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Spectral properties of non-homogeneous Timoshenko beam and its rest to rest controllability

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Abstract

The controllability of a slowly rotating non-homogeneous beam clamped to a disc is considered. It is assumed that at the beginning the beam remains at the position of rest and it is supposed to rotate by the given angle and stop. The movement is governed by the system of two differential equations with non-constant coefficients: mass density, flexural rigidity and shear stiffness. To solve the problem of controllability, the spectrum of the operator generating the dynamics of the model is studied. Then the problem of controllability is reduced to the moment problem that is, in turn, solved with the use of the asymptotics of the spectrum.

Keywords: Controllability; Minimal system; Moment problem; Neumann series; Timoshenko beam

1. Introduction

The problem of controllability of slowly rotating Timoshenko beam has been lately considered by many authors [6, 11,14]. There are different models studied by various authors. One of them is the beam clamped to a disc that is moved by an engine in horizontal plane. The controllability depends then on the relationship between the radius of the disc and the length of the beam—there are values of parameters for which controllability is not possible. Full analysis of the controllability of the above mentioned model was described by W. Krabs and G.M. Sklyar first in [5] and then in monograph [6] and in series of papers (for example, [3,7]). The research presented there is based on spectral model analysis and on non-Fourier trigonometric moment problem methods. Some other models of Timoshenko beam were studied by F. Woittennek, J. Rudolph [14] and S.W. Taylor, S.C.B. Yau [13].

In the present paper we consider problem of controllability of a beam clamped to a disc, same as in [6], but with non-homogeneous parameters: flexural rigidity, rotary inertia, shear stiffness and mass density. Values of those parameters depend on the point, where the cross section is considered. The research of such model was inspired by M. Shubov [10–12]. In those papers an extensive analysis of spectral properties of the operators associated with a

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certain model of Timoshenko beam is given. The essential advantage of M. Shubov's approach is that it allows to consider the case with dumping terms causing dissipativity of energy (model is not conservative as well). In [11] she also considers controllability problem for Timoshenko beam when the programming control is two-dimensional. In this case the problem of controllability is reduced to non-Fourier trigonometric moment problem with separable exponentials. Solvability of the latter problem is proceeded by the methods given in [1].

In the present work we extend the results of W. Krabs and G.M. Sklyar [6] to non-homogeneous case. In contrast to M. Shubov, the model considered in [6] and here has conservative boundary conditions. The main part of the paper is devoted to the solution of the eigenvalue problem of operators associated with the rotation of Timoshenko beam. We give careful proofs of asymptotic spectral distribution of eigenvalues also in the case, when analysis with methods proposed by M. Shubov do not work. Our method is based on transformation of the system of differential equations into the system of Volterra integral equations and then obtaining the solution via uniformly convergent Neumann series. Further on, we consider the problem of controllability from rest to rest with one-dimensional control. In contrast to the dissipative case [12], the problem of controllability leads here to a trigonometric moment problem with two asymptotically close families of exponentials. Solution to this problem is based on the methods from [4].

2. Timoshenko beam model

We consider the motion of a beam in a horizontal plane. The left end of the beam is clamped to the disk of a driving motor. We denote by r the radius of that disk and let $\theta = \theta(t)$ be the rotation angle considered as a function of time ($t \geqslant 0$). Further on, we assign to a (uniform) cross section at x, with $0 \leqslant x \leqslant 1$ the following: $E(x)$ which is the flexural rigidity, $K(x)$—shear stiffness, $\varrho(x)$—mass of the cross section and $R(x)$—rotary inertia. All of the above functions are assumed to be real and bounded by two positive numbers. We also assume that their first and second derivatives are bounded. The length of the beam is assumed to be 1. We denote by $w(x,t)$ the deflection of the center line of the beam and by $\xi(x,t)$ the rotation angle[1] of the cross section area at the location x and at the time t. Then w and ξ are governed by the following system of differential equations:

$$\begin{aligned}
&\varrho(x)\ddot{w}(x,t) - \big(K(x)\big(w'(x,t)+\xi(x,t)\big)\big)' = -\ddot{\theta}(t)\varrho(x)(x+r),\\
&R(x)\ddot{\xi}(x,t) - \big(E(x)\xi'(x,t)\big)' + K(x)\big(w'(x,t)+\xi(x,t)\big) = \ddot{\theta}(t)R(x).
\end{aligned} \tag{1}$$

Here for given function g of two variables t and x, we adopt the notation $\dot{g} = g_t$, $g' = g_x$ for the first derivative. In addition to (1) we impose the following initial:

$$w(x,0) = \dot{w}(x,0) = \xi(x,0) = \dot{\xi}(x,0) = 0 \quad \text{for } x \in [0,1] \quad \text{and} \quad \theta(0) = \dot{\theta}(0) = 0,$$

and boundary conditions

$$w(0,t) = \xi(0,t) = 0, \tag{2}$$

$$w'(1,t) + \xi(1,t) = 0, \quad \xi'(1,t) = 0 \tag{3}$$

for $t \geqslant 0$. The initial conditions mean that at $t = 0$ the beam is in the rest position. The physical meaning of the boundary conditions is as follows. The condition (2) means that there is no deformation at the clamped end. The condition (3) is a consequence of the energy balance law.

We define

$$\left\langle \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} y_2 \\ z_2 \end{pmatrix} \right\rangle = \int_0^1 \varrho(x) y_1(x)\overline{y_2(x)}\,dx + \int_0^1 R(x) z_1(x)\overline{z_2(x)}\,dx \tag{4}$$

[1] Due to the orientation of an angle, the sign of ξ may be opposite ($\xi = -\tilde{\xi}$). It leads to a different form of Eq. (1):

$$\begin{aligned}
&\varrho(x)\ddot{w}(x,t) + \big(K(x)\big(\tilde{\xi}(x,t) - w'(x,t)\big)\big)' = -\ddot{\theta}(t)\varrho(x)(x+r),\\
&R(x)\ddot{\tilde{\xi}}(x,t) - \big(E(x)\tilde{\xi}'(x,t)\big)' + K(x)\big(\tilde{\xi}(x,t) - w'(x,t)\big) = -\ddot{\theta}(t)R(x).
\end{aligned}$$

The equation of the rotational movement of the beam was derived in [6]. The derivation performed there works also for non-homogeneous case.

and consider the space H, whose underlying set is $L^2((0,1),\mathbb{C}^2)$ and with inner product (4). Due to the hypotheses imposed on ϱ, the norm generated by (4) is equivalent to the standard L^2 norm. Next, we define the linear operator $A : D(A) \to H$ by the formula

$$A\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{\varrho}(K(y'+z))' \\ -\frac{1}{R}((Ez')' - K(y'+z)) \end{pmatrix}, \tag{5}$$

where K, E, ϱ, R, y and z are functions of variable $x \in [0,1]$ and

$$D(A) = \left\{ \begin{pmatrix} y \\ z \end{pmatrix} \in H^2\big((0,1),\mathbb{C}^2\big)\colon \begin{matrix} y(0) = z(0) = 0, \\ y'(1) + z(1) = z'(1) = 0 \end{matrix} \right\} \subset H.$$

It is easy to see that $D(A)$ is dense in H. Using the defined operator A and putting

$$f_1(x,t) = -\ddot{\theta}(t)(r+x), \qquad f_2(x,t) = \ddot{\theta}(t), \tag{6}$$

we can rewrite Eqs. (1) in the vector form

$$\begin{pmatrix} \ddot{w}(x,t) \\ \ddot{\xi}(x,t) \end{pmatrix} + A\begin{pmatrix} w(x,t) \\ \xi(x,t) \end{pmatrix} = \begin{pmatrix} f_1(x,t) \\ f_2(x,t) \end{pmatrix}. \tag{7}$$

Proposition 1. *The operator $A : D(A) \to H$ is positive and invertible.*

Proof. It follows by an easy calculation that A is positive. To prove the invertibility, we write

$$A\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \quad \text{for } \begin{pmatrix} y \\ z \end{pmatrix} \in D(A) \text{ and } \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in H.$$

Consequently,

$$\begin{aligned} &\big(K(y'+z)\big)' = -\varrho g_1, \\ &(Ez')' - K(y'+z) = -Rg_2. \end{aligned}$$

After integrating both sides of the first equation with the use of boundary conditions we obtain

$$K(x)\big(y'(x) + z(x)\big) = \int_x^1 \varrho(s_1)g_1(s_1)\,ds_1$$

and soon after it

$$E(x)z'(x) = \int_x^1 R(s_2)g_2(s_2)\,ds_2 - \int_x^1\int_{s_2}^1 \varrho(s_1)g_1(s_1)\,ds_1\,ds_2.$$

Following some elementary computation we receive

$$z(x) = \int_0^x \frac{1}{E(s_3)}\int_{s_3}^1 R(s_2)g_2(s_2)\,ds_2\,ds_3 - \int_0^x \frac{1}{E(s_3)}\int_{s_3}^1\int_{s_2}^1 \varrho(s_1)g_1(s_1)\,ds_1\,ds_2\,ds_3$$

and

$$\begin{aligned} y(x) = {}& \int_0^x \frac{1}{K(s_4)}\int_{s_4}^1 \varrho(s_1)g_1(s_1)\,ds_1\,ds_4 - \int_0^x\int_0^{s_4} \frac{1}{E(s_3)}\int_{s_3}^1 R(s_2)g_2(s_2)\,ds_2\,ds_3\,ds_4 \\ & + \int_0^x\int_0^{s_4} \frac{1}{E(s_3)}\int_{s_3}^1\int_{s_2}^1 \varrho(s_1)g_1(s_1)\,ds_1\,ds_2\,ds_3\,ds_4. \end{aligned}$$

Thus we found formulas for the inverse operator. $\square$

Corollary 2. *The operator $A : D(A) \to H$ is surjective, self-adjoint and has a compact resolvent.*

Proof. The surjectivity of the operator A is an easy consequence of the proof of Proposition 1. Also, compactness of the integral operator on $L^2[0,1]$ implies that A^{-1} is compact. To prove the self-adjointness, it suffices to show that the domain of A is equal to the domain of A^*. The inclusion $D(A^*) \supset D(A)$ is always true, so we need to establish the reverse inclusion. Let $Y_0 \in D(A^*)$ and $Z_0 = A^* Y_0$. Then

$$\langle AY, Y_0\rangle = \langle Y, Z_0\rangle$$

for all $Y \in D(A)$. Due to surjectivity, one can find $Y_1 \in D(A)$ with $AY_1 = Z_0$. Therefore

$$\langle AY, Y_0\rangle = \langle Y, AY_1\rangle.$$

Using symmetricity of the operator A, we obtain readily $\langle AY, Y_0 - Y_1\rangle = 0$. But the operator A is surjective, so $Y_0 = Y_1$. Therefore $Y_0 \in D(A)$. Compactness of the resolvent $R_\lambda(A) = (A - \lambda I)^{-1}$, $\lambda \in \mathbb{C} \setminus \sigma(A)$ follows from the compactness of A^{-1} and Gilbert's identity

$$R_\lambda - A^{-1} = \lambda R_\lambda(A) A^{-1}.$$

It completes the proof. $\square$

Corollary 2 and the positiveness imply that A possesses an orthogonal sequence of eigenvectors $\binom{y_j}{z_j} \in D(A)$ $(j \in \mathbb{N})$ and corresponding sequence of eigenvalues of finite multiplicity satisfies

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \to \infty \quad \text{as } j \to \infty.$$

Therefore there exists the unique weak[2] solution to (1) given by

$$\begin{pmatrix} w(x,t) \\ \xi(x,t) \end{pmatrix} = \sum_{j=1}^{\infty} \frac{1}{\sqrt{\lambda_j}} \int_0^t \left\langle \begin{pmatrix} f_1(\cdot,s) \\ f_2(\cdot,s) \end{pmatrix}, \begin{pmatrix} y_j \\ z_j \end{pmatrix} \right\rangle \sin\sqrt{\lambda_j}(t-s)\, ds \begin{pmatrix} y_j(x) \\ z_j(x) \end{pmatrix}. \tag{8}$$

The inner product we use here is defined in (4), the functions f_1 and f_2 are defined in (6) and $\binom{y_j}{z_j}$ for $j \in \mathbb{N}$ are the eigenvectors of the operator A that correspond to eigenvalues λ_j. Also we notice, that the first (time) derivative of the above solution is

$$\begin{pmatrix} \dot w(x,t) \\ \dot\xi(x,t) \end{pmatrix} = \sum_{j=1}^{\infty} \int_0^t \left\langle \begin{pmatrix} f_1(\cdot,s) \\ f_2(\cdot,s) \end{pmatrix}, \begin{pmatrix} y_j \\ z_j \end{pmatrix} \right\rangle \cos\sqrt{\lambda_j}(t-s)\, ds \begin{pmatrix} y_j(x) \\ z_j(x) \end{pmatrix}. \tag{9}$$

For further work we need to study an asymptotic behavior of the eigenvalues of A.

3. Asymptotic behavior of eigenvalues

In order to study the location of eigenvalues of the operator A defined by (5), we need to consider the following system of equations:

$$\begin{aligned} &-\big(K(x)\big(y'(x) + z(x)\big)\big)' = \lambda \varrho(x) y(x), \\ &-\big(E(x) z'(x)\big)' + K(x)\big(y'(x) + z(x)\big) = \lambda R(x) z(x), \end{aligned} \tag{10}$$

with the boundary conditions

$$y(0) = z(0) = 0, \quad y'(1) + z(1) = 0, \quad z'(1) = 0. \tag{11}$$

The solution to the problem stated above is given by the following theorem.

[2] The solution x of the equation $\ddot x + Ax = f$ (considered in the Hilbert space H) is called weak if for any $y \in H$ the equation $\langle y, \ddot x\rangle + \langle y, Ax\rangle = \langle y, f\rangle$ holds. The reason for considering weak solution is that, in general, we are not certain of the existence of the strong solution, i.e., the existence of such twice differentiable, hence continuous function y, that satisfies the equation (we notice that f is a member of a Hilbert space and may not be continuous). However, if the strong solution exists, it is equal to the weak one.

Theorem 3. *Eigenvalues of the operator* A (*excluding at most finite number of them*) *are given by formulas*

$$\lambda_n^{(0)} = \left(\int_0^1 \sqrt{\frac{\varrho(t)}{K(t)}}\,dt\right)^{-2} \left(\frac{\pi}{2} + n\pi + \varepsilon_n^{(0)}\right)^2, \tag{12}$$

$$\lambda_n^{(1)} = \left(\int_0^1 \sqrt{\frac{R(t)}{E(t)}}\,dt\right)^{-2} \left(\frac{\pi}{2} + n\pi + \varepsilon_n^{(1)}\right)^2, \tag{13}$$

where $n \geqslant N$ *for some sufficiently large* N *and* $\varepsilon_n^{(0)}, \varepsilon_n^{(1)} \to 0$ *as* $n \to \infty$. *Thus the spectrum of the operator* A *asymptotically splits naturally into two sets—*$\Lambda^{(0)}$*, whose elements are described by* (12) *and* $\Lambda^{(1)}$ *containing all elements of the form* (13).

In order to prove the theorem we simplify, first, Eq. (10) by making the following substitution:

$$\begin{aligned} &y(x) = v_1(x), \qquad z(x) = v_2(x), \\ &K(x)\big(y'(x) + z(x)\big) = v_3(x), \qquad E(x)z'(x) = v_4(x). \end{aligned} \tag{14}$$

Taking (14) into account we receive the following system of equations:

$$\begin{aligned} v_1'(x) &= -v_2(x) + K(x)^{-1} v_3(x), \\ v_2'(x) &= E(x)^{-1} v_4(x), \\ v_3'(x) &= -\lambda \varrho(x) v_1(x), \\ v_4'(x) &= -\lambda R(x) v_2(x) + v_3(x), \end{aligned} \tag{15}$$

with the boundary conditions deducted from (11) and (15)

$$v_3'(0) = v_1(0) = v_2(0) = 0, \quad v_3(1) = v_4(1) = 0, \quad v_4'(0) = v_3(0). \tag{16}$$

Remark. The eigenvectors of the operator A are $\binom{v_1}{v_2}$. It is shown in [6] that in a homogeneous case of the Timoshenko beam, the eigenspaces are one-dimensional. In general, it is not difficult to show that the dimension of eigenspaces is less or equal to 2.

Eliminating v_1 and v_2 from (15) and taking into account boundary conditions (16) we arrive at

$$v_3''(x) - \left(\frac{\varrho(x)}{K(x)}\right)' \frac{K(x)}{2\varrho(x)} v_3'(x) + \lambda \frac{\varrho(x)}{K(x)} v_3(x) = F_3(x), \tag{17}$$

$$v_4''(x) - \left(\frac{R(x)}{E(x)}\right)' \frac{E(x)}{2R(x)} v_4'(x) + \lambda \frac{R(x)}{E(x)} v_4(x) = F_4(x), \tag{18}$$

where the right-hand side functions F_3 and F_4 are defined as follows:

$$F_3(x) = -\left(\frac{\varrho(x)}{K(x)}\right)' \frac{K(x)}{2\varrho(x)} v_3'(x) - \lambda \int_0^x \frac{\varrho'(x)}{K(t)} v_3(t)\,dt + \lambda \int_0^x \frac{\varrho(x) + (x-t)\varrho'(x)}{E(t)} v_4(t)\,dt, \tag{19}$$

$$F_4(x) = -\left(\frac{R(x)}{E(x)}\right)' \frac{E(x)}{2R(x)} v_4'(x) - \lambda \int_0^x \frac{\varrho(x)}{K(t)} v_3(t)\,dt + \lambda \int_0^x \frac{(x-t)\varrho(x) - R'(x)}{E(t)} v_4(t)\,dt. \tag{20}$$

Indeed, from the system (15) we get

$$v_3''(x) = -\lambda\big(\varrho(x) v_1(x)\big)' \quad \text{and} \quad v_1'(x) = \frac{v_3(x)}{K(x)} - \int_0^x \frac{v_4(t)}{E(t)}\,dt$$

as $v_2(0) = 0$. Therefore, taking into account that $v_1(0) = 0$, we have

$$v_1(x) = \int_0^x \frac{v_3(t)}{K(t)}\,dt - \int_0^x\int_0^t \frac{v_4(\tau)}{E(\tau)}\,d\tau\,dt \quad \text{and}$$

$$-\lambda\big(\varrho(x)v_1(x)\big)' = -\lambda\left(\varrho'(x)\int_0^x \frac{v_3(t)}{K(t)}\,dt - \varrho'(x)\int_0^x\int_0^t \frac{v_4(\tau)}{E(\tau)}\,d\tau\,dt + \frac{\varrho(x)}{K(x)}v_3(x) - \varrho(x)\int_0^x \frac{v_4(t)}{E(t)}\,dt\right).$$

The last expression immediately yields (17). We obtain (18) in a similar way.

Proposition 4. *Assume v, f and F are continuous on $[0,1]$, f is once and v is twice differentiable on $(0,1)$. The function v satisfies the equation*

$$v''(x) - \frac{f'(x)}{2f(x)}v'(x) + \lambda f(x)v(x) = F(x) \tag{21}$$

if and only if it satisfies the following integral equation:

$$v(x) = v(0)\cos J(\lambda,x) + \frac{v'(0)}{\sqrt{\lambda f(t)}}\sin J(\lambda,x) + \frac{1}{\sqrt{\lambda}}\int_0^x \frac{F(t)}{\sqrt{f(t)}}\sin\big(J(\lambda,x) - J(\lambda,t)\big)\,dt, \tag{22}$$

where $J(\lambda,x) = \sqrt{\lambda}\int_0^x \sqrt{f(t)}\,dt$.

The proof that v satisfying (21) satisfies also (22) is a straightforward computation. To show the opposite, one can use a standard method of solving differential equation.

In our case, F_3 and F_4 are functions dependent on v_3 and v_4, but the algorithm still may be applied.

To adjust the formula (22) to our case, we define

$$J_3(\lambda,x) = \sqrt{\lambda}\int_0^x \sqrt{\frac{\varrho(t)}{K(t)}}\,dt, \qquad J_4(\lambda,x) = \sqrt{\lambda}\int_0^x \sqrt{\frac{R(t)}{E(t)}}\,dt$$

and after applying (16) and Proposition 4 we obtain

$$v_3(x) = v_3(0)\cos J_3(\lambda,x) + \frac{1}{\sqrt{\lambda}}\int_0^x F_3(t)\sqrt{\frac{K(t)}{\varrho(t)}}\sin\big(J_3(\lambda,x) - J_3(\lambda,t)\big)\,dt, \tag{23}$$

$$v_4(x) = v_4(0)\cos J_4(\lambda,x) + \frac{1}{\sqrt{\lambda}}\sqrt{\frac{E(0)}{R(0)}}v_3(0)\sin J_4(\lambda,x) + \frac{1}{\sqrt{\lambda}}\int_0^x F_4(t)\sqrt{\frac{E(t)}{R(t)}}\sin\big(J_4(\lambda,x) - J_4(\lambda,t)\big)\,dt. \tag{24}$$

For further reference we define constant $\gamma_{43} = \sqrt{E(0)/R(0)}$. Because the functions F_3 and F_4 depend on the functions v_3 and v_4, we shall consider the system of two integral equations. But first, we need to get rid of derivatives of v_3 and v_4 from components of F_3 and F_4. Putting the first component of F_3 (Eq. (19)) into the integral from (23) we obtain

$$-\frac{1}{2\sqrt{\lambda}}\int_0^x \left(\frac{\varrho(t)}{K(t)}\right)'\left(\frac{K(t)}{\varrho(t)}\right)^{3/2}\sin\big(J_3(\lambda,x) - J_3(\lambda,t)\big)\,dv_3(t). \tag{25}$$

After integration by parts the formula (25) transforms into

$$\frac{1}{2\sqrt{\lambda}}\left(\frac{\varrho(t)}{K(t)}\right)'\Big|_{t=0}\left(\frac{\varrho(0)}{K(0)}\right)^{3/2}\sin J_3(\lambda,x)v_3(0) \tag{26}$$

$$+\int_0^x\left(\frac{p_3(x,t)}{\sqrt{\lambda}}-\left(\frac{\varrho(t)}{K(t)}\right)'\left(\frac{K(t)}{2\varrho(t)}\right)\cos\big(J_3(\lambda,x)-J_3(\lambda,t)\big)\right)v_3(t)\,dt \tag{27}$$

with

$$p_3(x,t)=\left(\left(\frac{\varrho(t)}{K(t)}\right)'\left(\frac{K(t)}{\varrho(t)}\right)^{3/2}\right)'\sin\big(J_3(\lambda,x)-J_3(\lambda,t)\big). \tag{28}$$

We proceed in a similar way with the first component of F_4 (Eq. (20)). The final result is

$$\frac{1}{2\sqrt{\lambda}}\left(\frac{R(t)}{E(t)}\right)'\Big|_{t=0}\left(\frac{R(0)}{E(0)}\right)^{3/2}\sin J_3(\lambda,x)v_4(0) \tag{29}$$

$$+\int_0^x\left(\frac{p_4(x,t)}{\sqrt{\lambda}}-\left(\frac{R(t)}{E(t)}\right)'\left(\frac{E(t)}{2R(t)}\right)\cos\big(J_4(\lambda,x)-J_4(\lambda,t)\big)\right)v_4(t)\,dt, \tag{30}$$

where

$$p_4(x,t)=\left(\left(\frac{R(t)}{E(t)}\right)'\left(\frac{E(t)}{R(t)}\right)^{3/2}\right)'\sin\big(J_3(\lambda,x)-J_3(\lambda,t)\big). \tag{31}$$

The functions p_3 and p_4 appearing in (28) and (31) are bounded with the bound independent on x. Therefore fractions $p_3/\sqrt{\lambda}$ and $p_4/\sqrt{\lambda}$ tend to zero as λ tends to infinity. Next, upon defining

$$\gamma_{33}=\left(\frac{\varrho(t)}{K(t)}\right)'\Big|_{t=0}\left(\frac{\varrho(0)}{2K(0)}\right)^{3/2},\qquad \gamma_{44}=\left(\frac{R(t)}{E(t)}\right)'\Big|_{t=0}\left(\frac{R(0)}{2E(0)}\right)^{3/2},$$

we have $\gamma_{33}/\sqrt{\lambda}$, $\gamma_{44}/\sqrt{\lambda}$ tend to 0 as $\lambda\to\infty$.

Now, we are going to elaborate the remaining, i.e., "integral," components of F_3 and F_4 next. To this purpose, we combine the appropriate parts of Eqs. (19), (20), (23) and (24). We are going to rule out $\sqrt{\lambda}$ in the following four integrals:

$$-\sqrt{\lambda}\int_0^x\int_0^t\sqrt{\frac{K(t)}{\varrho(t)}}\varrho'(t)\frac{v_3(\tau)}{K(\tau)}\sin\big(J_3(\lambda,x)-J_3(\lambda,t)\big)\,d\tau\,dt, \tag{32}$$

$$\sqrt{\lambda}\int_0^x\int_0^t\sqrt{\frac{K(t)}{\varrho(t)}}\big(\varrho(t)+(t-\tau)\varrho'(t)\big)\frac{v_4(\tau)}{E(\tau)}\sin\big(J_3(\lambda,x)-J_3(\lambda,t)\big)\,d\tau\,dt, \tag{33}$$

$$-\sqrt{\lambda}\int_0^x\int_0^t\sqrt{\frac{E(t)}{R(t)}}\varrho(t)\frac{v_3(\tau)}{K(\tau)}\sin\big(J_4(\lambda,x)-J_4(\lambda,t)\big)\,d\tau\,dt \tag{34}$$

and

$$\sqrt{\lambda}\int_0^x\int_0^t\sqrt{\frac{E(t)}{R(t)}}\big((t-\tau)\varrho(t)-R'(t)\big)\frac{v_4(\tau)}{E(\tau)}\sin\big(J_4(\lambda,x)-J_4(\lambda,t)\big)\,d\tau\,dt. \tag{35}$$

To achieve this we consider the integral of the form

$$\int_0^x\left(\int_0^t\frac{G(t,\tau)}{f(t)}v(\tau)\,d\tau\right)\big(\sqrt{\lambda f(t)}\sin\big(J(\lambda,x)-J(\lambda,t)\big)\big)\,dt, \tag{36}$$

where f and J are as before and $G(t,\tau)$ is a function that makes (36) one of integrals (32)–(35). We integrate (36) by parts to obtain

$$\int_0^x \left(\frac{G(x,t)}{f(x)} - \frac{G(t,t)}{f(t)}\cos\bigl(J(\lambda,x)-J(\lambda,t)\bigr)\right) v(t)\,dt$$
$$-\int_0^x\int_0^t \left(\frac{\partial}{\partial t}\frac{G(t,\tau)}{f(t)}\right)\cos\bigl(J(\lambda,x)-J(\lambda,t)\bigr)v(\tau)\,d\tau\,dt$$

and (32)–(35) take the following form:

$$-\int_0^x \left(\frac{\varrho'(x)K(x)}{\varrho(x)K(t)} - \frac{\varrho'(t)}{\varrho(t)}\cos\bigl(J_3(\lambda,x)-J_3(\lambda,t)\bigr)\right) v_3(t)\,dt$$
$$+\int_0^x\int_0^t \left(\frac{\varrho'(t)K(t)}{\varrho(t)}\right)'\frac{1}{K(\tau)}\cos\bigl(J_3(\lambda,x)-J_3(\lambda,t)\bigr)v_3(\tau)\,d\tau\,dt, \tag{32'}$$

$$\int_0^x \left(\frac{K(x)}{E(t)} + \frac{(x-t)\varrho'(x)K(x)}{\varrho(x)E(t)} - \frac{K(t)}{E(t)}\cos\bigl(J_3(\lambda,x)-J_3(\lambda,t)\bigr)\right) v_4(t)\,dt$$
$$-\int_0^x\int_0^t \left(\left(1+\frac{(t-\tau)\varrho'(t)}{\varrho(t)}\right)K(t)\right)'\frac{1}{E(\tau)}\cos\bigl(J_3(\lambda,x)-J_3(\lambda,t)\bigr)v_4(\tau)\,d\tau\,dt, \tag{33'}$$

$$-\int_0^x \left(\frac{\varrho(x)E(x)}{R(x)K(t)} - \frac{\varrho(t)E(t)}{R(t)K(t)}\cos\bigl(J_4(\lambda,x)-J_4(\lambda,t)\bigr)\right) v_3(t)\,dt$$
$$+\int_0^x\int_0^t \left(\frac{\varrho(t)E(t)}{R(t)}\right)'\frac{1}{K(\tau)}\cos\bigl(J_4(\lambda,x)-J_4(\lambda,t)\bigr)v_3(\tau)\,d\tau\,dt, \tag{34'}$$

$$\int_0^x \left(\frac{((x-t)\varrho(x)-R'(x))E(x)}{R(x)E(t)} + \frac{R'(t)}{R(t)}\cos\bigl(J_4(\lambda,x)-J_4(\lambda,t)\bigr)\right) v_4(t)\,dt$$
$$-\int_0^x\int_0^t \left(\frac{((t-\tau)\varrho(t)-R'(t))E(t)}{R(t)}\right)'\frac{1}{E(\tau)}\cos\bigl(J_4(\lambda,x)-J_4(\lambda,t)\bigr)v_4(\tau)\,d\tau\,dt. \tag{35'}$$

Now we are ready to set up the integral equations for v_3 and v_4. Thus we define

$$\mathfrak{G} = \begin{pmatrix} \cos J_3(\lambda,x) + \frac{\gamma_{33}}{\sqrt{\lambda}}\sin J_3(\lambda,x) & 0 \\ \frac{\gamma_{43}}{\sqrt{\lambda}}\sin J_4(\lambda,x) & \cos J_4(\lambda,x) + \frac{\gamma_{44}}{\sqrt{\lambda}}\sin J_4(\lambda,x) \end{pmatrix},$$
$$P_{33}(\lambda,x,t) = \frac{p_3(x,t)}{\sqrt{\lambda}} - \frac{\varrho'(x)K(x)}{\varrho(x)K(t)} + \left(\frac{\varrho'(t)}{\varrho(t)} - \left(\frac{\varrho(t)}{K(t)}\right)'\left(\frac{K(t)}{2\varrho(t)}\right)\right)\cos\bigl(J_3(\lambda,x)-J_3(\lambda,t)\bigr),$$
$$P_{34}(\lambda,x,t) = \frac{K(x)}{E(t)} + \frac{(x-t)\varrho'(x)K(x)}{\varrho(x)E(t)} - \frac{K(t)}{E(t)}\cos\bigl(J_3(\lambda,x)-J_3(\lambda,t)\bigr),$$
$$P_{43}(\lambda,x,t) = -\frac{\varrho(x)E(x)}{R(x)K(t)} + \frac{\varrho(t)E(t)}{R(t)K(t)}\cos\bigl(J_4(\lambda,x)-J_4(\lambda,t)\bigr),$$

$$P_{44}(\lambda,x,t)=\frac{p_4(x,t)}{\sqrt{\lambda}}+\frac{((x-t)\varrho(x)-R'(x))E(x)}{R(x)E(t)}$$
$$+\left(\frac{R'(t)}{R(t)}-\left(\frac{R(t)}{E(t)}\right)'\left(\frac{E(t)}{2R(t)}\right)\right)\cos\bigl(J_4(\lambda,x)-J_4(\lambda,t)\bigr),$$
$$N_{33}(\lambda,x,t,\tau)=\left(\frac{\varrho'(t)K(t)}{\varrho(t)}\right)'\frac{1}{K(\tau)}\cos\bigl(J_3(\lambda,x)-J_3(\lambda,t)\bigr),$$
$$N_{34}(\lambda,x,t,\tau)=-\left(\left(1+\frac{(t-\tau)\varrho'(t)}{\varrho(t)}\right)K(t)\right)'\frac{1}{E(\tau)}\cos\bigl(J_3(\lambda,x)-J_3(\lambda,t)\bigr),$$
$$N_{43}(\lambda,x,t,\tau)=\left(\frac{\varrho(t)E(t)}{R(t)}\right)'\frac{1}{K(\tau)}\cos\bigl(J_4(\lambda,x)-J_4(\lambda,t)\bigr),$$
$$N_{44}(\lambda,x,t,\tau)=-\left(\frac{((t-\tau)\varrho(t)-R'(t))E(t)}{R(t)}\right)'\frac{1}{E(\tau)}\cos\bigl(J_4(\lambda,x)-J_4(\lambda,t)\bigr).$$

For $i,j\in\{3,4\}$ let us define operators $\mathcal{P}_{ij}$ and $\mathcal{N}_{ij}$ by formulas

$$\mathcal{P}_{ij}(v)=\int_0^x P_{ij}(\lambda,x,t)v(t)\,dt,$$
$$\mathcal{N}_{ij}(v)=\int_0^x\int_0^t N_{ij}(\lambda,x,t,\tau)v(\tau)\,d\tau\,dt.$$

Next we define operators $\mathfrak{P}$ and $\mathfrak{N}$ by formulas

$$\mathfrak{P}=\begin{pmatrix}\mathcal{P}_{33}&\mathcal{P}_{34}\\ \mathcal{P}_{43}&\mathcal{P}_{44}\end{pmatrix}\quad\text{and}\quad\mathfrak{N}=\begin{pmatrix}\mathcal{N}_{33}&\mathcal{N}_{34}\\ \mathcal{N}_{43}&\mathcal{N}_{44}\end{pmatrix}.$$

Finally, we have the system of two integral equations (written in the operator form)

$$\begin{pmatrix}v_3\\ v_4\end{pmatrix}=\mathfrak{G}\begin{pmatrix}v_3(0)\\ v_4(0)\end{pmatrix}+(\mathfrak{P}+\mathfrak{N})\begin{pmatrix}v_3\\ v_4\end{pmatrix}. \tag{37}$$

To solve (37), we proceed by the standard method of solving the integral equation described for example in [8]. Namely, we form the Neumann series for this equation. This series will turn out to be uniformly convergent.

Lemma 5. *The series*

$$\sum_{k=0}^{\infty}(\mathfrak{P}+\mathfrak{N})^k\mathfrak{G} \tag{38}$$

is uniformly convergent, provided the kernels $P_{ij}(\lambda,x,t)$, $N_{ij}(\lambda,x,t,\tau)$ $(i,j\in\{3,4\})$ *are uniformly bounded with respect to all variables.*

Let M be a common bound of all kernels and let g be a common bound of matrix $\mathfrak{G}$ entries. Using mathematical induction one can prove that the terms of the kth component of (38) are majorized by $\frac{(4Mx)^k}{k!}g$. Because the series of such functions is uniformly convergent, also (38) uniformly converges.

Define $\Psi(\lambda,x)$ to be the sum of (38).

Remark. The entries of the matrix $\Psi(\lambda,x)$ are analytic functions of variable λ. Indeed, fixing x, we see that the entries of $(\mathfrak{P}+\mathfrak{N})^k\mathfrak{G}$ are analytical functions and the uniformly convergent series of analytical functions converges to an analytic function.

Corollary 6. *If the matrix* $\mathfrak{G}$ *from Lemma* 5 *is a one column matrix, the series* (38) *is uniformly convergent and its coordinates are bounded by* $\frac{(4Mx)^k}{k!}g$*, where* g *is a bound of coordinates of* $\mathfrak{G}$ *and* M *is a bound of kernels* $P_{ij}(\lambda, x, t)$ *and* $N_{ij}(\lambda, x, t, \tau)$.

The proof of the corollary follows immediately from the Lemma 5 and its proof.

Proposition 7. *If* $v_3(0) = v_4(0) = 0$*, then the solution to* (37) *is a trivial one.*

Assume $v_3(0) = v_4(0) = 0$. Then $\mathfrak{G}\binom{v_3(0)}{v_4(0)}$ appearing in (37) is a one column zero matrix. Therefore Eq. (37) for any positive integer k implies

$$\begin{pmatrix} v_3(x) \\ v_4(x) \end{pmatrix} = (\mathfrak{P} + \mathfrak{N})^k \begin{pmatrix} v_3(x) \\ v_4(x) \end{pmatrix}.$$

It means that $|v_j(x)| \leqslant \frac{(4Mx)^k}{k!}(|v_3(x)| + |v_4(x)|)$ for $j = 3, 4$ and any $x \in [0, 1]$, so $v_3 \equiv v_4 \equiv 0$.

Remark. If $v_3(0) = v_4(0) = 0$, then $v_3 \equiv v_4 \equiv 0$. The last and (15) imply that eigenvectors of the operator A (Eq. (5)) are equal to zero.

Proposition 8. *The continuous solutions of Eq.* (37) *form a two-dimensional vector space over* $\mathbb{C}$.

To prove the above statement, we first notice that the function $\Psi(\lambda, x)\binom{\alpha_3}{\alpha_4}$ is a solution of (37) for any complex numbers α_3, α_4. Indeed,

$$\Psi(\lambda, x)\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \mathfrak{G}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} + \sum_{k=1}^{\infty}(\mathfrak{P} + \mathfrak{N})^k \mathfrak{G}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \mathfrak{G}\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} + (\mathfrak{P} + \mathfrak{N})\Psi(\lambda, x)\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}.$$

Now, we show that any solution of (37) is uniquely determined by the initial conditions. It is the case, because if $\binom{f_3}{f_4}$ and $\binom{\tilde{f}_3}{\tilde{f}_4}$ are solutions of (37) with $f_3(0) = \tilde{f}_3(0)$ and $f_4(0) = \tilde{f}_4(0)$, then $\binom{f_3 - \tilde{f}_3}{f_4 - \tilde{f}_4}$ is a solution of (37) with initial conditions $(f_j - \tilde{f}_j)(0) = 0$ for $j = 3, 4$. By Proposition 7, it means $f_j = \tilde{f}_j$.

To complete the proof we define two base solutions $\binom{u_3^{(1)}}{u_4^{(1)}}$ and $\binom{u_3^{(2)}}{u_4^{(2)}}$ with $u_3^{(1)}(0) = u_4^{(2)}(0) = 1$ and $u_3^{(2)}(0) = u_4^{(1)}(0) = 0$. Let $\binom{v_3}{v_4}$ be any solution of (37). We consider the vector function

$$\begin{pmatrix} f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} v_3 \\ v_4 \end{pmatrix} - v_3(0)\begin{pmatrix} u_3^{(1)} \\ u_4^{(1)} \end{pmatrix} - v_4(0)\begin{pmatrix} u_3^{(2)} \\ u_4^{(2)} \end{pmatrix}.$$

It is a solution of (37) and $f_3(0) = f_4(0) = 0$. From Proposition 7 we gather that $f_3 \equiv f_4 \equiv 0$ and $\binom{v_3}{v_4}$ is a linear combination of the base solutions.

Corollary 9. *Any solution of* (37) *is given by*

$$\begin{pmatrix} v_3(x) \\ v_4(x) \end{pmatrix} = \Psi(\lambda, x)\begin{pmatrix} v_3(0) \\ v_4(0) \end{pmatrix}. \tag{39}$$

Remark. Because the eigenvectors are uniquely determined by $\binom{v_3}{v_4}$ and the last form a vector space of dimension 2, the eigenspaces are at most of dimension two.

We consider $\Psi(\lambda, x)$ now.

Proposition 10. *For any* $\varepsilon > 0$*, there exists* $\lambda_0 > 0$ *such that*

$$\Psi(\lambda, x) = \begin{pmatrix} \cos J_3(\lambda, x) + \eta_{33}(\lambda, x) & \eta_{34}(\lambda, x) \\ \eta_{43}(\lambda, x) & \cos J_4(\lambda, x) + \eta_{44}(\lambda, x) \end{pmatrix}, \tag{40}$$

with $|\eta_{ij}(\lambda, x)| < \varepsilon$.

For the proof, let ε be given. By Lemma 5, there exists K such that

$$\Psi(\lambda, x) = \sum_{k=0}^{K} (\mathfrak{P} + \mathfrak{N})^k \mathfrak{G}(\lambda, x) + \sum_{k=K+1}^{\infty} (\mathfrak{P} + \mathfrak{N})^k \mathfrak{G}(\lambda, x),$$

where the entries of the second summand are bounded by $\varepsilon/2$. Considering the first summand, let

$$\bar{\mathfrak{G}}(\lambda, x) = \mathfrak{G}(\lambda, x) - \mathfrak{G}_1(\lambda, x),$$

where

$$\mathfrak{G}_1(\lambda, x) = \begin{pmatrix} \cos J_3(\lambda, x) & 0 \\ 0 & \cos J_4(\lambda, x) \end{pmatrix}.$$

Then

$$\sum_{k=0}^{K} (\mathfrak{P} + \mathfrak{N})^k \mathfrak{G}(\lambda, x) = \mathfrak{G}_1(\lambda, x) + \sum_{k=0}^{K} (\mathfrak{P} + \mathfrak{N})^k \bar{\mathfrak{G}}(\lambda, x) + \sum_{k=1}^{K} (\mathfrak{P} + \mathfrak{N})^k \mathfrak{G}_1(\lambda, x).$$

We define $\Psi_1(\lambda, x)$ to be the second summand in the above formula and $\Psi_2(\lambda, x)$ to be the third one.

Let $\gamma = \max\{\gamma_{33}, \gamma_{43}, \gamma_{44}\}$. Then, by Corollary 6 each entry of $(\mathfrak{P} + \mathfrak{N})^k \mathfrak{G}(\lambda, x)$ is bounded by $\frac{(4M)^k \gamma}{\sqrt{\lambda} k!}$. Thus there exists λ_1, such that for $\lambda \geqslant \lambda_1$ we have $\frac{(4M)^k \gamma}{\sqrt{\lambda} k!} < \frac{\varepsilon}{2^{K+3}}$. It means that the entries of $\Psi_1(\lambda, x)$ are bounded by $\varepsilon/4$.

To estimate the entries of $\Psi_2(\lambda, x)$ we use the following lemma, whose proof is very similar to the one contained in [2].

Lemma 11. *Assume $S(x, t)$ is integrable with respect to t and f is strictly increasing function with $f'(x) > c$ for some positive constant c. Then*

$$\lim_{\lambda \to \infty} \int_0^x S(x, t) \cos \sqrt{\lambda} f(t)\, dt = 0.$$

Using the above lemma, we have the existence of λ_2, such that for $\lambda \geqslant \lambda_2$ the entries of $(\mathfrak{P} + \mathfrak{N})\mathfrak{G}_1(\lambda, x)$ are bounded by $\frac{\varepsilon}{2^{K+2} M_0}$, where $M_0 = \max\{(4M)^k/k!\colon\ 0 \leqslant k \leqslant K-1\}$. Because

$$\Psi_2(\lambda, x) = \sum_{k=0}^{K-1} (\mathfrak{P} + \mathfrak{N})^k \big((\mathfrak{P} + \mathfrak{N})\mathfrak{G}_1\big)(\lambda, x),$$

we have by Corollary 6 that each summand of $\Psi_2(\lambda, x)$ is bounded by $\varepsilon/2^{K+2}$. Therefore the entries of the whole $\Psi_2(\lambda, x)$ are bounded by $\varepsilon/4$.

Eventually, if $\lambda_0 = \max\{\lambda_1, \lambda_2\}$, then for $\lambda \geqslant \lambda_0$, we have (40) with the required conditions.

Now we are ready to complete the proof of Theorem 3. Using the boundary conditions $v_3(1) = v_4(1) = 0$, we have the following form of (39):

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} v_3(1) \\ v_4(1) \end{pmatrix} = \Psi(\lambda, 1) \begin{pmatrix} v_3(0) \\ v_4(0) \end{pmatrix}.$$

Therefore the determinant $\psi(\lambda) = \det \Psi(\lambda, 1)$ must be equal to zero because $\binom{v_3(0)}{v_4(0)} \neq \binom{0}{0}$ by Proposition 7. Additionally, $\psi(\lambda)$ is an analytic function and we may write it in the form

$$\psi(\lambda) = \cos J_3(\lambda, 1) \cos J_4(\lambda, 1) + \eta(\lambda),$$

where $\eta(\lambda) = \eta_{44}(\lambda, 1)\cos J_3(\lambda, 1) + \eta_{33}(\lambda, 1)\cos J_4(\lambda, 1) - \eta_{34}(\lambda, 1)\eta_{43}(\lambda, 1)$. By Rouche theorem, given any region R with boundary D such that,

$$\left|\cos J_3(\lambda, 1)\cos J_4(\lambda, 1)\right| > \left|\eta(\lambda)\right|$$

for $\lambda \in D$, then functions $\psi(\lambda)$ and $\cos J_3(\lambda, 1)\cos J_4(\lambda, 1)$ have the same number of zeroes (counting multiplicitly). Choosing the regions appropriately, we obtain that zeroes of function $\psi(\lambda)$ are at points given by (12) and (13). To be more exact, for the set D we may take a circle around zero of $\cos J_3(\lambda, 1)$ or $\cos J_4(\lambda, 1)$. If zeroes of those two functions are too close to each other, we take a circle about the two of them.

The proof of Theorem 3 is completed.

4. Solution to the problem of controllability

Given the beam, whose movement is described by (1), we want to rotate it from the state of rest at time $t = 0$ to the state of rest at the time $t = T > 0$. Thus we have the following boundary conditions:

$$\begin{aligned} &w(x, 0) = \dot{w}(x, 0) = \xi(x, 0) = \dot{\xi}(x, 0) = 0, \\ &w(x, T) = \dot{w}(x, T) = \xi(x, T) = \dot{\xi}(x, T) = 0 \end{aligned} \tag{41}$$

for $x \in [0, 1]$. The beam is controlled by motor that rotates it from angle 0 to θ_T. The control is given by angular acceleration "$\theta(t)$" and our goal is to find this function. The beginning position of rest means that the motor does not work, i.e., the control function θ is a member of $H_0^2(0, T)$, where

$$H_0^2(0, T) = \left\{f \in H^2(0, T):\ f(0) = \dot{f}(0) = 0\right\}.$$

At the end of the movement the beam is at the position $\theta(T) = \theta_T$ and the motor does not move, so $\dot{\theta}(T) = 0$.

Thus, to solve the problem of controllability from rest to rest, we need for given time $T > 0$ and angle $\theta_T \in \mathbb{R}$, $\theta_T \neq 0$ to find a function $\theta \in H_0^2(0, T)$ with

$$\theta(T) = \theta_T, \qquad \dot{\theta}(T) = 0. \tag{42}$$

Actually, the assumption $\theta_T = 0$ is not necessary, but we put it here to avoid triviality.

We are going to find conditions equivalent to (41) and (42) so the problem of controllability from rest to rest will be described in terms of some moment problem. To achieve this we consider the weak solution to (1) given by 8. Thus the conditions (41) are equivalent to

$$\frac{1}{\sqrt{\lambda_n}} \int_0^T \left\langle \begin{pmatrix} f_1(\cdot, t) \\ f_2(\cdot, t) \end{pmatrix}, \begin{pmatrix} y_n \\ z_n \end{pmatrix} \right\rangle \sin(T - t)\sqrt{\lambda_n}\, dt = 0,$$

and

$$\int_0^T \left\langle \begin{pmatrix} f_1(\cdot, t) \\ f_2(\cdot, t) \end{pmatrix}, \begin{pmatrix} y_n \\ z_n \end{pmatrix} \right\rangle \cos(T - t)\sqrt{\lambda_n}\, dt = 0$$

for all $n \in \mathbb{N}$. Here the eigenvalues λ_n are not yet distinguished on those that belong to $\Lambda^{(0)}$ and $\Lambda^{(1)}$. Therefore upon putting

$$a_n = \int_0^1 R(x)\overline{z_n(x)}dx - \int_0^1 \varrho(x)(r + x)\overline{y_n(x)}\, dx \tag{43}$$

we obtain

$$\left\langle \begin{pmatrix} f_1(x, t) \\ f_2(x, t) \end{pmatrix}, \begin{pmatrix} y_n(x) \\ z_n(x) \end{pmatrix} \right\rangle = a_n \ddot{\theta}(t)$$

for all positive integer n. We recall that the above inner product is defined by formula (4).

We remark here that for the controllability from rest to rest, the condition $a_n \neq 0$ for all positive integer n (the formulas for a_n are given by (43)) is not necessary. It becomes crucial while considering controllability from rest to arbitrary condition. The values of those parameters depend on the radius r of the disc (in general, on the ratio radius to the length of the disc). It was proved in [6] that in homogeneous case there are only countable many values of r for which some of a_n's are zeroes. We consider this problem in future papers while considering controllability from rest to arbitrary position.

Employing (43) we obtain

$$a_n \int_0^T \ddot{\theta}(t)\sin(T-t)\sqrt{\lambda_n}\,dt = 0,$$
$$a_n \int_0^T \ddot{\theta}(t)\cos(T-t)\sqrt{\lambda_n}\,dt = 0.$$

Using the well-known trigonometric formulas, we have the equivalence between the above and the following

$$a_n \int_0^T \ddot{\theta}(t)\sin t\sqrt{\lambda_n}\,dt = 0,$$
$$a_n \int_0^T \ddot{\theta}(t)\cos t\sqrt{\lambda_n}\,dt = 0. \tag{44}$$

Now we notice that if $\theta \in H_0^2(0,T)$, then the conditions (42) are equivalent to

$$\int_0^T \ddot{\theta}(t)\,dt = 0,$$
$$\int_0^T t\ddot{\theta}(t)\,dt = -\theta_T. \tag{45}$$

Indeed, after using the Leibnitz formula or integration by parts, we obtain

$$\int_0^T \ddot{\theta}(t)\,dt = \dot{\theta}(T) - \dot{\theta}(0) = 0$$

and

$$\int_0^T t\ddot{\theta}(t)\,dt = T\dot{\theta}(T) - \theta(T) - \theta(0) = \theta_T.$$

Gathering (44) and (45) we obtain that the problem of controllability from rest to rest is equivalent to the following moment problem.

Moment problem. *Find $u \in L^2(0,T)$ such that for all $n \in \mathbb{N}$ the conditions*

$$\int_0^T u(t)\cos t\sqrt{\lambda_n}\,dt = 0,$$

$$\int_0^T u(t)\sin t\sqrt{\lambda_n}\,dt = 0,$$

$$\int_0^T u(t)\,dt = 0,$$

$$\int_0^T tu(t)\,dt = -\theta_T$$

are satisfied.

We notice that once $u(t)$ is found, we also have $\theta(t) = \int_0^t (t-s)u(s)\,ds$.

Let $\sigma(A) = \{\mu_n\colon n \in \mathbb{N}\}$ be the set of all different eigenvalues λ_n. To find the solution to the stated moment problem we consider the system

$$\{t, 1, \cos t\sqrt{\mu_n}, \sin t\sqrt{\mu_n}\colon n \in \mathbb{N}\}. \tag{46}$$

Just for convenience, we rewrite the system (46) in the form

$$V \cup \{t\}, \quad \text{where } V = \{1, \cos t\sqrt{\mu_n}, \sin t\sqrt{\mu_n}\colon n \in \mathbb{N}\}.$$

Let W be the closure of the linear span over V. To prove the minimality of the system (46), we use the following theorem [9].

Theorem 12. *Suppose*

$$d(x) = \#\{n \in \mathbb{N}\colon \sqrt{\mu_n} < x\}.$$

The system $V \cup \{t\}$ is minimal in $L^2[0,T]$ (in particular $t \notin V$), if

$$\limsup_{x,y\to\infty} \frac{d(x+y) - d(x)}{y} < \frac{T}{2\pi}.$$

We notice that excluding some finite set, $\sigma(A)$ is the union of $\Lambda^{(0)}$ and $\Lambda^{(1)}$. That means that the value $d(x)$ defined in Theorem 12 is not greater than the number $\#\{n \in \mathbb{N}\colon \sqrt{\lambda_n^{(i)}} < x, i \in \{0,1\}\}$. We estimate $d(x+y) - d(x)$ now. We have

$$J^{(0)} = \int_0^1 \sqrt{\frac{\varrho(x)}{K(x)}}\,dx \quad \text{and} \quad J^{(1)} = \int_0^1 \sqrt{\frac{R(x)}{E(x)}}\,dx.$$

Then

$$\sqrt{\lambda_n^{(i)}} = \frac{1}{J^{(i)}}\left(\frac{\pi}{2} + n\pi + \varepsilon_n^{(i)}\right) \quad \text{for } n \geqslant N \text{ and } i \in \{0,1\},$$

where $\varepsilon_n^{(i)} \to 0$, as $n \to \infty$.

Assume

$$\frac{1}{J^{(i)}}\left(\frac{\pi}{2} + n\pi + \varepsilon_n^{(i)}\right) \approx x.$$

After transforming the above equation to compute n and disregarding $\varepsilon_n^{(i)}$, we get

$$d(x+y) - d(x) \leqslant \frac{J^{(0)} + J^{(1)}}{\pi} y.$$

And when we apply Theorem 12 we obtain

$$\frac{J^{(0)}+J^{(1)}}{\pi}<\frac{T}{2\pi},$$

so $T>2(J^{(0)}+J^{(1)})$. Thus for

$$T>2\big(J^{(0)}+J^{(1)}\big), \tag{47}$$

the system (46) is minimal.

We notice that the system (46) may be minimal for even less values of T than ones given in inequality (47).

Let $f(t)=t$. Considering T given in (47), we have the existence of exactly one $h_0\in W$ such that

$$\|h_0-f\|_2\leqslant\|h-f\|_2 \quad \text{for all } h\in W,$$

where $\|\cdot\|_2$ denotes the L^2-norm in $L^2(0,T)$. For that h_0 we have

$$\int_0^T \big(f(t)-h_0(t)\big)h(t)\,dt=0 \quad \text{for all } h\in W.$$

In particular, the above implies

$$\int_0^T \big(f(t)-h_0(t)\big)\,dt=0,$$

$$\int_0^T \big(f(t)-h_0(t)\big)\cos t\sqrt{\mu_n}\,dt=0, \quad n\in\mathbb{N},$$

$$\int_0^T \big(f(t)-h_0(t)\big)\sin t\sqrt{\mu_n}\,dt=0, \quad n\in\mathbb{N},$$

and

$$\int_0^T \big(f(t)-h_0(t)\big)f(t)\,dt=\|h_0-f\|_2^2>0.$$

Upon defining

$$u(t)=-\frac{\theta_T}{\|h_0-f\|_2^2}\big(f(t)-h_0(t)\big)$$

for $t\in[0,T]$ we receive $u\in L^2(0,T)$ that solves the moment problem.

Thus we have the following theorem:

Theorem 13. *The problem of controllability from rest to rest is solvable if*

$$T>T_0=2\left(\int_0^1\sqrt{\frac{\varrho(x)}{K(x)}}\,dx+\int_0^1\sqrt{\frac{R(x)}{E(x)}}\,dx\right).$$

5. Final remarks

The main problem that appears here is whether the eigenvalues are singular, i.e., whether the eigenspaces are one-dimensional. It is true in homogeneous case, when the parameters K, E, R and ϱ are constant functions—the careful proof of this fact is given in [6]. Mentioned proof cannot be generalized, because in homogeneous case one has the exact solution to the spectral equation (15). In present paper, only approximate solution is given. Still, if some additional conditions are imposed on eigenvectors, then the fact that the eigenspaces are one-dimensional follows from the uniqueness of solution to the integral equation given by Neumann series. Actually, what we need is to show that the matrix $\Psi(\lambda, 1)$ is not the zero one. It would imply that $v_3(0)$ is a multiplicity of $v_4(0)$ and the vector space described in Proposition 8 is in fact one-dimensional. However, if eigenvalues are double, we claim that controllability problem is solvable for $T > T_0$ (Theorem 13), but it may also be solvable for smaller values than T_0, because the system (46) is not complete then. If the system (46) is complete, then the problem of controllability has no solution if $T \leqslant T_0$.

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