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A research into the numerical method of Dirichlet's problem of complex Monge–Ampère equation on Cartan–Hartogs domain of the third type ☆

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Abstract

Monge–Ampère equation is a nonlinear equation with high degree, therefore its numerical solution is very important and very difficult. In present paper the numerical method of Dirichlet's problem of Monge–Ampère equation on Cartan–Hartogs domain of the third type is discussed by using the analytic method. Firstly, the Monge–Ampère equation is reduced to the nonlinear ordinary differential equation, then the numerical method of the Dirichlet problem of Monge–Ampère equation becomes the numerical method of two point boundary value problem of the nonlinear ordinary differential equation. Secondly, the solution of the Dirichlet problem is given in explicit formula under the special case, which can be used to check the numerical solution of the Dirichlet problem.
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1. Introduction

The numerical methods belong to the category of Computer Science, the authors research into them by using the several complex analysis. Therefore present paper is the intersect of Computer Science and Mathematics or is the intersect of numerical calculus and complex analysis.

Because of the great applications in many research areas such as deferential geometry, variational method, optimization, transfers problem, to study the complex Monge–Ampère equations becomes the research hotspot. By Yau's opinion, the complex Monge–Ampère operator is one of the five important differential operators in the differential

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geometry. The Yau's very important results are based on the existence of the solution of some types of complex Monge–Ampère equations. Therefore to study the complex Monge–Ampère equation has great scientific meaning.

S.-Y. Cheng, N.-M. Mok, S.-T. Yau proved [1,2] that there exists unique complete Kaehler–Einstein metric on any bounded pseudoconvex domain D in C^n. In fact they discuss the following Dirichlet's problem of the Monge–Ampère equation:

$$\begin{cases} \det(\frac{\partial^2 g}{\partial z_i \partial \bar{z}_j}) = e^{(n+1)g}, & z \in D, \\ g = \infty, & z \in \partial D. \end{cases}$$

The Monge–Ampère equation is the nonlinear equation, hence to get its solution is very difficult. By introducing some insightful techniques in differential geometry Yau et al. proved that there exists unique solution of the above problem, but they have not got the solution in explicit formula. Therefore mathematicians hope to get the solutions for that problem by using the numerical method. The first author asked many experts about the numerical method, they said that the numerical method is very important for the Monge–Ampère equation, but it is also very difficult, up to now the successful numerical method of Monge–Ampère equation is not appeared.

2. Presentation of the problem

In present paper the authors try to study the numerical solution of Dirichlet's problem of complex Monge–Ampère equation on Cartan–Hartogs domain of the third type. The Cartan–Hartogs domain of the third type is defined as follows:

$$Y_{III} = Y_{III}(N_3, q; K) := \{W \in C^{N_3},\ Z \in R_{III}(q)\colon |W|^{2K} < \det(I - Z\overline{Z}^t),\ K > 0\},$$

where $R_{III}(q)$ is the classical domain of the third type, that is

$$R_{III}(q) = \{Z \in C^{\frac{q(q-1)}{2}}\colon I - Z\overline{Z}^t > 0\},$$

where Z is skew symmetric matrix with q order, $Z > 0$ means that Z is the positive definite matrix, $\overline{Z}$ denotes the conjugate of Z, Z^t indicates the transpose of Z, det is the abbreviation of determinant. Due to its explicit formula the Bergman kernel function of Y_{III} has Bergman exhausticity [3]. Hence the domain is pseudoconvex domain. Therefore the solution of problem (1) exists and is unique. In present paper the numerical solution of following Dirichlet's problem of Monge–Ampère equation is discussed:

$$\begin{cases} \det(\frac{\partial^2 g}{\partial z_i \partial \bar{z}_j})_{1 \leqslant i,j \leqslant N} = e^{(N+1)g}, & z \in Y_{III}, \\ g = \infty, & z \in \partial Y_{III}, \end{cases} \tag{1}$$

where $N = N_3 + \frac{q(q-1)}{2}$ is the complex dimension of Y_{III}, and suppose $(Z, W) \in Y_{III}$, $W = (w_1, w_2, \ldots, w_{N_3})$,

$$Z = -Z^t = \begin{pmatrix} 0 & z_{12} & \cdots & z_{1q} \\ -z_{12} & 0 & \cdots & z_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ -z_{1q} & -z_{2q} & \cdots & 0 \end{pmatrix},$$

let

$$Z_1 = (z_1, z_2, \ldots, z_{\frac{q(q-1)}{2}}) = (z_{12}, \ldots, z_{1q}, z_{23}, \ldots, z_{2q}, \ldots, z_{q-1,q}),$$
$$Z_2 = (z_{\frac{q(q-1)}{2}+1}, z_{\frac{q(q-1)}{2}+2}, \ldots, z_N) = (w_1, w_2, \ldots, w_{N_3}),$$

then the point (Z, W) can be denoted by a vector z with N entries, that is

$$z = (Z_1, Z_2) = (z_1, z_2, \ldots, z_{\frac{q(q-1)}{2}}, z_{\frac{q(q-1)}{2}+1}, z_{\frac{q(q-1)}{2}+2}, \ldots, z_N).$$

Firstly, we reduce the Monge–Ampère equation in (1) to an ordinary differential equation; secondly, we get the analytic explicit formula of the solution of problem (1) under the special case. This explicit formula can be used to check the numerical solution of problem (1).

We write this paper from the point of view of numerical method of PDE and wish to show that the several complex analytic methods may be helpful to the numerical method of PDE. And the authors hope that the scholars, who study the numerical method of PDE, pay their attention to this matter. Perhaps this is the original idea for advocating the category subject.

3. Preliminaries

3.1. The following mappings are the holomorphic automorphism of $Y_{III}(N_3, q, K)$, the set of such mappings is denoted by $\mathrm{Aut}(Y_{III})$,

$$\begin{cases} W^* = UW\det\big(I - Z_0\overline{Z_0}^t\big)^{\frac{1}{2K}}\det\big(I - Z\overline{Z_0}^t\big)^{-\frac{1}{K}}, \\ Z^* = A(Z - Z_0)\big(I - \overline{Z_0}^t Z\big)^{-1}\overline{A}^{-1}, \end{cases} \tag{2}$$

where $\overline{A}^t A = (I - Z_0\overline{Z_0}^t)^{-1}$, $Z_0 \in R_{III}(q)$, U is the unitary matrix. Such mapping maps the point (Z_0, W) onto point $(0, W^*)$.

Proof. It is well known that $Z^* = A(Z - Z_0)(I - \overline{Z_0}^t Z)^{-1}\overline{A}^{-1}$ is the holomorphic automorphism of $R_{III}(q)$. By calculations one has (see [4])

$$I - Z^*\overline{Z}^{*t} = \big(\overline{A}^t\big)^{-1}\big(I - Z\overline{Z_0}^t\big)^{-1}\big(I - Z\overline{Z}^t\big)\big(I - Z_0\overline{Z}^t\big)^{-1}A^{-1},$$
$$\det\big(I - Z^*\overline{Z}^{*t}\big) = \det\big(I - Z_0\overline{Z_0}^t\big)\big|\det\big(I - Z\overline{Z_0}^t\big)\big|^{-2}\det\big(I - Z\overline{Z}^t\big),$$

and

$$W^*\overline{W}^{*t} = W\overline{W}^t\det\big(I - Z_0\overline{Z_0}^t\big)^{1/K}\big|\det\big(I - Z\overline{Z_0}^t\big)\big|^{-2/K},$$

hence

$$\det\big(I - Z^*\overline{Z}^{*t}\big) - \big|W^*\big|^{2K} = \det\big(I - Z_0\overline{Z_0}^t\big)\big|\det\big(I - Z\overline{Z_0}^t\big)\big|^{-2}\big[\det\big(I - Z\overline{Z}^t\big) - |W|^{2K}\big].$$

Therefore the above mapping is the holomorphic automorphism of $Y_{III}(N_3, q; K)$. See [3]. □

3.2. Let $X = X(Z, W) = |W|^2[\det(I - Z\overline{Z}^t)]^{-1/K}$, then X is invariant under the $\mathrm{Aut}(Y_{III})$ that is $X(Z^*, W^*) = X(Z, W)$.

Proof. That can be proved by the direct calculations, see [3]. Therefore any function $F(X)$ is also invariant under $\mathrm{Aut}(Y_{III})$.

From the definition of Y_{III}, one has $0 \leqslant X < 1$. □

3.3. If $g(z, \bar{z}) = g[(Z, W), \overline{(Z, W)}]$ is the solution of problem (1), then

$$ds^2 = \sum \frac{\partial^2 g}{\partial z_i \partial \bar{z}_j} dz_i \overline{dz_j}$$

is the Kaehler–Einstein metric of domain Y_{III}. The Kaehler–Einstein metric is invariant under the biholomorphic mapping. Therefore if mapping $(Z^*, W^*) = F(Z, W) = F(z)$ belongs to $\mathrm{Aut}(Y_{III})$, one has

$$\det\left(\frac{\partial^2 g[(Z_0, W), \overline{(Z_0, W)}]}{\partial z_i \partial \bar{z}_j}\right) = |\det J_F|^2 \det\left(\frac{\partial^2 g[(0, W^*), \overline{(0, W^*)}]}{\partial w_i^* \partial \bar{w}_j^*}\right), \tag{3}$$

where w^* is the image of z under $F(z)$, which is the vector with N entries, that is $w^* = (w_1^*, \ldots, w_N^*)$; let J_F denotes the Jacobian matrix of F, then

$$J_F = \begin{pmatrix} \partial Z^*/\partial Z & * \\ 0 & \partial W^*/\partial W \end{pmatrix}.$$

Hence

$$\left|\det(J_F)\right|^2_{Z_0=Z} = \left|\det\left(\partial W^*/\partial W\right)\det\left(\partial Z^*/\partial Z\right)\right|^2_{Z_0=Z}.$$

Due to the well-known theory of classical domain [4], one has

$$\left|\det\left(\partial Z^*/\partial Z\right)\right|^2_{Z_0=Z} = \det\left(I - Z\overline{Z}^t\right)^{-(q-1)}.$$

It is easy to get $|\det(\partial W^*/\partial W)_{Z_0=Z}|^2 = \det(I - Z\overline{Z}^t)^{-N_3/K}$. Therefore

$$\left|\det(J_F)\right|^2_{Z_0=Z} = \det\left(I - Z\overline{Z}^t\right)^{-(q-1+N_3/K)}.$$

If $Z = Z_0$, then the form (3) becomes

$$\det\left(\frac{\partial^2 g(z,\bar z)}{\partial z_i \partial \bar z_j}\right) = \det\left(I - Z\overline{Z}^t\right)^{-(q-1+N_3/K)} \det\left(\frac{\partial^2 g[(0, W^*), \overline{(0, W^*)}]}{\partial w_i^* \partial \bar w_j^*}\right). \tag{4}$$

In fact, only one needs to calculate the following form:

$$\det\left(\frac{\partial^2 g[(0, W^*), \overline{(0, W^*)}]}{\partial w_i^* \partial \bar w_j^*}\right). \tag{5}$$

If g is the solution of problem (1), then the right side of (4) must be equal to $e^{(N+1)g}$, that is

$$\det\left(I - Z\overline{Z}^t\right)^{-(q-1+N_3/K)} \det\left(\frac{\partial^2 g[(0, W^*), \overline{(0, W^*)}]}{\partial w_i^* \partial \bar w_j^*}\right) = e^{(N+1)g}. \tag{6}$$

Let

$$g = (N+1)^{-1}\log\left[G(X)\det\left(I - Z\overline{Z}^t\right)^{-(q-1+N_3/K)}\right], \tag{7}$$

then the form (6) is equivalent to

$$\det\left(\frac{\partial^2 g[(0, W^*), \overline{(0, W^*)}]}{\partial w_i^* \partial \bar w_j^*}\right) = G(X). \tag{8}$$

Therefore if the left side of the form (8) can be expressed by X, $G(X)$ and its derivatives, then the Monge–Ampère equation in (1) is equivalent to an ordinary differential equation. That is true, please see the next section for details.

4. Reduce the Monge–Ampère equation to an ordinary differential equation

For convenience, let W^*, w^* in the left side of (8) are denoted by W, z, respectively. Therefore one needs to calculate

$$\det\left(\frac{\partial^2 g[(0, W), \overline{(0, W)}]}{\partial z_i \partial \bar z_j}\right). \tag{9}$$

But

$$\frac{\partial^2 g[(0, W), \overline{(0, W)}]}{\partial z_i \partial \bar z_j} = \frac{\partial^2 g[(Z, W), \overline{(Z, W)}]}{\partial z_i \partial \bar z_j}\bigg|_{Z=0},$$

then from (7) one has

$$g = (N+1)^{-1}\log\left[G(X)\det\left(I - Z\overline{Z}^t\right)^{-(q-1+N_3/K)}\right].$$

Hence the form (9) is equal to

$$C\det\begin{pmatrix} \frac{\partial^2 \log[G(X)\det(I-Z\overline{Z}^t)^{-(q-1+N_3/K)}]}{\partial z_{\alpha\beta}\partial \bar z_{\sigma\tau}} & \frac{\partial^2 \log[G(X)\det(I-Z\overline{Z}^t)^{-(q-1+N_3/K)}]}{\partial z_{\alpha\beta}\partial \bar w_j} \\ \frac{\partial^2 \log[G(X)\det(I-Z\overline{Z}^t)^{-(q-1+N_3/K)}]}{\partial w_i\partial \bar z_{\sigma\tau}} & \frac{\partial^2 \log[G(X)\det(I-Z\overline{Z}^t)^{-(q-1+N_3/K)}]}{\partial w_i\partial \bar w_j} \end{pmatrix}\Bigg|_{Z=0}, \tag{10}$$

where

$$C=(N+1)^{-N},\quad 1\leqslant\alpha<\beta\leqslant q,\ 1\leqslant\sigma<\tau\leqslant q,\ 1\leqslant i,j\leqslant N_3.$$

Let

$$\log G(X)=M,\qquad \frac{d\log G(X)}{dX}=M',\qquad \frac{d^2\log G(X)}{dX^2}=M''. \tag{11}$$

By calculations one has

$$\begin{aligned}
&\frac{\partial X}{\partial z_{\alpha\beta}}\Big|_{z=0}=0,\qquad \frac{\partial X}{\partial \bar z_{\sigma\tau}}\Big|_{z=0}=0,\\
&\frac{\partial^2 X}{\partial z_{\alpha\beta}\partial \bar z_{\sigma\tau}}\Big|_{z=0}=\frac{1}{K}X\operatorname{tr}[I_{\alpha\beta}I_{\tau\sigma}],\qquad \frac{\partial X}{\partial w_i}\Big|_{z=0}=\bar w_i,\qquad \frac{\partial X}{\partial \bar w_j}\Big|_{z=0}=w_j,\\
&\frac{\partial^2 X}{\partial z_{\alpha\beta}\partial \bar w_q}\Big|_{z=0}=\frac{\partial^2 X}{\partial w_p\partial \bar z_{\sigma\tau}}\Big|_{z=0}=0,\qquad \frac{\partial^2 X}{\partial w_i\partial \bar w_j}\Big|_{z=0}=\delta_{ij},\\
&\frac{\partial^2\log\det(I-Z\overline{Z}^t)}{\partial z_{\alpha\beta}\partial \bar z_{\sigma\tau}}=-\operatorname{tr}(I_{\alpha\beta}I_{\tau\sigma})=-\delta_{\alpha\sigma}\cdot\delta_{\beta\tau},\\
&\frac{\partial^2\log\det(I-Z\overline{Z}^t)}{\partial z_{\alpha\beta}\partial \bar w_j}\Big|_{z=0}=\frac{\partial^2\log\det(I-Z\overline{Z}^t)}{\partial w_i\partial \bar z_{\sigma\tau}}\Big|_{z=0}=\frac{\partial^2\log\det(I-Z\overline{Z}^t)}{\partial w_i\partial \bar w_j}\Big|_{z=0}=0.
\end{aligned}$$

Where $I_{\alpha\beta}$ is $q\times q$ matrix, the entry located at the junction of the αth row and βth column is 1, the entry located at the junction of the βth row and αth column is -1, the other entries are 0. Therefore one has

$$\begin{aligned}
&\frac{\partial^2\log[G(X)\det(I-Z\overline{Z}^t)^{-(q-1+N_3/K)}]}{\partial z_{\alpha\beta}\partial \bar z_{\sigma\tau}}\Big|_{z=0}=2\frac{1}{K}M'X+\left(q-1+\frac{N_3}{K}\right)\operatorname{tr}(I_{\alpha\beta}I_{\tau\sigma}),\\
&\frac{\partial^2\log[G(X)\det(I-Z\overline{Z}^t)^{-(q-1+N_3/K)}]}{\partial z_{\alpha\beta}\partial \bar w_j}\Big|_{z=0}=\frac{\partial^2\log[G(X)\det(I-Z\overline{Z}^t)^{-(q-1+N_3/K)}]}{\partial w_i\partial \bar z_{\sigma\tau}}\Big|_{z=0}=0,\\
&\frac{\partial^2\log[G(X)\det(I-Z\overline{Z}^t)^{-(q-1+N_3/K)}]}{\partial w_i\partial \bar w_j}\Big|_{z=0}=M''\bar w_i w_j+M'\delta_{ij}.
\end{aligned}$$

Hence the form (10) is equal to

$$C\det\begin{pmatrix}2[\frac{1}{K}M'X+(q-1+\frac{N_3}{K})]I & 0\\ 0 & M'I+M''\overline{W}^tW\end{pmatrix}. \tag{12}$$

Let the W, z come back to W^*, w^*, respectively, and M, M', M'' are invariant functions, therefore the above form (12) is equal to

$$C\det\begin{pmatrix}2[\frac{1}{K}M'X+(q-1+\frac{N_3}{K})]I & 0\\ 0 & M'I+M''\overline{W}^{*t}W^*\end{pmatrix}, \tag{13}$$

which is equal to

$$C\left[\frac{2}{K}M'X+2\left(q-1+\frac{N_3}{K}\right)\right]^{\frac{q(q-1)}{2}}\det\bigl[M'I+M''\overline{W}^{*t}W^*\bigr].$$

Due to the following fact: for the any vector α, one always has

$$\det[I+\bar\alpha'\alpha]=1+\alpha\bar\alpha',$$

the form (13) is equal to

$$C\left[\frac{2}{K}M'X+2\left(q-1+\frac{N_3}{K}\right)\right]^{\frac{q(q-1)}{2}}(M')^{N_3}\left[1+\frac{M''}{M'}W^*\overline{W}^{*t}\right].$$

Due to $W^*\overline{W}^{*t} = X$, hence the above form is equal to

$$C\left[\frac{2}{K}M'X + 2\left(q - 1 + \frac{N_3}{K}\right)\right]^{\frac{q(q-1)}{2}} (M')^{N_3}\left[1 + \frac{M''}{M'}X\right].$$

Therefore the left side of the form (8) is equal to the above form, and due to the form (11), then the form (8) is equivalent to

$$(N+1)^{-N}\left[\frac{2X}{K}\frac{dG}{dX} + 2\left(q - 1 + \frac{N_3}{K}\right)G\right]^{\frac{q(q-1)}{2}}\left[G\frac{dG}{dX} + \left(G\frac{d^2G}{dX^2} - \left(\frac{dG}{dX}\right)^2\right)X\right]\left(\frac{dG}{dX}\right)^{N_3-1} = G^{N+2}.$$

Therefore the solution of problem (1) is

$$g = (N+1)^{-1}\log\left[G(X)\det\left(I - Z\overline{Z}^t\right)^{-(q-1+N_3/K)}\right],$$

where $G = G(X)$ is the solution of the following problem:

$$\begin{cases} (N+1)^{-N}2^{\frac{q(q-1)}{2}}\left[\frac{X}{K}G' + \left(q - 1 + \frac{N_3}{K}\right)G\right]^{\frac{q(q-1)}{2}}\left[GG' + \left(GG'' - (G')^2\right)X\right]\frac{(G')^{N_3-1}}{G^{N+1}} = G, \\ G(0) = \left(\frac{K}{2}\right)^{-\frac{q(q-1)}{2}}, \quad \lim\limits_{X\to 1} G(X) = \infty. \end{cases} \tag{14}$$

Therefore the Dirichlet's problem (1) of Monge–Ampère equation is reduced to the above problem (14) of an ordinary differential equation. Problem (14) is also called two point boundary value problem of an ordinary differential equation.

5. The explicit solution of the Dirichlet problem of Monge–Ampère equation

Under the special case, the solution of the problem (14) can be got in explicit formula. Let

$$G = G(X) = A(1 - X)^{-(N+1)}, \tag{15}$$

put it into the first equation of the form (14), then one can determine the constant A. By calculations, one has

$$G' = A(N+1)(1-X)^{-(N+2)}, \qquad G'X = A(N+1)(1-X)^{-(N+2)} - A(N+1)(1-X)^{-(N+1)},$$
$$\frac{1}{K}G'X = \frac{A}{K}(N+1)(1-X)^{-(N+2)} - \frac{A}{K}(N+1)(1-X)^{-(N+1)}.$$

Let

$$K = \frac{q(q-1)+2}{2(q-1)} = \frac{q}{2} + \frac{1}{q-1},$$

then one has

$$\left[\frac{1}{K}G'X + \left(q - 1 + \frac{N_3}{K}\right)G\right]^{\frac{q(q-1)}{2}} = \left[\frac{N+1}{K}A(1-X)^{-(N+2)}\right]^{\frac{q(q-1)}{2}}, \tag{16}$$

$$\frac{(G')^{N_3-1}}{G^{N+1}} = (N+1)^{N_3-1}A^{N_3-2-N}(1-X)^{(N+1)^2-(N+2)(N_3-1)}. \tag{17}$$

Then by the calculations

$$GG' = (N+1)A^2(1-X)^{-(2N+3)}, \qquad GG'' = (N+1)(N+2)A^2(1-X)^{-(2N+4)},$$
$$(G')^2 = (N+1)^2A^2(1-X)^{-(2N+4)}, \qquad GG'' - (G')^2 = (N+1)A^2(1-X)^{-(2N+4)}.$$

Therefore one has

$$\left(GG'' - (G')^2\right)X = (N+1)A^2(1-X)^{-(2N+4)} - (N+1)A^2(1-X)^{-(2N+3)},$$
$$GG' + \left(GG'' - (G')^2\right)X = (N+1)A^2(1-X)^{-(2N+4)}. \tag{18}$$

Due to the (15)–(18), the form (14) is equal to

$$\left(\frac{2}{K}\right)^{\frac{q(q-1)}{2}}(1-X)^{-(N+1)} = \left(\frac{4(q-1)}{q^2-q+2}\right)^{\frac{q(q-1)}{2}}(1-X)^{-(N+1)} = A(1-X)^{-(N+1)}.$$

Therefore

$$A = \left(\frac{4(q-1)}{q^2-q+2}\right)^{\frac{q(q-1)}{2}}.$$

At that time the $G(X)$ satisfies the first boundary condition, and

$$\lim_{X\to 1} G(X) = \lim_{X\to 1}\left[\left(\frac{4(q-1)}{q^2-q+2}\right)^{\frac{q(q-1)}{2}}(1-X)^{-(N+1)}\right] = \infty.$$

Therefore

$$G = G(X) = \left(\frac{4(q-1)}{q^2-q+2}\right)^{\frac{q(q-1)}{2}}(1-X)^{-(N+1)} \tag{19}$$

is the solution of the problem (14). Due to the (19) and $K = \frac{q(q-1)+2}{2(q-1)}$, the form (7) is equal to

$$g = \log\left[(1-X)^{-1}\det\bigl(I - Z\overline{Z}^t\bigr)^{-1/K}\left(\frac{2}{K}\right)^{\frac{q(q-1)}{2(N+1)}}\right], \tag{20}$$

which is equivalent to

$$g = \log\left[(1-X)^{-1}\det\bigl(I - Z\overline{Z}^t\bigr)^{-\frac{2(q-1)}{q^2-q+2}}\left(\frac{4(q-1)}{q^2-q+2}\right)^{\frac{q(q-1)}{2(N+1)}}\right]. \tag{20$'$}$$

Therefore the above g is the solution of the problem (1). In fact, one can prove that the g of form (7)

$$g = (N+1)^{-1}\log\bigl[G(X)\det\bigl(I - Z\overline{Z}^t\bigr)^{-(q-1+N_3/K)}\bigr]$$

is the solution of problem (1), if $G(X)$ is the solution of problem (14).

Firstly it is obvious that the g satisfies the complex Monge–Ampère equation of (1). Secondly g is also satisfying the boundary condition of (1) due to the following way:

If the point $(\widetilde{Z}, \widetilde{W}) \in \partial Y_{III}$, and $\widetilde{W} \neq 0$, when $(Z, W) \in Y_{III}$, and $(Z, W) \to (\widetilde{Z}, \widetilde{W})$, one has $X \to 1^-$, so $G(X) \to +\infty$, and $\det(I - Z\overline{Z}^t) \to |\widetilde{W}|^{2K} > 0$. Therefore $g \to +\infty$, $(Z, W) \to \partial Y_{III}$.
If $(\widetilde{Z}, \widetilde{W}) \in \partial Y_{III}$, and $\widetilde{W} = 0$, when $(Z, W) \in Y_{III}$, and $(Z, w) \to (\widetilde{Z}, 0)$, one has $G(X) \to (\frac{K}{2})^{-\frac{q(q-1)}{2}}$, $\det(I - Z\overline{Z}^t) \to 0$, that is $g \to +\infty$, $(Z, W) \to \partial Y_{III}$. Therefore the g of (7) is the solution of the problem (1).

Up to now the following theorem is proved.

Theorem. *If $G(X)$ is the solution of the problem* (14), *then*

$$g = (N+1)^{-1}\log\bigl[G(X)\det\bigl(I - Z\overline{Z}^t\bigr)^{-(q-1+N_3/K)}\bigr]$$

is the solution of the problem (1); *if $K = \frac{q(q-1)+2}{2(q-1)} = \frac{q}{2} + \frac{1}{q-1}$, then*

$$G(X) = \left(\frac{4(q-1)}{q^2-q+2}\right)^{\frac{q(q-1)}{2}}(1-X)^{-(N+1)}$$

satisfies the problem (14) *and accordingly*

$$g = (N+1)^{-1} \log\left[\left(\frac{4(q-1)}{q^2-q+2}\right)^{\frac{q(q-1)}{2}} (1-X)^{-(N+1)} \det\left(I - Z\overline{Z}^t\right)^{-(q-1+N_3/K)}\right]$$
$$= \log\left[(1-X)^{-1} \det\left(I - Z\overline{Z}^t\right)^{-\frac{2(q-1)}{q^2-q+2}} \left(\frac{4(q-1)}{q^2-q+2}\right)^{\frac{q(q-1)}{2(N+1)}}\right]$$

is the solution of the problem (1).

If $N_3 = 1$, *then the solution of the problem* (1) *is discussed in* [5]. *In present paper the* N_3 *is in general case, and the ordinary differential equation is different from that in* [5].

Remark. In problem (14) the condition $\lim_{X\to 1} G(X) = \infty$ is the nodus for the numerical method. One can use the following way to overcome this nodus.

Due to the form (8), $G(X)$ has to be positive, therefore

$$F(X) = \frac{1}{G(X)}$$

is determinate. And the problem (14) becomes the following problem:

$$\begin{cases} C\left[-XF' + \left(K(q-1) + N_3\right)F\right]^{\frac{q(q-1)}{2}} \left[-FF' - FF''X + (F')^2 X\right](-F')^{N_3-1} = F^N, \\ F(0) = \left(\dfrac{K}{2}\right)^{\frac{q(q-1)}{2}}, \quad \lim\limits_{X\to 1} F(X) = 0, \end{cases} \tag{21}$$

where $C = (N+1)^{-N}(\frac{2}{K})^{\frac{q(q-1)}{2}}$. Then one can use the numerical method to solve the problem (21).

The title of present paper is “A research into the numerical method of Dirichlet’s problem of complex Monge–Ampère equation on Cartan–Hartogs domain of the third type,” but up to now the numerical method has not appeared. In present paper the numerical method of the solution of Dirichlet’s problem (1) of Monge–Ampère equation is reduced to the numerical method of the problem (14) of the ordinary differential equation. The problem (14) is also called two point boundary value problem of an ODE. How to get the numerical solution of problem (14)? One can see the following book: Herbert B. Keller, Numerical Solution of Two Point Boundary Value Problems, Society for Industrial and Applied Mathematics, Philadelphia, 1976.

The numerical method provided in the above book can be used to solve the problem (14) or (21).

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