sigma(\omega_j)} - \sum_{j=1}^{[\alpha k]} \log_m q_{\sigma(\omega_j)} \\
&= \sum_{i=1}^{s} \sum_{d \in \Gamma_i} N_{[\alpha k]}(\omega, d) \log_m p_d + \sum_{i=1}^{s} \sum_{d \in \Gamma_i} N_k(\omega, d) \log_m q_{\sigma(d)} \\
&\quad - \sum_{i=1}^{s} \sum_{d \in \Gamma_i} N_{[\alpha k]}(\omega, d) \log_m q_{\sigma(d)} \\
&= \sum_{i=1}^{s} \sum_{d \in \Gamma_i} N_{[\alpha k]}(\omega, d) \log_m \frac{c_i}{n_i} + \sum_{i=1}^{s} \sum_{d \in \Gamma_i} N_k(\omega, d) \log_m c_i \\
&\quad - \sum_{i=1}^{s} \sum_{d \in \Gamma_i} N_{[\alpha k]}(\omega, d) \log_m c_i \\
&= \sum_{i=1}^{s} \sum_{d \in \Gamma_i} N_k(\omega, d) \log_m c_i - \sum_{i=1}^{s} \sum_{d \in \Gamma_i} N_{[\alpha k]}(\omega, d) \log_m n_i.
\end{aligned}$$

Therefore, by (1) and (4) we have

$$\lim_{k\to\infty}\frac{\log_m \tilde{\mu}_{\mathbf{p}}(Q_k(\omega))}{k}$$
$$=\lim_{k\to\infty}\frac{\sum_{i=1}^{s}\sum_{d\in\Gamma_i}N_k(\omega,d)\log_m c_i-\sum_{i=1}^{s}\sum_{d\in\Gamma_i}N_{[\alpha k]}(\omega,d)\log_m n_i}{k}$$
$$=\sum_{i=1}^{s}c_i\log_m c_i-\sum_{i=1}^{s}\alpha c_i\log_m n_i=\sum_{i=1}^{s}c_i\big(\log_m c_i-\log_m n_i^{\alpha}\big), \tag{5}$$

for each $\omega=(\omega_j)_{j=1}^{\infty}\in\Omega$. Denote $\gamma=\sum_{j=1}^{s}c_j(\log_m n_j^{\alpha}-\log_m c_j)$. Therefore, for each $\omega=(\omega_j)_{j=1}^{\infty}\in\Omega$ we have

$$\lim_{k\to\infty}\big(k\delta+\log_m\tilde{\mu}_{\mathbf{p}}\big(Q_k(\omega)\big)\big)=\lim_{k\to\infty}k\left(\delta+\frac{\log_m\tilde{\mu}_{\mathbf{p}}(Q_k(\omega))}{k}\right)=\begin{cases}+\infty & \text{if } \delta>\gamma,\\ -\infty & \text{if } \delta<\gamma,\end{cases}$$

by (5). This implies that $\dim_H K_T(\Omega)=\dim_P K_T(\Omega)=\sum_{j=1}^{s}c_j(\log_m n_j^{\alpha}-\log_m c_j)$ by Lemmas 2.1 and 2.2.

(II) We first recall that $\dim_H K(T,D)=\log_m\sum_{j=1}^{s}n_j^{\alpha}$ (cf. McMullen [6]), $\dim_P K(T,D)=\dim_B K(T,D)=\log_m(\#B^{1-\alpha}\#D^{\alpha})$ (cf. Peres [8]). Applying Lemma 2.4 to the probability vector $(c_1,c_2,\ldots,c_s)$ and $(\log_m n_i^{\alpha})_{i=1}^{s}$, we have

$$\dim_H K_T(\Omega)=\sum_{i=1}^{s}c_i\big(\log_m n_i^{\alpha}-\log_m c_i\big)\leqslant\log_m\sum_{i=1}^{s}n_i^{\alpha}=\dim_H K(T,D),$$

with equality if and only if $c_i=\frac{n_i^{\alpha}}{\sum_{j=1}^{s}n_j^{\alpha}}$ for all $1\leqslant i\leqslant s$.

(III) By taking all $\gamma_j=0$, Lemma 2.4 shows that for a probability vector $(u_1,u_2,\ldots,u_t)$

$$\sum_{j=1}^{t}-u_j\log u_j\leqslant\log t,$$

with equality if and only if all u_j are equal to t^{-1}. Then

$$\begin{aligned}\dim_P K_T(\Omega)&=\sum_{j=1}^{s}c_j\big(\log_m n_j^{\alpha}-\log_m c_j\big)\\ &=-\alpha\sum_{j=1}^{s}\sum_{i=1}^{n_j}\frac{c_j}{n_j}\log_m\frac{c_j}{n_j}-(1-\alpha)\sum_{j=1}^{s}c_j\log_m c_j\\ &\leqslant\alpha\log_m\#D+(1-\alpha)\log_m\#B=\dim_P K(T,D),\end{aligned}$$

where the equality holds if and only if all $\frac{c_j}{n_j}$ are equal for the first term, all c_j are equal for the second term. $\square$

Proof of Theorem 1.2. (I) We first prove the sufficiency. Denote $c=c_j$ and $r=n_j$. Then $sc=1$ and $\gamma=\alpha\log_m r-\log_m c$. Take $\mathbf{p}=(p_d)_{d\in D}$ with $p_d=\frac{c}{r}$ for all $d\in D$. Then by (2)

$$\begin{aligned}k\gamma+\log_m\tilde{\mu}_{\mathbf{p}}\big(Q_k(\omega)\big)&=k(\alpha\log_m r-\log_m c)+[\alpha k]\log_m\frac{c}{r}+\big(k-[\alpha k]\big)\log_m c\\ &=\big(\alpha k-[\alpha k]\big)\log_m r,\end{aligned}$$

for each $\omega\in\Omega$ and all $k\in\mathbf{N}$. So the conclusion is obtained by Lemmas 2.1(3) and 2.2(3).

The necessity is included in the following (II).

(II) The proof of this part will be obtained by applying Theorem 3(b) in [7] directly. Take $\mathbf{p} = (p_d)_{d\in D}$ with $p_d = \frac{c_j}{n_j}$ for $d \in \Gamma_j$, $1 \leqslant j \leqslant s$. Let

$$\Omega^* = \left\{\omega = (\omega_j)_{j=1}^{\infty} \in D^{\mathbf{N}}\colon \lim_{k\to\infty} \frac{N_k(\omega, d)}{k} = p_d \text{ for all } d \in D\right\}, \tag{6}$$

where $N_k(\omega, d)$ is defined as (3). O.A. Nielsen (cf. [7, Theorem 1]) proved that

$$\begin{aligned}\dim_H K_T(\Omega^*) = \dim_P K_T(\Omega^*) &= -\alpha \sum_{d\in D} p_d \log_m p_d - (1-\alpha)\sum_{j=1}^{s} q_{b_j} \log_m q_{b_j} \\ &= \sum_{i=1}^{s} c_i\bigl(\log_m n_i^{\alpha} - \log_m c_i\bigr) = \dim_H K_T(\Omega) = \dim_P K_T(\Omega) = \gamma.\end{aligned}$$

So $K_T(\Omega^*)$ and $K_T(\Omega)$ have the same dimensions.

A probability vector $\mathbf{u} = (u_d)_{d\in D}$ is said to be uniformly distributed on D if $u_d = \#D^{-1}$ for all $d \in D$ and D is said to have uniform horizontal fibers if $n_i = n_j$ for all $1 \leqslant i, j \leqslant s$. Theorem 3(b) in [7] shows that

$$\mathcal{H}^{\gamma}\bigl(K_T(\Omega^*)\bigr) = \mathcal{P}^{\gamma}\bigl(K_T(\Omega^*)\bigr) = +\infty,$$

if $\mathbf{p}$ is not uniformly distributed on D or if D does not have uniform horizontal fibers. This implies our result by the fact that $K_T(\Omega) \supseteq K_T(\Omega^*)$. $\square$

References

[1] T. Bedford, Crinkly curves, Markov partitions and box dimension in self-similar sets, PhD thesis, University of Warwick, 1984.
[2] K.J. Falconer, Techniques in Fractal Geometry, John Wiley & Sons, 1997.
[3] D. Gatzouras, S.P. Lalley, Hausdorff and box dimensions of certain self-affine fractals, Indiana Univ. Math. J. 41 (1992) 533–568.
[4] R. Kenyon, Y. Peres, Measures of full dimension on affine-invariant sets, Ergodic Theory Dynam. Systems 16 (1996) 307–323.
[5] R. Kenyon, Y. Peres, Hausdorff dimensions of sofic affine-invariant sets, Israel J. Math. 122 (1996) 540–574.
[6] C. McMullen, The Hausdorff dimension of general Sierpinski carpets, Nagoya Math. J. 96 (1984) 1–9.
[7] Ole A. Nielsen, The Hausdorff and packing dimensions of some sets related to Sierpinski carpets, Canad. J. Math. 51 (1999) 1073–1088.
[8] Y. Peres, The packing measure of self-affine carpets, Math. Proc. Cambridge Philos. Soc. 115 (1994) 437–450.
[9] Y. Peres, The self-affine carpets of McMullen and Bedford have infinite Hausdorff measure, Math. Proc. Cambridge Philos. Soc. 116 (1994) 513–526.