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Calderon reproducing formula associated with the Gegenbauer operator on the half-line

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Abstract

In this paper, we introduce the generalized shift operator generated by the Gegenbauer differential operator $D_\lambda = (x^2 - 1)^{\frac{1}{2}-\lambda} \frac{d}{dx} (x^2 - 1)^{\lambda+\frac{1}{2}} \frac{d}{dx}$, and define a generalized convolution $\otimes$ on the half-line corresponding to the Gegenbauer differential operator. We investigate the Calderon reproducing formula associated with the convolution $\otimes$ involving finite Borel measures, leading to results on the L_p-norm and pointwise approximation for functions on the half-line.

Keywords: Gegenbauer differential operator; Generalized shift operator; Generalized convolution; Calderon reproducing formula

1. Introduction

It is well known that the classical Calderon reproducing formula can be formulated as follows [1]. Let g and h be L_2-functions on $\mathbb{R}$, and

$$\int_0^\infty \hat{g}(a\lambda)\hat{h}(a\lambda)\frac{da}{a} = 1, \quad \text{for all } \lambda \in \mathbb{R} \setminus \{0\},$$

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where ˆ denotes the classical Fourier transform on $\mathbb{R}$. Set

$$g_a(x) = \frac{1}{a} g\left(\frac{x}{a}\right), \qquad h_a(x) = \frac{1}{a} h\left(\frac{x}{a}\right),$$

then

$$f = \int_0^\infty f * g_a * h_a \frac{da}{a}, \tag{1}$$

where $*$ denotes the usual convolution product on $\mathbb{R}$.

The formula (1) has many applications in applied mathematics, in particular in wavelet theory (see, for example, [2,4]).

Let μ be a finite Borel measure on the real line $\mathbb{R}$, then (1) has a natural generalization to

$$f = \int_0^\infty f * \mu_a \frac{da}{a}, \tag{2}$$

where μ_a is the dilated measure of μ. Under some restrictions on μ, the L_p-norm or a.e. convergence of (2) has been proved by B. Rubin and E. Shamir in [7]. A more general form of (2) has been investigated by B. Rubin in [6]. In [5], M.A. Mourou and K. Trimeche have studied the same problems when in (2) the classical convolution $*$ is replaced by a generalized convolution on the half-line generated by the Bessel differential operator

$$L_\alpha = \frac{d^2}{dx^2} + \frac{2\alpha+1}{x}\frac{d}{dx}, \quad \alpha > -\frac{1}{2}. \tag{3}$$

The aim of this paper is to study similar questions when in (2) the classical convolution $*$ is replaced by a generalized convolution $\otimes$ on the half-line generated by the Gegenbauer differential operator

$$D_\lambda = \left(x^2-1\right)^{\frac{1}{2}-\lambda} \frac{d}{dx} \left(x^2-1\right)^{\lambda+\frac{1}{2}} \frac{d}{dx}. \tag{4}$$

The Gegenbauer functions $P_\alpha^\lambda(x)$, $\alpha \in [1,\infty)$, $x \in [1,\infty)$, $\lambda \in (0, 1/2)$, are eigenfunctions of the operator D_λ that satisfy the following differential equation [3]:

$$\left\{\left(x^2-1\right)\frac{d^2}{dx^2} + (2\lambda+1)x\frac{d}{dx} - \alpha(\alpha+2\lambda)\right\} P_\alpha^\lambda(x) = 0, \tag{5}$$

$$P_\alpha^\lambda(x) = \frac{\Gamma(\alpha+2\lambda)\cos\pi\lambda}{\Gamma(\lambda)\Gamma(\alpha+\lambda+1)}(2x)^{-\alpha-2\lambda}{}_2F_1\left(\frac{\alpha}{2}+\lambda, \frac{\alpha}{2}+\lambda+\frac{1}{2}; \alpha+\lambda+1; x^{-2}\right), \tag{6}$$

and ${}_2F_1$ denotes the hypergeometric function.

The second linearly-independent solution of Eq. (5) is as follows

$$\begin{aligned} C_\alpha^\lambda(x) = {}& -\frac{\sin\pi\alpha\,\Gamma(\alpha+2\lambda)\Gamma(-\alpha-\lambda)}{\pi\Gamma(\lambda)}(2x)^{-\alpha-2\lambda} \\ & \times {}_2F_1\left(\frac{\alpha}{2}+\lambda, \frac{\alpha}{2}+\lambda+\frac{1}{2}; \alpha+\lambda+1; x^{-2}\right) \\ & + \frac{\Gamma(\alpha+\lambda)}{\Gamma(\lambda)\Gamma(\alpha+\lambda)}(2x)^{\alpha}{}_2F_1\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}+\frac{1}{2}; 1-\alpha-\lambda; x^{-2}\right), \end{aligned} \tag{7}$$

where $\alpha \in [1,\infty)$, $x \in [1,\infty)$ and $\lambda \in (0, 1/2)$.

We denote by $L_{p,\lambda}[0,\infty) = L_p(sh^{2\lambda} x\,dx)$, $1 \leqslant p \leqslant \infty$, is the set of all classes of measurable functions f on $[0,\infty)$ for which

$$\|f\|_{p,\lambda} \equiv \big\|f(ch)(\cdot)\big\|_{p,\lambda} = \left(\int_0^\infty \big|f(ch\,x)\big|^p\, sh^{2\lambda} x\,dx\right)^{\frac{1}{p}} < \infty, \quad 1 \leqslant p < \infty,$$

and

$$\|f\|_{\infty,\lambda} \equiv \big\|f(ch)(\cdot)\big\|_{\infty,\lambda} = \operatorname*{ess\,sup}_{x \geqslant 0} \big|f(ch\,x)\big| < \infty.$$

Denote by $M = M([0,\infty))$ the space of all finite Borel measures on the semi-axis $[0,\infty)$. For $\mu \in M$, we assume

$$\|\mu\| = |\mu|\big([0,\infty)\big) = \int_{[0,\infty)} d|\mu|(ch\,x),$$

where $|\mu|$ is the absolute value of μ.

Definition 1. For $f \in L_{1,\lambda}[0,\infty)$, the Gegenbauer transformations of f are defined by

$$\hat{f}_P(\alpha) = \int_0^\infty f(ch\,t) P_\alpha^\lambda(ch\,t)\, sh^{2\lambda} t\,dt, \quad \alpha \in [1,\infty),$$

$$\hat{f}_Q(\alpha) = \int_0^\infty f(ch\,t) Q_\alpha^\lambda(ch\,t)\, sh^{2\lambda} t\,dt, \quad \alpha \in [1,\infty),$$

where $P_\alpha^\lambda(ch\,x)$ is given by (6) and $Q_\alpha^\lambda(ch\,x) = \dfrac{\Gamma(2\alpha)\Gamma(\alpha+1)}{\Gamma(\alpha+2\lambda)} C_\alpha^\lambda(ch\,x)$.

Definition 2. The Gegenbauer transformation of the measure $\mu \in M$ is defined by the formula

$$\hat{\mu}_{\widetilde{P}}(\alpha) = \int_{[0,\infty)} \widetilde{P}_\alpha^\lambda(ch\,x)\, d\mu(ch\,x),$$

where $\widetilde{P}_\alpha^\lambda(ch\,x) = P_\alpha^\lambda(ch\,x)/P_\alpha^\lambda(1)$.

Definition 3.

(1) The generalized shift operator $A_{ch\,t}^\lambda$ is defined by

$$A_{ch\,t}^\lambda f(ch\,x) = \frac{\Gamma(\lambda+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} \int_0^\pi f(ch\,x\,ch\,t - sh\,x\,sh\,t\cos\varphi)(\sin\varphi)^{2\lambda-1}\,d\varphi.$$

(2) The generalized convolution $\otimes$ generated by the generalized shift operator is defined as follows

$$(\mu \otimes f)(ch\,x) = \int_{[0,\infty)} A_{ch\,t}^\lambda f(ch\,x)\, d\mu(ch\,t),$$

where $\mu \in M$ and f is a smooth function.

Properties.

(1) For the function $\widetilde{P}_\alpha^\lambda(ch\,x)$, the following "multiplication theorem" holds:

$$A_{ch\,t}^\lambda \widetilde{P}_\alpha^\lambda(ch\,x) = \widetilde{P}_\alpha^\lambda(ch\,t) Q_\alpha^\lambda(ch\,x).$$

(2) If $f \in L_{p,\lambda}[0,\infty)$, $1 \leqslant p \leqslant \infty$, then for all $t \geqslant 0$, $A_{ch\,t}^\lambda f \in L_{p,\lambda}$ and

$$\left\| A_{ch\,t}^\lambda f \right\|_{p,\lambda} \leqslant \|f\|_{p,\lambda}.$$

(3) For any $f \in L_{1,\lambda}[0,\infty)$, the following equality holds:

$$\big(\widehat{A_{ch\,t}^\lambda f}\big)_{\widetilde{P}}(\alpha) = \hat{f}_{\widetilde{P}}(\alpha) Q_\alpha^\lambda(ch\,t), \quad t \geqslant 0.$$

(4) If $\mu \in M$ and $f \in L_{p,\lambda}[0,\infty)$, $1 \leqslant p \leqslant \infty$, then

$$\mu \otimes f \in L_{p,\lambda} \quad \text{and} \quad \|\mu \otimes f\|_{p,\lambda} \leqslant \|\mu\| \|f\|_{p,\lambda}. \tag{8}$$

By using Minkowsky inequality and property (2), we obtain property (4):

$$\|\mu \otimes f\|_{p,\lambda} \leqslant \int_0^\infty \left\| A_{ch\,t}^\lambda f \right\|_{p,\lambda} d|\mu|(ch\,t) \leqslant \|f\|_{p,\lambda} \int_0^\infty d|\mu|(ch\,t) = \|\mu\| \|f\|_{p,\lambda}.$$

(5) For $\mu \in M$ and $f \in L_{1,\lambda}$, the equality

$$(\widehat{\mu \otimes f})_{\widetilde{P}}(\alpha) = \hat{f}_{\widetilde{P}}(\alpha) \hat{\mu}_Q(\alpha) \tag{9}$$

is valid.

In fact, by using the property (3) we obtain

$$\begin{aligned}
(\widehat{\mu \otimes f})_{\widetilde{P}}(\alpha) &= \int_0^\infty \left(\int_0^\infty A_{ch\,t}^\lambda f(ch\,u)\, d\mu(ch\,u) \right) \widetilde{P}_\alpha^\lambda(ch\,x)\, d\mu(ch\,x) \\
&= \int_0^\infty \left(\int_0^\infty \widetilde{P}_\alpha^\lambda(ch\,x) A_{ch\,t}^\lambda f(ch\,x)\, d\mu(ch\,x) \right) d\mu(ch\,u) \\
&= \int_0^\infty \big(\widehat{A_{ch\,t}^\lambda f}\big)_{\widetilde{P}}(\alpha)\, d\mu(ch\,u) = \hat{f}_{\widetilde{P}}(\alpha) \int_0^\infty Q_\alpha^\lambda(ch\,u)\, d\mu(ch\,u) = \hat{f}_{\widetilde{P}}(\alpha) \hat{\mu}_Q(\alpha).
\end{aligned}$$

Definition 4. Let $\mu \in M$ and $a > 0$. We define the dilated measure μ_a of μ by

$$\int_{[0,\infty)} \varphi(ch\,x)\, d\mu_a(ch\,x) = \int_{[0,\infty)} \varphi(ch\,ax)\, d\mu(ch\,x), \quad \varphi \in C_c\big([0,\infty)\big),$$

where $C_c([0,\infty))$ is the space of continuous functions with compact support.

Properties.

(1) Suppose $d\mu(ch\,x) = g(ch\,x)\,sh^{2\lambda}x\,dx$ with $g \in L_{1,\lambda}[0,\infty)$, then we give the measure μ_a, $a > 0$, by the function

$$g_a(ch\,x) = g\left(ch\,\frac{x}{a}\right) sh^{2\lambda}\frac{x}{a}\Big/\left(a\,sh^{2\lambda}x\right), \quad x > 0. \tag{10}$$

Hence

$$d\mu_a(ch\,x) = \frac{1}{a} g\left(ch\,\frac{x}{a}\right) sh^{2\lambda}\frac{x}{a}\,dx.$$

(2) For $\mu \in M$ and $f \in L_{p,\lambda}[0,\infty)$, $1 \leqslant p < \infty$, the following equality holds:

$$\lim_{a\to 0}\left\|\mu_a \otimes f - \mu\big([0,\infty)\big) f\right\|_{p,\lambda} = 0.$$

Proof. We have

$$(\mu_a \otimes f)(ch\,x) - \mu\big([0,\infty)\big) f(ch\,x) = \int_0^\infty \left[A^\lambda_{ch\,t} f(ch\,x) - f(ch\,x)\right] d\mu(ch\,t).$$

By the Minkowsky inequality we get

$$\begin{aligned}\left\|\mu_a \otimes f - \mu\big([0,\infty)\big) f\right\|_{p,\lambda} &\leqslant \int_0^\infty \left\|A^\lambda_{ch\,t} f - f\right\|_{p,\lambda} d|\mu|_a(ch\,t)\\ &\leqslant \int_0^\infty \omega_f(at)_{p,\lambda}\, d|\mu|(ch\,t),\end{aligned} \tag{11}$$

where $\omega_f(\delta)_{p,\lambda} = \sup_{0<t\leqslant\delta}\|A^\lambda_{ch\,t} f - f\|_{p,\lambda}$ and $\omega_f(\delta)_{p,\lambda} \to 0$ as $\delta \to 0$.

Since $\|A^\lambda_{ch\,t} f - f\|_{p,\lambda} \leqslant 2\|f\|_{p,\lambda}$, the last integral on the right-hand side in (11) tends to zero by the Lebesgue's dominated convergence theorem. □

(3) Let $f \in L_{1,\lambda}[0,\infty)$ and $g \in L_{p,\lambda}([0,\infty))$, $1 < p < \infty$, then

$$\lim_{a\to\infty}\|f \otimes g_a\|_{p,\lambda} = 0. \tag{12}$$

Proof. From Young's inequality

$$\|f \otimes g_a\|_{p,\lambda} \leqslant \|f\|_{1,\lambda}\|g_a\|_{p,\lambda} = \|f\|_{1,\lambda}\|g\|_{p,\lambda} a^{\frac{1-p}{p}} \to 0 \quad \text{as } a \to \infty,$$

then we get (12). □

2. Calderon's reproducing formula associated with the Gegenbauer operator

Lemma 1. *For $\mu \in M$ and $0 < \varepsilon < \delta < \infty$, assume*

$$G_{\varepsilon,\delta}(x) = \mu[x/\delta, x/\varepsilon]\Big/\left(x\,sh^{2\lambda}x\right), \quad x > 0, \tag{13}$$

and

$$K_{\varepsilon,\delta}(\alpha) = \int_{\varepsilon}^{\delta} \left(\int_{[0,\infty)} \widetilde{P}_{\alpha}^{\lambda}\big(ch(tu)\big)\, d\mu(ch\,u) \right) \frac{dt}{t}. \tag{14}$$

Then $G_{\varepsilon,\delta}(x) \in L_{1,\lambda}[0,\infty)$ *and*

$$\widehat{(G_{\varepsilon,\delta})}_{\widetilde{P}}(\alpha) = K_{\varepsilon,\delta}(\alpha) - \mu\big(\{0\}\big) \log \frac{\delta}{\varepsilon}. \tag{15}$$

Proof. By the definition, we have

$$\begin{aligned}
\|G_{\varepsilon,\delta}\|_{1,\lambda} &= \int_0^\infty \big|G_{\varepsilon,\delta}(ch\,x)\big|\, sh^{2\lambda} x\, dx \leqslant \int_0^\infty \left(\int_{x/\delta}^{x/\varepsilon} d|\mu|(ch\,t) \right) \frac{dx}{x} \\
&= \int_0^\infty d|\mu|(ch\,t) \int_{\varepsilon t}^{\delta t} \frac{dx}{x} = |\mu|\big([0,\infty)\big) \log \frac{\delta}{\varepsilon} < \infty.
\end{aligned}$$

Further, applying Fubini's theorem we obtain

$$\begin{aligned}
\widehat{(G_{\varepsilon,\delta})}_{\widetilde{P}}(\alpha) &= \int_{(0,\infty)} \left(\int_{[x/\delta, x/\varepsilon]} d\mu(ch\,t) \right) \widetilde{P}_{\alpha}^{\lambda}(ch\,x) \frac{dx}{x} \\
&= \int_{(0,\infty)} \left(\int_{\varepsilon t}^{\delta t} \widetilde{P}_{\alpha}^{\lambda}(ch\,x) \frac{dx}{x} \right) d\mu(ch\,t) \\
&= \int_{(0,\infty)} \left(\int_{\varepsilon}^{\delta} \widetilde{P}_{\alpha}^{\lambda}\big(ch(tx)\big) \frac{dx}{x} \right) d\mu(ch\,t) \\
&= \int_{\varepsilon}^{\delta} \left(\int_{(0,\infty)} \widetilde{P}_{\alpha}^{\lambda}\big(ch(tx)\big)\, d\mu(ch\,t) \right) \frac{dx}{x} \\
&= \int_{\varepsilon}^{\delta} \left(\int_{[0,\infty)} \widetilde{P}_{\alpha}^{\lambda}\big(ch(tx)\big)\, d\mu(ch\,t) - \mu\big(\{0\}\big) \right) \frac{dx}{x} \\
&= K_{\varepsilon,\delta}(\alpha) - \mu\big(\{0\}\big) \log \frac{\delta}{\varepsilon}.
\end{aligned}$$

Hence Lemma 1 is proved. □

Lemma 2. *Let* $\mu \in M$, $f \in L_{p,\lambda}[0,\infty)$, $1 \leqslant p \leqslant \infty$, *and* $0 < \varepsilon < \delta < \infty$, *then the function*

$$f^{\varepsilon,\delta}(ch\,x) = \int_{\varepsilon}^{\delta} (f \otimes \mu_a)(ch\,x) \frac{da}{a} \tag{16}$$

belongs to $L_{p,\lambda}[0,\infty)$ *and can be represented as follows*

$$f^{\varepsilon,\delta}(ch\,x) = (f \otimes G_{\varepsilon,\delta})(ch\,x) + \mu(\{0\}) f(ch\,x) \log\frac{\delta}{\varepsilon}. \tag{17}$$

Proof. Applying Fubini's theorem we obtain

$$\begin{aligned}
f^{\varepsilon,\delta}(ch\,x) &= \int_{\varepsilon}^{\delta}\left(\int_{[0,\infty)} A^{\lambda}_{ch(at)} f(ch\,x)\,d\mu(ch\,t)\right)\frac{da}{a} \\
&= \int_{[0,\infty)}\left(\int_{\varepsilon}^{\delta} A^{\lambda}_{ch(at)} f(ch\,x)\frac{da}{a}\right) d\mu(ch\,t) \\
&= \mu(\{0\}) f(ch\,x)\log\frac{\delta}{\varepsilon} + \int_{(0,\infty)}\left(\int_{\varepsilon t}^{\delta t} A^{\lambda}_{ch\,a} f(ch\,x)\frac{da}{a}\right) d\mu(ch\,t) \\
&= \mu(\{0\}) f(ch\,x)\log\frac{\delta}{\varepsilon} + \int_{(0,\infty)} A^{\lambda}_{ch\,a} f(ch\,x)\left(\int_{a/\delta}^{a/\varepsilon} d\mu(ch\,t)\right)\frac{da}{a} \\
&= \mu(\{0\}) f(ch\,x)\log\frac{\delta}{\varepsilon} + (f \otimes G_{\varepsilon,\delta})(ch\,x).
\end{aligned}$$

Hence from (8) and Lemma 1 it follows that $f^{\varepsilon,\delta} \in L_{p,\lambda}[0,\infty)$ and the equality (17) holds. □

Lemma 3. *Let* $\mu \in M$. *Then for* $f \in L_{2,\lambda}[0,\infty)$ *the equality*

$$\widehat{\left(f^{\varepsilon,\delta}\right)}_{\widetilde{P}}(\alpha) = \hat{f}_Q(\alpha)K_{\varepsilon,\delta} + \mu(\{0\})\left(\hat{f}_{\widetilde{P}}(\alpha) - \hat{f}_Q(\alpha)\right)\log\frac{\delta}{\varepsilon}$$

holds, where $K_{\varepsilon,\delta}$ *is the function defined by* (14).

Proof. In fact, from (9), (15) and (17) it can be easily seen that

$$\begin{aligned}
\widehat{\left(f^{\varepsilon,\delta}\right)}_{\widetilde{P}}(\alpha) &= (f \widehat{\otimes G}_{\varepsilon,\delta})(\alpha) + \mu(\{0\})\hat{f}_{\widetilde{P}}(\alpha)\log\frac{\delta}{\varepsilon} \\
&= \hat{f}_Q(\alpha)\widehat{(G_{\varepsilon,\delta})}(\alpha) + \mu(\{0\})\hat{f}_{\widetilde{P}}(\alpha)\log\frac{\delta}{\varepsilon} \\
&= \hat{f}_Q(\alpha)K_{\varepsilon,\delta} + \mu(\{0\})\left(\hat{f}_{\widetilde{P}}(\alpha) - \hat{f}_Q(\alpha)\right)\log\frac{\delta}{\varepsilon}. \qquad \square
\end{aligned}$$

Theorem 1. *Let* $\mu \in M$ *and*

$$C_\mu = \int_0^\infty \left|\mu([0,x])\right|\frac{dx}{x} < \infty. \tag{18}$$

If $f \in L_{p,\lambda}[0,\infty)$, $1 < p < \infty$, *then*

$$\lim_{\substack{\varepsilon\to 0\\ \delta\to\infty}} \left\| f^{\varepsilon\delta} - C_\mu f\right\|_{p,\lambda} = 0. \tag{19}$$

Proof. From (13) we have

$$G_{\varepsilon,\delta}(x) = G_\varepsilon(x) - G_\delta(x),$$

where

$$G(x) = \mu\big([0,x]\big)\big/\big(x\, sh^{2\lambda} x\big), \quad G_\varepsilon(x) = \mu\big([0,x/\varepsilon]\big)\big/\big(x\, sh^{2\lambda} x\big), \quad x > 0.$$

Hence from (17) we obtain

$$f^{\varepsilon,\delta}(ch\, x) = (f \otimes G_\varepsilon)(ch\, x) - (f \otimes G_\delta)(ch\, x). \tag{20}$$

Now we have

$$\big\| f^{\varepsilon,\delta} - C_\mu f \big\|_{p,\lambda} \leqslant \| f \otimes G_\varepsilon - C_\mu f\|_{p,\lambda} + \| f \otimes G_\delta\|_{p,\lambda} \tag{21}$$

and also

$$(f \otimes G_\varepsilon)(ch\, x) - f(ch\, x) \int_0^\infty \mu\big([0,t]\big) \frac{dt}{t}$$
$$= \int_0^\infty \big[A^\lambda_{ch\, t} f(ch\, x) - f(ch\, x)\big] \mu\big([0,t/\varepsilon]\big) \frac{dt}{t}.$$

By the Minkowsky inequality we have

$$\| f \otimes G_\varepsilon - f C_\mu\|_{p,\lambda} \leqslant \int_0^\infty \big\| A^\lambda_{ch\, t} - f \big\|_{p,\lambda} \big|\mu\big([0,t/\varepsilon]\big)\big| \frac{dt}{t}$$
$$\leqslant \int_0^\infty \omega_f(\varepsilon t)_{p,\lambda} \big|\mu\big([0,t]\big)\big| \frac{dt}{t} \leqslant 2\|f\|_{p,\lambda} \int_0^\infty \big|\mu\big([0,t]\big)\big| \frac{dt}{t} < \infty.$$

By the Lebesgue's dominated convergence theorem we get

$$\lim_{\varepsilon\to 0} \| f \otimes G_\varepsilon - C_\mu f\|_{p,\lambda} = 0. \tag{22}$$

It remains to prove that

$$\lim_{\delta\to\infty} \| f \otimes G_\delta\|_{p,\lambda} = 0. \tag{23}$$

By the Young's inequality we have

$$\| f \otimes G_\delta\|_{p,\lambda} \leqslant \|f\|_{1,\lambda} \cdot \|G_\delta\|_{p,\lambda} \leqslant \delta^{\frac{1-p}{p}} \|f\|_{1,\lambda} \|G\|_{p,\lambda} \to 0 \quad \text{as } \delta \to \infty.$$

Thus the proof of the theorem is completed. □

Definition 5. Let f be locally integrable function on $[0,\infty)$, a point $x \in [0,\infty)$ is a Lebesgue point of f if

$$\lim_{t\to 0} \frac{1}{t^{2\lambda+1}} \int_0^t \big|A^\lambda_{ch\, t} f(ch\, x) - f(ch\, x)\big| sh^{2\lambda} u\, du = 0.$$

It is well known that if x_0 is a continuity point of the function f, then it is a Lebesgue point of f. Further, if $f \in L_{1,\lambda}[0,\infty)$, then almost every points of the interval $[0,\infty)$ are the Lebesgue points of f.

We will need the following lemma being analogue of I.P. Natanson's lemma [8, p. 304] later.

Lemma 4. *Let $f \in L_{1,\lambda}[0,\infty)$ be a nonnegative function and assume the following condition holds*:

$$\sup_{0<t<\infty} t^{-2\lambda-1} \int_0^t f(ch\, u)\, sh^{2\lambda} u\, du = M < \infty. \tag{24}$$

Then for any nonnegative function $g \in L_{1,\lambda}[0,\infty)$ decreasing on $[0,\infty)$, the integral

$$\int_0^\infty f(ch\, t) g(ch\, t)\, sh^{2\lambda} t\, dt \tag{25}$$

exists and the inequality

$$\int_0^\infty f(ch\, t) g(ch\, t)\, sh^{2\lambda} t\, dt \leqslant M(2\lambda+1) \int_0^\infty g(ch\, t)\, sh^{2\lambda} t\, dt \tag{26}$$

holds.

Proof. We note that if $g(1) = \infty$, then the integral (25) exists as a singular integral. If $g(1) < \infty$, then the function $g(ch\, t)$ is bounded and therefore the integral (25) exists as an ordinary Lebesgue integral.

Without loss of generality we assume $g(\infty) = 0$. In fact, if it were not so, then we would take $\tilde{g}(ch\, t)$, instead of $g(ch\, t)$ which is defined by

$$\tilde{g}(ch\, t) = \begin{cases} g(ch\, t), & 0 < t < \infty, \\ 0, & t = \infty. \end{cases}$$

If we take $\tilde{g}(ch\, t)$ instead of $g(ch\, t)$, then the value of this function at any isolated point does not effect the value of the integral. So we assume $g(\infty) = 0$. Let $0 < \alpha < \infty$. The function $g(ch\, t)$ is bounded on the ray $[\alpha, \infty)$ and therefore the integral

$$\int_\alpha^\infty f(ch\, t) g(ch\, t)\, sh^{2\lambda} t\, dt \tag{27}$$

exists. Assuming

$$F(t) = \int_0^t f(ch\, u)\, sh^{2\lambda} u\, du,$$

we can write the integral (27) in the following form:

$$\int_{\alpha}^{\infty} f(ch\,t)g(ch\,t)\,sh^{2\lambda}t\,dt = \int_{\alpha}^{\infty} g(ch\,t)\,dF(t)$$
$$= -g(ch\,\alpha)F(\alpha) + \int_{\alpha}^{\infty} F(t)\,d\bigl[-g(ch\,t)\bigr]. \tag{28}$$

By using (24) we get

$$0 \leqslant F(t) \leqslant Mt^{2\lambda+1}, \quad t > 0, \tag{29}$$

and

$$\int_{0}^{\alpha} g(ch\,t)\,sh^{2\lambda}t\,dt \geqslant g(ch\,\alpha)\int_{0}^{\alpha} sh^{2\lambda}t\,dt$$
$$\geqslant g(ch\,\alpha)\int_{0}^{\alpha} t^{2\lambda}\,dt = g(ch\,\alpha)\frac{\alpha^{2\lambda+1}}{2\lambda+1},$$

since $g(ch\,t)$ decreases. Hence, it follows that

$$g(ch\,\alpha) \leqslant \frac{2\lambda+1}{\alpha^{2\lambda+1}}\int_{0}^{\alpha} g(ch\,t)\,sh^{2\lambda}t\,dt. \tag{30}$$

From (29) and (30) we have

$$0 \leqslant F(\alpha)g(ch\,\alpha) \leqslant M(2\lambda+1)\int_{0}^{\alpha} g(ch\,t)\,sh^{2\lambda}t\,dt, \tag{31}$$

further

$$\int_{\alpha}^{\infty} F(t)\,d\bigl[-g(ch\,t)\bigr] \leqslant M\int_{\alpha}^{\infty} t^{2\lambda+1}\,d\bigl[-g(ch\,t)\bigr]$$
$$= M\alpha^{2\lambda+1}g(ch\,\alpha) + M(2\lambda+1)\int_{\alpha}^{\infty} t^{2\lambda}g(ch\,t)\,dt$$
$$\leqslant M(2\lambda+1)\int_{0}^{\infty} g(ch\,t)\,sh^{2\lambda}t\,dt. \tag{32}$$

Now from (28), (31) and (32) we have

$$\int_{\alpha}^{\infty} f(ch\,t)g(ch\,t)\,sh^{2\lambda}t\,dt \leqslant M(2\lambda+1)\int_{0}^{\infty} g(ch\,t)\,sh^{2\lambda}t\,dt.$$

Taking limit as $\alpha \to 0$, we obtain the assertion of the lemma. □

Theorem 2. *Let $f \in L_{p,\lambda}[0,\infty)$, $1 < p < \infty$, and let $g(ch(\cdot))$ be a measurable function on $[0,\infty)$ such that g^* belongs to $L_{1,\lambda}[0,\infty)$, where $g^*(ch\,x) = \operatorname{ess\,sup}_{t \geqslant x} |g(ch\,t)|$. Then for every Lebesgue point $x \in [0,\infty)$ of f, the following equality holds*:

$$\lim_{a\to+0}(f \otimes g_a)(ch\,x) = f(ch\,x)\int_0^\infty g(ch\,t)\,sh^{2\lambda} t\,dt.$$

Proof. We have

$$g_a^*(ch\,x) = \frac{1}{a} g^*\left(ch\,\frac{x}{a}\right) sh^{2\lambda}\,\frac{x}{a} \Big/ sh^{2\lambda}\,x.$$

Since

$$\int_0^\infty dg_a(ch\,t) = \int_0^\infty \frac{1}{a} g\left(ch\,\frac{t}{a}\right) sh^{2\lambda}\,\frac{t}{a}\,dt = \int_0^\infty g(ch\,t)\,sh^{2\lambda} t\,dt,$$

we get

$$\begin{aligned}
&\left|(f \otimes g_a)(ch\,x) - f(ch\,x)\int_0^\infty dg_a(ch\,t)\right| \\
&\quad = \left|\int_0^\infty A^\lambda_{ch\,t} f(ch\,x)\,dg_a(ch\,t) - f(ch\,x)\int_0^\infty dg_a(ch\,t)\right| \\
&\quad = \left|\int_0^\infty \left[A^\lambda_{ch\,t} f(ch\,x) - f(ch\,x)\right]\frac{1}{a} g\left(ch\,\frac{t}{a}\right) sh^{2\lambda}\,\frac{t}{a}\,dt\right| \\
&\quad = \left|\int_0^\infty \left[A^\lambda_{ch\,t} f(ch\,x) - f(ch\,x)\right] g_a(ch\,t)\,sh^{2\lambda} t\,dt\right| = J_a(x).
\end{aligned} \tag{33}$$

Since x is a Lebesgue point, then for $\forall a > 0$, $\exists \delta(a) > 0$, $0 < t \leqslant \delta$, we have

$$\int_0^t \left|A^\lambda_{ch\,u} f(ch\,x) - f(ch\,x)\right| sh^{2\lambda} u\,du < a t^{2\lambda+1}. \tag{34}$$

Now, divide the integral J into two parts as the following

$$J_a(x) = \int_0^\delta + \int_\delta^\infty = J_{1,a}(x) + J_{2,a}(x). \tag{35}$$

Firstly, we estimate the integral $J_{1,a}$,

$$\left|J_{1,a}(x)\right| \leqslant \int_0^\delta \left|A^\lambda_{ch\,t} f(ch\,x) - f(ch\,x)\right|\left|g_a(ch\,t)\right| sh^{2\lambda} t\,dt$$

$$\leqslant \int_0^\delta \left|A_{ch\,t}^\lambda f(ch\,x) - f(ch\,x)\right| g_a^*(ch\,t)\, sh^{2\lambda} t\, dt$$
$$= \int_0^\delta \left|A_{ch\,t}^\lambda f(ch\,x) - f(ch\,x)\right| \frac{1}{a} g^*\left(ch\,\frac{t}{a}\right) sh^{2\lambda}\,\frac{t}{a}\, dt.$$

Since g^* is a decreasing function, we obtain from Lemma 4 the following

$$\begin{aligned}\left|J_{1,a}(x)\right| &\leqslant a(2\lambda+1)\delta^{2\lambda+1}\int_0^\delta \frac{1}{a} g^*\left(ch\,\frac{t}{a}\right) sh^{2\lambda}\,\frac{t}{a}\, dt\\ &= a(2\lambda+1)\delta^{2\lambda+1}\int_0^{\delta/a} g^*(ch\,u)\, sh^{2\lambda} u\, du\\ &\leqslant a(2\lambda+1)\delta^{2\lambda+1}\left|g^*\right|_{1,\lambda} \to 0 \quad \text{as } a \to 0,\end{aligned} \tag{36}$$

for all Lebesgue point $x \in [0, \infty)$.

Now consider the integral J_2. By the Hölder inequality we have

$$\begin{aligned}\left|J_{2,a}(x)\right| &\leqslant \int_\delta^\infty \left|A_{ch\,t}^\lambda f(ch\,x) - f(ch\,x)\right| g_a^*(ch\,t)\, sh^{2\lambda} t\, dt\\ &\leqslant \left(\int_\delta^\infty \left|A_{ch\,t}^\lambda f(ch\,x) - f(ch\,x)\right| g_a^*(ch\,t)\, sh^{2\lambda} t\, dt\right)^{1/p}\left(\int_\delta^\infty g_a^*(ch\,t)\, sh^{2\lambda} t\, dt\right)^{1/p'}\\ &= A_{\delta,a}(x)\cdot B_{\delta,a},\end{aligned} \tag{37}$$

where $1/p + 1/p' = 1$.

We get

$$\begin{aligned}A_{\delta,a}(x) &\leqslant g_a^*(ch\,\delta)\left(\int_\delta^\infty \left|A_{ch\,t}^\lambda f(ch\,x) - f(ch\,x)\right|^p sh^{2\lambda} t\, dt\right)^{1/p}\\ &\leqslant g_a^*(ch\,\delta)\left\|A_{ch\,t}^\lambda f - f\right\|_{p,\lambda} \leqslant 2g_a^*(ch\,\delta)\|f\|_{p,\lambda},\end{aligned} \tag{38}$$

since $g_a^*(ch\,t)$ is a decreasing function.

By inequality (30) we have

$$\begin{aligned}g_a^*(ch\,\delta) &\leqslant \frac{2\lambda+1}{\delta^{2\lambda+1}}\int_0^\delta g_a^*(ch\,t)\, sh^{2\lambda} t\, dt = \frac{2\lambda+1}{\delta^{2\lambda+1}}\int_0^\delta \frac{1}{a} g^*\left(ch\,\frac{t}{a}\right) sh^{2\lambda}\,\frac{t}{a}\, dt\\ &= \frac{2\lambda+1}{\delta^{2\lambda+1}}\int_0^{\delta/a} g^*(ch\,t)\, sh^{2\lambda} t\, dt < \frac{2\lambda+1}{\delta^{2\lambda+1}}\left\|g^*\right\|_{1,\lambda}.\end{aligned} \tag{39}$$

From (38) and (39) we obtain

$$A_{\delta,a}(x) \leqslant \frac{2(2\lambda+1)}{\delta^{2\lambda+1}} \|f\|_{p,\lambda} \|g^*\|_{1,\lambda}. \tag{40}$$

Furthermore, we have

$$\begin{aligned} B^q_{\delta,a} &= \int_\delta^\infty g^*_a(ch\,t)\, sh^{2\lambda}\, t\, dt = \int_\delta^\infty \frac{1}{a} g^*\left(ch\,\frac{t}{a}\right) sh^{2\lambda}\, \frac{t}{a}\, dt \\ &= \int_{\delta/a}^\infty g^*(ch\,t)\, sh^{2\lambda}\, t\, dt \to 0 \quad \text{as } a \to 0, \end{aligned} \tag{41}$$

since $g^* \in L_{1,\lambda}[0,\infty)$.

Hence, for all Lebesgue point $x \in [0,\infty)$ we get

$$\lim_{a\to 0} J_{2,a}(x) = 0. \tag{42}$$

By combining (36) and (42), we get the proof of the theorem. □

Theorem 3. *Let $f \in L_{p,\lambda}[0,\infty)$, $1 \leqslant p < \infty$, and assume $\mu \in M$ satisfies the following conditions*:

$$\int_0^1 |\mu|\big([0,x]\big)\frac{dx}{x} < \infty \tag{43}$$

and

$$\int_1^\infty \big|\mu\big([0,x]\big)\big|\frac{dx}{x} < \infty. \tag{44}$$

If $G^* \in L_{1,\lambda}[0,\infty)$, where $G^*(ch\,x) = \operatorname{ess\,sup}_{t\geqslant x} |G(ch\,t)|$, then for every Lebesgue point $x \in [0,\infty)$ of f, the following equality holds

$$\lim_{\substack{\varepsilon\to 0\\ \delta\to\infty}} f^{\varepsilon,\delta}(ch\,x) = C_\mu f(ch\,x).$$

Proof. From (20) and Theorem 2 it is sufficient to show that $(f \otimes G_\delta)(ch\,x) \to 0$ as $\delta \to \infty$. From condition (43) it follows that $\mu(\{0\}) = 0$, then we have $\lim_{\delta\to\infty} \mu([0,t/\delta]) = 0$.

Further,

$$\begin{aligned} (f \otimes G_\delta)(ch\,x) &= \int_0^1 A^\lambda_{ch\,t} f(ch\,x)\mu\big([0,t/\delta]\big)\frac{dt}{t} + \int_1^\infty A^\lambda_{ch\,t} f(ch\,x)\mu\big([0,t/\delta]\big)\frac{dt}{t} \\ &= A_{\delta,a}(x)(x) + B_{\delta,a}(x). \end{aligned} \tag{45}$$

For $\delta \geqslant 1$,

$$\big|A^\lambda_{ch\,t} f(ch\,x)\mu\big([0,t/\delta]\big)/t\big| \leqslant A^\lambda_{ch\,t}|f|(ch\,x)|\mu|\big([0,t/\delta]\big)\cdot\frac{1}{t}$$

and from (43), we have

$$\int_0^1 A^{\lambda}_{ch\,t}|f|(ch\,x)|\mu|\big([0,t]\big)\frac{dt}{t} < \infty.$$

Hence from the general theorem of convergence it follows that

$$\lim_{\delta\to\infty} A_{\delta,a}(x) = 0 \tag{46}$$

for all Lebesgue point $x \in [0,\infty)$. From (44)

$$\begin{aligned}\int_1^\infty \big|A^{\lambda}_{ch\,t} f(ch\,x)\big|\big|\mu\big([0,t]\big)\big|\frac{dt}{t} &\leqslant \int_1^\infty \big\|A^{\lambda}_{ch\,t} f\big\|_{p,\lambda}\big|\mu\big([0,t]\big)\big|\frac{dt}{t} \\ &\leqslant \|f\|_{p,\lambda}\int_1^\infty \big|\mu\big([0,t]\big)\big|\frac{dt}{t} < \infty.\end{aligned}$$

Finally, since $A^{\lambda}_{ch\,t} f \in L_{p,\lambda}[0,\infty)$ for all $f \in L_{p,\lambda}[0,\infty)$, $1 \leqslant p < \infty$, it follows from the general theorem of convergence that

$$\lim_{\delta\to\infty} B_{\delta,a}(x) = 0 \tag{47}$$

for all Lebesgue point $x \in [0,\infty)$. From (46) and (47), the proof of the theorem is completed. □

Lemma 5. *Let $f(ch(\cdot))$ be an essentially bounded function on $[0,\infty)$, and*

$$\lim_{t\to\infty} t^{-2\lambda-1}\int_0^t A^{\lambda}_{ch\,u} f(ch\,x)\, sh^{2\lambda} u\, du = 0. \tag{48}$$

If $g \in L_{1,\lambda}[0,\infty)$, then

$$\lim_{a\to\infty} (f \otimes g_a)(ch\,x) = 0 \tag{49}$$

uniformly for $x \in [0,\infty)$.

Proof. It is sufficient to consider the case $f \geqslant 0$ and $g(ch(\cdot))$ is continuous on $[0,R]$, $R > 0$. Then we have

$$\begin{aligned}\big|(f \otimes g_a)(ch\,x)\big| &= \left|\int_0^{aR} A^{\lambda}_{ch\,t} f(ch\,x) g_a(ch\,t)\, sh^{2\lambda} t\, dt\right| \\ &= \left|\frac{1}{a}\int_0^{aR} A^{\lambda}_{ch\,t} f(ch\,x) g\left(ch\,\frac{t}{a}\right) sh^{2\lambda}\,\frac{t}{a}\, dt\right|. \end{aligned}\tag{50}$$

Note that, $sh\,\frac{t}{a} \leqslant \frac{1}{a}\, sh\,t$ for all $a \geqslant 1$. Then by virtue of (48), we have

$$\left|(f \otimes g_a)(ch\,x)\right| \leqslant \left|\frac{1}{a^{2\lambda+1}} \int_0^{aR} A^\lambda_{ch\,t} f(ch\,x) g\left(ch\,\frac{t}{a}\right) sh^{2\lambda}\,t\,dt\right|$$
$$\leqslant \|g\|_{\infty,\lambda} R^{2\lambda+1} \left(\frac{1}{(Ra)^{2\lambda+1}} \int_0^{aR} A^\lambda_{ch\,t} f(ch\,x)\, sh^{2\lambda}\,t\,dt\right) \to 0 \quad \text{as } a \to \infty.$$

The proof of the lemma is completed. □

Theorem 4. *Assume $\mu \in M$ satisfies the condition*

$$\int_0^\infty \left|\mu\big([0,x]\big)\right| \frac{dx}{x} < \infty,$$

$f \in L_{\infty,\lambda}[0,\infty)$, and Eq. (48) *holds. If $G^* \in L_{1,\lambda}[0,\infty)$, where $G^*(ch\,x) = \operatorname{ess\,sup}_{t \geqslant x} |G(ch\,t)|$, then at each Lebesgue point x the following equality holds*

$$\lim_{\substack{\varepsilon \to 0 \\ \delta \to \infty}} f^{\varepsilon,\delta}(ch\,x) = C_\mu f(ch\,x).$$

Proof. The proof of the theorem follows from equality (20), Theorem 3 and Lemma 5. □

Lemma 6. *Let $\mu \in M$ and $\mu([0,\infty)) = 0$. Then the following assertions are equivalent*

(1) $\int_0^1 |\mu|([0,t]) \frac{dt}{t} < \infty$ *and* $\int_1^\infty |\mu|((t,\infty)) \frac{dt}{t} < \infty$;
(2) $\int_{[0,\infty)} |\log x|\, d|\mu|(x) < \infty$.

Proof. We have

$$\int_0^1 |\mu|\big([0,t]\big) \frac{dt}{t} = \int_{[0,1]} |\log x|\, d|\mu|(x), \tag{51}$$

further, since

$$\mu\big([0,\infty)\big) = 0 \quad \Leftrightarrow \quad \mu\big([0,t]\big) = -\mu\big((t,\infty)\big),$$

then

$$\int_1^\infty |\mu|\big((t,\infty)\big) \frac{dt}{t} = \int_{(1,\infty)} \log x\, d|\mu|(x). \tag{52}$$

From (51) and (52), we get the assertion of the lemma. □

Theorem 5. *Let $\mu \in M$, $\mu([0,\infty)) = 0$ and let*

$$\int_{[0,\infty)} |\log x|\, d|\mu|(x) < \infty. \tag{53}$$

Then the integral C_μ given in (18) *is finite and can be represented as follows*

$$C_\mu = \int\limits_{[0,\infty)} \log \frac{1}{x}\, d\mu(x). \tag{54}$$

Proof. From (51) we have

$$\int\limits_0^1 \mu\big([0,t]\big)\frac{dt}{t} = \int\limits_{[0,1]} \log \frac{1}{x}\, d\mu(x). \tag{55}$$

On the other hand

$$\int\limits_1^\infty \mu\big([0,t]\big)\frac{dt}{t} = \int\limits_{(1,\infty)} \log \frac{1}{x}\, d\mu(ch\, x). \tag{56}$$

Combining (55) and (56), we get the assertion of the theorem. □

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