Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 341 (2008) 906–921
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Lie group symmetry analysis of transport in porous media with variable transmissivity

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Received 26 October 2006
Available online 29 September 2007
Submitted by B. Straughan

Abstract

We determine the Lie group symmetries of the coupled partial differential equations governing a novel problem for the transient flow of a fluid containing a solidifiable gel, through a hydraulically isotropic porous medium. Assuming that the permeability (K^*) of the porous medium is a function of the gel concentration (c^*), we determine a number of exact solutions corresponding to the cases where the concentration-dependent permeability is either arbitrary or has a power law variation or is a constant. Each case admits a number of distinct Lie symmetries and the solutions corresponding to the optimal systems are determined. Some typical concentration and pressure profiles are illustrated and a specific moving boundary problem is solved and the concentration and pressure profiles are displayed.

Keywords: Pollutants; Porous media; Variable permeability; Coupled partial differential equations; Lie symmetries; Exact solutions

1. Introduction

The migration of hazardous chemicals and other contaminants in porous geological media is a topic of considerable importance to the general area of environmental geosciences, and is of particular interest to geo-environmental engineering [1–4]. In attempting to prevent migration of pollutants from an underground repository or in attempting to minimize the groundwater flow to underground structures and excavations, one strategy is to inject a fluid that contains a chemical gel into the earth surrounding the area. The gel subsequently solidifies and thereby alters the permeability of the porous medium to the extent of preventing any transport of pollutants or groundwater influx. The transport mechanisms can include both advective and diffusive processes that are governed, respectively, by velocities of the fluid in the pore space and chemical concentration gradients. Although the fundamental processes governing these basic modes of transport are highly non-linear and dependent on the micro-structural morphology of the porous medium and the geochemistry of the system, plausible mathematical theories can be constructed to provide a methodology for examining the quantities and rates at which chemicals are being transported within the geosphere. For example,

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.09.042

in the simplest mathematical representations, the advective transport of the chemical is related to a flow velocity that can be either steady or transient and governed by Darcy flow, while the diffusive transport processes are governed by Fick's law. Depending upon the characteristics of the flow and diffusive processes, either process can dominate, or both processes can occur simultaneously.

The assessment of the chemical transport process is, of course, an important aspect of the overall problem. In recent years, however, attention has also been focused on the mitigation of contaminant migration through the provision of either temporary or permanent barriers. The containment strategy is regarded as a viable alternative to large scale collection and treatment of the groundwater. While the methods for creating barriers to groundwater flow are diverse, the focus of this paper is to examine the general class of mathematical problems that arise during the injection of gel-type fluids to minimize the fluid transport characteristics of the porous medium. The gels can range from cementitious grouts to thixotropic materials that can either reduce the pore space or undergo swelling with time to reduce the hydraulic conductivity of the porous medium. In such situations, the effective hydraulic conductivity of the porous medium can depend on both the concentration of "barrier material" that has filled the pore space and time. The class of advective transport problems examined here also have potential applications to the modelling of resin migration in the modelling of a poroelastic material such as bone. Here, the resin injection procedures are proposed as a technique for improving the load bearing characteristics of cancellous bone susceptible to osteoporosis.

In this paper, we present for the first time a novel theoretical model that describes the coupled processes of transient advective flow in a porous medium and migration of the barrier material, defined in terms of its concentration, as an advective transport process. The strong non-linearity in the problem stems from the dependency of advective flow velocities on both time and the concentration of the barrier material. While the choice of this variation invariably depends on the particular barrier material-porous medium combination, a plausible specific variation is assumed to make the coupled non-linear problem mathematically tractable. The resulting non-linear problem is examined through Lie group symmetries of the resulting coupled partial differential equations [5–9]. Specifically, when the dependency of the permeability of the porous medium on the concentration of the migrating barrier material is either arbitrary, or has a power law variation, it is shown that these correspond to two distinct families of Lie symmetries for which we compile an extensive table of exact solutions. Specifically, we examine the Lie group symmetries of the resulting coupled partial differential equations. Assuming the permeability K^* to be a function of the gel concentration c^*, we show that the cases $K^*(c^*)$ arbitrary and $K^*(c^*) = K_0 c^{*m}$ give rise to two distinct families of Lie symmetries.

In particular, as an initial study, attention is restricted to domains where the porous medium is of infinite extent and bounded internally by either spherical or cylindrical cavities. Discussion is also focused on a comprehensive Lie symmetry analysis of the non-linear coupled one-dimensional problem giving rise to optimal systems. In the subsequent sections, we examine a number of particular cases that admit further simplifications and analysis; some of the resulting solutions are shown graphically.

In the following section we formulate the underlying model coupled partial differential equations. This coupled system, although similar in form to coupled systems arising in heat and mass transfer problems, possesses a number of novel features and has not been previously studied in the literature. In the next section we undertake a comprehensive Lie symmetry analysis for one-dimensional flow, giving rise to the optimal systems. In Section 6 of the paper we examine a specific moving boundary value problem and the resulting solutions for the concentration and pressure are shown graphically.

2. Basic coupled system of equations

We consider the problem of a porous medium of infinite extent, such that the fluid flow through the medium is governed by Darcy's law, which takes the form

$$\mathbf{v} = -\frac{K^*\gamma_w}{\mu}\nabla p^* \tag{1}$$

where $\mathbf{v}$ is the fluid velocity vector, p^* is the fluid pressure, γ_w is the unit weight of the fluid, μ is the dynamic viscosity of the fluid and K^* is the permeability of the porous medium. We consider the modelling, where the fluid that migrates through the porous medium contains a barrier material, which solidifies as a function of its concentration $c^*(\mathbf{x}^*, t^*)$ and time t^*, and we assume that this process alters the permeability K^*. The analysis of fluid pressure transients in a porous medium requires the coupled analysis of pressure-induced fluid flow and the deformability of the porous skeleton. The complete analysis of this aspect is beyond the scope of this paper. Instead, we consider a simplified model of the transient processes of fluid flow governed by the elastic drive equation (see, e.g. Selvadurai [10])

$$\frac{\partial p^*}{\partial t^*} = \nabla \cdot \left(\frac{K^*}{(nC_w + C_e)\mu} \nabla p^* \right), \tag{2}$$

where n is the porosity of the porous medium, C_w is the compressibility of the migrating fluid and C_e is the compressibility of the porous solid skeleton. The advective transport of the barrier material is governed by the advective transport equation arising from its mass conservation; thus

$$\frac{\partial c^*}{\partial t^*} = \nabla \cdot \left(\frac{K^* \gamma_w}{\mu} c^* \nabla p^* \right). \tag{3}$$

The non-linearity in the initial boundary value problems arises as a result of the time- and concentration-dependency of the permeability K^*. It could be argued that the compressibility of the porous medium itself could be altered by the presence of the barrier material. This is regarded to be of secondary importance in relation to the influence of c^* and t^* on the permeability K^*. Although we make no further use of this, we note that a plausible expression for $K^*(c^*, t^*)$ could be of the form

$$K^*(c^*, t^*) = K_0\big[1 - \exp(-\lambda_c c^*)\big]\big[1 - \exp(\lambda_t t^*)\big], \tag{4}$$

where K_0, λ_c and λ_t denote certain constants, and to a first approximation, we might adopt the following expression for $K^*(c^*, t^*)$:

$$K^*(c^*, t^*) = K_0 \exp\big[-(\alpha_c c^* + \alpha_t t^*)\big],$$

where α_c and α_t are constants. Further discussion of related aspects is given in [11]. The case $K^* = K^*(c^*, t^*)$ is reserved for a future investigation and here we assume K^* to be a function of $c^*(x^*, t^*)$ only.

For the one-dimensional problem referred to the Cartesian coordinate system the two coupled partial differential equations (2) and (3) reduce to

$$\frac{\partial p^*}{\partial t^*} = \frac{\partial}{\partial x^*}\left\{ \frac{K^*}{\mu(nC_w + C_e)} \frac{\partial p^*}{\partial x^*} \right\}, \qquad \frac{\partial c^*}{\partial t^*} = \frac{\partial}{\partial x^*}\left\{ \frac{K^* c^* \gamma_w}{\mu} \frac{\partial p^*}{\partial x^*} \right\}, \tag{5}$$

where $K^* = K^*(c^*)$. By an appropriate non-dimensionalization of the physical variables, we may deduce the following system of partial differential equations

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}\left\{ K(c) \frac{\partial p}{\partial x} \right\}, \qquad \frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left\{ cK(c) \frac{\partial p}{\partial x} \right\}, \tag{6}$$

which we adopt as the model system for a Lie group symmetry analysis and the subsequent tabulation of an extensive number of exact solutions.

3. Lie group symmetry analysis and reductions

We are interested in constructing exact solutions to the coupled system of partial differential equations (6) where the permeability K is taken as any arbitrary function of the concentration c, the only constraint on K being that it is always positive. Classical Lie group theory is used to determine the symmetries of the general class of coupled partial differential equations (6). The existence of symmetries for arbitrary and special cases of K will lead to a reduction in the number of independent variables and hence to a coupled system of ordinary differential equations. These systems of ordinary differential equations may or may not be solvable in closed form, but at least are readily amenable to standard numerical packages, such as those provided in MAPLE.

We consider the infinitesimal transformations of the dependent variables p and c and the independent variables x and t, where the infinitesimal generator (see for example [5,6]) is given by

$$\Gamma = \mathcal{X}\frac{\partial}{\partial x} + \mathcal{T}\frac{\partial}{\partial t} + \mathcal{C}\frac{\partial}{\partial c} + \mathcal{P}\frac{\partial}{\partial p}.$$

The invariance of (6)(a) and (6)(b) under the infinitesimal transformation and the fact that the derivatives of c and p are independent leads to a set of determining equations for the symmetry group of the governing equations (6).

These determining equations are linear partial differential equations in $\mathcal{C}(x,t,c,p)$, $\mathcal{P}(x,t,c,p)$, $\mathcal{X}(x,t,c,p)$ and $\mathcal{T}(x,t,c,p)$.

The most general infinitesimal symmetry of the governing equations is found by solving this system of linear partial differential equations. The current analysis was performed using the symmetry-finding package Dimsym [12] under REDUCE.

The symmetries

$$\Gamma_1 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}, \qquad \Gamma_2 = \frac{\partial}{\partial x}, \qquad \Gamma_3 = \frac{\partial}{\partial p}, \qquad \Gamma_4 = \frac{\partial}{\partial t}$$

apply for all forms of $K(c)$. Dimsym flags that the rescaled simplified forms

$$K(c) = \ln(c)^m, \qquad K(c) = c^m, \qquad K(c) = 1$$

may admit additional symmetries. Symmetry analysis of (6) with $K(c) = \ln(c)^m$ reveals that this "special case" admits four symmetries which are the same as those listed for $K(c)$ arbitrary. That is, even though this case is flagged as a special case, further investigation shows that this is not so.

Symmetry analysis of the governing equations with $K(c) = c^m$ shows that this case admits the additional symmetry

$$\Gamma_5 = c\frac{\partial}{\partial c} - mt\frac{\partial}{\partial t}$$

while symmetry analysis of the governing equation with $K(c) = 1$ shows that this case admits the additional symmetry

$$\Gamma_5 = c\frac{\partial}{\partial c}$$

which is the same as the symmetries for $K(c) = c^m$ with $m = 0$.

The main use of symmetries is to obtain a reduction of variables as described by Sander et al. [13]. A reduction of variables of the coupled equations (6) will result in a coupled system of ordinary differential equations, which may or may not be solvable. The similarity variables used in the reduction of order follow from the solution of the characteristic equations

$$\frac{dx}{\mathcal{X}} = \frac{dt}{\mathcal{T}} = \frac{dc}{\mathcal{C}} = \frac{dp}{\mathcal{P}}. \tag{7}$$

Reductions may be obtained from any symmetry that is any arbitrary combination of the type

$$\begin{aligned} &a_1\Gamma_1 + a_2\Gamma_2 + a_3\Gamma_3 + a_4\Gamma_4 + a_5\Gamma_5, \quad \text{for } K(c) = 1 \text{ or } c^m, \\ &a_1\Gamma_1 + a_2\Gamma_2 + a_3\Gamma_3 + a_4\Gamma_4, \quad \text{otherwise.} \end{aligned} \tag{8}$$

To ensure that a minimal complete set of reductions is found from the symmetries of the governing equations, the optimal system [6,7] is found for each of the cases for $K(c)$ given in (8). If we let Γ_i and Γ_j be two group operators, then commutator $[\Gamma_i, \Gamma_j]$ is given by

$$[\Gamma_i, \Gamma_j] = \Gamma_i\Gamma_j - \Gamma_j\Gamma_i.$$

The commutator table for the case $K(c) = c^m$ is given in Table 1. We note that the commutator table for the case $K(c) = 1$ may be obtained by setting $m = 0$ in Table 1. Now using the adjoint operator

$$\mathrm{Ad}\big(\exp(\epsilon\Gamma_i)\big)\Gamma_j = e^{\epsilon\Gamma_i}\Gamma_j e^{\epsilon\Gamma_i} = \Gamma_j - \epsilon[\Gamma_i, \gamma_j] + \frac{1}{2}\epsilon^2\big[\Gamma_i, [\Gamma_i, \Gamma_j]\big] - \cdots,$$

Table 1
Commutator table for $K(c) = c^m$

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5
Γ_1	0	$-\Gamma_2$	0	$-2\Gamma_1$	0
Γ_2	Γ_2	0	0	0	0
Γ_3	0	0	0	0	0
Γ_4	$2\Gamma_4$	0	0	0	$-m\Gamma_4$
Γ_5	0	0	0	$m\Gamma_4$	0

Table 2
Adjoint table for $K(c) = c^m$

Ad	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5
Γ_1	Γ_1	$e^{\epsilon}\Gamma_2$	Γ_3	$e^{2\epsilon}\Gamma_1$	Γ_5
Γ_2	$\Gamma_1 - \epsilon\Gamma_2$	Γ_2	Γ_3	Γ_4	Γ_5
Γ_3	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5
Γ_4	$\Gamma_1 - 2\epsilon\Gamma_4$	Γ_2	Γ_3	Γ_4	$\Gamma_5 + m\epsilon\Gamma_4$
Γ_5	Γ_1	Γ_2	Γ_3	$e^{-m\epsilon}\Gamma_4$	Γ_5

Table 3
Optimal system for coupled equations (6)(a) and (6)(b)

$K(c)$	Optimal system
Arbitrary	$\Gamma_1 + \alpha\Gamma_3,\ \Gamma_2 \pm \Gamma_3 + \alpha\Gamma_4,\ \Gamma_2 \pm \Gamma_4,\ \Gamma_2,\ \Gamma_3 \pm \Gamma_4,\ \Gamma_3,\ \Gamma_4$
c^m	$\Gamma_1 + \alpha\Gamma_3 \pm \Gamma_4 + \beta\Gamma_5,\ \Gamma_1 + \alpha\Gamma_3 + \beta\Gamma_5,\ \pm\Gamma_2 + \alpha\Gamma_3 + \Gamma_5,\ \Gamma_2 + \alpha\Gamma_3 \pm \Gamma_4,\ \Gamma_2 \pm \Gamma_3,\ \Gamma_3,\ \alpha\Gamma_3 + \Gamma_5,\ \Gamma_3 \pm \Gamma_4,\ \Gamma_3,\ \Gamma_4$
1	$\Gamma_1 + \alpha\Gamma_3 + \beta\Gamma_5,\ \Gamma_2 + \alpha\Gamma_3 \pm \Gamma_4 + \beta\Gamma_5,\ \Gamma_2 + \alpha\Gamma_3 \pm \Gamma_5,\ \Gamma_2 \pm \Gamma_3,\ \Gamma_2,\ \Gamma_3 \pm \Gamma_4 + \alpha\Gamma_5,\ \Gamma_3 + \alpha\Gamma_5,\ \Gamma_4 \pm \Gamma_5,\ \Gamma_4,\ \Gamma_5$

Table 4
Reduced ordinary differential equations for $K(c)$ arbitrary

Γ	Reduced equations
$\Gamma_1 + \alpha\Gamma_3$	$\frac{d}{d\rho}[2KG'] + \rho G' - \alpha = 0,\ \frac{d}{d\rho}[2FKG'] + \rho F' = 0$, with $c = F(\rho)$, $p = \frac{\alpha}{2}\ln t + G(\rho)$, $\rho = \frac{x}{\sqrt{t}}$
$\Gamma_2 + \Gamma_3 + \alpha\Gamma_4, \alpha \neq 0$	$\frac{d}{d\rho}[\alpha KG'] + G' - 1 = 0,\ \frac{d}{d\rho}[\alpha FKG'] + F' = 0$, with $c = F(\rho)$, $p = \frac{t}{\alpha} + G(\rho)$, $\rho = x - \frac{t}{\alpha}$
$\Gamma_2 + \Gamma_3$	$G' = 0$, $F' = 0$, with $c = F(\rho)$, $p = G(\rho) + x$, $\rho = t$
$\Gamma_2 + \Gamma_4$	$\frac{d}{d\rho}[KG'] - G' = 0$, $\frac{d}{d\rho}[FKG'] - F' = 0$, with $c = F(\rho)$, $p = G(\rho)$, $\rho = x + t$
$\Gamma_3 + \Gamma_4$	$\frac{d}{d\rho}[KG'] - 1 = 0$, $\frac{d}{d\rho}[FKG'] = 0$, with $c = F(\rho)$, $p = G(\rho) + t$, $\rho = x$

we may construct the adjoint table for $K(c) = c^m$. We note that the adjoint table for $K(c) = 1$ may be obtained by setting $m = 0$ in Table 2. The optimal system may be determined by taking a general element

$$\Gamma = a_1\Gamma_1 + a_2\Gamma_2 + a_3\Gamma_3 + a_4\Gamma_4 + a_5\Gamma_5$$

in the Lie algebra and simplifying it as much as possible by subjecting it to judiciously chosen adjoint transformations. Although the case $K(c) = 1$ has the same symmetries as the case $K(c) = c^m$ with $m = 0$, the optimal systems for these two cases are not equivalent because the adjoint table for $K(c) = 1$ allows less freedom when simplifying Γ.

A minimal set of generators for the symmetry algebra under the action of the symmetry group in the adjoint representation for each of the three cases in (8) is given in Table 3. The optimal systems in Table 3 may be further reduced if we allow discrete symmetries. For example, for $K(c)$ arbitrary, if we allow the discrete symmetry $(x, t, c, p) \mapsto (x, -t, c, p)$, $\Gamma_2 + \Gamma_4$ maps to $\Gamma_2 - \Gamma_4$. Other entries in Table 1 may be reduced in a similar fashion.

For each of the cases $K(c)$ arbitrary, $K(c) = c^m$ and $K(c) = 1$ of the governing equations (6), the corresponding optimal system given in Table 3 can be used to construct a minimal complete set of reductions. Reduction of the governing ordinary differential equations by two members of the same symmetry conjugacy classes is equivalent by a change of variables as described by [6,7]. Therefore we need only to consider reduction by members of the optimal system of symmetries, as these are representatives of the conjugacy classes. The symmetry variables used in the reductions follow from the solution of the characteristic equations (7). The reduced equations and the corresponding functional forms for the three cases $K(c)$ arbitrary, $K(c) = c^m$ and $K(c) = 1$ are listed in Tables 4, 5 and 6. We note that G' and F' denote differentiations of G and F with respect to ρ.

Table 5
Reduced ordinary differential equations for $K(c)=c^m$

Γ	Reduced equations
$\Gamma_1+\alpha\Gamma_3+\Gamma_4+\beta\Gamma_5$,	$\frac{d}{d\rho}[\rho^{\beta m}F^mG']+\rho G'=0$,
$2-\beta m\neq 0$	$\frac{d}{d\rho}[\rho^{\beta m}F^{m+1}G']+\beta\rho^{\beta m-1}F^{m+1}G'+\rho F'=0$,
	with $c=F(\rho)(x+\alpha)^\beta$, $p=G(\rho)$, $\rho=\frac{x+\alpha}{[(2-\beta m)t+1]^{1/(2-\beta m)}}$
$\Gamma_1+\alpha\Gamma_3+\Gamma_4+\beta\Gamma_5$,	$\frac{d}{d\rho}[\rho^2F^mG']-\rho G'=0$,
$2-\beta m=0$	$\frac{d}{d\rho}[m\rho^2F^{m+1}G']+2\rho F^{m+1}G'-m\rho F'=0$,
	with $c=F(\rho)(x+\alpha)^{2/m}$, $p=G(\rho)$, $\rho=(x+\alpha)e^t$
$\Gamma_1+\alpha\Gamma_3+\beta\Gamma_5$,	$\frac{d}{d\rho}[(\beta m-2)\rho^{\beta m}F^mG']-\rho G'=0$,
$2-\beta m\neq 0$	$\frac{d}{d\rho}[(\beta m-2)\rho^{\beta m}F^{m+1}G']+\beta(\beta m-2)\rho^{\beta m-1}F^{m+1}G'-\rho F'=0$,
	with $c=F(\rho)(x+\alpha)^\beta$, $p=G(\rho)$, $\rho=\frac{x+\alpha}{t^{1/(2-\beta m)}}$
$\Gamma_1+\alpha\Gamma_3+\beta\Gamma_5$,	$F'=0$, $G'=0$,
$2-\beta m=0$	with $c=F(\rho)(x+\alpha)^{2/m}$, $p=G(\rho)$, $\rho=t$
$\Gamma_2+\alpha\Gamma_3+\Gamma_5$	$\frac{d}{d\rho}[mF^mG']+\alpha-G'=0$, $\frac{d}{d\rho}[mF^{m+1}G']+F-F'=0$,
	with $c=F(\rho)t^{-1/m}$, $p=G(\rho)-\frac{\alpha}{m}\ln t$, $\rho=x+\frac{\ln t}{m}$
$\alpha\Gamma_3+\Gamma_5$	$\frac{d}{d\rho}[mF^mG']+\alpha=0$, $\frac{d}{d\rho}[mF^{m+1}G']+F=0$,
	with $c=F(\rho)t^{-1/m}$, $p=G(\rho)-\frac{\alpha}{m}\ln t$, $\rho=x$
$\Gamma_2+\alpha\Gamma_3+\Gamma_4$	$\frac{d}{d\rho}[F^mG']+G'-\alpha=0$, $\frac{d}{d\rho}[F^{m+1}G']+F'=0$,
	with $c=F(\rho)$, $p=G(\rho)+\alpha t$, $\rho=x-t$
$\Gamma_2+\Gamma_3$	$F'=0$, $G'=0$, with $c=F(\rho)$, $p=G(\rho)+x$, $\rho=t$
$\Gamma_3+\Gamma_4$	$\frac{d}{d\rho}[F^mG']-1=0$, $\frac{d}{d\rho}[F^{m+1}G']=0$, with $c=F(\rho)$, $p=G(\rho)+t$, $\rho=x$

Table 6
Reduced ordinary differential equations for $K(c)=1$

Γ	Reduced equations
$\Gamma_1+\alpha\Gamma_3+\beta\Gamma_5$	$2G''+\rho G'-\alpha=0$, $\frac{d}{d\rho}[2FG']+\rho F'-\beta F=0$,
	with $c=F(\rho)t^{\beta/2}$, $p=G(\rho)+\frac{\alpha}{2}\ln t$, $\rho=\frac{x}{\sqrt{t}}$
$\Gamma_2+\alpha\Gamma_3+\Gamma_4+\beta\Gamma_5$	$\frac{d}{d\rho}[G'+G-\alpha\rho]=0$, $\frac{d}{d\rho}[FG']+F'-\beta F=0$,
	with $c=F(\rho)e^{\beta t}$, $p=G(\rho)+\alpha t$, $\rho=x-t$
$\Gamma_2+\alpha\Gamma_3+\Gamma_5$	$G'=0$, $F'-\alpha F=0$, with $c=F(\rho)e^x$, $p=G(\rho)+\alpha x$, $\rho=t$
$\Gamma_2+\Gamma_3$	$G'=0$, $F'=0$, with $c=F(\rho)$, $p=G(\rho)+x$, $\rho=t$
$\Gamma_3+\Gamma_4+\alpha\Gamma_5$	$G''-1=0$, $\frac{d}{d\rho}[FG']-\alpha F=0$, with $c=F(\rho)e^{\alpha t}$, $p=G(\rho)+t$, $\rho=x$
$\Gamma_4+\Gamma_5$	$G''=0$, $\frac{d}{d\rho}[FG']-F=0$, with $c=F(\rho)e^t$, $p=G(\rho)$, $\rho=x$

4. Exact solutions

We are interested in determining which of the pairs of ordinary differential equations from Tables 4, 5 and 6 may be solved. Some systems of equations may be partially solved while other cases can be solved completely. Results from each of the three cases $K(c)$ arbitrary, $K(c)=c^m$ and $K(c)=1$ are presented separately.

We note that cases with solutions of the form $F=c_1$, $G=c_2$ have not been included in the following discussion.

Case 1. $K(c)$ arbitrary.

Case 1.1. $\Gamma_1 + \alpha\Gamma_3$

The first equation may be rearranged to give

$$G = \frac{\alpha}{2}\int\left[\frac{1}{K}e^{-\int\frac{\rho}{2K}\,d\rho}\left[\int e^{\int\frac{\rho}{2K}\,d\rho}\,d\rho\right] + \frac{c_1}{K}e^{-\int\frac{\rho}{2K}\,d\rho}\right]d\rho + c_2.$$

We note that although K is arbitrary, we cannot integrate the above expression as K is any function of c, and from the corresponding functional form from Table 2, $c = F$, which is still undetermined. Alternatively, a neater expression for G is

$$G' = \frac{\rho F' + \alpha F}{\rho F - 2KF'}.$$

The second equation gives

$$\left[4\alpha K^2 F + 2\rho^2 KF\right]F'' + \left[2\rho^2 K' F - 8\alpha K^2 - 4K^2 - 4\rho^2 K\right](F')^2 + \left[2\alpha\rho K' F^2 + 2\alpha\rho KF + \rho^3 F\right]F' - 2\alpha KF^2 = 0.$$

Case 1.2. $\Gamma_2 + \Gamma_3 + \alpha\Gamma_4, \alpha \neq 0$

$$G = \rho - \frac{c_2}{F} + c_1 + 1 \quad\text{and}\quad \rho = \int\frac{\alpha c_2 K}{c_2 F - \alpha K - 1}\,dF + c_3$$

giving a closed form solution for F and hence G once the function $K(F)$ has been chosen. In terms of the original variables, the solution is

$$c = F(\rho), \qquad p = x - \frac{t}{\alpha} - \frac{c_2}{F} + c_1 + 1, \qquad \rho = x - \frac{t}{\alpha}.$$

Case 1.3. $\Gamma_2 + \Gamma_4$

$$G = 1 - c_1 + \frac{c_2}{F} \quad\text{with}\quad \int\frac{K}{F(F - c_2)}\,dF = -\frac{\rho}{c_2} + c_3.$$

In terms of the original variables, the solution is

$$c = F(\rho), \qquad p = 1 - c_1 + \frac{c_2}{F}, \qquad \rho = x + t.$$

Case 1.4. $\Gamma_3 + \Gamma_4$

$$F = \frac{c_2}{c_1 + \rho} \quad\text{with } G = \int\frac{c_1 + \rho}{K}\,d\rho + c_3$$

which gives an integral purely in terms of ρ once the form of $K(F)$ has been selected. In terms of the original variables, the solution is

$$c = \frac{c_2}{c_1 + x}, \qquad p = G(\rho) + t, \qquad \rho = x.$$

Case 2. $K(c) = c^m$.

Case 2.1. $\Gamma_1 + \alpha\Gamma_3 + \Gamma_4 + \beta\Gamma_5, 2 - \beta m \neq 0$

$$G' = \frac{c_1}{\rho^{\beta m}F^m}\exp\left[\int\frac{F^m}{\rho^{\beta m - 1}}\,d\rho\right]$$

with

$$\left[\beta\rho^{2\beta m}F^{2m} - \rho^{\beta m + 2}F^m\right]F'' + \left[(1-\beta)\rho^{2\beta m}F^{2m-1} - (m-2)\rho^{\beta m + 2}F^{m-1}\right](F')^2 + \left[2\beta\rho^{2\beta m - 1}F^{2m} - \beta(m-1)\rho^{\beta m + 1}F^m - \rho^3\right]F' = 0.$$

Case 2.2. $\Gamma_1 + \alpha\Gamma_3 + \Gamma_4 + \beta\Gamma_5$, $2 - \beta m = 0$

$$G' = \frac{c_1}{\rho^2 F^m} \exp\left[\int \frac{1}{\rho F^m}\, d\rho\right]$$

with

$$\left[2F^{2m} + mF^m\right]\rho F'' + (m-2)\left[F^{2m-1} + mF^{m-1}\right]\rho (F')^2 + \left[4F^{2m} + 2(m-1)F^m - m\right]F' = 0$$

which can be rearranged to give

$$-\frac{1}{2}\ln\rho + c_1 = \int \frac{F^{m-2}[F^m + \frac{m}{2}]^{-\frac{(m-2)}{2m}}\, dF}{2\int F^{-2}(F^m - 1)[F^m + \frac{m}{2}]^{-\frac{(m-2)}{2m}}\, dF + c_2} \tag{9}$$

with the imposed condition

$$m\rho F^m F' + 2F^{m+1} + mF \neq 0. \tag{10}$$

Choosing the value $m = 2$ reduces Eq. (9) to

$$-\frac{1}{2}\ln\rho + c_1 = \frac{1}{2}\int \frac{F\, dF}{F^2 + \frac{1}{2}c_2 F + 1}$$

and hence F may be found in closed form for the following cases for c_2:

$$-\ln\rho + c_1 = \ln(F+1) + \frac{1}{F+1}, \quad c_2 = 4,$$

$$-\ln\rho + c_1 = \ln(F-1) - \frac{1}{F-1}, \quad c_2 = -4,$$

$$-\ln\rho + c_1 = \frac{1}{2}\ln\left[F^2 + c_2F + 1\right] - \frac{c_2}{\sqrt{16 - c_2^2}}\tan^{-1}\left(\frac{2F + c_2}{\sqrt{16 - c_2^2}}\right), \quad |c_2| < 4\ (c_2 \neq 0),$$

$$-\ln\rho + c_1 = \frac{1}{2}\ln\left(F^2 + c_2F + 1\right) - \frac{c_2}{2\sqrt{c_2^2 - 16}}\ln\left[\frac{4F + c_2 - \sqrt{c_2^2 - 16}}{4F + c_2 + \sqrt{c_2^2 - 16}}\right], \quad |c_2| > 4. \tag{11}$$

Choosing the value $m = 1$ reduces Eq. (9) to

$$-3\ln\rho + c_1 = \ln\left[4\left(F + \frac{1}{2}\right)^{3/2} + c_2F\right] - \int \frac{c_2\, dF}{4(F + \frac{1}{2})^{3/2} + c_2F}.$$

We note that the case $c_2 = 0$ violates condition (10), hence we require $c_2 \neq 0$. Choosing $c_2 = -8$ leads to the closed form solution

$$-3\ln\rho + c_1 = \ln\left[4\left(F + \frac{1}{2}\right)^{3/2} - 8F\right] + \ln\left[\frac{\left(F - \frac{1}{2} - \sqrt{F + \frac{1}{2}}\right)^2}{\left(\sqrt{F + \frac{1}{2}} - 1\right)^4}\right] - \frac{2}{\sqrt{5}}\ln\left[\frac{2\sqrt{F + \frac{1}{2}} - 1 - \sqrt{5}}{\sqrt{F + \frac{1}{2}} - 1 + \sqrt{5}}\right].$$

Other choices of c_2 would lead to similar types of implicit solutions for F.

Case 2.3. $\Gamma_1 + \alpha\Gamma_3 + \beta\Gamma_5$, $2 - \beta m \neq 0$

$$G' = \frac{c_1}{\rho^{\beta m} F^m} \exp\left[\int \frac{1}{(\beta m - 2)\rho^{\beta m - 1} F^m}\, d\rho\right]$$

with

$$\begin{aligned}
&\left[-\beta(\beta m - 2)^2 \rho^{2\beta m} F^{2m} - (\beta m - 2)\rho^{\beta m + 2} F^m\right]F'' \\
&\quad + \left[(\beta m - 2)^2(\beta - 1)\rho^{2\beta m} F^{2m-1} - (\beta m - 2)(m-2)\rho^{\beta m + 2} F^{m-1}\right](F')^2 \\
&\quad + \left[-2\beta(\beta m - 2)^2 \rho^{2\beta m - 1} F^{2m} - \beta(\beta m - 2)(m-1)\rho^{\beta m + 1} F^m + \rho^3\right]F' = 0.
\end{aligned}$$

Case 2.4. $\Gamma_2 + \alpha\Gamma_3 + \Gamma_5$

$$G = -c_1 - \alpha e^{\int \frac{1}{mF^m}\,d\rho} \int \frac{\rho}{mF^m} e^{-\int \frac{1}{mF^m}\,d\rho}\,d\rho + c_2 e^{\int \frac{1}{mF^m}\,d\rho}$$

or

$$G' = \frac{F' + (\alpha - 1)F}{mF^m F' + F}$$

with

$$\big[-m^2(\alpha-1)F^{2m} + mF^m\big]F'' + \big[m^2(2\alpha-1)F^{2m-1} + m(m-2)F^{m-1}\big](F')^2 + \big[m(\alpha m + \alpha - m + 1)F^m - 1\big]F' + F = 0.$$

Case 2.5. $\alpha\Gamma_3 + \Gamma_5$

$$F = c_2(\alpha\rho - c_1)^{\frac{1-\alpha}{\alpha}}, \quad G = -\frac{(\alpha\rho - c_1)^{2+m-m/\alpha}}{m\alpha c_2^m (2 + m - \frac{m}{\alpha})} + c_3, \quad 2 + m - \frac{m}{\alpha} \neq 0,$$

or

$$F = c_2(\alpha\rho - c_1)^{\frac{1-\alpha}{\alpha}}, \quad G = \frac{\alpha - 1}{2\alpha^2 c_2^{2\alpha/(1-\alpha)}} \ln(\alpha\rho - c_1) + c_3, \quad 2 + m - \frac{m}{\alpha} = 0.$$

In terms of the original variables, the solutions are

$$c = c_2 t^{-1/m}(\alpha x - c_1)^{\frac{1-\alpha}{\alpha}}, \quad p = -\frac{(\alpha x - c_1)^{2+m-m/\alpha}}{m\alpha c_2^m (2 + m - \frac{m}{\alpha})} - \frac{\alpha}{m}\ln t + c_3, \quad 2 + m - \frac{m}{\alpha} \neq 0,$$

or

$$c = c_2 t^{-1/m}(\alpha x - c_1)^{\frac{1-\alpha}{\alpha}}, \quad p = \frac{\alpha - 1}{2\alpha^2 c_2^{2\alpha/(1-\alpha)}} \ln(\alpha x - c_1) - \frac{\alpha}{m}\ln t + c_3, \quad 2 + m - \frac{m}{\alpha} = 0.$$

Case 2.6. $\Gamma_2 + \alpha\Gamma_3 + \Gamma_4$

$$G = -\frac{c_2}{F} + c_1 + \alpha\rho + 1$$

with the closed form solution

$$\frac{\rho}{c_2} = \int \frac{F^m}{c_2 F - F^2 - \alpha F^{m+2}}\,dF$$

where m is arbitrary. In terms of the original variables, the solution is

$$c = F(\rho), \qquad p = G(\rho) + \alpha t, \qquad \rho = x - t.$$

For example, in the case $\alpha = 0$ and $m = 1$,

$$F = c_2 - c_3 e^{-\rho/c_2}, \qquad G = \frac{c_2}{c_3 e^{-\rho/c_2} - c_2} + c_1 + 1.$$

In terms of the original variables, the solutions are

$$c = c_2 - c_3 e^{-(x-t)/c_2}, \qquad p = \frac{c_2}{c_3 e^{-(x-t)/c_2} - c_2} + c_1 + 1. \tag{12}$$

In the case $\alpha \neq 0$ and $m = 1$,

$$F = \frac{1}{2\alpha}\left[\sqrt{4c_2\alpha + 1}\tanh\left(\frac{\sqrt{4c_2\alpha + 1}(\rho + c_2 c_3)}{2c_2}\right) - 1\right]$$

and corresponding forms for G, c and p can be found. Note that closed form solutions may be found for other values of m.

Case 2.7. $\Gamma_3 + \Gamma_4$

$$F = \frac{c_2}{\rho + c_1}, \quad G = \frac{(\rho + c_1)^{m+2}}{(m+2)c_2^m} + c_3, \quad m + 2 \neq 0,$$

or

$$F = \frac{c_2}{\rho + c_1}, \quad G = c_2^2 \ln(\rho + c_1) + c_3, \quad m + 2 = 0.$$

In terms of the original variables, the solutions are

$$c = \frac{c_2}{x + c_1}, \quad p = \frac{(x + c_1)^{m+2}}{(m+2)c_2^m} + t + c_3, \quad m + 2 \neq 0,$$

or

$$c = \frac{c_2}{x + c_1}, \quad p = c_2^2 \ln(x + c_1) + t + c_3, \quad m + 2 = 0.$$

Case 3. $K(c) = 1$.

Case 3.1. $\Gamma_1 + \alpha\Gamma_3 + \beta\Gamma_5$

$$G' = \frac{\alpha}{2} e^{-\frac{\rho^2}{4}} \int e^{\frac{\rho^2}{4}} \, d\rho + c_1 e^{-\frac{\rho^2}{4}}, \qquad F = c_3 \exp\left[\int \frac{\beta + \rho G' - \alpha}{2G' + \rho} \, d\rho\right].$$

Since α and β are completely arbitrary, we can choose $\alpha = 0$ and hence obtain

$$G = c_1 \sqrt{\pi} \operatorname{erf}\left(\frac{\rho}{2}\right) + c_2$$

giving the expected error function solution for the linear form of (6) where c_1 and c_2 are arbitrary constants. Hence the expression for F takes the form

$$F = c_3 \exp\left[\int \frac{\beta + c_1 \rho e^{-\rho^2/4}}{2c_1 e^{-\rho^2/4} + \rho} \, d\rho\right].$$

This may be integrated when the value $\beta = -1$ is chosen to give

$$F = \frac{c_3}{2c_1 e^{-\rho^2/4} + \rho}.$$

In terms of the original variables, the solution is

$$c = \frac{c_3}{2c_1 \sqrt{t} e^{-x^2/4t} + x}, \qquad p = c_1 \sqrt{\pi} \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right) + c_2.$$

Case 3.2. $\Gamma_2 + \alpha\Gamma_3 + \Gamma_4 + \beta\Gamma_5$

$$G = \alpha\rho + c_1 e^{-\rho} + c_2, \quad F = \frac{c_3 e^{\rho}}{[(\alpha + 1)e^{\rho} - c_1]^{\frac{\alpha - \beta + 1}{\alpha + 1}}}, \quad \alpha + 1 \neq 0,$$

or

$$G = \alpha\rho + c_1 e^{-\rho} + c_2, \quad F = c_3 \exp\left[-\frac{\beta}{c_1} e^{\rho} + \rho\right], \quad \alpha + 1 = 0.$$

In terms of the original variables, the solutions are

$$c = \frac{c_3 e^{x + \beta t - t}}{[(\alpha + 1)e^{x - t} - c_1]^{\frac{\alpha - \beta + 1}{\alpha + 1}}}, \quad p = \alpha x + c_1 e^{-x + t} + c_2, \quad \alpha + 1 \neq 0,$$

or

$$c = c_3 \exp\left[-\frac{\beta}{c_1} e^{x - t} + x + \beta t - t\right], \quad p = \alpha x + c_1 e^{-x + t} + c_2, \quad \alpha + 1 = 0.$$

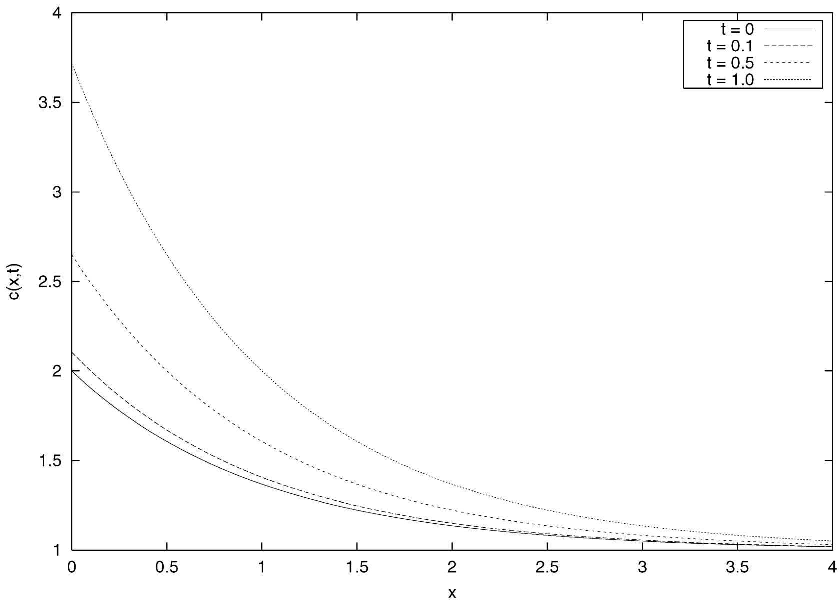

Fig. 1. Concentration $c(x,t)$ against distance x for $t = 0, 0.1, 0.5, 1.0$ for Case 2.6, with $m = 1$, $\alpha = 0$, $c_1 = c_2 = 1$, $c_3 = -1$.

Case 3.3. $\Gamma_2 + \alpha\Gamma_3 + \Gamma_5$

$$G = c_1, \qquad F = c_2 e^{\alpha t}.$$

In terms of the original variables, the solution is

$$c = c_2 e^{x+\alpha t}, \qquad p = c_1 + \alpha x.$$

Case 3.4. $\Gamma_3 + \Gamma_4 + \alpha\Gamma_5$

$$G = \frac{\rho^2}{2} + c_2\rho + c_1, \qquad F = c_3(\rho + c_2)^{\alpha-1}.$$

In terms of the original variables, the solution is

$$c = c_3 e^{\alpha t}(x + c_2)^{\alpha-1}, \qquad p = \frac{x^2}{2} + c_2 x + c_1 + t.$$

Case 3.5. $\Gamma_4 + \Gamma_5$

$$G = c_2\rho + c_1, \qquad F = c_3 e^{\rho/c_2}.$$

In terms of the original variables, the solution is

$$c = c_3 e^{t+x/c_2}, \qquad p = c_2 x + c_1.$$

A number of exact solutions have been constructed in this section. In particular, when $K(c) = c^m$ we have found many closed form solutions for arbitrary values of m. Explicit solutions (12) for the concentration and pressure are given in Case 2.6 with $m = 1$ and $\alpha = 0$. For the purposes of illustration, Figs. 1 and 2 give the concentration and pressure respectively for time $t = 0, 0.1, 0.5, 1.0$ with the constants $c_1 = c_2 = 1$ and $c_3 = -1$. We note that the concentration and pressure are functions of time t at the boundary and both steadily increase as time increases.

5. A moving boundary problem

Traditionally, similarity solutions are well-known to offer the prospect of obtaining exact solutions to moving boundary value problems. Here for a slightly artificial moving boundary problem, we illustrate our solutions with

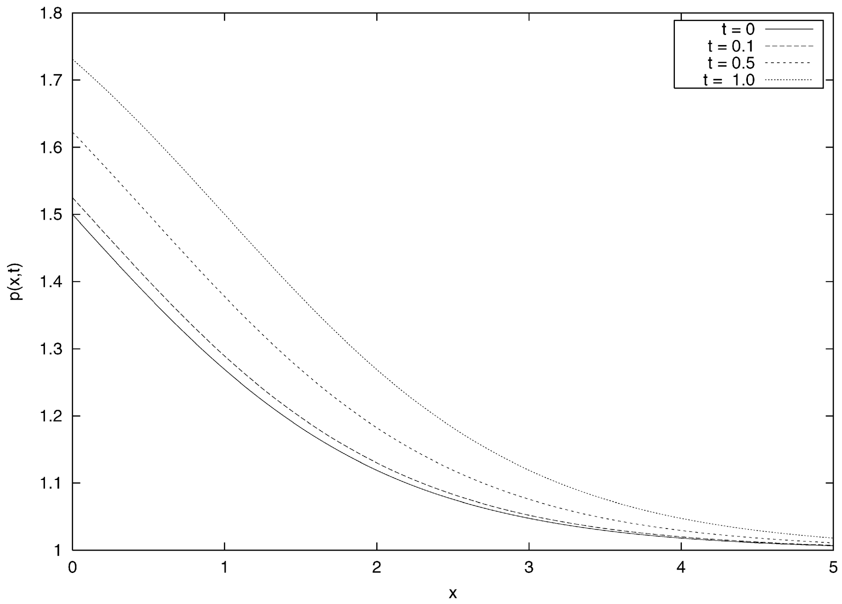

Fig. 2. Pressure $p(x,t)$ against distance x for $t = 0$, 0.1, 0.5, 1.0 for Case 2.6, with $m = 1$, $\alpha = 0$, $c_1 = c_2 = 1$, $c_3 = -1$.

reference to family 1 of Table 4 for the constant α non-zero and assuming constant permeability K. Specifically, we suppose that at $x = 0$ we inject fluid with constant gel concentration c_0 by means of a time dependent pressure ($t > 0$)

$$p(0,t) = \frac{\alpha}{2}\log t + p_0, \tag{13}$$

where α and p_0 are constants. For family 1 of Table 4 we have for $t > 0$

$$c(x,t) = F(\rho), \qquad p(x,t) = \frac{\alpha}{2}\log t + G(\rho), \qquad \rho = \frac{x}{\sqrt{t}},$$

so that at $x = 0$ we have the boundary conditions

$$F(0) = c_0, \qquad G(0) = p_0. \tag{14}$$

We further suppose that the moving concentration front $X(t) = \gamma\sqrt{t}$, moves into a region of zero gel concentration and pressure $(\alpha/2)\log t$, where γ is a constant to be determined. From Eq. (1) we may show that the velocity of the moving concentration front is given by

$$\frac{dX}{dt} = -K(c)\frac{\partial p}{\partial x}. \tag{15}$$

Thus, at the moving front we have the two conditions

$$F(\gamma) = 0, \quad \gamma = -2KG'(\gamma), \tag{16}$$

where throughout this section primes denote differentiation with respect to ρ.

Now for the case of constant permeability, we may deduce from Table 4 the two coupled differential equations

$$2KG'' + \rho G' = \alpha, \qquad F'(2KG' + \rho) = (\rho G' - \alpha)F, \tag{17}$$

the first of which may be integrated to yield

$$G'(\rho) = \frac{\alpha}{2K}e^{-\rho^2/4K}\int_0^{\rho} e^{\xi^2/4K}\,d\xi + Ce^{-\rho^2/4K}, \tag{18}$$

where C is the constant of integration. Thus the Stefan condition $(16)_2$ becomes

$$\gamma e^{\gamma^2/4K} + \alpha \int_0^\gamma e^{\xi^2/4K}\, d\xi = -2CK, \tag{19}$$

while from $(17)_2$ we have

$$F'(\rho) = \frac{(\rho G'(\rho) - \alpha)F(\rho)}{2KG'(\rho) + \rho}. \tag{20}$$

Now

$$2KG'(\rho) + \rho = \rho - \gamma e^{(\gamma^2-\rho^2)/4K} - \alpha e^{-\rho^2/4K} \int_\rho^\gamma e^{\xi^2/4K}\, d\xi, \tag{21}$$

which evidently vanishes at $\rho = \gamma$, as it must because of the Stefan condition $(16)_2$. However, since both numerator and denominator of (20) vanish for $\rho = \gamma$, it means that we may apply L'Hopital's rule to (20) to deduce that the constant α is given by

$$\alpha = -\frac{1}{2}\left(1 + \frac{\gamma^2}{K}\right), \tag{22}$$

and hence the requirement to assume α non-zero and the time dependent pressure (13) at $x = 0$.

From (21) we may deduce

$$G'(\rho) = -\frac{e^{-\rho^2/4K}}{2K}\left\{\gamma e^{\gamma^2/4K} + \alpha \int_\rho^\gamma e^{\xi^2/4K}\, d\xi\right\}, \tag{23}$$

and therefore from the boundary condition $(14)_2$ we have

$$G(\rho) = p_0 - \frac{1}{2K}\int_0^\rho e^{-\sigma^2/4K}\left\{\gamma e^{\gamma^2/4K} + \alpha \int_\sigma^\gamma e^{\xi^2/4K}\, d\xi\right\} d\sigma. \tag{24}$$

Further, from (20) we have

$$F(\rho) = c_0 \exp\left\{\int_0^\rho \frac{\sigma G'(\sigma) - \alpha}{2KG'(\sigma) + \sigma}\, d\sigma\right\}, \tag{25}$$

noting that the condition $F(\gamma) = 0$ is satisfied because the integrand in (25) is singular at $\sigma = \gamma$. But observe that

$$\frac{d}{d\sigma}\left\{2KG'(\sigma) + \sigma\right\} = 1 - \left\{\sigma G'(\sigma) - \alpha\right\},$$

so that if we set $H(\sigma) = 2KG'(\sigma) + \sigma$, we may simplify (25) to yield

$$F(\rho)H(\rho) = 2c_0 K G'(0) \exp\left\{\int_0^\rho \frac{d\sigma}{H(\sigma)}\right\}. \tag{26}$$

From this equation it is clear that the integral in (26) must tend to $-\infty$ in the limit ρ tending to γ. Now if at the moving boundary we prescribe the pressure $(\alpha/2)\log t$, then the transcendental equation for γ is obtained from (24), as

$$\int_0^1 e^{-\beta u^2}\left\{e^\beta + \alpha \int_u^1 e^{\beta v^2}\, dv\right\} du = \frac{p_0}{2\beta},$$

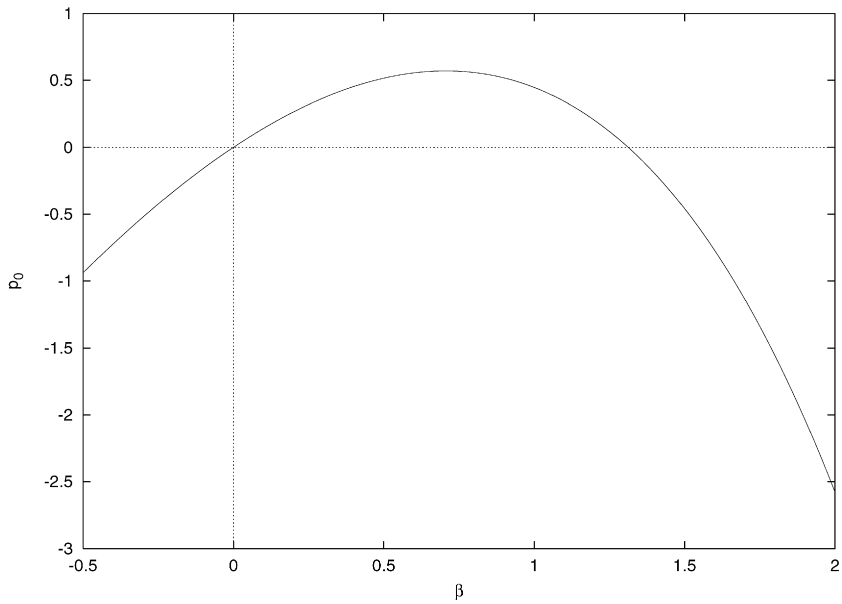

Fig. 3. Relationship between initial pressure p_0 and $\beta(\gamma)$.

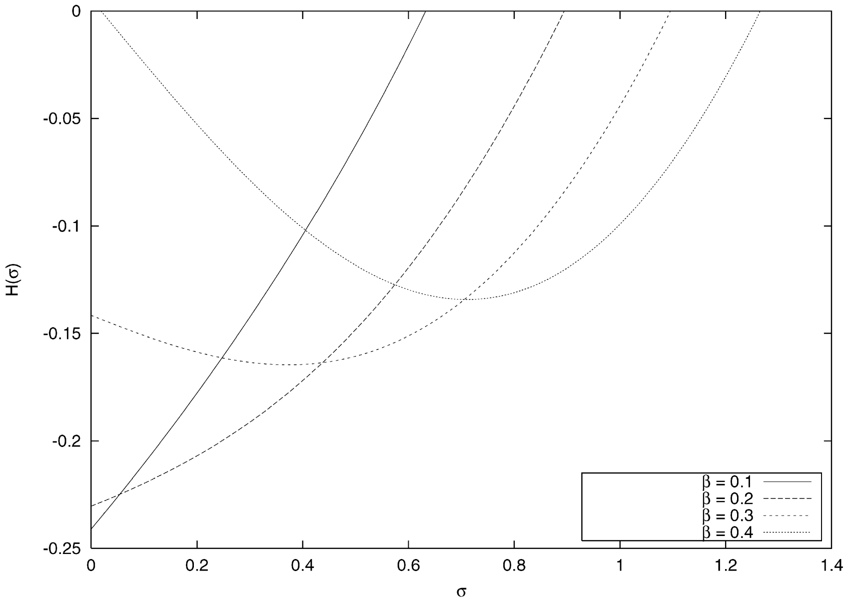

Fig. 4. $H(\sigma)$ against σ for $\beta = 0.1,\ 0.2,\ 0.3,\ 0.4$.

where we have used $\sigma = \gamma u$, $\xi = \gamma \upsilon$ and $\beta = \gamma^2/4K$. Notice that from (22) the final equation for the determination of $\beta\ (\gamma)$ becomes

$$\int_0^1 e^{-\beta u^2} \left\{ e^{\beta} - \frac{1}{2}(1+4\beta) \int_u^1 e^{\beta \upsilon^2}\, d\upsilon \right\} du = \frac{p_0}{2\beta}. \tag{27}$$

Figure 3 shows the relationship between p_0 and β. We expect that $\beta \geqslant 0$, but note that p_0 could be positive or negative. From the expression for $H(\sigma)$ we have $H(\gamma) = 0$, however, for values of $\beta > 0.4$ with $K = 1$ there is a second zero for $0 \leqslant \sigma < \gamma$ which leads to a singularity in $F(\rho)$. Figure 4 illustrates $H(\sigma)$ for $\beta = 0.1,\ 0.2,\ 0.3$

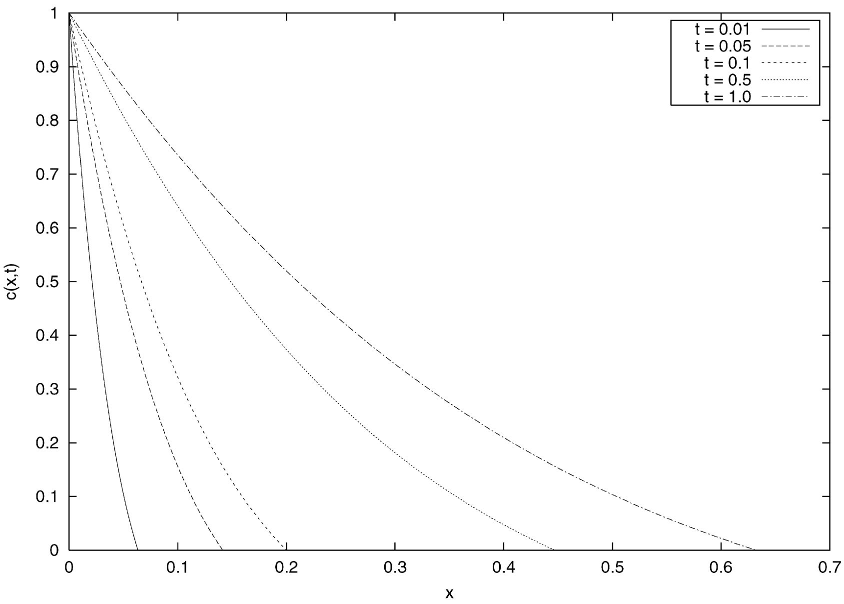

Fig. 5. Concentration $c(x,t)$ with $\beta = 0.1$ for $t = 0.01,\ 0.05,\ 0.1,\ 0.5,\ 1.0$.

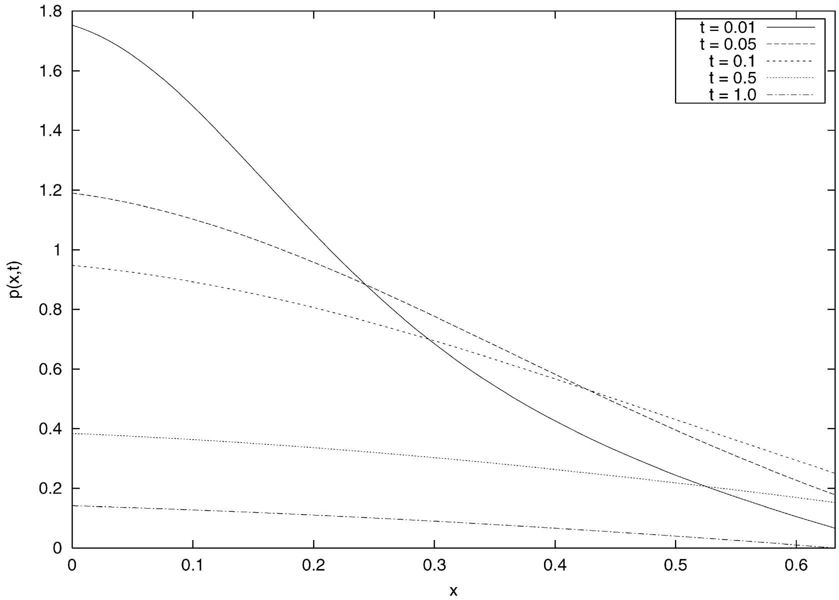

Fig. 6. Pressure $p(x,t)$ with $\beta = 0.1$ for $t = 0.01,\ 0.05,\ 0.1,\ 0.5,\ 1.0$.

and 0.4. The second positive zero for $H(\sigma)$ appears for $\beta \approx 0.3969$. Accordingly for the purposes of illustration, we choose $\beta = 0.1$ for the remainder of this discussion.

In terms of $c(x,t)$ and $p(x,t)$ for concentration and pressure, the corresponding solutions are given in Figs. 5 and 6 respectively for $t = 0.01, 0.05, 0.1, 0.5$ and 1.0 with $\beta = 0.1$ and $K = 1$ and initial concentration $c_0 = 1$. From Fig. 5, we see that the concentration front moves steadily to the right as time increases. The shown concentration profile is typical for values of β between 0 and 0.3969. Figure 6 shows the change in pressure for $t = 0.01, 0.05, 0.1, 0.5$ and 1.0 with $\beta = 0.1$ and $K = 1$, showing that the pressure at the boundary $x = 0$ steadily decreases as t increases.

6. Concluding remarks

A realistic formulation of problems involving contaminant migration in porous media has to address two aspects. The first relates to non-linearities that are present in the physical processes governing flow and transport in a porous medium and the second relates to the development of computational schemes that can determine the time- and position-dependent contaminant migration patterns in three-dimensional problems. The first aspect is a difficult problem in material characterisation that can only be addressed through discerning efforts in laboratory testing and field studies. The second usually involves the development of efficient and reliable computational schemes that can handle arbitrary types of non-linearities, and which can give unconditionally stable results for contaminant transport profiles when the governing partial differential equations are non-linear. Although efficient computational algorithms have been developed for the computational treatment of non-linear parabolic equations, there exist opportunities to improve the reliability of the overall computational approach. The choice of mesh discretisation and time-integration in computational approaches such as the finite element method is not universal and depends on a number of parameters influenced by the form of the non-linearity present. Analytical solutions that address the non-linearities in coupled parabolic partial differential equations therefore form useful bench-marking tools that can aid the advancement of computational schemes. This paper shows that the Lie group symmetry analysis of the one-dimensional problem with the flow field being influenced by the concentration, can give rise to a wide range of analytical solutions of the closed form type that can serve as a calibration exercise. Admittedly, not all the treated forms of non-linearities may have their practical applicability; nonetheless they will provide a framework for examining the reliability of computational schemes in terms of accuracy and stability. The present work also illustrates that these analytical solutions can be easily obtained through the judicious use of symbolic computational approaches currently available in the mathematics and scientific literature.

In this paper we have formulated a novel coupled pressure and concentration approach for modelling transport in porous media with variable transmissivity. For a wide variety of concentration permeability $K(c)$ we have determined the Lie group symmetries and we have tabulated the resulting coupled ordinary differential equations. One typical analytical solution is shown graphically. Finally, a specific moving boundary is formulated and shown to admit one of the analytical solutions previously derived. This solution is displayed graphically and shown to behave in a physically reasonable manner.

Acknowledgments

This work is partially funded by the Discovery Project scheme of the Australian Research Council, and the authors gratefully acknowledge this support. J.M.H. is grateful to the Australian Research Council for the provision of an Australian Professorial Fellowship.

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