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Boundary-value problem for density–velocity model of collective motion of organisms

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ABSTRACT

The collective motion of organisms is observed at almost all levels of biological systems. In this paper the density–velocity model of the collective motion of organisms is analyzed. This model consists of a system of nonlinear parabolic equations, a forced Burgers equation for velocity and a mass conservation equation for density. These equations are supplemented with the Neumann boundary conditions for the density and the Dirichlet boundary conditions for the velocity. The existence, uniqueness and regularity of solution for the density–velocity problem is proved in a bounded 1D domain. Moreover, *a priori* estimates for the solutions are established, and existence of an attractor is proved. Finally, some numerical approximations for asymptotical behavior of the density–velocity model are presented.

1. Introduction

The collective motion is a common phenomenon inherent to a variety of biological species at different spatial scales, from microscopic bacterial colonies [7,14], phytoplankton aggregations [2], insect swarms [6,15,17,23], to macroscopic fish schools [1,3,5,17,18,22,27], bird flocks [17], and others. The mutual separation distance and the alignment between the neighbors in a group of organisms are the main factors in their collective motion [16,18,22]. The analysis of these factors can be performed at different spatial scales using "Individual-based" and "Population-based" models [7]. Individual-based models [5,11,16,26] provide a useful tool for studying relatively small groups, but become impractical when the number of individuals approaches the sizes of real biological groups. On the other hand, population-based models [1–4,6,12–15, 23–25] are particularly useful when the number of individuals is large. They can be regarded as the continuum limit of individual-based models.

In population-based models a spatially distributed population is often described by the conservation mass equation in the following form [17]

$$\rho_t + \operatorname{div}(\rho u) = \mu \Delta \rho, \tag{1.1}$$

where $\rho = \rho(t, x)$ is the density of particles at time t and position x, and $u = u(t, x)$ is the average velocity. Assuming a Newtonian law for the motion of individuals and using a Lagrangian framework, the equation for velocity becomes forced Burgers equation [3]

$$\frac{du}{dt} = u_t + (u \cdot \nabla)u = \nu \Delta u + F \tag{1.2}$$

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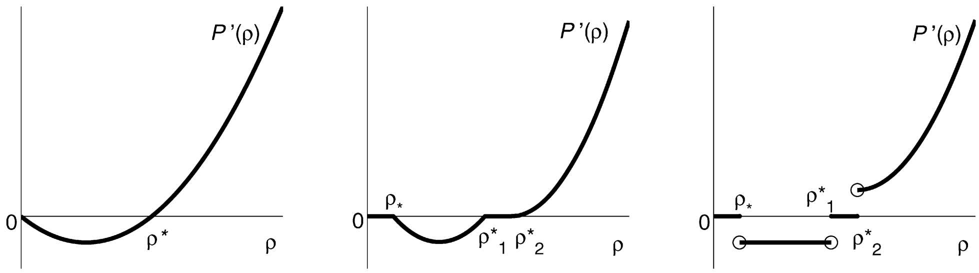

Fig. 1. Examples for the function $P'(\rho)$.

with the force F composed of the internal force, $-P'(\rho)\nabla\rho$, representing the interaction between particles or fish and the external force, f, incorporating environmental effects, such as temperature gradient, chemical gradient, as well as food and predator densities

$$F = -P'(\rho)\nabla\rho + f. \tag{1.3}$$

The central idea in the modelling of group formations in the collective motion of organisms is to model the internal force as a "*pressure*" resulting from the behavior of individuals trying to achieve and maintain a prescribed level of density. In such case the internal force $-P'(\rho)\nabla\rho$ acts in the direction of the density gradient $\nabla\rho$ or in the opposite direction depending on the local density. When the density of organisms is larger than some prescribed density, the distance between nearest neighbors is too small, and the organisms tend to repulse from each other. In this case the internal force acts in the direction opposite to the density gradient. On the other hand, when the density of organisms is smaller than the prescribed density, then the organisms attract to each other, and the direction of the internal force is the same as the gradient of the density [3,22,25]. This behavior of the system of organisms defines the intervals of positivity and negativity for the function $P'(\rho)$ as shown in Fig. 1, see also [3]. Clearly, this function is positive for very large densities and negative for the density levels smaller than the prescribed density. Fig. 1 shows that the prescribed density may be equal to either one density value ρ^*, or to any density value from the interval $[\rho_1^*, \rho_2^*]$. When the density of organisms is very low, the distance between nearest neighbors is too high to sense each other, so in this case the function $P'(\rho)$ takes on the values that are either very small, or zero.

The first steps in theoretical analysis of the one-dimensional density–velocity model governed by Eqs. (1.1)–(1.3) were undertaken in [3]. In particular, it was shown the existence of global attractor, and demonstrated via numerical simulations that the shape of the density function asymptotically converges to density patterns that correspond well to observed shapes of groups of organisms such as fish, birds and insects.

In this paper theoretical study of velocity–density model (1.1)–(1.3) for collective motion of organisms is continued. The following initial–boundary value problem for this density–velocity model in 1D is considered:

$$\begin{cases} \rho_t + (\rho u)_x = \mu\rho_{xx}, \qquad u_t + u\cdot u_x + \big(P(\rho)\big)_x = \nu u_{xx} + f, \quad (t,x)\in G^T = (0,T]\times\Omega, \\ \rho_x(t,0) = \rho_x(t,L) = 0, \qquad u(t,0) = u(t,L) = 0, \quad t\in(0,T], \\ \rho(0,x) = \rho_0(x), \qquad u(0,x) = u_0(x), \quad x\in\bar\Omega. \end{cases} \tag{1.4}$$

Here $\Omega = (0,L)$, $L>0$, the unknowns $\rho = \rho(t,x)$ and $u = u(t,x)$ are functions of the time $t\in[0,T]$ and position $x\in\bar\Omega$, ν and μ are positive constants, and functions P and f are given and defined on $[0,+\infty)$ and G^T, respectively.

The results of this study are organized into the following sections. In Section 2, some preliminary results and definitions are presented for Problem (1.4). In particular, Hilbert spaces H and V are introduced, and the variational formulation of the boundary-value problem for the density–velocity system of collective motion of organisms is presented. The uniqueness theorem is proven in Section 3. Using *a priori* estimates for Problem (1.4) proven in Section 4, the existence theorem is proved in Section 5 by applying the Faedo–Galerkin method. In Section 6 the regularity of solutions for Problem (1.4) are proved.e Further estimates of solutions and nonnegativity of density are established in Section 7. Finally, in Section 8 the theorem about attractors is proved and some numerical approximations for attractors are presented.

2. Variational formulation of the problem

The goal of this section is the variational formulation of Problem (1.4). To this aim we will introduce the functional spaces to study this problem in Section 1.1, examine the nonlinear forms generated by the problem in Section 1.2, which will finally lead to the variational formulation of the problem in Section 1.3.

2.1. Functional spaces and notations

We use $L^p(\Omega)$ to denote the scalar L^p space on $\Omega = (0,L)\subset\mathbb{R}^1$, $0<L<+\infty$, $1\leqslant p\leqslant\infty$, supplemented with the norm $\|\cdot\|_p$. By $\|\cdot\|$ and $(\cdot,\cdot)$ we denote the standard norm and scalar product in the space $L^2(\Omega)$.

Let $\mathcal{V}_1 = C^\infty(\Omega)$, $\mathcal{V}_2 = C_0^\infty(\Omega)$ and $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$ be the spaces (without topology). We consider the spaces $H = H_1 \times H_2$ and $V = V_1 \times V_2$, where $H_1 = H_2 = L^2(\Omega)$, $V_1 = H^1(\Omega)$ and $V_2 = H_0^1(\Omega)$. The space H is endowed with the norm $\|\cdot\|_H = (\|\cdot\|_{H_1}^2 + \theta\|\cdot\|_{H_2}^2)^{1/2}$, and with the scalar product $(\cdot,\cdot)_H = (\cdot,\cdot)_{H_1} + \theta(\cdot,\cdot)_{H_2}$, where $\theta > 0$, $\|\cdot\|_{H_i} = \|\cdot\|$ and $(\cdot,\cdot)_{H_i} = (\cdot,\cdot)$, $i = 1, 2$. The space V is endowed with the norm $\|\cdot\|_V = (\|\cdot\|_{V_1}^2 + \theta\|\cdot\|_{V_2}^2)^{1/2}$, and with the scalar product $(\cdot,\cdot)_V = (\cdot,\cdot)_{V_1} + \theta(\cdot,\cdot)_{V_2}$, where $\|\cdot\|_{V_i} = \|\cdot\|_{H^1(\Omega)}$, and $(\cdot,\cdot)_{V_i} = (\cdot,\cdot)_{H^1(\Omega)}$, $i = 1, 2$.

Note that the space V is compactly imbedded into the space H. Moreover, the spaces H_1, H_2, H and V_1, V_2 and V are the closures of $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}$ with respect to the corresponding norms.

2.2. Analysis of the nonlinear forms

In this section, the nonlinear forms appearing in the Problem (1.4) will be introduced and examined, that is, two bilinear forms a_1 and a_2, two trilinear forms b and d, and nonlinear form c. The properties of nonlinear forms will be used in the subsequent sections for the variational formulation of the problem, uniqueness and existence theorems. Here we denote by $c_1, c_2, \ldots$ the constants that independent of the functions u, v, w, ρ and ϕ.

2.2.1. Bilinear forms a_1 and a_2

Let us define two bilinear forms $a_i : \mathcal{V}_i \times \mathcal{V}_i \to \mathbb{R}$, $i = 1, 2$, as

$$a_i(v, w) = (\partial_x v, \partial_x w), \quad v, w \in \mathcal{V}_i. \tag{2.1}$$

The forms a_1 and a_2 are bilinear and continuous on $V_1 \times V_1$ and $V_2 \times V_2$, respectively [9,19]. Moreover, the form a_2 is V_2-coercive, and the form a_1 is V_1-coercive with respect to H_1, see [9]. Note that for each fixed $v \in V_i$ the map $w \mapsto a_i(v, w)$ is a linear functional on V_i, $i = 1, 2$. It follows that we can define two linear operators $A_1 : V_1 \to V_1'$ and $A_2 : V_2 \to V_2'$, such that

$$\langle A_i v, w\rangle_{V_i} = a_i(v, w), \quad v \in V_i,\ \forall w \in V_i,\ i = 1, 2. \tag{2.2}$$

Since the bilinear forms a_1 and a_2 are also defined and continuous on $H^2(\Omega) \times L^2(\Omega)$, the operators A_1 and A_2 are linear and continuous if they are defined from $H^2(\Omega)$ to $L^2(\Omega)$. Furthermore, the following lemma is true

Lemma 2.1.

(i) *If $u \in L^2(0, T; V_i)$, then $A_i u \in L^2(0, T; V_i')$, $i = 1, 2$.*
(ii) *If $v \in L^2(0, T; H^2(\Omega))$, then $A_i v \in L^2(0, T; L^2(\Omega))$, $i = 1, 2$.*

2.2.2. Trilinear forms b and d

Dealing with the nonlinear terms in Problem (1.4) in a similar way to (2.1), we define two trilinear forms $b : \mathcal{V}_2 \times \mathcal{V}_2 \times \mathcal{V}_2 \to \mathbb{R}$ and $d : \mathcal{V}_1 \times \mathcal{V}_2 \times \mathcal{V}_1 \to \mathbb{R}$ as follows

$$b(u, v, w) = \int_0^L u(\partial_x v) w\, dx, \tag{2.3}$$

$$d(\rho, u, \phi) = \int_0^L \partial_x(\rho u)\phi\, dx. \tag{2.4}$$

Let u, v and w belong to V_2, then u, $w \in L^4(\Omega)$, $\partial_x v \in L^2(\Omega)$, and by Hölder inequality $u(\partial_x v)w \in L^1(\Omega)$. Therefore, the form b is defined on $V_2 \times V_2 \times V_2$. Similarly, we can show that the form d is defined on $V_1 \times V_2 \times V_1$. The continuity of the trilinear forms b and d in these spaces follows from the inequalities

$$\bigl|b(u, v, w)\bigr| \leqslant \|u\|_\infty \|\partial_x v\| \|w\| \leqslant \sqrt{c^A}\|u\|^{1/2}\|\partial_x u\|^{1/2}\|\partial_x v\|\|w\|, \tag{2.5}$$

$$\begin{aligned}\bigl|d(\rho, u, \phi)\bigr| &\leqslant \|\partial_x u\|\|\rho\phi\| + \|\partial_x \rho\|\|u\phi\| \leqslant \|\phi\|\bigl(\|\partial_x u\|\|\rho\|_\infty + \|\partial_x\rho\|\|u\|_\infty\bigr)\\ &\leqslant \sqrt{c^A}\|\phi\|\bigl(\|\partial_x u\|\|\partial_x\rho\|^{1/2}\|\rho\|^{1/2} + \|\partial_x\rho\|\|\partial_x u\|^{1/2}\|u\|^{1/2}\bigr),\end{aligned} \tag{2.6}$$

where c^A is the constant from the Agmon inequality: $\|v\|_\infty^2 \leqslant c^A\|v\|\|\partial_x v\|$. Using integration by parts in the form b, we can easily show that

$$b(u, u, v) = b(v, u, u) = -\frac{1}{2} b(u, v, u), \quad \forall u, v \in V_2, \tag{2.7}$$

therefore, from (2.5) and (2.7) we infer that

$$b(u,u,u)=0, \quad \forall u \in V_2, \tag{2.8}$$

$$\left|b(u,u,v)\right| \leqslant \frac{\sqrt{c^A}}{2}\|u\|^{3/2}\|\partial_x u\|^{1/2}\|\partial_x v\|, \quad \forall u,v \in V_2. \tag{2.9}$$

Also, using integration by parts and the Agmon inequality we get

$$\left|d(\rho,u,\phi)\right| \leqslant \|\rho u\|\|\partial_x\phi\| \leqslant \begin{cases} \sqrt{c^A}\|\rho\|^{1/2}\|\partial_x\rho\|^{1/2}\|u\|\|\partial_x\phi\|, \\ \sqrt{c^A}\|u\|^{1/2}\|\partial_x u\|^{1/2}\|\rho\|\|\partial_x\phi\|, \end{cases} \tag{2.10}$$

$$d(\rho,u,\rho)=\frac{1}{2}\int_0^L \rho^2\partial_x u\,dx, \quad \forall \rho \in V_1,\ u \in V_2. \tag{2.11}$$

Let us denote by $B(u,v)$ the linear continuous form on V_2 and by $D(\rho,u)$ the linear continuous form on V_1 defined by

$$\left\langle B(u,v),w\right\rangle_{V_2}=b(u,v,w), \quad u,v \in V_2,\ \forall w \in V_2, \tag{2.12}$$

$$\left\langle D(\rho,u),\phi\right\rangle_{V_1}=d(\rho,u,\phi), \quad (\rho,u) \in V,\ \forall \phi \in V_1. \tag{2.13}$$

For $u=v$, we write

$$B(u)=B(u,u), \quad u \in V_2. \tag{2.14}$$

Finally, using inequalities (2.5), (2.6), (2.9) and (2.10), we can prove the following, see also [20].

Lemma 2.2.

(i) *If $u \in L^2(0,T;V_2)$, then $B(u) \in L^1(0,T;V_2')$.*
(ii) *If $u \in L^2(0,T;V_2) \cap L^\infty(0,T,H_2)$, then $B(u) \in L^4(0,T;V_2')$.*
(iii) *If $u \in L^\infty(0,T;V_2)$, then $B(u) \in L^2(0,T;H_2)$.*
(iv) *If $(\rho,u) \in L^2(0,T;V)$, then $D(\rho,u) \in L^1(0,T;V_1')$.*
(v) *If $(\rho,u) \in L^2(0,T;V) \cap L^\infty(0,T,H)$, then $D(\rho,u) \in L^4(0,T;V_1')$.*
(vi) *If $(\rho,u) \in L^\infty(0,T;V)$, then $D(\rho,u) \in L^2(0,T;H_1)$.*

Proof. If $u \in L^2(0,T;V_2)$, then for almost all t, $B(u(t))$ is an element of V_2' and the function $t \in [0,T] \to B(u(t)) \in V_2'$ is a measurable. Then from (2.9) we get (i) and (ii) by showing that the integrals $\int_0^T \|B(u(t))\|_{V_2'}\,dt$ and $\int_0^T \|B(u(t))\|_{V_2'}^4\,dt$ are finite.

If $u \in L^\infty(0,T;V_2)$, then for almost all t, $B(u(t))$ is an element of H_2' and the function $t \in [0,T] \to B(u(t)) \in H_2' \equiv H_2$ is a measurable. From (2.5) we obtain that the integral $\int_0^T \|B(u(t))\|^2\,dt$ is finite, which implies (iii).

Similarly, if $(\rho,u) \in L^2(0,T;V)$ then for almost all t, $D(\rho(t),u(t))$ is an element of V_1'. The function $t \in [0,T] \to D(\rho(t),u(t)) \in V_1'$ is a measurable. From (2.10) we get (iv) and (v) by showing that the integrals $\int_0^T \|D(\rho(t),u(t))\|_{V_1'}\,dt$ and $\int_0^T \|D(\rho(t),u(t))\|_{V_1'}^4\,dt$ are finite.

If $(\rho,u) \in L^\infty(0,T;V)$, then for almost all t, $D(\rho(t),u(t))$ is an element of H_1', and the function $t \in [0,T] \to D(\rho(t),u(t)) \in H_1' \equiv H_1$ is a measurable. From (2.6) we obtain that the integral $\int_0^T \|D(\rho(t),u(t))\|^2\,dt$ is finite, and, therefore, (vi) holds. □

2.2.3. Nonlinear form e

For each ρ from $\mathcal{V}_1$ and w from $\mathcal{V}_2$ we can define the nonlinear form

$$e(\rho,w)=-\int_0^L P(\rho)\cdot\partial_x w\,dx. \tag{2.15}$$

Let us state the following assumptions for the function $P(\rho)$:

($\mathcal{P}_1^0$) $\quad \left|P(\rho)\right| \leqslant p_1\left(|\rho|^m+1\right), \quad m \geqslant 0.$

($\mathcal{P}_2^0$) $\quad \left|P(\rho)-P(\phi)\right| \leqslant p_2\left(|\rho|^l+|\phi|^l+1\right)|\rho-\phi|, \quad l \geqslant 0.$

($\mathcal{P}_3^0$) $\quad \exists P'(\rho)$ such that $\left|P'(\rho)\right| \leqslant p_3\left(|\rho|^k+1\right), \quad k \geqslant 0.$

Clearly, Hypothesis $(\mathcal{P}_1^0)$ follows from Hypothesis $(\mathcal{P}_2^0)$ with $l \leqslant m-1$, and Hypothesis $(\mathcal{P}_2^0)$ follows from Hypothesis $(\mathcal{P}_3^0)$ with $k \leqslant l$.

If Hypothesis $(\mathcal{P}_1^0)$ holds, then using the Hölder and Agmon inequalities, we get

$$\left|e(\rho, w)\right| \leqslant \|\partial_x w\| \left\|P(\rho)\right\| \leqslant c_1 \|\partial_x w\| \left(\|\rho\|^{m_1} \|\partial_x \rho\|^{m_2} + 1\right) \tag{2.16}$$

with $m_1 = \max\{1, (m+1)/2\}$, $m_2 = \max\{0, (m-1)/2\}$. Therefore, since the form e is linear with respect to the second argument, inequality (2.16) ensures the continuity of e in the second argument on V_2. Moreover, if Hypothesis $(\mathcal{P}_2^0)$ holds, then the continuity of e in the first argument on V_1 follows from inequalities

$$\begin{aligned}\left|e(\rho, w) - e(\phi, w)\right| &\leqslant \|\partial_x w\| \left\|P(\rho) - P(\phi)\right\| \\ &\leqslant c_2 \|\partial_x w\| \left\|\partial_x(\rho - \phi)\right\|^{1/2} \|\rho - \phi\|^{1/2} \left(\|\rho\|^{l_1} \|\partial_x \rho\|^{l_2} + \|\phi\|^{l_1} \|\partial_x \phi\|^{l_2} + 1\right)\end{aligned} \tag{2.17}$$

with $l_1 = \max\{1, (l+1)/2\}$, $l_2 = \max\{0, (l-1)/2\}$.

If Hypothesis $(\mathcal{P}_3^0)$ is satisfied then for all $\rho \in V_1$ and $w \in V_2$, using integration by parts, we get

$$e(\rho, w) = \int_0^L P'(\rho) \partial_x \rho \cdot w \, dx, \tag{2.18}$$

$$\left|e(\rho, w)\right| \leqslant \|w\| \|\partial_x \rho\| \left\|P'(\rho)\right\|_\infty \leqslant c_3 \|w\| \|\partial_x \rho\| \left(\|\partial_x \rho\|^{\frac{k}{2}} \|\rho\|^{\frac{k}{2}} + 1\right). \tag{2.19}$$

Denote by $E(\rho)$ the linear continuous form on V_2 for $\rho \in V_1$ defined by

$$\left\langle E(\rho), w\right\rangle_{V_2} = e(\rho, w), \quad \rho \in V_1, \ \forall w \in V_2,$$

then the following results hold.

Lemma 2.3.

(i) *Let Hypothesis* $(\mathcal{P}_1^0)$ *hold with* $0 \leqslant m \leqslant 2$. *If* $\rho \in L^2(0, T; V_1)$, *then* $E(\rho) \in L^1(0, T; V_2')$.
(ii) *Let Hypothesis* $(\mathcal{P}_1^0)$ *hold with* $0 \leqslant m \leqslant 3$. *If* $\rho \in L^2(0, T; V_1) \cap L^\infty(0, T, H_1)$, *then* $E(\rho) \in L^2(0, T; V_2')$.
(iii) *Let Hypothesis* $(\mathcal{P}_3^0)$ *hold. If* $\rho \in L^\infty(0, T; V_1)$, *then* $E(\rho) \in L^2(0, T; H_2)$.

Proof. If ρ belongs to $L^2(0, T; V_1)$, then for almost all t, $E(\rho(t))$ is an element of V_2' and the function $t \in [0, T] \to E(\rho(t)) \in V_2'$ is a measurable. Statement (i) of the lemma follows from inequality (2.16), in view of

$$\int_0^T \left\|E\big(\rho(t)\big)\right\|_{V_2'} dt \leqslant \int_0^T c_1 \left(\left\|\rho(t)\right\|_{V_1}^2 + 1\right) dt < +\infty, \quad 1 \leqslant m \leqslant 2.$$

A similar argument using inequality (2.16) gives (ii) for $1 \leqslant m \leqslant 3$.

If $\rho \in L^\infty(0, T; V_1)$, then for almost all t, $E(\rho(t))$ is an element of H_2' and the function $t \in [0, T] \to E(\rho(t)) \in H_2' \equiv H_2$ is a measurable. Then from inequality (2.19), we get (iii) for $k \geqslant 1$. □

2.3. Variational formulation of the problem

Let us give the weak formulation of Problem (1.4). Clearly, if (ρ, u) is a classical solution of Problem (1.4), say $\rho, u \in C_{t,x}^{1,2}(\bar{G}^T)$, and (ϕ, w) denotes any element of $\mathcal{V}$, then

$$\left(\frac{\partial \rho}{\partial t}, \phi\right)_{H_1} = -\mu \cdot a_1(\rho, \phi) - d(\rho, u, \phi),$$

$$\left(\frac{\partial u}{\partial t}, w\right)_{H_2} = -\nu \cdot a_2(u, w) - b(u, u, w) - e(\rho, w) + (f, w)_{H_2}.$$

By continuity argument, these equalities hold also for each $\phi \in V_1$ and $w \in V_2$. We can identify H_i and H_i', then the following injections hold

$$V_i \subset H_i \equiv H_i' \subset V_i', \quad i = 1, 2,$$

where each space is dense in the following one and the injections are continuous. Consequently, the scalar product in H_i of $v \in H_i$ and $\psi \in V_i$ is the same as the scalar product of v and ψ in the duality between V_i' and V_i, $i = 1, 2$ [20]. Therefore, for each $v \in H_i$ and $\psi \in V_i$

$$(v, \psi)_{H_i} = \langle v, \psi\rangle_{V_i}, \qquad \left(\frac{\partial v}{\partial t}, \psi\right)_{H_i} = \frac{d}{dt}(v, \psi)_{H_i} = \frac{d}{dt}\langle v, \psi\rangle_{V_i}.$$

This suggests the following weak formulation of Problem (1.4): *For given* $f \in L^2(0,T;V_2')$ *and* $(\rho_0,u_0) \in H$, *to find* (ρ,u), *satisfying*

$$(\rho,u) \in L^2(0,T;V) \cap L^\infty(0,T;H),$$

and

$$\begin{cases} \rho' + \mu A_1\rho + D(\rho,u) = 0, \qquad u' + \nu A_2 u + B(u) + E(\rho) = f, \quad t \in (0,T); \\ \rho(0) = \rho_0, \qquad u(0) = u_0. \end{cases} \tag{2.20}$$

Let Hypothesis $(\mathcal{P}_1^0)$ hold with $0 \leqslant m \leqslant 3$. Then using Lemmas 2.1(i), 2.2(ii), (v) and 2.3(ii) we obtain from (2.20) that $\rho' \in L^2(0,T;V_1')$ and $u' \in L^2(0,T;V_2')$. Therefore, (ρ,u) is almost everywhere equal to continuous vector function from $[0,T]$ into H [20], see also [10], and the initial conditions in the weak formulation of the problem make sense.

Note also that the mass conservation can be obtained by choosing the test function $\phi = 1$ in the weak formulation of the problem, specifically,

$$\int_0^L \rho(t,x)\,dx = \int_0^L \rho_0(x)\,dx = \bar{\rho}, \quad \text{a.e.} \tag{2.21}$$

To analyze Problem (1.4) we will need the following hypotheses for P:

$(\mathcal{P}_1)$ $\quad \lim_{\rho\to+\infty} \frac{P(\rho)}{\rho^2} = P_\infty > 0$ and $\left|P(\rho) - \rho^2 P_\infty\right| \leqslant P_1|\rho| + P_0,$

$(\mathcal{P}_2)$ $\quad \left|P'(\rho)\right| \leqslant P_3\rho^2 + P_2.$

3. Uniqueness

In this section we shall prove the following uniqueness result for Problem (1.4).

Theorem 3.1 *(Uniqueness). Assume that Hypothesis* $(\mathcal{P}_2^0)$ *holds with* $0 \leqslant l \leqslant 3$. *Then Problem* (1.4) *has at most one solution.*

Proof. Let us assume that (ρ_1,u_1) and (ρ_2,u_2) be two solutions of Problem (1.4), and let $\rho = \rho_2 - \rho_1$ and $u = u_2 - u_1$. Then (ρ,u) satisfies the problem:

$$\rho' + \mu\cdot A_1\rho = -D(\rho_2,u_2) + D(\rho_1,u_1), \qquad u' + \nu\cdot A_2 u = -B(u_2) + B(u_1) - E(\rho_2) + E(\rho_1), \quad t \in (0,T);$$
$$\rho(0) = 0, \qquad u(0) = 0.$$

We take a.e. in t the scalar product of these equations and (ρ,u) in the duality between V and V', then we get

$$\begin{aligned} &\frac{d}{dt}\left(\left\|\rho(t)\right\|^2 + \left\|u(t)\right\|^2\right) + 2\mu\left\|\partial_x\rho(t)\right\|^2 + 2\nu\left\|\partial_x u(t)\right\|^2 \\ &\quad = 2\big(e(\rho_1,u) - e(\rho_2,u) + b(u_1,u_1,u) - b(u_2,u_2,u) + d(\rho_1,u_1,\rho) - d(\rho_2,u_2,\rho)\big). \end{aligned}$$

From inequality (2.17) we obtain

$$2\left|e(\rho_2,u) - e(\rho_1,u)\right| \leqslant \nu\|\partial_x u\|^2 + \mu\|\partial_x\rho\|^2 + +c\|\rho\|^4\big(\|\rho_2\|_{H_1}^6\|\rho_2\|_{V_1}^2 + \|\rho_1\|_{H_1}^6\|\rho_1\|_{V_1}^2 + 1\big).$$

Using trilinear property and equality (2.7), we get

$$b(u_1,u_1,u) - b(u_2,u_2,u) = b(2u_1 - u_2,u,u),$$

then from inequality (2.5)

$$2\left|b(2u_1 - u_2,u,u)\right| \leqslant \nu\|\partial_x u\|^2 + \frac{c_4}{\nu}\|u\|^2\big(\|u_1\|_{V_2}^2 + \|u_2\|_{V_2}^2\big).$$

It is clear from linearity, that

$$2d(\rho_1,u_1,\rho) - 2d(\rho_2,u_2,\rho) = -2d(\rho,u_1,\rho) - 2d(\rho_2,u,\rho),$$

then, in view of (2.10), we get

$$2\left|d(\rho,u_1,\rho)\right| + 2\left|d(\rho_2,u,\rho)\right| \leqslant \mu\|\partial_x\rho\|^2 + \frac{2c^A}{\mu}\|\rho\|^2\,\|u_1\|_{V_2}^2 + \frac{2c^A}{\mu}\|u\|^2\,\|\rho_2\|_{V_1}^2.$$

Hence,

$$\frac{d}{dt}\left(\|\rho\|^2+\|u\|^2\right)\leqslant c_5 S(t)\left(\|\rho\|^2+\|u\|^2\right),$$

where the function

$$t\to S(t)=\left(1+\|\rho_1\|_{H_1}^8+\|\rho_2\|_{H_1}^8\right)\left(1+\|\rho_1\|_{V_1}^2+\|u_1\|_{V_2}^2+\|\rho_2\|_{V_1}^2+\|u_2\|_{V_2}^2\right)$$

is integrable for $(\rho_i,u_i)\in L^2(0,T;V)\cap L^\infty(0,T;H)$, $i=1,2$. This shows, that

$$\frac{d}{dt}\left\{\left(\left\|\rho(t)\right\|^2+\left\|u(t)\right\|^2\right)\exp\left(-c_5\int_0^t S(\tau)\,d\tau\right)\right\}\leqslant 0.$$

Integrating and applying the homogeneous initial conditions, we find

$$\left\|\rho(t)\right\|^2+\left\|u(t)\right\|^2\leqslant 0,\quad \forall t\in[0,T].$$

Therefore, $\rho_1=\rho_2$ and $u_1=u_2$. $\square$

Note that if Hypothesis $(\mathcal{P}_2)$ is satisfied then Hypothesis $(\mathcal{P}_2^0)$ of the lemma is true for $l=2$, therefore, the uniqueness theorem can be reformulated by replacing Hypothesis $(\mathcal{P}_2^0)$ by Hypothesis $(\mathcal{P}_2)$.

4. *A priori* estimates

In this section we will prove some estimates for the solution of Problem (1.4). These estimates will be used to show the existence and asymptotic behavior of solutions.

Theorem 4.1. *Let (ρ,u) be a solution of Problem* (1.4)*. If Hypothesis $(\mathcal{P}_1)$ holds, then for $(\rho_0,u_0)\in H$ and $f\in L^2(0,T;V_2')$ with $0<T<\infty$,*

$$\left\|\rho(t)\right\|^2+\theta\left\|u(t)\right\|^2\leqslant C_1,\quad \forall t\in[0,T],\tag{4.1}$$

$$\int_0^T\left\|\partial_x\rho(s)\right\|^2+\theta\left\|\partial_x u(s)\right\|^2 ds\leqslant C_2,\tag{4.2}$$

where $\theta=1/(2P_\infty)$, C_1 and C_2 depend on L, T, ν, μ, P_∞, P_0, P_1, $\|u_0\|$, $\|\rho_0\|$, $\|f\|_{L^2(0,T;V_1')}$.

Furthermore, if the product of positive coefficients μ and ν is large enough and $f\in L^\infty(0,+\infty;V_2')$, then there exist such positive numbers K_1, K_2 and k that

$$\left\|\rho(t)\right\|_{H_1}^2+\theta\left\|u(t)\right\|_{H_1}^2\leqslant K_1\left(1-e^{-kt}\right)+\left(\|\rho_0\|_{H_1}^2+\theta\|u_0\|_{H_1}^2\right)e^{-kt},\tag{4.3}$$

$$\int_t^{t+1}\left\|\partial_x\rho(s)\right\|^2+\theta\left\|\partial_x u(s)\right\|^2 ds\leqslant K_2\left(1+\left(\|\rho_0\|_{H_1}^2+\theta\|u_0\|_{H_1}^2\right)e^{-kt}\right).\tag{4.4}$$

Proof. Let us choose the functions $\rho(t)$ and $u(t)$ as the test functions in Problem (2.20). Then summing the first equation of Problem (2.20) with the second equation multiplied by $\theta>0$, and applying the equalities (2.1), (2.8), (2.11), and

$$2\left\langle\rho'(t),\rho(t)\right\rangle_{V_1}=\frac{d}{dt}\left\|\rho(t)\right\|^2,\qquad 2\left\langle u'(t),u(t)\right\rangle_{V_2}=\frac{d}{dt}\left\|u(t)\right\|^2,$$

see [20, p. 260], we get

$$\frac{d}{dt}\left(\left\|\rho(t)\right\|^2+\theta\left\|u(t)\right\|^2\right)+2\mu\left\|\partial_x\rho(t)\right\|^2+2\theta\nu\left\|\partial_x u(t)\right\|^2=\int_0^L\left(2\theta P(\rho)-\rho^2\right)\partial_x u\,dx+2\theta\left\langle f(t),u(t)\right\rangle_{V_2}.\tag{4.5}$$

In view of Hypothesis $\mathcal{P}_1$ and the Poincaré inequalities

$$\|\rho\|^2\leqslant c_1^P\|\partial_x\rho\|^2+\frac{1}{L}\bar{\rho}^2,\qquad \|u\|^2\leqslant c_0^P\|\partial_x u\|^2,\quad \forall\rho\in V_1,\ \forall u\in V_2,$$

we obtain for $\theta = 1/(2P_\infty)$:

$$2\int_0^L \big(P(\rho) - P_\infty \rho^2\big)\partial_x u \, dx \leqslant 2\int_0^L \big(P_1|\rho| + P_0\big)|\partial_x u| \, dx \leqslant (\tau_1 + \tau_2)\|\partial_x u\|^2 + \frac{P_1^2}{\tau_1}\|\rho\|^2 + \frac{P_0^2 L}{\tau_2},$$

$$2\big|\langle f, u\rangle_{V_2}\big| \leqslant 2\|u\|_{V_2}\|f\|_{V_2'} \leqslant 2\big(\|u\| + \|\partial_x u\|\big)\|f\|_{V_2'} \leqslant \tau_3\|\partial_x u\|^2 + 2\frac{c_0^P + 1}{\tau_3}\|f\|_{V_2'}^2.$$

Then from (4.5):

$$\frac{d}{dt}\big(\big\|\rho(t)\big\|^2 + \theta\big\|u(t)\big\|^2\big) + 2\mu\big\|\partial_x \rho(t)\big\|^2 - \frac{\theta P_1^2}{\tau_1}\big\|\rho(t)\big\|^2(2\nu - \tau_1 - \tau_2 - \tau_3)\theta\big\|\partial_x u(t)\big\|^2$$
$$\leqslant \frac{2\theta(c_0^P + 1)}{\tau_3}\big\|f(t)\big\|_{V_2'}^2 + \frac{\theta P_0^2 L}{\tau_2} = F. \tag{4.6}$$

Therefore, inequalities (4.1) and (4.2) can be easily deduced.

Using the Poincaré inequality for $\|\partial_x \rho\|^2$ and $\|\partial_x u\|^2$ in (4.6), we obtain

$$\frac{d}{dt}\big(\big\|\rho(t)\big\|^2 + \theta\big\|u(t)\big\|^2\big) + k\big(\big\|\rho(t)\big\|^2 + \theta\big\|u(t)\big\|^2\big) \leqslant F + \frac{2\mu}{c_1^P}\bar{\rho}^2,$$

where $k = \min\{(2\mu\tau_1 - \theta P_1^2 c_1^P)/(\tau_1 c_1^P), (2\nu - \tau_1 - \tau_2 - \tau_3)/c_0^P\}$. Assuming that $\mu\nu > c_1^P P_1^2/(8P_\infty)$, we infer that there exist positive constants τ_1, τ_2 and τ_3 such that $k > 0$. Applying the Gronwall lemma, we obtain

$$\big\|\rho(t)\big\|^2 + \theta\big\|u(t)\big\|^2 \leqslant \big(\|\rho_0\|^2 + \theta\|u_0\|^2\big)e^{-kt} + K_1\big(1 - e^{-kt}\big),$$

where

$$K_1 = \frac{1}{k}\left(\frac{c_0^P + 1}{P_\infty \tau_3}\|f\|_{L^\infty(0,\infty;V_2')}^2 + \frac{2\mu}{c_1^P}\bar{\rho}^2 + \frac{P_0^2 L}{2P_\infty \tau_2}\right).$$

Finally, inequality (4.4) can be shown by integrating of inequality (4.6) from t to $t+1$, and applying inequality (4.3). The constant K_2 depends on the same arguments as K_1. $\square$

Let us consider the Hilbert spaces X_0, X and X_1 with

$$X_0 \subset X \subset X_1,$$

where the injection of X_0 into X is compact. Denote by $\hat{v}$ the Fourier transform of the function v from $\mathbb{R}$ into X_1,

$$\hat{v}(\tau) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{-i\tau\cdot t} v(t)\, dt,$$

Then the derivative in t of order γ of v is the inverse Fourier transform of $(-i\tau)^\gamma \hat{v}$

$$\widehat{\big(D_t^\gamma v\big)}(\tau) = (-i\tau)^\gamma \cdot \hat{v}(\tau).$$

For given $\gamma > 0$, we define the Hilbert space

$$\mathcal{H}^\gamma(\mathbb{R}; X_0, X_1) = \big\{v \in L^2(\mathbb{R}; X_0),\ D_t^\gamma v \in L^2(\mathbb{R}; X_1)\big\},$$

supplemented with the norm

$$\|v\|_{\mathcal{H}^\gamma(\mathbb{R}; X_0, X_1)} = \big(\|v\|_{L^2(\mathbb{R}; X_0)}^2 + \big\||\tau|^\gamma \hat{v}\big\|_{L^2(\mathbb{R}; X_1)}^2\big)^{1/2}.$$

Associate with any set $K \subset \mathbb{R}$, the subspace $\mathcal{H}_K^\gamma$ of $\mathcal{H}^\gamma$ defined as the set of functions u in $\mathcal{H}^\gamma$ with support contained in K:

$$\mathcal{H}_K^\gamma(\mathbb{R}; X_0, X_1) = \big\{u \in \mathcal{H}^\gamma(\mathbb{R}; X_0, X_1),\ \text{support } u \subset K\big\}.$$

The following compactness theorem was proved in [20].

Proposition 4.2. *(See [20, p. 274].) For any bounded set K and any $\gamma > 0$, the injection of $\mathcal{H}_K^\gamma(\mathbb{R}; X_0, X_1)$ into $L^2(\mathbb{R}; X)$ is compact.*

Clearly, choosing $X_0 = V_i$, $X = X_1 = H_i$, $i = 1, 2$, we obtain that for any $\gamma > 0$, the injection of $\mathcal{H}^{\gamma}_{[0,T]}(\mathbb{R}; V_i, H_i)$ into $L^2(\mathbb{R}; H_i)$ is compact.

Let $\tilde{v}$ denote the function from $\mathbb{R}$ into V_i, $i = 1, 2$, which is equal to v on $[0, T]$ and 0 on the complement of this interval, and let $\hat{v}$ denote the Fourier transform of $\tilde{v}$. Then the following statement is valid.

Theorem 4.3. *Let (ρ, u) be a solution of Problem (1.4) with $(\rho_0, u_0) \in H$, $f \in L^2(0, T; V_2')$. If Hypothesis $(\mathcal{P}_1)$ holds, then for each γ: $(0 < \gamma < 1/4)$ there exists a positive number C such that*

$$\|\tilde{\rho}(t)\|^2_{\mathcal{H}^{\gamma}(\mathbb{R}; V_1, H_1)} + \theta\|\tilde{u}(t)\|^2_{\mathcal{H}^{\gamma}(\mathbb{R}; V_2, H_2)} \leqslant C_3,$$

where $\theta = 1/(2P_\infty)$, and C_3 depends on γ, L, T, ν, μ, P_0, P_1, P_∞, $\bar{\rho}$, $\|\rho_0\|$, $\|u_0\|$ and $\|f\|_{L^2(0,T;V_2')}$.

Proof. Let us rewrite Problem (1.4) as follows

$$\left(\frac{d}{dt}\tilde{\rho}(t), \phi\right) = \langle \tilde{g}, \phi\rangle_{V_1} + (\rho_0, \phi)\delta_0 - \big(\rho(T), \phi\big)\delta_T, \quad \forall \phi \in V_1,$$
$$\left(\frac{d}{dt}\tilde{u}(t), w\right) = \langle \tilde{f}, w\rangle_{V_2} + (u_0, w)\delta_0 - \big(u(T), w\big)\delta_T, \quad \forall w \in V_2,$$

where δ_0 and δ_T are Dirac distributions at $t = 0$ and $t = T$, and

$$\tilde{g} = -\mu A_1\rho - D(\rho, u), \qquad \tilde{f} = f - \nu A_2 u - B(u) - E(\rho)$$

on $[0, T]$ and 0 outside this interval. By applying the Fourier transform we get

$$-i\tau\big(\hat{\rho}(\tau), \Phi\big) = \langle \hat{g}, \Phi\rangle_{V_1} + \frac{1}{\sqrt{2\pi}}(\rho_0, \Phi) - \frac{1}{\sqrt{2\pi}}\big(\rho(T), \Phi\big)e^{-i\tau T}, \quad \forall \Phi \in V_1,$$
$$-i\tau\big(\hat{u}(\tau), U\big) = \langle \hat{f}, U\rangle_{V_2} + \frac{1}{\sqrt{2\pi}}(u_0, U) - \frac{1}{\sqrt{2\pi}}\big(u(T), U\big)e^{-i\tau T}, \quad \forall U \in V_2.$$

Choosing $\hat{\rho}(\tau)$ and $\hat{u}(\tau)$ as testing functions and adding the first equation with the second multiplied by a positive number θ,

$$\begin{aligned} -i\tau\big(\|\hat{\rho}(\tau)\|^2 + \theta\|\hat{u}(\tau)\|^2\big) &= \langle \hat{g}, \hat{\rho}(\tau)\rangle_{V_1} + \theta\langle \hat{f}, \hat{u}(\tau)\rangle_{V_2} \\ &\quad + \frac{1}{\sqrt{2\pi}}\Big[\big(\rho_0, \hat{\rho}(\tau)\big) + \theta\big(u_0, \hat{u}(\tau)\big) - \Big[\big(\rho(T), \hat{\rho}(\tau)\big) + \theta\big(u(T), \hat{u}(\tau)\big)\Big]e^{-i\tau T}\Big]. \end{aligned} \tag{4.7}$$

In view of Lemmas 2.1(i), 2.2(i), (iv) and 2.3(i) and inequality (4.2) of Theorem 4.1, we obtain

$$\int_{-\infty}^{+\infty} \|\tilde{g}(t)\|_{V_1'}\,dt = \int_0^T \|g(t)\|_{V_1'}\,dt < c_1, \qquad \int_{-\infty}^{+\infty} \|\tilde{f}(t)\|_{V_2'}\,dt = \int_0^T \|\tilde{f}(t)\|_{V_2'}\,dt < c_2,$$

where c_1, c_2 depend on the same arguments as C_1 in Theorem 4.1. Therefore,

$$\sup_{\tau\in\mathbb{R}}\|\hat{g}(\tau)\|_{V_1'} < c_1, \qquad \sup_{\tau\in\mathbb{R}}\|\hat{f}(\tau)\|_{V_2'} < c_2.$$

If $(\rho_0, u_0) \in H$, then from (4.1) of Theorem 4.1,

$$\|\rho(T)\| \leqslant c_3, \qquad \|u(T)\| \leqslant c_3.$$

Then we deduce from (4.7), that

$$|\tau|\big(\|\hat{\rho}(\tau)\|^2 + \theta\|\hat{u}(\tau)\|^2\big) \leqslant c_4\big(\|\hat{\rho}(\tau)\|_{V_1} + \theta\|\hat{u}(\tau)\|_{V_2}\big).$$

For fixed positive γ, such that $\gamma < 1/4$, we observe that

$$\begin{aligned} I_1 &= \int_{\mathbb{R}\setminus[-1.1]} |\tau|^{2\gamma}\big(\|\hat{\rho}(\tau)\|^2 + \theta\|\hat{u}(\tau)\|^2\big)\,d\tau \leqslant c_4 \int_{\mathbb{R}\setminus[-1.1]} |\tau|^{2\gamma-1}\big(\|\hat{\rho}(\tau)\|_{V_1} + \theta\|\hat{u}(\tau)\|_{V_2}\big)\,d\tau \\ &\leqslant c_4 \int_{\mathbb{R}\setminus[-1.1]} |\tau|^{4\gamma-2} + \big(\|\hat{\rho}(\tau)\|^2 + \theta\|\hat{u}(\tau)\|^2\big)\,d\tau \leqslant 2c_4\int_1^{+\infty}|\tau|^{4\gamma-2}\,d\tau + c_4\int_{-\infty}^{+\infty}\big(\|\hat{\rho}(\tau)\|^2 + \theta\|\hat{u}(\tau)\|^2\big)\,d\tau, \end{aligned}$$

$$I_2 = \int_{-1}^{1} |\tau|^{2\gamma}\big(\|\hat{\rho}(\tau)\|^2 + \theta\|\hat{u}(\tau)\|^2\big)\,d\tau \leqslant \int_{-1}^{1}\big(\|\hat{\rho}(\tau)\|^2 + \theta\|\hat{u}(\tau)\|^2\big)\,d\tau \leqslant \int_{-\infty}^{+\infty}\big(\|\hat{\rho}(\tau)\|^2 + \theta\|\hat{u}(\tau)\|^2\big)\,d\tau.$$

Therefore, from Theorem 4.1 we obtain that

$$I_1 + I_2 = \int_{-\infty}^{+\infty} |\tau|^{2\gamma} \big(\|\hat{\rho}(\tau)\|^2 + \theta \|\hat{u}(\tau)\|^2 \big)\, d\tau < C_3,$$

where C_3 depends on the same arguments as C_1 in Theorem 4.1. □

5. Existence

The existence of solutions of Problem (1.4) is ensured by the following theorem.

Theorem 5.1 *(Existence). Suppose that Hypotheses $(\mathcal{P}_1)$ and $(\mathcal{P}_2^0)$ with $0 \leqslant l \leqslant 3$ hold. Then, for $(\rho_0, u_0) \in H$ and $f \in L^2(0, T; V_2')$, there exists at least one solution (ρ, u) of Problem (1.4). Moreover, (ρ, u) is almost everywhere equal to a function continuous from $[0, T]$ into H and*

$$\big(\rho(t), u(t)\big) \to (\rho_0, u_0) \quad \text{in } H, \ \text{as } t \to 0.$$

Proof. To prove the existence theorem we use the Faedo–Galerkin method. The space $V = V_1 \times V_2$ is a separable space, therefore, there exists a sequence of linearly independent elements $\{(\phi_i, w_i)\}$, $i = 1, 2, \ldots,$ that is total in V. For each m we define an approximate solution (ρ_m, u_m) of Problem (1.4) in the form

$$\rho_m(t, x) = \sum_{i=1}^{m} \rho_m^i(t)\phi_i(x), \qquad u_m(t, x) = \sum_{i=1}^{m} u_m^i(t) w_i(x),$$

where the functions $\rho_m^i(t)$ and $u_m^i(t)$ are determined by choosing the test functions ϕ_j and w_j, $j = 1, 2, \ldots, m$, that is from

$$\big(\rho_m', \phi_j\big) + \mu a_1(\rho_m, \phi_j) + d(\rho_m, u_m, \phi_j) = 0, \tag{5.1}$$

$$\big(u_m', w_j\big) + \nu a_2(u_m, w_j) + b(u_m, u_m, w_j) + e(\rho_m, w_j) = \langle f, w_j \rangle, \tag{5.2}$$

for $t \in [0, T]$. These equations form a nonlinear system of ordinary differential equations for the functions $\rho_m^i(t)$ and $u_m^i(t)$ in the form

$$\sum_{i=1}^{m} (\phi_i, \phi_j) \frac{d\rho_m^i}{dt} + \mu \sum_{i=1}^{m} a_1(\phi_i, \phi_j)\rho_m^i + \sum_{i,k=1}^{m} d(\phi_k, w_i, \phi_j) u_m^i \rho_m^k = 0,$$

$$\sum_{i=1}^{m} (w_i, w_j) \frac{du_m^i}{dt} + \nu \sum_{i=1}^{m} a_2(w_i, w_j) u_m^i + \sum_{i,k=1}^{m} b(w_i, w_k, w_j) u_m^i u_m^k + e\left(\sum_{i=1}^{m} \rho_m^i \phi_i, w_j \right) = \langle f, w_j \rangle.$$

Inverting the nonsingular matrices with elements (ϕ_i, ϕ_j) and (w_i, w_j), we obtain the system of differential equations

$$\frac{d\rho_m^i}{dt} + \sum_{j=1}^{m} \eta_{ij} \rho_m^j + \sum_{j,k=1}^{m} \theta_{ijk} u_m^j \rho_m^k = 0,$$

$$\frac{du_m^i}{dt} + \sum_{j=1}^{m} \alpha_{ij} u_m^j + \sum_{j,k=1}^{m} \beta_{ijk} u_m^j u_m^k + \gamma_i\big(\rho_m^1, \ldots, \rho_m^m\big) = \sum_{j=1}^{m} \zeta_{ij} \langle f, w_j \rangle.$$

We supplement this system with the initial conditions

$$\rho_m^i(0) = \rho_{0m}^i, \qquad u_m^i(0) = u_{0m}^i,$$

where

$$\rho_m(0) = \rho_{0m} = \sum_{i=1}^{m} \rho_{0m}^i \phi_i, \qquad u_m(0) = u_{0m} = \sum_{i=1}^{m} u_{0m}^i w_i,$$

and (ρ_{0m}, u_{0m}) is the orthogonal projection in H of (ρ_0, u_0) onto the space spanned by $\{(\phi_i, w_i)\}$, $i = 1, 2, \ldots, m$.

Note that there exists a solution of the above problem for $\rho_m^i(t)$ and $u_m^i(t)$, which is defined on some interval $[0, t_m]$.

Using the same method of estimation as in Theorems 4.1 and 4.3, we infer that:

- the sequence (ρ_m, u_m) remains in a bounded set of $L^2(0, T; V)$,
- the sequence (ρ_m, u_m) remains in a bounded set of $L^\infty(0, T; H)$,
- the sequence $(\tilde{u}_m, \tilde{\rho}_m)$ remains in a bounded set of $\mathcal{H}^\gamma(\mathbb{R}; V, H)$ for each γ, $0 < \gamma < 1/4$.

Therefore, $t_m = T$, and there exists such subsequence, say (ρ_m, u_m), that

$$(\rho_m, u_m) \to (\rho, u) \quad \text{strongly in } L^2(0,T;H), \quad \text{weakly star in } L^\infty(0,T;H), \quad \text{weakly in } L^2(0,T;V);$$

moreover, $(\rho, u) \in L^2(0,T;V) \cap L^\infty(0,T;H)$.

To show that (ρ, u) is a solution of Problem (1.4) in the form (2.20) we will need the following convergence result proved in Appendix A.

Lemma 5.2 *(Convergence lemma).*

(i) *If u_α converges to u in $L^2(0,T;V_2)$ weakly and in $L^2(0,T;H_2)$ strongly, then for any function $w \in C^1(\bar{G}^T)$, $G^T = (0,T] \times \Omega$,*

$$\int_0^T b\big(u_\alpha(t), u_\alpha(t), w(t)\big)\,dt \to \int_0^T b\big(u(t), u(t), w(t)\big)\,dt.$$

(ii) *If (ρ_α, u_α) converges to (ρ, u) in $L^2(0,T;V)$ weakly and in $L^2(0,T;H)$ strongly, then for any function $\phi \in C^1(\bar{G}^T)$,*

$$\int_0^T d\big(\rho_\alpha(t), u_\alpha(t), \phi(t)\big)\,dt \to \int_0^T d\big(\rho(t), u(t), \phi(t)\big)\,dt.$$

(iii) *Assume that Hypothesis $(\mathcal{P}_2^0)$ holds with $0 \leqslant l \leqslant 3$. If ρ_α converges to ρ in $L^2(0,T;V_1)$ weakly, in $L^\infty(0,T;H_1)$ weakly star and in $L^2(0,T;H_1)$ strongly, then for any function $w \in C^1(\bar{G}^T)$*

$$\int_0^T e\big(\rho_\alpha(t), w(t)\big)\,dt \to \int_0^T e\big(\rho(t), w(t)\big)\,dt.$$

Let us consider a vector function (ψ, v) continuously differentiable on $[0,T]$ and such that $\psi(T) = 0$ and $v(T) = 0$. By multiplying (5.1) by $\psi(t)$ and (5.2) by $v(t)$ and integrating over $(0,T)$ we obtain

$$-(\rho_{0m}, \phi_j)\psi(0) - \int_0^T \big(\rho_m(t), \phi_j \psi'(t)\big)\,dt + \mu \int_0^T a_1\big(\rho_m(t), \phi_j \psi(t)\big)\,dt + \int_0^T d\big(\rho_m(t), u_m(t), \phi_j \psi(t)\big)\,dt = 0,$$

$$-(u_{0m}, w_j)v(0) - \int_0^T \big(u_m(t), w_j v'(t)\big)\,dt + \nu \int_0^T a_2\big(u_m(t), w_j v(t)\big)\,dt$$
$$+ \int_0^T b\big(u_m(t), u_m(t), w_j v(t)\big)\,dt + \int_0^T e\big(\rho_m(t), w_j v(t)\big)\,dt = \int_0^T \big\langle f(t), w_j v(t)\big\rangle\,dt.$$

Let us pass to the limit as $m \to \infty$. Clearly,

$$\rho_{0m} \to \rho_0 \quad \text{in } H_1, \qquad u_{0m} \to u_0 \quad \text{in } H_2, \quad \text{as } m \to \infty.$$

Then using Lemma 5.2, we obtain that the equations

$$-(\rho_0, \phi)\psi(0) - \int_0^T \big(\rho(t), \phi\psi'(t)\big)\,dt + \mu \int_0^T a_1\big(\rho(t), \phi\psi(t)\big)\,dt + \int_0^T d\big(\rho(t), u(t), \phi\psi(t)\big)\,dt = 0,$$

$$-(u_0, w)v(0) - \int_0^T \big(u(t), w v'(t)\big)\,dt + \nu \int_0^T a_2\big(u(t), w v(t)\big)\,dt$$
$$+ \int_0^T b\big(u(t), u(t), w v(t)\big)\,dt + \int_0^T e\big(\rho(t), w v(t)\big)\,dt = \int_0^T \big\langle f(t), w v(t)\big\rangle\,dt$$

hold for $(\phi, w) \in \{(\phi_j, w_j)\}$. By the linearity argument, these equations also hold for any finite linear combination of the (ϕ_j, w_j)'s. Moreover, by the continuity argument these equations are still true for any $(\phi, w) \in V$. Writing with components (ψ, v) in $\mathcal{D}(0, T)$, we see that (ρ, u) satisfies the equations of Problem (1.4) in the distributional sense.

To show that the initial conditions are hold for (ρ, u), we multiply Eqs. (2.20) by $\psi(t)$ and $v(t)$ as above, and integrate over interval $(0, T)$. Using integration by parts, we obtain

$$-\big(\rho(0), \phi\big)\psi(0) - \int_0^T \big(\rho(t), \phi\psi'(t)\big)\,dt + \mu \int_0^T a_1\big(\rho(t), \phi\psi(t)\big)\,dt + \int_0^T d\big(\rho(t), u(t), \phi\psi(t)\big)\,dt = 0,$$

$$-\big(u(0), w\big)v(0) - \int_0^T \big(u(t), wv'(t)\big)\,dt + \nu \int_0^T a_2\big(u(t), wv(t)\big)\,dt$$
$$+ \int_0^T b\big(u(t), u(t), wv(t)\big)\,dt + \int_0^T e\big(\rho(t), wv(t)\big)\,dt = \int_0^T \big\langle f(t), wv(t)\big\rangle\,dt.$$

Then for each $\phi \in V_1$ and $w \in V_2$,

$$\big(\rho(0) - \rho_0, \phi\big)\psi(0) = 0, \qquad \big(u(0) - u_0, w\big)v(0) = 0.$$

Choosing $\psi(0) = 1$ and $v(0) = 1$ we see that the initial conditions are valid.

Finally, in view of Lemmas 2.1(i), 2.2(ii), (v) and 2.3(ii),

$$\rho' \in L^2\big(0, T; V_1'\big), \qquad u' \in L^2\big(0, T; V_2'\big),$$

therefore, (ρ, u) is almost everywhere equal to continuous vector function from $[0, T]$ into H, see [10,20]. □

6. Regularity

In this section we will prove the regularity of the solution of Problem (1.4) whose existence and uniqueness are ensured by Theorems 5.1 and 3.1. We will also derive some *a priori* estimates of solutions.

Theorem 6.1. *Let Hypotheses* $(\mathcal{P}_1)$ *and* $(\mathcal{P}_2)$ *hold. If* $f \in L^2(0, T; H_2)$ *and* $(\rho_0, u_0) \in H$*, then there exists a unique solution of Problem* (1.4) *such that* $(\sqrt{t}\rho, \sqrt{t}u) \in L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega))$*, and for any* $\varepsilon > 0$ *there are such numbers* $C_4 = C_4(\varepsilon)$ *and* $C_5 = C_5(\varepsilon)$ *that*

$$\big\|\partial_x \rho(t)\big\|^2 + \theta\big\|\partial_x u(t)\big\|^2 \leqslant C_4(\varepsilon), \quad \forall t\colon t \in [\varepsilon, T],\ \theta > 0, \tag{6.1}$$

$$\big\|\partial_x^2 \rho\big\|^2_{L^2(\varepsilon,T;L^2(\Omega))} + \theta\big\|\partial_x^2 u\big\|^2_{L^2(\varepsilon,T;L^2(\Omega))} \leqslant C_5(\varepsilon). \tag{6.2}$$

Moreover, if $(\rho_0, u_0) \in V$*, then* $(\rho, u) \in L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega))$*, and* $(\rho', u') \in L^2(0, T; H)$*.*

Furthermore, if the product of positive coefficients μ *and* ν *is large enough,* $f \in L^\infty(0, +\infty; V_2')$*, for any* $t > 0$: $\int_t^{t+1} \|f(t)\|^2\,dx \leqslant C_f$*, and* $(\rho_0, u_0) \in H$*. Then there are such positive numbers* K_3*,* K_4 *and* t_1 *that*

$$\big\|\partial_x \rho(t)\big\|^2 + \theta\big\|\partial_x u(t)\big\|^2 \leqslant K_3^2, \quad \forall t \geqslant t_1, \tag{6.3}$$

$$\int_t^{t+1} \big\|\partial_x^2 \rho(s)\big\|^2 + \theta\big\|\partial_x^2 u(s)\big\|^2\,ds \leqslant K_4^2, \quad \forall t \geqslant t_1. \tag{6.4}$$

Proof. We consider again Galerkin approximation (5.1) and (5.2), where ϕ_j's are the eigenfunctions of the operator A_1 with the eigenvalues λ_j^1, and the w_j's are the eigenfunctions of the operator A_2 with the eigenvalues λ_j^2. In the case when $(\rho_0, u_0) \in V$, we assume that $\rho_{0m} \in \mathrm{Sp}[\phi_1, \ldots, \phi_m]$ and $u_{0m} \in \mathrm{Sp}[w_1, \ldots, w_m]$ are chosen so that

$$(\rho_{0m}, u_{0m}) \to (\rho_0, u_0) \quad \text{strongly in } V \text{ as } m \to \infty.$$

Note that

$$a_1(\phi_j, \phi) = \langle A_1\phi_j, \phi\rangle_{V_1} = \big\langle \lambda_j^1 \phi_j, \phi\big\rangle_{V_1} = \lambda_j^1(\phi_j, \phi)_{H_1},$$
$$a_2(w_j, w) = \langle A_2 w_j, w\rangle_{V_2} = \big\langle \lambda_j^2 w_j, w\big\rangle_{V_2} = \lambda_j^2(w_j, w)_{H_2}.$$

Then, multiplying Eq. (5.1) by λ_j^1, and Eq. (5.2) by λ_j^2, we write

$$a_1(\rho_m', \phi_j) + \mu(A_1\rho_m, A_2\phi_j) + d(\rho_m(t), u_m, A_1\phi_j) = 0,$$
$$a_2(u_m', w_j) + \nu(A_2u_m, A_1w_j) + b(u_m(t), u_m, A_2w_j) + e(\rho_m, A_2w_j) = (f, A_2w_j),$$

where $t \in [0, T]$ and $j = 1, \dots, m$. Summing with respect to j these equalities multiplied by $\rho_m^j(t)$ and $u_m^j(t)$, respectively, see Theorem 5.1, we obtain

$$a_1(\rho_m', \rho_m) + \mu(A_1\rho_m, A_1\rho_m) + d(\rho_m(t), u_m, A_1\rho_m) = 0, \tag{6.5}$$
$$a_2(u_m', u_m) + \nu(A_2u_m, A_2u_m) + b(u_m, u_m, A_2u_m) + e(\rho_m, A_2u_m) = (f, A_2u_m). \tag{6.6}$$

In view of inequalities (2.5), (2.6) and (2.19),

$$2\left|d(\rho_m, u_m, A_1\rho_m)\right| \leqslant \mu\|A_1\rho_m\|^2 + c_6\left(\|\partial_x\rho_m\|^4 + \|\partial_xu_m\|^4 + \|\partial_x\rho_m\|^2\|\rho_m\|^2 + \|\partial_xu_m\|^2\|u_m\|^2\right),$$
$$2\left|b(u_m, u_m, A_1u_m)\right| \leqslant \frac{\nu}{3}\|A_2u_m\|^2 + c_7\left(\|\partial_xu_m\|^4 + \|\partial_xu_m\|^2\|u_m\|^2\right),$$
$$2\left|e(\rho_m, A_2u_m)\right| \leqslant \frac{\nu}{3}\|A_2u_m\|^2 + c_8\|\partial_x\rho_m\|^2\left(\|\partial_x\rho_m\|^2\|\rho_m\|^2 + 1\right),$$
$$2\left|\left(f(t), A_2u_m\right)\right| \leqslant 2\left\|f(t)\right\|\|A_2u_m\| \leqslant \frac{\nu}{3}\|A_2u_m\|^2 + \frac{3}{\nu}\left\|f(t)\right\|^2.$$

Then from (6.5) and (6.6), using

$$2a_1(\rho_m'(t), \rho_m(t)) = \frac{d}{dt}\left\|\partial_x\rho_m(t)\right\|^2, \qquad 2a_2(u_m'(t), u_m(t)) = \frac{d}{dt}\left\|\partial_xu_m(t)\right\|^2,$$
$$(A_1\rho_m(t), A_1\rho_m(t)) = \left\|A_1\rho_m(t)\right\|^2, \qquad (A_2u_m(t), A_2u_m(t)) = \left\|A_2u_m(t)\right\|^2,$$

we obtain the following inequality for $t \geqslant 0$

$$\frac{dy_m(t)}{dt} + r_m(t) \leqslant g_m(t)y_m(t) + q_m(t), \tag{6.7}$$

where $\theta > 0$

$$y_m(t) = \|\partial_x\rho_m\|^2 + \theta\|\partial_xu_m\|^2, \qquad r_m(t) = \mu\|A_1\rho_m\|^2 + \nu\theta\|A_2u_m\|^2,$$
$$g_m(t) = c_9\left(\|\partial_x\rho_m\|^2\|\rho_m\|^2 + \|\partial_x\rho_m\|^2 + \|\rho_m\|^2 + \|\partial_xu_m\|^2 + \|u_m\|^2 + 1\right),$$
$$q_m(t) = \frac{3}{\nu}\left\|f(t)\right\|^2.$$

In view of Theorems 4.1 and 5.1 for each $\varepsilon > 0$, we can estimate the following integrals for all t in $[0, T - \varepsilon]$:

$$\int_t^{t+\varepsilon} y_m(t)\,dt \leqslant Y(\varepsilon), \qquad \int_t^{t+\varepsilon} g_m(t)\,dt \leqslant G(\varepsilon), \qquad \int_t^{t+\varepsilon} q_m(t)\,dt \leqslant Q(\varepsilon),$$

where Y, G and Q depend on ε, T, L, μ, ν, θ, P_∞, $P_0, \dots, P_3$, $\bar{\rho}$, $\|\rho_0\|_{H_1}$, $\|u_0\|_{H_2}$ and $\|f\|_{L^2(0,T;V_2')}$, but they are independent of m. By applying the uniform Gronwall lemma [21], we get

$$\left\|\partial_x\rho_m(t)\right\|^2 + \theta\left\|\partial_xu_m(t)\right\|^2 \leqslant C_4(\varepsilon) = \left(\frac{Y(\varepsilon)}{\varepsilon} + Q(\varepsilon)\right)e^{G(\varepsilon)}, \quad \forall t \in [\varepsilon, T],$$

where $C_4(\varepsilon)$ is independent of m. Since, (ρ_m, u_m) converges to (ρ, u), then $(\rho, u) \in L^\infty(\varepsilon, T; V)$.

Integrating (6.7) with respect to t on the interval $[\varepsilon, T]$, it is easy to see that

$$\int_\varepsilon^T \left\|\partial_x^2\rho(t)\right\|^2 + \theta\left\|\partial_x^2u(t)\right\|^2 dt \leqslant \frac{1}{\min\{\mu, \nu\}}\int_\varepsilon^T r_m(t)\,dt \leqslant C_5(\varepsilon)$$

uniformly in m, hence, $(\rho, u) \in L^2(\varepsilon, T; H^2(\Omega))$.

Let us multiply (6.7) by t. Choosing

$$z_m(t) = t \cdot y_m(t), \qquad q_m^1(t) = t \cdot q_m(t) + y_m(t) \leqslant T \cdot q_m(t) + y_m(t),$$

we get the differential inequality

$$\frac{dz_m(t)}{dt} \leqslant g_m(t)z_m(t) + q_m^1(t), \quad \text{for } t \geqslant 0,$$

with zero initial conditions

$$z_m(0) = 0.$$

In view of the Gronwall inequality, the sequence z_m remains bounded in $L^\infty(0,T)$. Hence, the sequence $(\sqrt{t}\rho_m, \sqrt{t}u_m)$ remains bounded in $L^\infty(0,T;V)$. In this case $(\sqrt{t}\rho_m, \sqrt{t}u_m)$ converges to $(\sqrt{t}\rho, \sqrt{t}u) \in L^\infty(0,T;V)$. The sequence $(\sqrt{t}\rho_m, \sqrt{t}u_m)$ also remains bounded in $L^2(0,T;\ H^2(\Omega))$, and $(\sqrt{t}\rho_m,\ \sqrt{t}u_m)$ converges to $(\sqrt{t}\rho, \sqrt{t}u)$, with $(\sqrt{t}\rho, \sqrt{t}u) \in L^2(0,T;\ H^2(\Omega))$.

Assume that $(\rho_0, u_0) \in V$, then by Gronwall method there exists a positive number ε such that the sequence y_m defined by (6.7) is uniformly bounded in $L^\infty(0,\varepsilon)$. Combining with (6.1) and (6.2) we get uniform boundedness of the sequence $(\rho_m,\ u_m)$ in $L^\infty(0,T;V)$. Hence, (ρ_m, u_m) converges to $(\rho,u) \in L^\infty(0,T;V)$. Now, from (6.7) we get that the sequences $\|A_1\rho_m(t)\|$ and $\|A_2u_m(t)\|$ and, therefore, $\|\partial_x^2\rho_m(t)\|$ and $\|\partial_x^2 u_m(t)\|$, are uniformly bounded in $L^2(0,T)$. Then, from Theorem 4.1, we conclude that (ρ_m,u_m) converges to $(\rho,u) \in L^2(0,T;H^2(\Omega))$. Finally, Lemmas 2.1(ii), 2.2(iii), (vi) and 2.3(iii) imply that

$$\rho' = -\mu \cdot A_1\rho - D(\rho,u) \in L^2(0,T;H_1);$$
$$u' = -\nu \cdot A_2 u - B(u) - E(\rho) + f \in L^2(0,T;H_2).$$

Let now $f \in L^\infty(0,\infty;V_2')$ and $\mu\nu > c_1^P P_1^2/(8P_\infty)$, then from Theorem 4.1 for each fixed $\bar{\rho}$, there exists a time $t_0 = t_0(\|\rho_0\|_{H_1}^2 + \theta\|u_0\|_{H_2}^2)$ such that for $t > t_0$,

$$\|\rho_m(t)\|_{H_1}^2 + \theta\|u_m\|_{H_1}^2 \leqslant 2K_1, \qquad \int_t^{t+1} \|\partial_x\rho_m(s)\|_{H_1}^2 + \theta\|\partial_x u_m(s)\|_{H_2}^2\,ds \leqslant 2K_2.$$

Using the uniform Gronwall lemma to (6.7) and $\int_t^{t+1}\|f(t)\|^2\,dx \leqslant C_f$, we get

$$\|\partial_x\rho_m(t)\|^2 + \theta\|\partial_x u_m(t)\|^2 \leqslant K_3, \quad \forall t > t_1 = t_0 + 1,\ m = 1,2,\ldots,$$

therefore, estimate (6.3) holds.

Integrating (6.7) with respect to t on the interval $[t,t+1]$, we get (6.4), since there exists such independent of m constant K_4 that

$$\int_t^{t+1} \|\partial_x^2\rho_m(s)\|^2 + \theta\|\partial_x^2 u_m(s)\|^2\,ds \leqslant K_4, \quad \forall t > t_1,\ m = 1,2,\ldots. \qquad \square$$

7. Positivity and boundedness of the density ρ

To show positivity and boundedness of the solution ρ for Problem (1.4), we first establish an integral identity for ρ and u. Multiplying the first equation of (1.4) by a piecewise differentiable function $m(\rho) = l'(\rho)$ and integrating over $x \in \Omega$, we get

$$\frac{d}{dt}\int_0^L l(\rho)\,dx + \int_0^L \partial_x u \cdot \bigl(l'(\rho)\rho - l(\rho)\bigr)\,dx = -\mu\int_0^L (\partial_x\rho)^2 l''(\rho)\,dx. \tag{7.1}$$

Choosing $l(\rho) = H(\rho)$, where $H(\rho)$ is the nonnegative cut-off function in the form

$$H(y) = \begin{cases} y^2, & y < 0,\\ 0, & y \geqslant 0,\end{cases}$$

we can easily obtain the integral inequality

$$\frac{dz(t)}{dt} + q(t)z(t) \leqslant 0, \tag{7.2}$$

where $z(t) = \int_0^L H(\rho(t,x))\,dx$, and the function $q(t) = \operatorname{ess\,inf}_{x\in\Omega} u_x(t,x)$ belongs to $L^1(0,T)$, see Theorems 4.1 and 6.1.

Clearly, for any nonnegative initial function $\rho_0(x)$,

$$z(0) = \int_0^L H\bigl(\rho_0(x)\bigr)\,dx = 0$$

therefore, the solution $z(t)$ of the differential inequality (7.2) is non-positive. Thus, $\int_0^L H(\rho(t,x))\,dx = 0$ for any $t \in [0,T]$ and $\rho(t,x) \geqslant 0$ a.e. for $x \in \Omega$ and $t \in [0,T]$.

The obtained result can be summarized in the form.

Theorem 7.1. *Let Hypotheses* $(\mathcal{P}_1)$ *and* $(\mathcal{P}_2)$ *be satisfied,* $f \in L^2(0,T;H_2)$ *and* $(\rho_0, u_0) \in V$. *Then, if the initial function* ρ_0 *is nonnegative in* Ω, *the solution of Problem* (1.4) *is such that* $\rho(t,x)$ *is nonnegative for* $(t,x) \in [0,T] \times \Omega$.

Now we proceed to more precise result on the boundedness of the density ρ.

Theorem 7.2. *Let Hypotheses* $(\mathcal{P}_1)$ *and* $(\mathcal{P}_2)$ *be satisfied,* $f \in L^2(0,T;H_2)$ *and* $(\rho_0, u_0) \in V$. *Then the solution of Problem* (1.4) *satisfies the inequality*

$$\rho(t,x) \leqslant \exp\left\{-\int_0^t q(s)\,ds\right\} \operatorname*{ess\,sup}_{x\in\Omega} \rho_0(x), \tag{7.3}$$

where $q(t) = \operatorname{ess\,inf}_{x\in\Omega} u_x(t,x)$. *Moreover, if the initial function* ρ_0 *is nonnegative in* Ω, *then*

$$\rho(t,x) \geqslant \exp\left\{-\int_0^t g(s)\,ds\right\} \operatorname*{ess\,sup}_{x\in\Omega} \rho_0(x), \tag{7.4}$$

where $g(t) = \operatorname{ess\,sup}_{x\in\Omega} u_x(t,x)$.

Proof. To prove this theorem we first choose $l(\rho) = H(\rho/s)$ in (7.1) with $s > 0$, where $H(y)$ is the cut-off function in the form

$$H(y) = \begin{cases} (y-1)^2, & y > 1, \\ 0, & y \leqslant 1. \end{cases}$$

Then identity (7.1) can be rewritten as follows

$$\frac{\partial}{\partial t}\int_0^L H(\rho/s)\,dx + \int_0^L \big(\rho/s \cdot H'(\rho/s) - H(\rho/s)\big)\cdot \partial_x u\,dx = -\mu \int_0^L H''(\rho/s)\cdot(\partial_x\rho)^2/s^2\,dx. \tag{7.5}$$

Integrating both parts of identity (7.5) with respect to s in (λ, ∞), we get

$$\frac{\partial}{\partial t}\int_\lambda^\infty ds \int_0^L H(\rho/s)\,dx + \int_\lambda^\infty ds\int_0^L \big(\rho/s \cdot H'(\rho/s) - H(\rho/s)\big)\cdot \partial_x u\,dx = -\mu \int_\lambda^\infty ds \int_0^L H''(\rho/s)\cdot(\partial_x\rho)^2/s^2\,dx. \tag{7.6}$$

Note that the integral over s in (7.6) is actually taken within finite limits, since, by Theorems 4.1 and 6.1, the solution of Problem (1.4) for ρ is bounded, and $H(\rho/s) = 0$ in some neighborhood of $s = \infty$.

Integrating by parts in (7.6) and taking in account that the left-hand side of this identity is non-positive, we get

$$\frac{\partial}{\partial t}\int_\lambda^\infty ds \int_0^L H(\rho/s)\,dx + \lambda \int_0^L \partial_x u \cdot H(\rho/\lambda)\,dx \leqslant 0.$$

Let

$$q(t) = \operatorname*{ess\,inf}_{x\in\Omega} u_x(t,x), \quad \text{and} \quad v(t,\lambda) = \int_\lambda^\infty ds \int_0^L H\big(\rho(t,x)/s\big)\,dx,$$

then the following differential inequality can be deduced from (7.6)

$$\frac{\partial v(t,\lambda)}{\partial t} - \lambda \cdot q(t)\frac{\partial v(t,\lambda)}{\partial \lambda} \leqslant 0. \tag{7.7}$$

Making in (7.7) the change of variables $(t,\lambda) \to (\tau,\xi)$, where

$$\tau = t, \qquad \xi = \lambda \cdot \exp\left\{\int_0^t q(\sigma)\,d\sigma\right\}; \qquad \hat{v}(\tau,\xi) = v\big(t(\tau,\xi), \lambda(\tau,\xi)\big),$$

we get

$$\frac{\partial v(t,\lambda)}{\partial t} - \lambda \cdot q(t)\frac{\partial v(t,\lambda)}{\partial \lambda} = \frac{\partial \hat{v}(\tau,\xi)}{\partial \tau} \leqslant 0.$$

Thus

$$0 \leqslant v\left(\tau, \xi \exp\left\{-\int_0^\tau q(\sigma)\,d\sigma\right\}\right) = \hat{v}(\tau,\xi) \leqslant \hat{v}(0,\xi) = v(0,\xi).$$

Since $H(\rho_0/s) = 0$ for $\rho_0/s \leqslant 1$, then $v(0,\xi) = 0$ for $\xi > \operatorname{ess\,sup}_{x\in\Omega}\rho_0(x)$, therefore, $v(t,\lambda) = 0$ for $\lambda \geqslant \operatorname{ess\,sup}_{x\in\Omega}\rho_0(x) \times \exp\{-\int_0^t q(\sigma)\,d\sigma\}$. This completes the proof of inequality (7.3).

To show that inequality (7.4) holds, we choose the cut-off function in the form

$$H(y) = \begin{cases} (y-1)^2, & y < 1, \\ 0, & y \geqslant 1. \end{cases} \qquad \square$$

8. Asymptotic behavior of solutions

In this section we apply the *a priori* estimates obtained for solutions of Problem (1.4) in Theorems 4.1 and 6.1 to show their asymptotic behavior.

Let us introduce the metric spaces H^a and $\mathcal{H}^\alpha$, endowed with the norm $\|\cdot\|_H$, where

$$H^a = \left\{(\phi, w) \in H\colon \frac{1}{L}\int_0^L \phi(x)\,dx = a\right\}, \qquad \mathcal{H}^\alpha = \bigcup_{0\leqslant a\leqslant\alpha} H^a.$$

Clearly, the spaces H^a and $\mathcal{H}^\alpha$ are convex. The following theorem about the asymptotic behavior of solutions holds, see also [3].

Theorem 8.1. *Let Hypotheses $\mathcal{P}_1$ and $\mathcal{P}_2$ be satisfied and let the function $f(t,x)$ be independent of t and $f \in H_2$. Then, for every $\alpha \geqslant 0$, the semigroup $S(t)$ associated with Problem (1.4) maps $\mathcal{H}^\alpha$ into itself. It possesses a maximal attractor $\mathcal{A}^\alpha$ in $\mathcal{H}^\alpha$ that is bounded in V, compact and connected in H. Moreover, the semigroup $S(t)$ maps H^α into itself and possesses in H^α a maximal attractor A^α that is compact.*

Proof. The proof of theorem relies on the following proposition.

Proposition 8.2. *(See [8,21].) Let the continuous operators $S(t)$, $t \geqslant 0$, given on the metric space X satisfy the semigroup property, that is, $S(t+\tau) = S(t)S(\tau)$ for all nonnegative t and τ, and $S(0) = I$ with the identity operator I on X. Also let $S(t)$ be uniformly compact for large t. If there exists a bounded set $\mathcal{B}$ of X such that $\mathcal{B}$ is absorbing in X, then the ω-limit set of $\mathcal{B}$,*

$$\mathcal{A} = \omega(\mathcal{B}) = \bigcap_{s\geqslant 0}\overline{\bigcup_{t\geqslant s} S(t)\mathcal{B}},$$

is a compact attractor which attracts the bounded sets of X. It is the maximal bounded attractor in X. Furthermore, if X is a convex set of a Banach space H, and the mapping $t \to S(t)u_0$ is continuous from $[0,+\infty)$ into X, for every u_0 in X, then $\mathcal{A}$ is connected.

The existence of the continuous operators $S(t)$, $t \geqslant 0$, from $X = \mathcal{H}^\alpha$ into itself for fixed $\alpha \geqslant 0$ associated with Problem (1.4) is ensured by Theorem 5.1. Clearly, the operators $S(t)$ satisfy the semigroup property.

Theorem 4.1 guaranties the existence of the absorbing set $\mathcal{B} = \mathcal{B}^\alpha$ in $\mathcal{H}^\alpha$. Moreover, Theorem 6.1 implies the existence of the absorbing set in $\mathcal{H}^\alpha \cap V$ and the uniformly compactness of $S(t)$ for large t. Therefore, the conclusions of Theorem 8.1 are inferred by applying Proposition 8.2. $\square$

The theorem asserts for each fixed $\alpha \geqslant 0$ there exists a compact attractor on the set $\mathcal{H}^\alpha$. This attractor belongs to the absorbing set $\mathcal{B}^\alpha$ which absorbs all bounded subsets of $\mathcal{H}^\alpha$. However, in view of inequality (4.3), this absorbing set does not absorb all bounded sets of H. So, the obtained attractors are not attractors in the usual sense in H. This is a consequence of conservation of mass property in Problem (1.4).

Fig. 2 illustrates numerical approximations of attractors for homogeneous 1D density–velocity model (1.4). There are trivial and nontrivial attractors for the density and velocity. In the trivial attractor the density and velocity are constant over all domain. Fig. 2 shows the examples of the nontrivial attractors for the density and velocity. From this figure we can observe that if the total initial density is large enough, the solutions to Problem (1.4) for the density ρ are attracted to the step functions with the same nonzero density level. The position of the steps can vary from the boundaries to interior of the spatial domain. In the case when the total initial density is too small the density is attracted to the shapes that are similar to the peaks. The height of these peaks is smaller than the nonzero level of the steps corresponding to the case with large total initial density.

Moreover, it is worth noting that the nontrivial density attractors shown in Fig. 2 are similar to the density contours of fish schools, insect swarms, bird flocks with fixed distance between nearest neighbors. This distance corresponds to the nonzero constant density level in the density attractors.

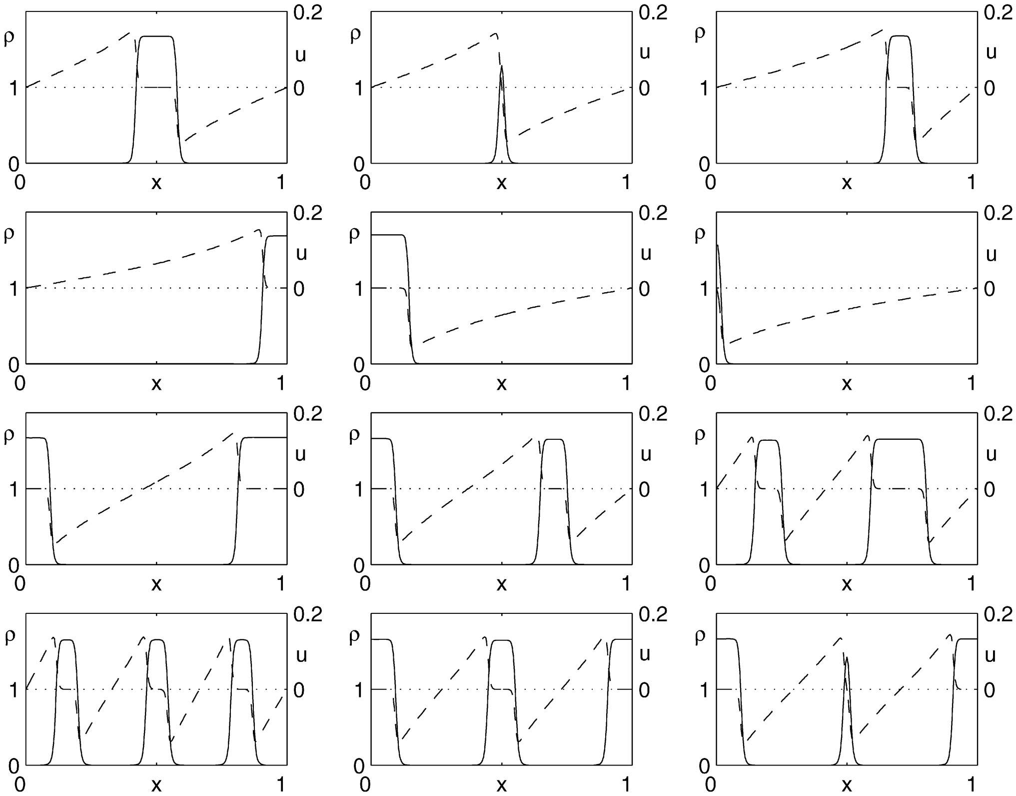

Fig. 2. Examples of nontrivial attractors for the density "—" and velocity "- - -" in the model governed by Eqs. (1.4) with $P'(\rho) = 2|\rho - 0.5| - 1$, $\mu = 0.001$, $\nu = 0.05$ and $f = 0$.

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Appendix A. Proof of Lemma 5.2

Proof. (i) By virtue of (2.7), we get

$$\int_0^T b(u_\alpha, u_\alpha, w)\,dt = -\frac{1}{2}\int_0^T dt \int_0^L (u_\alpha)^2 \partial_x w\,dx.$$

This integral converges to

$$-\frac{1}{2}\int_0^T dt \int_0^L u^2 \partial_x w\,dx = \int_0^T b(u, u, w)\,dt.$$

(ii) It is clear that

$$\int_0^T d(\rho_\alpha, u_\alpha, \phi)\,dt = -\int_0^T dt \int_0^L \rho_\alpha u_\alpha \partial_x \phi\,dx.$$

This integral converges to

$$-\int_0^T dt \int_0^L \rho u \partial_x \phi\,dx = \int_0^T d(\rho, u, \phi)\,dt.$$

(iii) Consider

$$\left| \int_0^T e(\rho_\alpha, w) - e(\rho, w)\,dt \right| \leqslant \|\partial_x w\|_{L^\infty(G^T)} \int_0^T dt \int_0^L \left|P(\rho_\alpha) - P(\rho)\right| dx.$$

Next, for $0 \leqslant l \leqslant 3$:

$$\int_0^T dt \int_0^L \left|P(\rho_\alpha) - P(\rho)\right| dx \leqslant \int_0^T dt \int_0^L p_2\big(|\rho|^l + |\rho_\alpha|^l + 1\big) \cdot |\rho_\alpha - \rho|\,dx$$

$$\leqslant \|\rho_\alpha - \rho\|_{L^2(G^T)} \sqrt{\int_0^T dt \int_0^L p_2^2\big(|\rho|^3 + |\rho_\alpha|^3 + 3\big)^2 dx} \leqslant C\|\rho_\alpha - \rho\|_{L^2(G^T)},$$

because, by the Agmon inequality

$$\int_0^T dt \int_0^L \rho^6\,dx \leqslant \|\rho\|^2_{L^\infty(0,T;H_1)} \int_0^T \left\|\rho(t)\right\|^4_\infty dt \leqslant \left(c^A\right)^2 \|\rho\|^2_{L^\infty(0,T;H_1)} \int_0^T \left\|\rho(t)\right\|^2 \left\|\partial_x \rho(t)\right\|^2 dt$$

$$\leqslant \left(c^A\right)^2 \|\rho\|^4_{L^\infty(0,T;H_1)} \|\rho\|^2_{L^2(0,T;V_1)} < \infty.$$

This completes the lemma. □

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