J. Math. Anal. Appl. 345 (2008) 825–844

Contents lists available at ScienceDirect

J. Math. Anal. Appl.

www.elsevier.com/locate/jmaa

Sharp estimates of the Kobayashi metric and Gromov hyperbolicity

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ARTICLE INFO

Article history:
Received 23 April 2008
Available online 30 April 2008
Submitted by Steven G. Krantz

Keywords:
Almost complex structure
Kobayashi metric
Gromov hyperbolic space

ABSTRACT

Let $D = \{\rho < 0\}$ be a smooth relatively compact domain in a four-dimensional almost complex manifold (M, J), where ρ is a J-plurisubharmonic function on a neighborhood of $\overline{D}$ and strictly J-plurisubharmonic on a neighborhood of ∂D. We give sharp estimates of the Kobayashi metric. Our approach is based on an asymptotic quantitative description of both the domain D and the almost complex structure J near a boundary point. Following Z.M. Balogh and M. Bonk [Z.M. Balogh, M. Bonk, Gromov hyperbolicity and the Kobayashi metric on strictly pseudoconvex domains, Comment. Math. Helv. 75 (2000) 504–533], these sharp estimates provide the Gromov hyperbolicity of the domain D.

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0. Introduction

One can define different notions of hyperbolicity on a given manifold, based on geometric structures, and it seems natural to try to connect them. For instance, the links between the symplectic hyperbolicity and the Kobayashi hyperbolicity were studied by A.-L. Biolley [3]. In the article [1], Z.M. Balogh and M. Bonk established deep connections between the Kobayashi hyperbolicity and the Gromov hyperbolicity, based on sharp asymptotic estimates of the Kobayashi metric. Since the Gromov hyperbolicity may be defined on any geodesic space, it is natural to understand its links with the Kobayashi hyperbolicity in the most general manifolds on which the Kobayashi metric can be defined, namely the almost complex manifolds. As emphasized by [1], it is necessary to study precisely the Kobayashi metric. Since there is no exact expression of this pseudometric, except for particular domains where geodesics can be determined explicitly, we are interested in the boundary behaviour of the Kobayashi metric and in its asymptotic geodesics. One can note that boundary estimates of this invariant pseudometric, whose existence is directly issued from the existence of pseudoholomorphic discs proved by A. Nijenhuis and W. Woolf [20], is also a fundamental tool for the study of the extension of diffeomorphisms and for the classification of manifolds.

The first results in this direction are due to I. Graham [12], who gave boundary estimates of the Kobayashi metric near a strictly pseudoconvex boundary point, providing the (local) complete hyperbolicity near such a point. Considering an L^2-theory approach, D. Catlin [5] obtained similar estimates on pseudoconvex domains of finite type in $\mathbb{C}^2$. A crucial progress in the strictly pseudoconvex case is due to D. Ma [18], who gave an optimal asymptotic description of this metric. His approach is based on a localization principle given by F. Forstneric and J.-P. Rosay [9] using some purely complex analysis arguments as peak holomorphic functions. The estimates proved by D. Ma were used in [1] to prove the Gromov hyperbolicity of relatively compact strictly pseudoconvex domains. The aim of this paper is to obtain sharp estimates of the Kobayashi metric on strictly pseudoconvex domains in four almost complex manifolds.

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doi:10.1016/j.jmaa.2008.04.057

Theorem A. *Let D be a relatively compact strictly J-pseudoconvex smooth domain in a four-dimensional almost complex manifold (M, J). Then for every $\varepsilon > 0$, there exist $0 < \varepsilon_0 < \varepsilon$ and positive constants C and s such that for every $p \in D \cap N_{\varepsilon_0}(\partial D)$ and every $v = v_n + v_t \in T_pM$ we have*

$$e^{-C\delta(p)^s}\left(\frac{|v_n|^2}{4\delta(p)^2} + \frac{\mathcal{L}_J\rho(\pi(p), v_t)}{2\delta(p)}\right)^{\frac{1}{2}} \leqslant K_{(D,J)}(p, v) \leqslant e^{C\delta(p)^s}\left(\frac{|v_n|^2}{4\delta(p)^2} + \frac{\mathcal{L}_J\rho(\pi(p), v_t)}{2\delta(p)}\right)^{\frac{1}{2}}. \tag{0.1}$$

In the above theorem, $\delta(p) := \operatorname{dist}(p, \partial D)$, where dist is taken with respect to a Riemannian metric. For p sufficiently close to the boundary the point $\pi(p)$ denotes the unique boundary point such that $\delta(p) = \|p - \pi(p)\|$. Moreover $N_{\varepsilon_0}(\partial D) := \{q \in M,\ \delta(q) < \varepsilon_0\}$. We point out that the splitting $v = v_n + v_t \in T_pM$ in tangent and normal components in (0.1) is understood to be taken at $\pi(p)$.

As a corollary of Theorem A, we obtain:

Theorem B.

(1) *Let D be a relatively compact strictly J-pseudoconvex smooth domain in an almost complex manifold (M, J) of dimension four. Then the domain D endowed with the Kobayashi integrated distance $d_{(D,J)}$ is a Gromov hyperbolic metric space.*
(2) *Each point in a four-dimensional almost complex manifold admits a basis of Gromov hyperbolic neighborhoods.*

The paper is organized as follows. In Section 1, we give general facts about almost complex manifolds. In Section 2, we show how to deduce Theorem B from Theorem A. Finally, Section 3 is devoted to the proof of our main result, namely Theorem A.

1. Preliminaries

We denote by Δ the unit disc of $\mathbb{C}$ and by Δ_r the disc of $\mathbb{C}$ centered at the origin of radius $r > 0$.

1.1. Almost complex manifolds and pseudoholomorphic discs

An *almost complex structure* J on a real smooth manifold M is a $(1,1)$ tensor field which satisfies $J^2 = -\mathrm{Id}$. We suppose that J is smooth. The pair (M, J) is called an *almost complex manifold*. We denote by J_{st} the standard integrable structure on $\mathbb{C}^n$ for every n. A differentiable map $f : (M', J') \to (M, J)$ between two almost complex manifolds is said to be (J', J)-*holomorphic* if $J(f(p)) \circ d_pf = d_pf \circ J'(p)$, for every $p \in M'$. In case $M' = \Delta \subset \mathbb{C}$, such a map is called a *pseudoholomorphic disc*. If $f : (M, J) \to M'$ is a diffeomorphism, we define an almost complex structure, f_*J on M' as the direct image of J by f:

$$f_*J(q) := d_{f^{-1}(q)}f \circ J\big(f^{-1}(q)\big) \circ d_qf^{-1},$$

for every $q \in M'$.

The following lemma (see [10]) states that locally any almost complex manifold can be seen as the unit ball of $\mathbb{C}^n$ endowed with a small smooth perturbation of the standard integrable structure J_{st}.

Lemma 1.1. *Let (M, J) be an almost complex manifold, with J of class C^k, $k \geqslant 0$. Then for every point $p \in M$ and every $\lambda_0 > 0$ there exist a neighborhood U of p and a coordinate diffeomorphism $z : U \to \mathbb{B}$ centered at p (i.e. $z(p) = 0$) such that the direct image of J satisfies $z_*J(0) = J_{\mathrm{st}}$ and $\|z_*(J) - J_{\mathrm{st}}\|_{C^k(\bar{\mathbb{B}})} \leqslant \lambda_0$.*

This is simply done by considering a local chart $z : U \to \mathbb{B}$ centered at p (i.e. $z(p) = 0$), composing it with a linear diffeomorphism to insure $z_*J(0) = J_{\mathrm{st}}$ and dilating coordinates.

So let J be an almost complex structure defined in a neighborhood U of the origin in $\mathbb{R}^{2n}$, and such that J is sufficiently closed to the standard structure in uniform norm on the closure $\overline{U}$ of U. The J-holomorphy equation for a pseudoholomorphic disc $u : \Delta \to U \subseteq \mathbb{R}^{2n}$ is given by

$$\frac{\partial u}{\partial y} - J(u)\frac{\partial u}{\partial x} = 0. \tag{1.1}$$

According to [20], for every $p \in M$, there is a neighborhood V of zero in T_pM, such that for every $v \in V$, there is a J-holomorphic disc u satisfying $u(0) = p$ and $d_0u(\partial/\partial x) = v$.

1.2. Splitting of the tangent space

Assume that J is a diagonal almost complex structure defined in a neighborhood of the origin in $\mathbb{R}^4$ and such that $J(0) = J_{st}$. Consider a basis (ω_1, ω_2) of $(1,0)$ differential forms for the structure J in a neighborhood of the origin. Since J is diagonal, we may choose

$$\omega_j = dz^j - B_j(z)\,d\bar{z}^j, \quad j = 1, 2.$$

Denote by (Y_1, Y_2) the corresponding dual basis of $(1,0)$ vector fields. Then

$$Y_j = \frac{\partial}{\partial z^j} - \beta_j(z)\frac{\partial}{\partial \overline{z^j}}, \quad j = 1, 2.$$

Moreover $B_j(0) = \beta_j(0) = 0$ for $j = 1, 2$. The basis $(Y_1(0), Y_2(0))$ simply coincides with the canonical $(1,0)$ basis of $\mathbb{C}^2$. In particular $Y_1(0)$ is a basis vector of the complex tangent space $T_0^J(\partial D)$ and $Y_2(0)$ is normal to ∂D. Consider now for $t \geqslant 0$ the translation $\partial D - t$ of the boundary of D near the origin. Consider, in a neighborhood of the origin, a $(1,0)$ vector field X_1 (for J) such that $X_1(0) = Y_1(0)$ and $X_1(z)$ generates the J-invariant tangent space $T_z^J(\partial D - t)$ at every point $z \in \partial D - t$, $0 \leqslant t << 1$. Setting $X_2 = Y_2$, we obtain a basis of vector fields (X_1, X_2) on D (restricting D if necessary). Any complex tangent vector $v \in T_z^{(1,0)}(D, J)$ at point $z \in D$ admits the unique decomposition $v = v_t + v_n$ where $v_t = \alpha_1 X_1(z)$ is the tangent component and $v_n = \alpha_2 X_2(z)$ is the normal component. Identifying $T_z^{(1,0)}(D, J)$ with $T_z D$ we may consider the decomposition $v = v_t + v_n$ for each $v \in T_z(D)$. Finally we consider this decomposition for points z in a neighborhood of the boundary.

1.3. Levi geometry

Let ρ be a $\mathcal{C}^2$ real valued function on a smooth almost complex manifold (M, J). We denote by $d_J^c\rho$ the differential form defined by

$$d_J^c\rho(v) := -d\rho(Jv), \tag{1.2}$$

where v is a section of TM. The *Levi form* of ρ at a point $p \in M$ and a vector $v \in T_pM$ is defined by

$$\mathcal{L}_J\rho(p, v) := d\big(d_J^c\rho\big)(p)\big(v, J(p)v\big) = dd_J^c\rho(p)\big(v, J(p)v\big).$$

In case $(M, J) = (\mathbb{C}^n, J_{st})$, then $\mathcal{L}_{J_{st}}\rho$ is, up to a positive multiplicative constant, the usual standard Levi form:

$$\mathcal{L}_{J_{st}}\rho(p, v) = 4\sum \frac{\partial^2\rho}{\partial z_j \partial \overline{z_k}} v_j \overline{v_k}.$$

We investigate now how close is the Levi form with respect to J from the standard Levi form. For $p \in M$ and $v \in T_pM$, we easily get

$$\mathcal{L}_J\rho(p, v) = \mathcal{L}_{J_{st}}\rho(p, v) + d\big(d_J^c - d_{J_{st}}^c\big)\rho(p)\big(v, J(p)v\big) + dd_{J_{st}}^c\rho(p)\big(v, \big(J(p) - J_{st}\big)v\big). \tag{1.3}$$

In local coordinates $(t_1, t_2, \ldots, t_{2n})$ of $\mathbb{R}^{2n}$, (1.3) may be written as follows

$$\mathcal{L}_J\rho(p, v) = \mathcal{L}_{J_{st}}\rho(p, v) + {}^t v\big(A - {}^t A\big)J(p)v + {}^t\big(J(p) - J_{st}\big)vDJ_{st}v + {}^t\big(J(p) - J_{st}\big)vD\big(J(p) - J_{st}\big)v \tag{1.4}$$

where

$$A := \left(\sum_i \frac{\partial u}{\partial t_i}\frac{\partial J_{i,j}}{\partial t_k}\right)_{1\leqslant j,k\leqslant 2n} \quad \text{and} \quad D := \left(\frac{\partial^2 u}{\partial t_j \partial t_k}\right)_{1\leqslant j,k\leqslant 2n}.$$

Let f be a (J', J)-biholomorphism from (M', J') to (M, J). Then for every $p \in M$ and every $v \in T_pM$:

$$\mathcal{L}_{J'}\rho(p, v) = \mathcal{L}_J\rho \circ f^{-1}\big(f(p), d_pf(v)\big).$$

This expresses the invariance of the Levi form under pseudobiholomorphisms.

The next proposition is useful in order to compute the Levi form (see [15]).

Proposition 1.2. *Let $p \in M$ and $v \in T_pM$. Then*

$$\mathcal{L}_J\rho(p, v) = \Delta(\rho \circ u)(0),$$

where $u : \Delta \to (M, J)$ is any J-holomorphic disc satisfying $u(0) = p$ and $d_0u(\partial/\partial_x) = v$.

Proposition 1.2 leads to the following proposition–definition.

Proposition 1.3. *The two statements are equivalent*:

(1) $\rho \circ u$ *is subharmonic for any* J*-holomorphic disc* $u : \Delta \to M$.
(2) $\mathcal{L}_J \rho(p, v) \geqslant 0$ *for every* $p \in M$ *and every* $v \in T_p M$.

If one of the previous statements is satisfied we say that ρ is *J-plurisubharmonic*. We say that ρ is *strictly J-plurisubharmonic* if $\mathcal{L}_J \rho(p, v)$ is positive for any $p \in M$ and any $v \in T_p M \setminus \{0\}$. Plurisubharmonic functions play a very important role in almost complex geometry: they give attraction and localization properties for pseudoholomorphic discs. For this reason the construction of J-plurisubharmonic functions is crucial.

Similarly to the integrable case, one may define the notion of pseudoconvexity in almost complex manifolds. Let D be a domain in (M, J). We denote by $T^J \partial D := T\partial D \cap JT\partial D$ the J-invariant subbundle of $T\partial D$.

Definition 1.4.

(1) The domain D is J-pseudoconvex (respectively is strictly J-pseudoconvex) if $\mathcal{L}_J \rho(p, v) \geqslant 0$ (respectively > 0) for any $p \in \partial D$ and $v \in T_p^J \partial D$ (respectively $v \in T_p^J \partial D \setminus \{0\}$).
(2) A J-pseudoconvex region is a domain $D = \{\rho < 0\}$ where ρ is a $\mathcal{C}^2$ defining function, J-plurisubharmonic on a neighborhood of $\overline{D}$.

We recall that a defining function for D satisfies $d\rho \neq 0$ on ∂D.
We need the following lemma due to E. Chirka [6].

Lemma 1.5. *Let J be an almost complex structure of class $\mathcal{C}^1$ defined in the unit ball $\mathbb{B}$ of $\mathbb{R}^{2n}$ satisfying $J(0) = J_{\mathrm{st}}$. Then there exist positive constants ε and $A_\varepsilon = O(\varepsilon)$ such that the function $\log\|z\|^2 + A_\varepsilon\|z\|$ is J-plurisubharmonic on $\mathbb{B}$ whenever $\|J - J_{\mathrm{st}}\|_{\mathcal{C}^1(\bar{\mathbb{B}})} \leqslant \varepsilon$.*

Proof. This is due to the fact that for $p \in \mathbb{B}$ and $\|J - J_{\mathrm{st}}\|_{\mathcal{C}^1(\bar{\mathbb{B}})}$ sufficiently small, we have

$$\begin{aligned}\mathcal{L}_J A\|z\|(p, v) &\geqslant A\left(\frac{1}{\|p\|} - \frac{2}{\|p\|}\big\|J(p) - J_{\mathrm{st}}\big\| - 2\big(1 + \big\|J(p) - J_{\mathrm{st}}\big\|\big)\|J - J_{\mathrm{st}}\|_{\mathcal{C}^1(\bar{\mathbb{B}})}\right)\|v\|^2 \\ &\geqslant \frac{A}{2\|p\|}\|v\|^2\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_J \ln\|z\|(p, v) &\geqslant \left(-\frac{2}{\|p\|^2}\big\|J(p) - J_{\mathrm{st}}\big\| - \frac{1}{\|p\|^2}\big\|J(p) - J_{\mathrm{st}}\big\|^2 - \frac{2}{\|p\|}\|J - J_{\mathrm{st}}\|_{\mathcal{C}^1(\bar{\mathbb{B}})}\right. \\ &\quad \left. - \frac{2}{\|p\|}\big\|J(p) - J_{\mathrm{st}}\big\|\|J - J_{\mathrm{st}}\|_{\mathcal{C}^1(\bar{\mathbb{B}})}\right)\|v\|^2 \\ &\geqslant -\frac{6}{\|p\|}\|J - J_{\mathrm{st}}\|_{\mathcal{C}^1(\bar{\mathbb{B}})}\|v\|^2.\end{aligned}$$

So taking $A = 24\|J - J_{\mathrm{st}}\|_{\mathcal{C}^1(\bar{\mathbb{B}})}$ the Chirka's lemma follows. □

The strict J-pseudoconvexity of a relatively compact domain D implies that there is a constant $C \geqslant 1$ such that:

$$\frac{1}{C}\|v\|^2 \leqslant \mathcal{L}_J \rho(p, v) \leqslant C\|v\|^2, \tag{1.5}$$

for $p \in \partial D$ and $v \in T_p^J(\partial D)$.

Let ρ be a defining function for D, J-plurisubharmonic on a neighborhood of $\overline{D}$ and strictly J-plurisubharmonic on a neighborhood of the boundary ∂D. Consider the one-form $d_J^c \rho$ defined by (1.2) and let α be its restriction on the tangent bundle $T\partial D$. It follows that $T^J \partial D = \operatorname{Ker}\alpha$. Due to the strict J-pseudoconvexity of ρ, the two-form $\omega := dd_J^c \rho$ is a symplectic form (i.e. nondegenerate and closed) on a neighborhood of ∂D, that tames J. This implies that

$$g_R := \frac{1}{2}\big(\omega(., J.) + \omega(J., .)\big) \tag{1.6}$$

defines a Riemannian metric. We say that $T^J \partial D$ is a *contact structure* and α is *contact form* for $T^J \partial D$. Consequently vector fields in $T^J \partial D$ span the whole tangent bundle $T\partial D$. Indeed if $v \in T^J \partial D$, it follows that $\omega(v, Jv) = \alpha([v, Jv]) > 0$ and thus $[v, Jv] \in T\partial D \setminus T^J \partial D$. We point out that in case $v \in T^J \partial D$, the vector fields v and Jv are orthogonal with respect to the Riemannian metric g_R.

1.4. The Kobayashi pseudometric

The existence of local pseudoholomorphic discs proved by A. Nijenhuis and W. Woolf [20] allows to define the *Kobayashi–Royden pseudometric*, abusively called the *Kobayashi pseudometric*, $K_{(M,J)}$ for $p \in M$ and $v \in T_pM$:

$$K_{(M,J)}(p,v) := \inf\left\{\frac{1}{r} > 0,\ u : \Delta \to (M,J)\, J\text{-holomorphic},\ u(0) = p,\ d_0u(\partial/\partial x) = rv\right\}$$
$$= \inf\left\{\frac{1}{r} > 0,\ u : \Delta_r \to (M,J),\ \ J\text{-holomorphic},\ u(0) = p,\ d_0u(\partial/\partial x) = v\right\}.$$

Since the composition of pseudoholomorphic maps is still pseudoholomorphic, the Kobayashi pseudometric satisfies the decreasing property.

Proposition 1.6. *Let $f : (M', J') \to (M, J)$ be a (J', J)-holomorphic map. Then for any $p \in M'$ and $v \in T_pM'$ we have*

$$K_{(M,J)}\big(f(p), d_pf(v)\big) \leqslant K_{(M',J')}(p,v).$$

Since the structures we consider are smooth enough, we may define the integrated pseudodistance $d_{(M,J)}$ of $K_{(M,J)}$:

$$d_{(M,J)}(p,q) := \inf\left\{\int_0^1 K_{(M,J)}\big(\gamma(t), \dot\gamma(t)\big)\, dt,\ \gamma : [0,1] \to M,\ \gamma(0) = p,\ \gamma(1) = q\right\}.$$

Similarly to the standard integrable case, B. Kruglikov [16] proved that the integrated pseudodistance of the Kobayashi pseudometric coincides with the Kobayashi pseudodistance defined by chains of pseudoholomorphic discs.

We now define the Kobayashi hyperbolicity.

Definition 1.7.

(1) The manifold (M, J) is Kobayashi hyperbolic if the Kobayashi pseudodistance $d_{(M,J)}$ is a distance.
(2) The manifold (M, J) is local Kobayashi hyperbolic at $p \in M$ if there exist a neighborhood U of p and a positive constant C such that

$$K_{(M,J)}(q,v) \geqslant C\|v\|,$$

for every $q \in U$ and every $v \in T_qM$.
(3) A Kobayashi hyperbolic manifold (M, J) is complete hyperbolic if it is complete for the distance $d_{(M,J)}$.

2. Gromov hyperbolicity

In this section we give some backgrounds about Gromov hyperbolic spaces. Furthermore, according to Z.M. Balogh and M. Bonk [1], proving that a domain D with some curvature is Gromov hyperbolic reduces to providing sharp estimates for the Kobayashi metric $K_{(D,J)}$ near the boundary of D.

2.1. Gromov hyperbolic spaces

Let (X, d) be a metric space.

Definition 2.1. The Gromov product of two points $x, y \in X$ with respect to the basepoint $\omega \in X$ is defined by

$$(x|y)_\omega := \frac{1}{2}\big(d(x,\omega) - d(y,\omega) - d(x,y)\big).$$

The Gromov product measures the failure of the triangle inequality to be an equality and is always nonnegative.

Definition 2.2. The metric space X is Gromov hyperbolic if there is a nonnegative constant δ such that for any $x, y, z, \omega \in X$ one has

$$(x|y)_\omega \geqslant \min\big((x|z)_\omega, (z|y)_\omega\big) - \delta. \tag{2.1}$$

We point out that (2.1) can also be written as follows:

$$d(x, y) + d(z, \omega) \leqslant \max\big(d(x, z) + d(y, \omega), d(x, \omega) + d(y, z)\big) + 2\delta, \tag{2.2}$$

for $x, y, z, \omega \in X$.

There is a family of metric spaces for which Gromov hyperbolicity may be defined by means of geodesic triangles. A metric space (X, d) is said to be *geodesic space* if any two points $x, y \in X$ can be joined by a *geodesic segment*, that is the image of an isometry $g : [0, d(x, y)] \to X$ with $g(0) = x$ and $g(d(x, y)) = y$. Such a segment is denoted by $[x, y]$. A *geodesic triangle* in X is the subset $[x, y] \cup [y, z] \cup [z, x]$, where $x, y, z \in X$. For a geodesic space (X, d), one may define equivalently (see [11]) the Gromov hyperbolicity as follows:

Definition 2.3. The geodesic space X is Gromov hyperbolic if there is a nonnegative constant δ such that for any geodesic triangle $[x, y] \cup [y, z] \cup [z, x]$ and any $\omega \in [x, y]$ one has

$$d\big(\omega, [y, z] \cup [z, x]\big) \leqslant \delta.$$

2.2. *Gromov hyperbolicity of strictly pseudoconvex domains in almost complex manifolds of dimension four*

Let $D = \{\rho < 0\}$ be a relatively compact J-strictly pseudoconvex smooth domain in an almost complex manifolds (M, J) of dimension four. Although the boundary of a compact complex manifold with pseudoconvex boundary is always connected, this is not the case in almost complex setting. Indeed D. McDuff obtained in [19] a compact almost complex manifold (M, J) of dimension four, with a disconnected J-pseudoconvex boundary. Since D is globally defined by a smooth function, J-plurisubharmonic on a neighborhood of $\overline{D}$ and strictly J-plurisubharmonic on a neighborhood of the boundary ∂D, it follows that the boundary ∂D of D is connected. Moreover this also implies that there are no J-complex line contained in D and so that $(D, d_{D,J})$ is a metric space.

A $\mathcal{C}^1$ curve $\alpha : [0, 1] \to \partial D$ is *horizontal* if $\dot\alpha(s) \in T^J_{\alpha(s)}\partial D$ for every $s \in [0, 1]$. This is equivalent to $\dot\alpha_n \equiv 0$. Thus we define the *Levi length* of a horizontal curve by

$$\mathcal{L}_J\rho - \text{length}(\alpha) := \int_0^1 \mathcal{L}_J\rho\big(\alpha(s), \dot\alpha(s)\big)^{\frac{1}{2}}\, ds.$$

We point out that, due to (1.6),

$$\mathcal{L}_J\rho - \text{length}(\alpha) = \int_0^1 g_R\big(\alpha(s), \dot\alpha(s)\big)^{\frac{1}{2}}\, ds.$$

Since $T^J\partial D$ is a *contact structure*, a theorem due to Chow [7] states that any two points in ∂D may be connected by a $\mathcal{C}^1$ horizontal curve. This allows to define the *Carnot–Carathéodory metric* as follows:

$$d_H(p, q) := \big\{\mathcal{L}_J\rho - \text{length}(\alpha),\ \alpha : [0, 1] \to \partial D \text{ horizontal},\ \alpha(0) = p,\ \alpha(1) = q\big\}.$$

Equivalently, we may define locally the *Carnot–Carathéodory metric* by means of vector fields as follows. Consider two g_R-orthogonal vector fields $v, Jv \in T^J\partial D$ and the *sub-Riemannian metric* associated to v, Jv:

$$g_{SR}(p, w) := \inf\big\{a_1^2 + a_2^2,\ a_1 v(p) + a_2 (Jv)(p) = w\big\}.$$

For a horizontal curve α, we set

$$g_{SR} - \text{length}(\alpha) := \int_0^1 g_{SR}\big(\alpha(s), \dot\alpha(s)\big)^{\frac{1}{2}}\, ds.$$

Thus we define

$$d_H(p, q) := \big\{g_{SR} - \text{length}(\alpha),\ \alpha : [0, 1] \to \partial D \text{ horizontal},\ \alpha(0) = p,\ \alpha(1) = q\big\}.$$

We point out that for a small horizontal curve α, we have

$$\dot\alpha(s) = a_1(s) v\big(\alpha(s)\big) + a_2(s) J\big(\alpha(s)\big) v\big(\alpha(s)\big).$$

Consequently

$$g_R\big(\alpha(s), \dot\alpha(s)\big) = \big[a_1^2(s) + a_2^2(s)\big] g_R\big(\alpha(s), v\big(\alpha(s)\big)\big).$$

Although the role of the bundle $T^J\partial D$ is crucial, it is not essential to define the Carnot–Carathéodory metric with g_{SR} instead of g_R. Actually, two Carnot–Carathéodory metrics defined with different Riemannian metrics are bi-Lipschitz equivalent (see [14]).

According to A. Bellaiche [2] and M. Gromov [14] and since $T\partial D$ is spanned by vector fields of $T^J\partial D$ and Lie Brackets of vector fields of $T^J\partial D$, balls with respect to the Carnot–Carathéodory metric may be anisotropically approximated. More precisely

Proposition 2.4. *There exists a positive constant C such that for ε small enough and $p \in \partial D$:*

$$\operatorname{Box}\left(p, \frac{\varepsilon}{C}\right) \subseteq \mathbb{B}_H(p,\varepsilon) \subseteq \operatorname{Box}(p, C\varepsilon), \tag{2.3}$$

where $\mathbb{B}_H(p,\varepsilon) := \{q \in \partial D,\ d_H(p,q) < \varepsilon\}$ and $\operatorname{Box}(p,\varepsilon) := \{p + v \in \partial D,\ |v_t| < \varepsilon,\ |v_n| < \varepsilon^2\}$.

The splitting $v = v_t + v_n$ is taken at p. We point out that choosing local coordinates such that $p = 0$, $J(0) = J_{\mathrm{st}}$ and $T_0^J\partial D = \{z_1 = 0\}$, then $\operatorname{Box}(p,\varepsilon) = \partial D \cap Q(0,\epsilon)$, where $Q(0,\epsilon)$ is the classical polydisc $Q(0,\epsilon) := \{z \in \mathbb{C}^2,\ |z_1| < \varepsilon^2,\ |z_2| < \varepsilon\}$.

As proved by Z.M. Balogh and M. Bonk [1], (2.3) allows to approximate the Carnot–Carathéodory metric by a Riemannian anisotropic metric:

Lemma 2.5. *There exists a positive constant C such that for any positive κ,*

$$\frac{1}{C} d_\kappa(p,q) \leqslant d_H(p,q) \leqslant C d_\kappa(p,q),$$

whenever $d_H(p,q) \geqslant 1/\kappa$ for $p, q \in \partial D$. Here, the distance $d_\kappa(p,q)$ is taken with respect to the Riemannian metric g_κ defined by

$$g_\kappa(p,v) := \mathcal{L}_J\rho(p, v_h) + \kappa^2|v_n|^2,$$

for $p \in \partial D$ and $v = v_t + v_n \in T_p\partial D$.

The crucial idea of Z.M. Balogh and M. Bonk [1] to prove the Gromov hyperbolicity of D is to introduce a function on $D \times D$, using the Carnot–Carathéodory metric, which satisfies (2.1) and which is roughly similar to the Kobayashi distance.

For $p \in D$ we define a boundary projection map $\pi : D \to \partial D$ by

$$\delta(p) = \left\| p - \pi(p) \right\| = \operatorname{dist}(p, \partial D).$$

We notice that $\pi(p)$ is uniquely determined only if $p \in D$ is sufficiently close to the boundary. We set

$$h(p) := \delta(p)^{\frac{1}{2}}.$$

Then we define a map $g : D \times D \to [0, +\infty)$ by

$$g(p,q) := 2\log\left(\frac{d_H(\pi(p), \pi(q)) + \max\{h(p), h(q)\}}{\sqrt{h(p)h(q)}}\right),$$

for $p, q \in D$. The map π is uniquely determined only near the boundary. But an other choice of π gives a function g that coincides up to a bounded additive constant that will not disturb our results. The motivation of introducing the map g is related with the Gromov hyperbolic space $\operatorname{Con}(Z)$ defined by M. Bonk and O. Schramm in [4] (see also [13]) as follows. Let (Z, d) be a bounded metric space which does not consist of a single point and set

$$\operatorname{Con}(Z) := Z \times \big(0, \operatorname{diam}(Z)\big].$$

Let us define a map $\widetilde{g} : \operatorname{Con}(Z) \times \operatorname{Con}(Z) \to [0, +\infty)$ by

$$\widetilde{g}\big((z,h), (z',h')\big) := 2\log\left(\frac{d(z,z') + \max\{h,h'\}}{\sqrt{hh'}}\right).$$

M. Bonk and O. Schramm in [4] proved that $(\operatorname{Con}(Z), \widetilde{g})$ is a Gromov hyperbolic (metric) space.

In our case the map g is not a metric on D since two different points $p \neq q \in D$ may have the same projection; nevertheless

Lemma 2.6. *The function g satisfies* (2.2) (*or equivalently* (2.1)) *on D.*

Proof. Let r_{ij} be real nonnegative numbers such that

$$r_{ij} = r_{ji} \quad \text{and} \quad r_{ij} \leqslant r_{ik} + r_{kj},$$

for $i, j, k = 1, \dots, 4$. Then

$$r_{12}r_{34} \leqslant 4 \max(r_{13}r_{24}, r_{14}r_{23}). \tag{2.4}$$

Consider now four points $p_i \in D$, $i = 1, \dots, 4$. We set $h_i = \delta(p_i)^{\frac{1}{2}}$ and $d_{i,j} = d_{(H,J)}(\pi(p_i), \pi(p_j))$. Then applying (2.4) to $r_{ij} = d_{i,j} + \min(h_i, h_j)$, we obtain

$$\begin{aligned} &\big(d_{1,2} + \min(h_1, h_2)\big)\big(d_{3,4} + \max(h_3, h_4)\big) \\ &\quad \leqslant 4 \max\Big(\big(d_{1,3} + \max(h_1, h_3)\big)\big(d_{2,4} + \min(h_2, h_4)\big), \big(d_{1,4} + \min(h_1, h_4)\big)\big(d_{2,3} + \max(h_2, h_3)\big)\Big). \end{aligned}$$

Then:

$$g(p_1, p_2) + g(p_3, p_4) \leqslant \max\big(g(p_1, p_3) + g(p_2, p_4), g(p_1, p_4) + g(p_2, p_3)\big) + 2 \log 4,$$

which proves the desired statement. □

As a direct corollary, if a metric d on D is roughly similar to g, then the metric space (D, d) is Gromov hyperbolic.

Corollary 2.7. *Let d be a metric on D verifying*

$$-C + g(p,q) \leqslant d(p,q) \leqslant g(p,q) + C \tag{2.5}$$

for some positive constant C, and every $p, q \in D$. Then d satisfies (2.2) *and so the metric space (D, d) is Gromov hyperbolic.*

Z.M. Balogh and M. Bonk [1] proved that if the Kobayashi metric (with respect to J_{st}) of a bounded strictly pseudoconvex domain satisfies (0.1), then the Kobayashi distance is rough similar to the function g. Their proof is purely metric and does not use complex geometry or complex analysis. We point out that the strict pseudoconvexity is only needed to obtain (1.5) or the fact that $T\partial D$ is spanned by vector fields of $T^{J_{\mathrm{st}}}\partial D$ and Lie Brackets of vector fields of $T^{J_{\mathrm{st}}}\partial D$. In particular their proof remains valid in the almost complex setting and, consequently, Theorem A implies.

Theorem 2.8. *Let D be a relatively compact strictly J-pseudoconvex smooth domain in an almost complex manifold (M, J) of dimension four. There is a nonnegative constant C such that for any $p, q \in D$,*

$$g(p,q) - C \leqslant d_{(D,J)}(p,q) \leqslant g(p,q) + C.$$

According to Corollary 2.7 we finally obtain the following theorem (see also (1) of Theorem B).

Theorem 2.9. *Let D be a relatively compact strictly J-pseudoconvex smooth domain in an almost complex manifolds (M, J) of dimension four. Then the metric space $(D, d_{(D,J)})$ is Gromov hyperbolic.*

Example 2.10. There exist a neighborhood U of p and a diffeomorphism $z : U \to \mathbb{B} \subseteq \mathbb{R}^4$, centered at p, such that the function $\|z\|^2$ is strictly J-plurisubharmonic on U and $\|z_*(J) - J_{\mathrm{st}}\|_{\mathcal{C}^2(U)} \leqslant \lambda_0$. Hence the unit ball $\mathbb{B}$ equipped with the metric $d_{(\mathbb{B}(0,1), z_* J)}$ is Gromov hyperbolic.

As a direct corollary of Example 2.10 we have (see also (2) of Theorem B):

Corollary 2.11. *Let (M, J) be a four-dimensional almost complex manifold. Then every point $p \in M$ has a basis of Gromov hyperbolic neighborhoods.*

3. Sharp estimates of the Kobayashi metric

In this section we give a precise localization principle for the Kobayashi metric and we prove Theorem A.

Let $D = \{\rho < 0\}$ be a domain in an almost complex manifold (M, J), where ρ is a smooth defining strictly J-plurisubharmonic function. For a point $p \in D$ we define

$$\delta(p) := \mathrm{dist}(p, \partial D), \tag{3.1}$$

and for p sufficiently close to ∂D, we define $\pi(p) \in \partial D$ as the unique boundary point such that:

$$\delta(p) = \|p - \pi(p)\|. \tag{3.2}$$

For $\varepsilon > 0$, we introduce

$$N_\varepsilon := \{p \in D,\ \delta(p) < \varepsilon\}. \tag{3.3}$$

3.1. Sharp localization principle

F. Forstneric and J.-P. Rosay [9] obtained a sharp localization principle of the Kobayashi metric near a strictly J_{st}-pseudoconvex boundary point of a domain $D \subset \mathbb{C}^n$. However their approach is based on the existence of some holomorphic peak function at such a point; this is purely complex and cannot be generalized in the nonintegrable case. The sharp localization principle we give is based on some estimates of the Kobayashi length of a path near the boundary.

Proposition 3.1. *There exists a positive constant r such that for every $p \in D$ sufficiently close to the boundary and for every sufficiently small neighborhood U of $\pi(p)$ there is a positive constant c such that for every $v \in T_pM$:*

$$K_{(D\cap U,J)}(p,v) \geqslant \left(1 - c\delta(p)^r\right)K_{(D\cap U,J)}(p,v). \tag{3.4}$$

We will give later a more precise version of Proposition 3.1, where the constants c and r are given explicitly (see Lemma 3.4).

Proof. We consider a local diffeomorphism z centered at $\pi(p)$ from a sufficiently small neighborhood U of $\pi(p)$ to $z(U)$ such that

(1) $z(p) = (\delta(p), 0)$,
(2) the structure z_*J satisfies $z_*J(0) = J_{\mathrm{st}}$ and is diagonal,
(3) the defining function $\rho \circ z^{-1}$ is locally expressed by

$$\rho \circ z^{-1}(z) = -2\,\Re e\, z_1 + 2\,\Re e \sum \rho_{j,k} z_j z_k + \sum \rho_{j,\bar{k}} z_j \overline{z_k} + O\left(\|z\|^3\right),$$

where $\rho_{j,k}$ and $\rho_{j,\bar{k}}$ are constants satisfying $\rho_{j,k} = \rho_{k,j}$ and $\rho_{j,\bar{k}} = \overline{\rho}_{k,\bar{j}}$.

According to Lemma 4.8 in [17], there exists a positive constant c_1 ($C_{1/4}$ in the notations of [17]), independent of p, such that, shrinking U if necessary, for any $q \in D \cap U$ and any $v \in T_q\mathbb{R}^4$:

$$K_{(D,J)}(q,v) \geqslant c_1 \frac{\|d_q\chi(v)\|}{\chi(q)},$$

where $\chi(q) := |z_1(q)|^2 + |z_2(q)|^4$.

Let $u : \Delta \to D$ be a J-holomorphic discs satisfying $u(0) = p \in D$. Assume that $u(\Delta) \not\subset D \cap U$ and let $\zeta \in \Delta$ such that $u(\zeta) \in D \cap \partial U$. We consider a $\mathcal{C}^\infty$ path $\gamma : [0;1] \to D$ from $u(\zeta)$ to the point p; so $\gamma(0) = u(\zeta)$ and $\gamma(1) = p$. Without loss of generality we may suppose that $\gamma([0,1[) \subseteq D \cap U$. From this we get that the Kobayashi length of γ satisfies

$$L_{(D,J)}(\gamma) := \int_0^1 K_{(D,J)}\left(\gamma(t), \dot{\gamma}(t)\right) dt \geqslant c_1 \int_0^1 \frac{\|d_{\gamma(t)}\chi(\dot{\gamma}(t))\|}{\chi(\gamma(t))}\, dt.$$

This leads to

$$L_{(D,J)}(\gamma) \geqslant c_1 \int_{\chi(p)}^{\chi(u(s\zeta))} \frac{dt}{t} = c_1 \left|\log \frac{\chi(u(s\zeta))}{\chi(p)}\right| = c_1 \log \frac{\chi(u(s\zeta))}{\chi(p)},$$

for p sufficiently small. Since there exists a positive constant $c_2(U)$ such that for all $z \in D \cap \partial U$:

$$\chi(z) \geqslant c_2(U),$$

and since $\chi(p) = \delta(p)^2$ it follows that

$$L_{(D,J)}(\gamma) \geqslant c_1 \log \frac{c_2(U)}{\delta(p)^2}. \tag{3.5}$$

We set $c_3(U) = c_1 \log(c_2(U))$.

According to the decreasing property of the Kobayashi distance, we have

$$d_{(D,J)}\big(p, u(\zeta)\big) \leqslant d_{(\Delta, J_{\mathrm{st}})}(0,\zeta) = \log \frac{1+|\zeta|}{1-|\zeta|}. \tag{3.6}$$

Due to (3.5) and (3.6) we have

$$\frac{e^{c_3(U)} - \delta(p)^{2c_1}}{e^{c_3(U)} + \delta(p)^{2c_1}} \leqslant |\zeta|,$$

and so for p sufficiently close to its projection point $\pi(p)$:

$$1 - 2e^{-c_3(U)}\delta(p)^{2c_1} \leqslant |\zeta|.$$

This finally proves that

$$u(\Delta_s) \subset D \cap U$$

with $s := 1 - 2e^{-c_3(U)}\delta(p)^{2c_1}$. $\square$

3.2. Sharp estimates of the Kobayashi metric

In this subsection we give the proof of Theorem A.

Proof of Theorem A. Let $p \in D \cap N_{\varepsilon_0}$ with ε_0 small enough and set $\delta := \delta(p)$. Considering a local diffeomorphism $z : U \to z(U) \subset \mathbb{R}^4$ such that Proposition 3.1 holds, me may assume that:

(1) $\pi(p) = 0$ and $p = (\delta, 0)$.
(2) $D \cap U \subset \mathbb{R}^4$.
(3) The structure J is diagonal and coincides with J_{st} on the complex tangent space $\{z_1 = 0\}$:

$$J_{\mathbb{C}} = \begin{pmatrix} a_1 & \overline{b_1} & 0 & 0 \\ b_1 & \overline{a_1} & 0 & 0 \\ 0 & 0 & a_2 & \overline{b_2} \\ 0 & 0 & a_2 & \overline{a_2} \end{pmatrix}, \tag{3.7}$$

with

$$\begin{cases} a_l = i + O\big(\|z_1\|^2\big), \\ b_l = O\big(\|z_1\|\big), \end{cases}$$

for $l = 1, 2$.
(4) The defining function ρ is expressed by

$$\rho(z) = -2\,\Re e\, z_1 + 2\,\Re e \sum \rho_{j,k} z_j z_k + \sum \rho_{j,\bar k} z_j \overline{z_k} + O\big(\|z\|^3\big),$$

where $\rho_{j,k}$ and $\rho_{j,\bar k}$ are constants satisfying $\rho_{j,k} = \rho_{k,j}$ and $\rho_{j,\bar k} = \overline{\rho}_{k,\bar j}$.

Since the structure J is diagonal, the Levi form of ρ at the origin with respect to the structure J coincides with the Levi form of ρ at the origin with respect to the structure J_{st} on the complex tangent space. It follows essentially from [10].

Lemma 3.2. *Let $v_2 = (0, v_2) \in \mathbb{R}^4$ be a tangent vector to ∂D at the origin. We have*

$$\rho_{2,\bar 2}|v_2|^2 = \mathcal{L}_{J_{\mathrm{st}}}\rho(0, v_2) = \mathcal{L}_J \rho(0, v_2). \tag{3.8}$$

Proof. Let $u : \Delta \to \mathbb{C}^2$ be a J-holomorphic disc such that $u(0) = 0$ and tangent to v_2,

$$u(\zeta) = \zeta v_2 + \mathcal{O}\big(|\zeta|^2\big).$$

Since J is a diagonal structure, the J-holomorphy equation leads to

$$\frac{\partial u_1}{\partial \bar\zeta} = q_1(u) \overline{\frac{\partial u_1}{\partial \zeta}}, \tag{3.9}$$

where $q_1(z) = O(\|z\|)$. Moreover, since $d_0 u_1 = 0$, (3.9) gives

$$\frac{\partial^2 u_1}{\partial\zeta\partial\bar{\zeta}}(0) = 0.$$

This implies that

$$\frac{\partial^2 \rho \circ u}{\partial\zeta\partial\bar{\zeta}}(0) = \rho_{2,\bar{2}}|v_t|^2.$$

Thus, the Levi form with respect to J coincides with the Levi form with respect to J_{st} on the complex tangent space of ∂D^δ at the origin. $\square$

Remark 3.3. More generally, even if $J(0) = J_{\mathrm{st}}$, the Levi form of a function ρ with respect to J at the origin does not coincide with the Levi form of ρ with respect to J_{st}. According to Lemma 3.2 if the structure is diagonal then they are equal at the origin on the complex tangent space; but in real dimension greater than four, the structure cannot be (generically) diagonal. K. Diederich and A. Sukhov [8] proved that if the structure J satisfies $J(0) = J_{\mathrm{st}}$ and $d_z J = 0$ (which is always possible by a local diffeomorphism in arbitrary dimensions), then the Levi forms coincide at the origin (for all the directions).

Lemma 3.2 implies that since the domain D is strictly J pseudoconvex at $\pi(p) = 0$, we may assume that $\rho_{2,\bar{2}} = 1$.

Consider the following biholomorphism Φ (for the standard structure J_{st}) that removes the harmonic term $2\Re e(\rho_{2,2}z_2^2)$:

$$\Phi(z_1, z_2) := \big(z_1 - \rho_{2,2}z_2^2, z_2\big). \tag{3.10}$$

The complexification of the structure $\Phi_* J$ admits the following matricial representation:

$$(\Phi_* J)_{\mathbb{C}} = \begin{pmatrix} a_1(\Phi^{-1}(z)) & \overline{b_1(\Phi^{-1}(z))} & c_1(z) & \overline{c_2(z)} \\ b_1(\Phi^{-1}(z)) & \overline{a_1(\Phi^{-1}(z))} & c_2(z) & \overline{c_1(z)} \\ 0 & 0 & a_2(\Phi^{-1}(z)) & \overline{b_2(\Phi^{-1}(z))} \\ 0 & 0 & b_2(\Phi^{-1}(z)) & \overline{a_2(\Phi^{-1}(z))} \end{pmatrix}, \tag{3.11}$$

where

$$\begin{cases} c_1(z) := 2\rho_{2,2}z_2\big(a_1\big(\Phi^{-1}(z)\big) - a_2\big(\Phi^{-1}(z)\big)\big), \\ c_2(z) := 2\rho_{2,2}z_2 b_1\big(\Phi^{-1}(z)\big) - \overline{\rho_{2,2}z_2} b_2\big(\Phi^{-1}(z)\big). \end{cases}$$

In what follows, we need a quantitative version of Proposition 3.1. So we consider the following polydisc $Q_{(\delta,\alpha)} := \{z \in \mathbb{C}^2,\ |z_1| < \delta^{1-\alpha},\ |z_2| < c\delta^{\frac{1-\alpha}{2}}\}$ centered at the origin, where c is chosen such that

$$\Phi(D \cap U) \cap \partial Q_{(\delta,\alpha)} \subset \big\{z \in \mathbb{C}^2,\ |z_1| = \delta^{1-\alpha}\big\}. \tag{3.12}$$

Lemma 3.4. *Let $0 < \alpha < 1$ be a positive number. There is a positive constant β such that for every sufficiently small δ we have*

$$K_{(D\cap U, J)}(p, v) = K_{(\Phi(D\cap U), \Phi_* J)}(p, v) \geqslant \big(1 - 2\delta^\beta\big) K_{(\phi(D\cap U)\cap Q_{(\delta,\alpha)}, \Phi_* J)}(p, v), \tag{3.13}$$

for $p = (\delta, 0)$ and every $v \in T_p\mathbb{R}^4$.

Proof. The proof is a quantitative repetition of the proof of Proposition 3.1; we only notice that according to (3.12) we have $c_2 = \delta^{1-\alpha}$, implying $\beta = 2\alpha c_1$. $\square$

Let $0 < \alpha < \alpha' < 1$ to be fixed later, independently of δ. For every sufficiently small δ, we consider a smooth cut off function $\chi : \mathbb{R}^4 \to \mathbb{R}$:

$$\begin{cases} \chi \equiv 1 & \text{on } Q_{(\delta,\alpha)}, \\ \chi \equiv 0 & \text{on } \mathbb{R}^4 \setminus Q_{(\delta,\alpha')}, \end{cases}$$

with $\alpha' < \alpha$. We point out that χ may be chosen such that

$$\|d_z \chi\| \leqslant \frac{c}{\delta^{1-\alpha'}}, \tag{3.14}$$

for some positive constant c independent of δ. We consider now the following endomorphism of $\mathbb{R}^4$:

$$q'(z) := \chi(z) q(z),$$

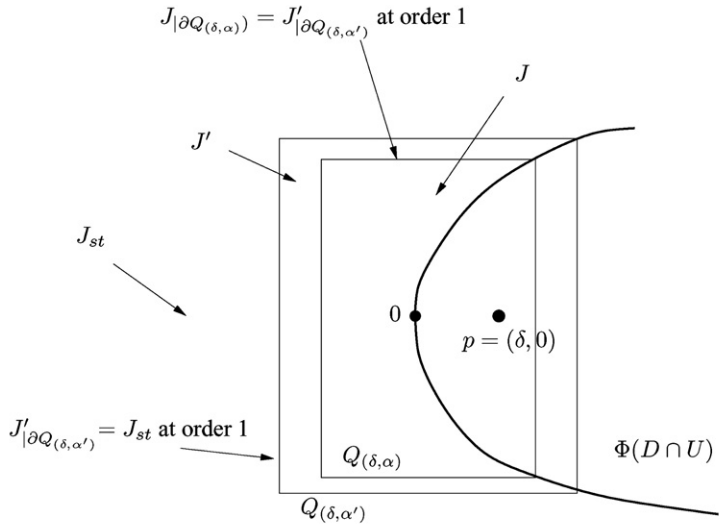

Fig. 1. Extension of the almost complex structure J.

for $z \in Q_{(\delta,\alpha')}$, where

$$q(z) := \big(\Phi_* J(z) + J_{\mathrm{st}}\big)^{-1}\big(\Phi_* J(z) - J_{\mathrm{st}}\big).$$

According to the fact that $q(z) = O(|z_1 + \rho_{2,2} z_2^2|)$ (see (3.11)) and according to (3.14), the differential of q' is upper bounded on $Q_{(\delta,\alpha')}$, independently of δ. Moreover the $dz_2 \otimes \frac{\partial}{\partial z_1}$ and the $dz_2 \otimes \frac{\partial}{\partial \overline{z_1}}$ components of the structure $\Phi_* J$ are $O(|z_1 + \rho_{2,2} z_2^2||z_2|)$ by (3.11); this is also the case for the endomorphism q'. We define an almost complex structure on the whole space $\mathbb{R}^4$ by

$$J'(z) = J_{\mathrm{st}}\big(\mathrm{Id} + q'(z)\big)\big(\mathrm{Id} - q'(z)\big)^{-1},$$

which is well defined since $\|q'(z)\| < 1$. It follows that the structure J' is identically equal to $\Phi_* J$ in $Q_{(\delta,\alpha)}$ and coincides with J_{st} on $\mathbb{R}^4 \setminus Q_{(\delta,\alpha')}$ (see Fig. 1). Notice also that since $\chi \equiv d\chi \equiv 0$ on $\partial Q_{(\delta,\alpha')}$, J' coincides with J_{st} at first order on $\partial Q_{(\delta,\alpha')}$. Finally the structure J' satisfies

$$J' = J_{\mathrm{st}} + O\big(\big|z_1 + \rho_{2,2} z_2^2\big|\big)$$

on $Q_{(\delta,\alpha')}$. To fix the notations, the almost complex structure J' admits the following matricial interpretation:

$$J'_{\mathbb{C}} = \begin{pmatrix} a'_1 & \overline{b'_1} & c'_1 & \overline{c'_2} \\ b'_1 & \overline{a'_1} & c'_2 & \overline{c'_1} \\ 0 & 0 & a'_2 & \overline{b'_2} \\ 0 & 0 & b'_2 & \overline{a'_2} \end{pmatrix}, \tag{3.15}$$

with

$$\begin{cases} a'_l = i + O\big(\|z\|^2\big), \\ b'_l = O\big(\|z\|\big), \\ c'_l = O\big(|z_2|\|z\|\big), \end{cases}$$

for $l = 1, 2$.

Furthermore, according to the decreasing property of the Kobayashi metric we have for $p = (\delta, 0)$:

$$K_{(\Phi(D\cap U)\cap Q_{(\delta,\alpha)},\Phi_* J)}(p, v) = K_{(\Phi(D\cap U)\cap Q_{(\delta,\alpha)},J')}(p, v) \geqslant K_{(\Phi(D\cap U)\cap Q_{(\delta,\alpha')},J')}(p, v). \tag{3.16}$$

Finally, (3.13) and (3.16) lead to

$$K_{(D\cap U,J)}(p, v) \geqslant \big(1 - 2\delta^{\beta}\big) K_{(\Phi(D\cap U)\cap Q_{(\delta,\alpha')},J')}(p, v). \tag{3.17}$$

This implies that in order to obtain the lower estimate of Theorem A it is sufficient to prove lower estimates for $K_{(\Phi(D\cap U)\cap Q_{(\delta,\alpha')},J')}(p,v)$.

We set $\Omega := \Phi(D\cap U)\cap Q_{(\delta,\alpha')}$. Let T_δ be the translation of $\mathbb{C}^2$ defined by

$$T_\delta(z_1,z_2) := (z_1-\delta, z_2),$$

and let φ_δ be a linear diffeomorphism of $\mathbb{R}^4$ such that the direct image of J' by $\varphi_\delta\circ T_\delta\circ\Phi$, denoted by J'^δ, satisfies

$$J'^\delta(0) = J_{\mathrm{st}}. \tag{3.18}$$

To do this we consider a linear diffeomorphism such that its differential at the origin transforms the basis $(e_1, (T_\delta\circ\Phi)_*J'(0)(e_1), e_3, (T_\delta\circ\Phi)_*J'(0)e_3)$ into the canonical basis (e_1,e_2,e_3,e_4) of $\mathbb{R}^4$. According to (3.10) and (3.11), we have

$$(T_\delta\circ\Phi)_*J'(0) = \Phi_*J'(\delta,0) = J'(\delta,0).$$

This means that the endomorphism $(T_\delta\circ\Phi)_*J'(0)$ is block diagonal. This and the fact that $J'(\delta,0) = J'_{\mathrm{st}} + O(\delta)$ imply that the desired diffeomorphism is expressed by

$$\varphi_\delta(z) := \big(z_1 + O\big(\delta|z_1|\big), z_2 + O\big(\delta|z_2|\big)\big), \tag{3.19}$$

for $z\in T_\delta(\Omega)$, and that:

$$\big(J'^\delta\big)_{\mathbb{C}}(z) = \begin{pmatrix} a'_{1,\delta}(z) & \overline{b'_{1,\delta}(z)} & c'_{1,\delta}(z) & \overline{c'_{2,\delta}(z)} \\ b'_{1,\delta}(z) & \overline{a'_{1,\delta}(z)} & c'_{2,\delta}(z) & \overline{c'_{1,\delta}(z)} \\ 0 & 0 & a'_{2,\delta}(z) & \overline{b'_{2,\delta}(z)} \\ 0 & 0 & b'_{2,\delta}(z) & \overline{a'_{2,\delta}(z)} \end{pmatrix}, \tag{3.20}$$

where

$$\begin{cases} a'_{k,\delta}(z) := a'_k\big(\Phi^{-1}\circ T_\delta^{-1}\circ\varphi_\delta^{-1}(z)\big) + O(\delta), \\ b'_{k,\delta}(z) := b'_k\big(\Phi^{-1}\circ T_\delta^{-1}\circ\varphi_\delta^{-1}(z)\big) + O(\delta), \\ c'_{k,\delta}(z) := c'_k\big(T_\delta^{-1}\circ\varphi_\delta^{-1}(z)\big) + O(\delta), \end{cases}$$

for $k=1,2$. Furthermore we notice that the structure J'^δ is constant and equal to $J_{\mathrm{st}} + O(\delta)$ on $\mathbb{R}^4\setminus(\varphi_\delta\circ T_\delta\circ(\Omega))$.

We consider now the following anisotropic dilation Λ_δ of $\mathbb{C}^2$:

$$\Lambda_\delta(z_1,z_2) := \left(\frac{z_1}{z_1+2\delta}, \frac{\sqrt{2\delta}z_2}{z_1+2\delta}\right).$$

Its inverse is given by

$$\Lambda_\delta^{-1}(z) = \left(2\delta\frac{z_1}{1-z_1}, \sqrt{2\delta}\frac{z_2}{1-z_1}\right). \tag{3.21}$$

Let

$$\Psi_\delta := \Lambda_\delta\circ\varphi_\delta\circ T_\delta.$$

We have the following matricial representation for the complexification of the structure $\widetilde{J}^\delta := (\Lambda_\delta)_*J^\delta$:

$$\begin{pmatrix} A'_{1,\delta}(z) & \overline{B'_{1,\delta}(z)} & C'_{1,\delta}(z) & \overline{C'_{2,\delta}(z)} \\ B'_{1,\delta}(z) & \overline{A'_{1,\delta}((z)} & C'_{2,\delta}(z) & \overline{C'_{1,\delta}(z)} \\ D'_{1,\delta}(z) & \overline{D'_{2,\delta}(z)} & A'_{2,\delta}(z) & \overline{B'_{2,\delta}(z)} \\ D'_{2,\delta}(z) & \overline{D'_{1,\delta}(z)} & B'_{2,\delta}(z) & \overline{A'_{2,\delta}(z)} \end{pmatrix}, \tag{3.22}$$

with

$$\begin{cases} A'_{1,\delta}(z) := a'_{1,\delta}\big(\Lambda_\delta^{-1}(z)\big) + \dfrac{1}{\sqrt{2\delta}} z_2 c'_{1,\delta}\big(\Lambda_\delta^{-1}(z)\big), \\ A'_{2,\delta}(z) := a'_{2,\delta}\big(\Lambda_\delta^{-1}(z)\big) - \dfrac{1}{\sqrt{2\delta}} z_2 c'_{1,\delta}\big(\Lambda_\delta^{-1}(z)\big), \\ B'_{1,\delta}(z) := \dfrac{(1-\overline{z_1})^2}{(1-z_1)^2} b'_{1,\delta}\big(\Lambda_\delta^{-1}(z)\big) + \dfrac{1}{\sqrt{2\delta}} \dfrac{(1-\overline{z_1})^2 z_2}{(1-z_1)^2} c'_{2,\delta}\big(\Lambda_\delta^{-1}(z)\big), \\ B'_{2,\delta}(z) := \dfrac{1-\overline{z_1}}{1-z_1} b'_{2,\delta}\big(\Lambda_\delta^{-1}(z)\big) - \dfrac{1}{\sqrt{2\delta}} \dfrac{(1-\overline{z_1})\overline{z_2}}{1-z_1} c'_{2,\delta}\big(\Lambda_\delta^{-1}(z)\big), \\ C'_{1,\delta}(z) := \dfrac{1}{\sqrt{2\delta}}(1-z_1) c'_{1,\delta}\big(\Lambda_\delta^{-1}(z)\big), \\ C'_{2,\delta}(z) := \dfrac{1}{\sqrt{2\delta}} \dfrac{(1-\overline{z_1})^2}{1-z_1} c'_{2,\delta}\big(\Lambda_\delta^{-1}(z)\big), \\ D'_{1,\delta}(z) := \dfrac{z_2}{1-z_1}\big(a'_{2,\delta}\big(\Lambda_\delta^{-1}(z)\big) - a'_{1,\delta}\big(\Lambda_\delta^{-1}(z)\big)\big) - \dfrac{1}{\sqrt{2\delta}} \dfrac{z_2^2}{1-z_1} c'_{1,\delta}\big(\Lambda_\delta^{-1}(z)\big), \\ D'_{2,\delta}(z) := \dfrac{1-\overline{z_1}}{(1-z_1)^2}\big(z_2 b'_{2,\delta}\big(\Lambda_\delta^{-1}(z)\big) - \overline{z_2} b'_{1,\delta}\big(\Lambda_\delta^{-1}(z)\big)\big) - \dfrac{1}{\sqrt{2\delta}} \dfrac{(1-\overline{z_1})|z_2|^2}{(1-z_1)^2} c'_{2,\delta}\big(\Lambda_\delta^{-1}(z)\big). \end{cases}$$

Direct computations lead to

$$\begin{cases} A'_{1,\delta}(z) = a'_1\big(\widetilde{z_1} + \rho_{2,2}\widetilde{z_2}^2, \widetilde{z_2}\big) + \dfrac{1}{\sqrt{2\delta}} z_2 O\big(|\widetilde{z_2}|\big|\widetilde{z_1} + \rho_{2,2}\widetilde{z_2}^2\big|\big) + O(\sqrt{\delta}), \\ B'_{1,\delta}(z) = \dfrac{(1-\overline{z_1})^2}{(1-z_1)^2} b'_1\big(\widetilde{z_1} + \rho_{2,2}\widetilde{z_2}^2, \widetilde{z_2}\big) + \dfrac{1}{\sqrt{2\delta}} \dfrac{(1-\overline{z_1})^2}{1-z_1^2} z_2 O\big(|\widetilde{z_2}|\big|\widetilde{z_1} + \rho_{2,2}\widetilde{z_2}^2\big|\big) + O(\sqrt{\delta}), \\ C'_{1,\delta}(z) = \dfrac{1}{\sqrt{2\delta}}(1-z_1) O\big(|\widetilde{z_2}|\big|\widetilde{z_1} + \rho_{2,2}\widetilde{z_2}^2\big|\big) + O(\sqrt{\delta}), \\ D'_{1,\delta}(z) = \dfrac{z_2}{1-z_1}\big[\big(a'_2 - a'_1\big)\big(\widetilde{z_1} + \rho_{2,2}\widetilde{z_2}^2, \widetilde{z_2}\big)\big] + \dfrac{1}{\sqrt{2\delta}} \dfrac{z_2^2}{1-z_1} O\big(|\widetilde{z_2}|\big|\widetilde{z_1} + \rho_{2,2}\widetilde{z_2}^2\big|\big) + O(\sqrt{\delta}), \end{cases}$$

where

$$\begin{cases} \widetilde{z_1} := 2\delta \dfrac{z_1}{1-z_1} + \delta + O\left(\delta^2 \left|\dfrac{z_1}{1-z_1}\right|\right), \\ \widetilde{z_2} := \sqrt{2\delta} \dfrac{z_2}{1-z_1} + O\left(\delta^{3/2} \left|\dfrac{z_2}{1-z_1}\right|\right). \end{cases}$$

Notice that:

$$\begin{cases} \dfrac{\partial}{\partial z_1}\widetilde{z_1} := 2\delta \dfrac{1}{(1-z_1)^2} + \dfrac{\partial}{\partial z_1} O\left(\delta^2 \left|\dfrac{z_1}{1-z_1}\right|\right), \\ \dfrac{\partial}{\partial z_1}\widetilde{z_2} := -\sqrt{2\delta} \dfrac{z_2}{(1-z_1)^2} + \dfrac{\partial}{\partial z_1} O\left(\delta^{3/2} \left|\dfrac{z_2}{1-z_1}\right|\right). \end{cases}$$

The crucial step is to control $\|\widetilde{J'^\delta} - J_{\mathrm{st}}\|_{\mathcal{C}^1(\overline{\Psi_\delta(\Omega)})}$ by some positive power of δ. Working on a small neighborhood of the unit ball $\mathbb{B}$ (see next Lemma 3.5), it is sufficient to prove that the differential of $\widetilde{J'^\delta}$ is controlled by some positive constant of δ. We first need to determine the behaviour of a point $z = (z_1, z_2) \in \Psi_\delta(\Omega)$ near the infinite point $(1, 0)$. Let $\omega = (\omega_1, \omega_2) \in \Omega$ be such that $\Psi_\delta(\omega) = z$; then:

$$z_1 = \frac{\omega_1 - \delta + O(\delta|\omega_1 - \delta|)}{\omega_1 + \delta + O(\delta|\omega_1 - \delta|)},$$

where the two terms $O(\delta|\omega_1 - \delta|)$ are equal, and so

$$\left|\frac{1}{1-z_1}\right| = \left|\frac{\omega_1 + \delta + O(\delta|\omega_1 - \delta|)}{2\delta}\right| \leqslant c_1 \delta^{-\alpha'}, \tag{3.23}$$

for some positive constant c_1 independent of z. Moreover there is a positive constant c_2 such that

$$|z_2| = \sqrt{2\delta}\left|\frac{\omega_2 + O(\delta|\omega_2|)}{\omega_1 + \delta + O(\delta|\omega_1 - \delta|)}\right| \leqslant c_2 \delta^{\alpha'/2}. \tag{3.24}$$

All the behaviours being equivalent, we focus for instance on the derivative $\frac{\partial}{\partial z_1} D'_{1,\delta}(z)$. In this computation we focus only on terms that play a crucial role:

$$\begin{aligned}\frac{\partial}{\partial z_1} D'_{1,\delta}(z) = {}& -\frac{z_2}{(1-z_1)^2}\big[(a'_2 - a'_1)(\widetilde{z_1} + \rho_{2,2}\widetilde{z_2}^2, \widetilde{z_2})\big] + \frac{z_2}{(1-z_1)}\left[\frac{\partial}{\partial z_1}(a'_2 - a'_1).\left(2\delta\frac{1}{(1-z_1)^2} - 4\rho_{2,2}\delta\frac{z_2^2}{(1-z_1)^3}\right)\right] \\ &+ \frac{z_2}{(1-z_1)}\left[\frac{\partial}{\partial z_2}(a'_2 - a'_1).\sqrt{2\delta}\frac{z_2}{(1-z_1)^2}\right] + \frac{-1}{\sqrt{2\delta}}\frac{z_2^2}{(1-z_1)^2}O\big(|\widetilde{z_2}||\widetilde{z_1} + \rho_{2,2}\widetilde{z_2}^2|\big) \\ &+ \frac{1}{\sqrt{2\delta}}\frac{z_2^2}{1-z_1}\frac{\partial}{\partial z_1}O\big(|\widetilde{z_2}||\widetilde{z_1} + \rho_{2,2}\widetilde{z_2}^2|\big) + R(z).\end{aligned}$$

According to (3.23), (3.24) and the fact that $(a'_2 - a'_1)(z) = O|z|$, it follows that for α' small enough

$$\left|\frac{\partial}{\partial z_1} D'_{1,\delta}(z)\right| \leqslant c\delta^s$$

for positive constants c and s. By similar arguments on other derivatives, it follows that there are positive constants, still denoted by c and s such that

$$\|d\widetilde{J'^\delta}\|_{\mathcal{C}^0(\overline{\Psi_\delta(\Omega)})} \leqslant c\delta^s.$$

In view of the next Lemma 3.5, since $\Psi_\delta(\Omega)$ is bounded, this also proves that

$$\|\widetilde{J'^\delta} - J_{\mathrm{st}}\|_{\mathcal{C}^1(\overline{\Psi_\delta(\Omega)})} \leqslant c\delta^s. \tag{3.25}$$

Moreover on $\mathbb{B}(0,2)\setminus\Psi_\delta(\Omega)$, by similar and easier computations we see that $\|\widetilde{J'^\delta} - J_{\mathrm{st}}\|_{\mathcal{C}^1(\overline{\mathbb{B}(0,2)\setminus\Psi_\delta(\Omega)})}$ is also controlled by some positive constant of δ. This finally implies the crucial control:

$$\begin{cases}\widetilde{J'^\delta}(0) = J_{\mathrm{st}}, \\ \|\widetilde{J'^\delta} - J_{\mathrm{st}}\|_{\mathcal{C}^1(\overline{\mathbb{B}(0,2)})} \leqslant c\delta^s.\end{cases} \tag{3.26}$$

In order to obtain estimates of the Kobayashi metric, we need to localize the domain $\Psi_\delta(\Omega) = \Psi_\delta(\Phi(D\cap U)\cap\Phi(Q_{(\delta,\alpha')}))$ between two balls. This technical result is essentially due to D. Ma [18].

Lemma 3.5. *There exists a positive constant C such that*:

$$\mathbb{B}\big(0, e^{-C\delta^{\alpha'}}\big) \subset \Psi_\delta(\Omega) \subset \mathbb{B}\big(0, e^{C\delta^{\alpha'}}\big).$$

Proof. We have

$$\Psi_\delta(z) = \left(\frac{z_1 - \delta + O(\delta|z_1-\delta|)}{z_1 + \delta + O(\delta|z_1-\delta|)}, \sqrt{2\delta}\frac{z_2 + O(\delta|z_2|)}{z_1 + \delta + O(\delta|z_1-\delta|)}\right). \tag{3.27}$$

Consider the following expression:

$$\begin{aligned}L(z) &:= \big|z_1 + \delta + O\big(\delta|z_1-\delta|\big)\big|^2\big(\|\Psi_\delta(z)\|^2 - 1\big) \\ &= \big|z_1 - \delta + O\big(\delta|z_1-\delta|\big)\big|^2 + 2\delta\big|z_2 + O\big(\delta|z_2|\big)\big|^2 - \big|z_1 + \delta + O\big(\delta|z_1-\delta|\big)\big|^2.\end{aligned}$$

Since $O(\delta|z_1-\delta|)$ in the first and last terms of the right-hand side of the previous equality are equal, this leads to

$$L(z) = 2\delta M(z) + \delta^2 O\big(|z_1|\big) + \delta^2 O\big(|z_2|^2\big),$$

where

$$M(z) := -2\,\Re e\, z_1 + |z_2|^2.$$

Let $z \in \Omega = \Phi(D\cap U)\cap Q_{(\delta,\alpha')}$. For δ small enough, we have

$$\begin{aligned}\big|z_1 + \delta + O\big(\delta|z_1-\delta|\big)\big|^2 &\geqslant |z_1|^2 + \delta^2 + \delta^2 O\big(|z_1| + \delta\big) + \delta O\big(|z_1|^2 + \delta|z_1|\big) + \delta^2 O\big(|z_1| + \delta\big)^2 + 2\delta\,\Re e\, z_1 \\ &\geqslant |z_1|^2 + \delta^2 + \delta O\big(|z_1|^2\big) + \delta^2 O\big(|z_1|\big) + O\big(\delta^3\big) + 2\delta\,\Re e\, z_1 \\ &\geqslant \frac{3}{4}\big(|z_1|^2 + \delta^2\big) + 2\delta\,\Re e\, z_1.\end{aligned} \tag{3.28}$$

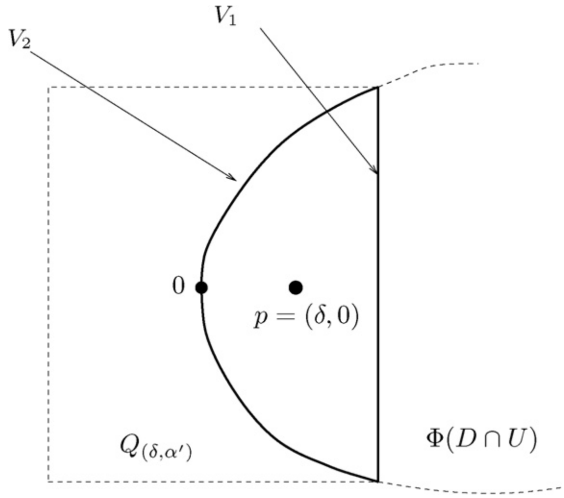

Fig. 2. Boundary of Ω.

Moreover

$$2\,\Re e\, z_1 > 2\,\Re e\, \rho_{1,1} z_1^2 + 2\,\Re e\, \rho_{1,2} z_1 z_2 + \sum \rho_{j,\bar k} z_j \overline{z_k} + O\big(\|z\|^3\big).$$

Since the defining function ρ is strictly J-plurisubharmonic, we know that, for z small enough, $\sum \rho_{j,\bar k} z_j \overline{z_k} + O(\|z\|^3)$ is nonnegative. Hence:

$$2\,\Re e\, z_1 \geqslant 2\,\Re e\, \rho_{1,1} z_1^2 + 2\,\Re e\, \rho_{1,2} z_1 z_2$$

for z sufficiently small and so there is a positive constant C_1 such that:

$$2\,\Re e\, z_1 \geqslant -C_1 |z_1| \|z\|. \tag{3.29}$$

Finally, (3.28) and (3.29) lead to

$$\big|z_1 + \delta + O\big(\delta|z_1 - \delta|\big)\big|^2 \geqslant \frac{1}{2}\big(|z_1|^2 + \delta^2\big)$$

for z small enough. Hence we have

$$\big|\|\Psi_\delta(z)\|^2 - 1\big| = \frac{|L(z)|}{|z_1 + \delta + O(\delta|z_1 - \delta|)|^2} \leqslant \frac{4\delta|M(z)| + \delta^2 O(|z_1|) + \delta^2 O(|z_2|^2)}{|z_1|^2 + \delta^2}. \tag{3.30}$$

The boundary of Ω is equal to $V_1 \cup V_2$ (see Fig. 2), where

$$\begin{cases} V_1 := \Phi(\overline{D \cap U}) \cap \partial Q_{(\delta,\alpha')}, \\ V_2 := \Phi\big(\partial(D \cap U)\big) \cap Q_{(\delta,\alpha')}. \end{cases}$$

Let $z \in V_1$. According (3.30) we have

$$\big|\|\Psi_\delta(z)\|^2 - 1\big| \leqslant \frac{4\delta|M(z)| + \delta^2 O(|z_1|) + \delta^2 O(|z_2|^2)}{|z_1|^2 + \delta^2} \leqslant \frac{4\delta|z_1| + 4\delta|z_2|^2 + C_2\delta^{3-\alpha'}}{\delta^{2-2\alpha'} + \delta^2} \leqslant \frac{C_3\delta^{2-\alpha'}}{\delta^{2-2\alpha'} + \delta^2} \leqslant C_4\delta^{\alpha'},$$

for some positive constants C_1, C_2, C_3 and C_4, and for α' small enough.

If $z \in V_2$, then

$$M(z) = -2\,\Re e\, z_1 + |z_2|^2 = O\big(|z_2|^3 + |z_1|\|z\|\big)$$

and so there is a positive constant C_5 such that:

$$M(z) \leqslant C_5 \delta^{\frac{3}{2}(1-\alpha')}. \tag{3.31}$$

We finally obtain from (3.30) and (3.31):

$$\big|\|\Psi_\delta(z)\|^2 - 1\big| \leqslant 2C_5 \frac{\delta^{\frac{5-3\alpha'}{2}}}{|z_1|^2 + \delta^2} + C_2 \frac{\delta^{3-\alpha'}}{|z_1|^2 + \delta^2} \leqslant 2C_5\delta^{\frac{1-3\alpha'}{2}} + C_2\delta^{1-\alpha'} \leqslant (2C_5 + C_2)\delta^{\frac{1-3\alpha'}{2}}.$$

This proves that:

$$\mathbb{B}\big(0, 1 - C\delta^{\alpha'}\big) \subset \Psi_\delta(\Omega) \subset \mathbb{B}\big(0, 1 + C\delta^{\alpha'}\big),$$

for some positive constant C. □

Lemma 3.5 provides for every $v \in T_0\mathbb{C}^2$:

$$K_{(\mathbb{B}(0,e^{C\delta^{\alpha'}}), \widetilde{J'^\delta})}(0, v) \leqslant K_{(\Psi_\delta(\Omega), \widetilde{J'^\delta})}(0, v) \leqslant K_{(\mathbb{B}(0,e^{-C\delta^{\alpha'}}), \widetilde{J'^\delta})}(0, v). \tag{3.32}$$

Lower estimate

In order to give a lower estimate of $K_{(\mathbb{B}(0,e^{C\delta^{\alpha'}}), \widetilde{J'^\delta})}(0, v)$ we need the following proposition:

Proposition 3.6. *Let $\widetilde{J}$ be an almost complex structure defined on $\mathbb{B} \subseteq \mathbb{C}^2$ such that $\widetilde{J}(0) = J_{\mathrm{st}}$. There exist positive constants ε and $A_\varepsilon = O(\varepsilon)$ such that if $\|\widetilde{J} - J_{\mathrm{st}}\|_{\mathcal{C}^1(\mathbb{B})} \leqslant \varepsilon$ then we have*

$$K_{(\mathbb{B},\widetilde{J})}(0, v) \geqslant \exp\left(-\frac{A_\varepsilon}{2}\right)\|v\|. \tag{3.33}$$

Proof. Due to Lemma 1.5, there exist positive constants ε and $A_\varepsilon = O(\varepsilon)$ such that the function $\log\|z\|^2 + A_\varepsilon\|z\|$ is $\widetilde{J}$-plurisubharmonic on $\mathbb{B}$ if $\|\widetilde{J} - J_{\mathrm{st}}\|_{\mathcal{C}^1(\mathbb{B})} \leqslant \varepsilon$. Consider the function Ψ defined by

$$\Psi(z) := \|z\|^2 e^{A_\varepsilon\|z\|}.$$

Let $u : \Delta \to \mathbb{B}$ be a $\widetilde{J}$-holomorphic disc such that $u(0) = 0$ and $d_0u(\partial/\partial x) = rv$ where $v \in T_q\mathbb{C}^2$ and $r > 0$. For ζ sufficiently close to 0 we have

$$u(\zeta) = q + d_0u(\zeta) + \mathcal{O}\big(|\zeta|^2\big).$$

Setting $\zeta = \zeta_1 + i\zeta_2$ and using the $\widetilde{J}$-holomorphy condition $d_0u \circ J_{\mathrm{st}} = \widetilde{J} \circ d_0u$, we may write

$$d_0u(\zeta) = \zeta_1 d_0u\left(\frac{\partial}{\partial x}\right) + \zeta_2\widetilde{J}\left(d_0u\left(\frac{\partial}{\partial x}\right)\right).$$

This implies

$$\big|d_0u(\zeta)\big| \leqslant |\zeta|\|I + \widetilde{J}\|\left\|d_0u\left(\frac{\partial}{\partial x}\right)\right\|. \tag{3.34}$$

We now consider the following function:

$$\phi(\zeta) := \frac{\Psi(u(\zeta))}{|\zeta|^2} = \frac{\|u(\zeta)\|^2}{|\zeta|^2}\exp\big(A_\varepsilon\big|u(\zeta)\big|\big),$$

which is subharmonic on $\Delta \setminus \{0\}$ since $\log\phi$ is subharmonic. According to (3.34) $\limsup_{\zeta\to 0}\phi(\zeta)$ is finite. Moreover setting $\zeta_2 = 0$ we have

$$\limsup_{\zeta\to 0}\phi(\zeta) \geqslant \left\|d_0u\left(\frac{\partial}{\partial x}\right)\right\|^2.$$

Applying the maximum principle to a subharmonic extension of ϕ on Δ we obtain the inequality:

$$\left\|d_0u\left(\frac{\partial}{\partial x}\right)\right\|^2 \leqslant \exp A_\varepsilon.$$

Hence, by definition of the Kobayashi infinitesimal metric, we obtain for every $q \in D \cap V$, $v \in T_qM$:

$$K_{(D,\widetilde{J})}(q, v) \geqslant \exp\left(-\frac{A_\varepsilon}{2}\right)\|v\|. \tag{3.35}$$

This gives the desired estimate (3.33). □

In order to apply Proposition 3.6 to the structure $\widetilde{J'^\delta}$, it is necessary to dilate isotropically the ball $\mathbb{B}(0, e^{C\delta^{\alpha'}})$ to the unit ball $\mathbb{B}$. So consider the dilation of $\mathbb{C}^2$:

$$\Gamma(z) = e^{-C\delta^{\alpha'}} z,$$
$$K_{(\mathbb{B}(0,e^{C\delta^{\alpha'}}),\widetilde{J'^{\delta}})}(0,v) = e^{-C\delta^{\alpha'}} K_{(\mathbb{B},\Gamma_*\widetilde{J'^{\delta}})}(0,v). \tag{3.36}$$

According to (3.32) we obtain

$$e^{-C\delta^{\alpha'}} K_{(\mathbb{B},\Gamma_*\widetilde{J'^{\delta}})}(0,v) \leqslant K_{(\Psi_\delta(\Omega),\widetilde{J'^{\delta}})}(0,v). \tag{3.37}$$

Then applying Proposition 3.6 to the structure $\Gamma_*\widetilde{J'^{\delta}} = \widetilde{J'^{\delta}}(e^{C\delta^{\alpha'}}.)$ and to $\varepsilon = c\delta^s$ (see (3.26)) provides the existence of a positive constant C_1 such that:

$$K_{(\mathbb{B},\Gamma_*\widetilde{J'^{\delta}})}(0,v) \geqslant e^{-C_1\delta^s}\|v\|. \tag{3.38}$$

Moreover

$$K_{(\Omega,J')}\big((\delta,0),v\big) = K_{(\Psi_\delta(\Omega),\widetilde{J'^{\delta}})}\big(0, d_{(\delta,0)}\Psi_\delta(v)\big), \tag{3.39}$$

where

$$d_{(\delta,0)}\Psi_\delta(v) = d_0\Lambda_\delta \circ d_0\varphi_\delta \circ d_{(\delta,0)}T_\delta(v) = \left(\frac{1}{2\delta}\big(v_1 + O(\delta)v_1\big), \frac{1}{\sqrt{2\delta}}\big(v_2 + O(\delta)v_2\big)\right).$$

According to (3.17), (3.38), (3.37) and (3.39), we finally obtain

$$K_{(D,J)}(p,v) \geqslant e^{-C_2\delta^{\beta''}}\left(\frac{|v_1|^2}{4\delta^2} + \frac{|v_2|^2}{2\delta}\right)^{\frac{1}{2}}, \tag{3.40}$$

for some positive constant C_2 and β''.

Upper estimate

Now, we want to prove the existence of a positive constant C_3 such that

$$K_{(D,J)}(p,v) \leqslant e^{C_3\delta^{\alpha'}}\left(\frac{|v_1|^2}{4\delta^2} + \frac{|v_2|^2}{2\delta}\right)^{\frac{1}{2}}.$$

According to the decreasing property of the Kobayashi metric it is sufficient to give an upper estimate for $K_{(\Phi(D\cap U)\cap Q_{(\delta,\alpha)},J)}(p,v)$. Moreover, due to (3.32) and (3.39) it is sufficient to prove

$$K_{(\mathbb{B}(0,e^{-C\delta^{\alpha'}}),\widetilde{J^{\delta}})}(0,v) \leqslant e^{C_4\delta^{\alpha'}}\|v\|. \tag{3.41}$$

In that purpose we need to deform quantitatively a standard holomorphic disc contained in the ball $\mathbb{B}(0,e^{-C\delta^{\alpha'}})$ into a $\widetilde{J^{\delta}}$-holomorphic disc, controlling the size of the new disc, and consequently its derivative at the origin. As previously by dilating isotropically the ball $\mathbb{B}(0,e^{-C\delta^{\alpha'}})$ into the unit ball $\mathbb{B}$, we may suppose that we work on the unit ball endowed with $\widetilde{J^{\delta}}$ satisfying (3.26).

We define for a map g with values in a complex vector space, continuous on $\overline{\Delta}$, and for $z \in \Delta$ the *Cauchy–Green operator* by

$$T_{CG}(g)(z) := \frac{1}{\pi}\int_{\Delta} \frac{g(\zeta)}{z-\zeta}\,dx\,dy.$$

We consider now the operator $\Phi_{\widetilde{J^{\delta}}}$ from $\mathcal{C}^{1,r}(\overline{\Delta},\mathbb{B}(0,2))$ into $\mathcal{C}^{1,r}(\overline{\Delta},\mathbb{R}^4)$ by

$$\Phi_{\widetilde{J^{\delta}}}(u) := \left(\mathrm{Id} - T_{CG}\,q_{\widetilde{J^{\delta}}}(u)\frac{\partial}{\partial z}\right)u,$$

which is well defined since $\widetilde{J^{\delta}}$ satisfying (3.26). Let $u:\Delta\to\mathbb{B}$ be a $\widetilde{J^{\delta}}$-holomorphic disc in $\mathcal{C}^{1,r}(\overline{\Delta},\mathbb{B})$. According to the continuity of the Cauchy–Green operator from $\mathcal{C}^{r}(\overline{\Delta},\mathbb{R}^4)$ into $\mathcal{C}^{1,r}(\overline{\Delta},\mathbb{R}^4)$ and since $\widetilde{J^{\delta}}$ satisfies (3.26), we get

$$\left\|T_{CG}\,q_{\widetilde{J^{\delta}}}(u)\frac{\partial}{\partial z}u\right\|_{\mathcal{C}^{1,r}(\overline{\Delta})} \leqslant c\left\|q_{\widetilde{J^{\delta}}}(u)\frac{\partial}{\partial z}u\right\|_{\mathcal{C}^{r}(\overline{\Delta})} \leqslant c\|q_{\widetilde{J^{\delta}}}\|_{\mathcal{C}^1(\overline{\mathbb{B}})}\|u\|_{\mathcal{C}^{1,r}(\overline{\Delta})} \leqslant c'\big\|\widetilde{J^{\delta}} - J_{\mathrm{st}}\big\|_{\mathcal{C}^1(\overline{\mathbb{B}})}\|u\|_{\mathcal{C}^{1,r}(\overline{\Delta})} \leqslant c''\delta^s\|u\|_{\mathcal{C}^{1,r}(\overline{\Delta})},$$

for some positive constants c, c' and c''. Hence

$$\big(1 - c''\delta^s\big)\|u\|_{\mathcal{C}^{1,r}(\overline{\Delta})} \leqslant \big\|\Phi_{\widetilde{J^{\delta}}}(u)\big\|_{\mathcal{C}^{1,r}(\overline{\Delta})} \leqslant \big(1 + c''\delta^s\big)\|u\|_{\mathcal{C}^{1,r}(\overline{\Delta})}, \tag{3.42}$$

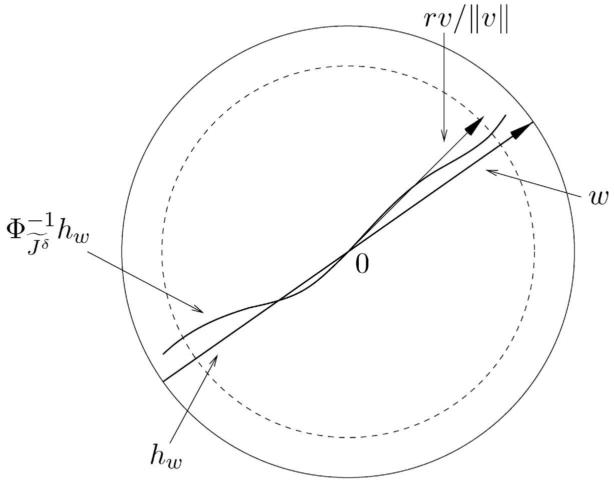

Fig. 3. Deformation of a standard holomorphic disc.

for any $\widetilde{J}^\delta$-holomorphic disc $u:\Delta\to\mathbb{B}$. This implies that the map $\Phi_{\widetilde{J}^\delta}$ is a $\mathcal{C}^1$ diffeomorphism from $\mathcal{C}^{1,r}(\overline{\Delta},\mathbb{B})$ onto $\Phi_{\widetilde{J}^\delta}(\mathcal{C}^{1,r}(\overline{\Delta},\mathbb{B}))$. Furthermore the following property is classical: the disc u is $\widetilde{J}^\delta$-holomorphic if and only if $\Phi_{\widetilde{J}^\delta}(u)$ is J_{st}-holomorphic. According to (3.42), there exists a positive constant c_3 such that for $w\in\mathbb{R}^4$ with $\|w\|=1-c_3\delta^s$, the map $h_w:\Delta\to\mathbb{B}(0,1-c_3\delta^s)$ defined by $h_w(\zeta)=\zeta w$ belongs to $\Phi_{\widetilde{J}^\delta}(\mathcal{C}^{1,r}(\overline{\Delta},\mathbb{B}))$. In particular, the map $\Phi^{-1}_{\widetilde{J}^\delta}(h_w)$ is a $\widetilde{J}^\delta$-holomorphic disc from Δ to the unit ball $\mathbb{B}$.

Consider now $w\in\mathbb{R}^4$ such that $\|w\|=1-c_3\delta^s$, and h_w the associated standard holomorphic disc. Let us estimate the derivative of the $\widetilde{J}^\delta$-holomorphic disc $u:=\Phi^{-1}_{\widetilde{J}^\delta}(h_w)$ at the origin:

$$w=\frac{\partial h}{\partial x}(0)=\frac{\partial}{\partial x}\big(\Phi_{\widetilde{J}^\delta}(u)\big)(0)=\frac{\partial}{\partial x}u(0)+\frac{\partial}{\partial x}T_{CG}q_{\widetilde{J}^\delta}(u)\frac{\partial u}{\partial z}=\frac{\partial}{\partial x}u(0)+T_{CZ}\left(q_{\widetilde{J}^\delta}(u)\frac{\partial u}{\partial z}\right)(0), \tag{3.43}$$

where T_{CZ} denotes the *Calderon–Zygmund* operator. This is defined by

$$T_{CZ}(g)(z):=\frac{1}{\pi}\int_{\Delta}\frac{g(\zeta)}{(z-\zeta)^2}\,dx\,dy,$$

for a map g with values in a complex vector space, continuous on $\overline{\Delta}$ and for $z\in\Delta$, with the integral in the sense of principal value. Since T_{CZ} is a continuous operator from $\mathcal{C}^r(\overline{\Delta},\mathbb{R}^4)$ into $\mathcal{C}^r(\overline{\Delta},\mathbb{R}^4)$, we have

$$\left\|T_{CZ}\left(q_{\widetilde{J}^\delta}(u)\frac{\partial u}{\partial z}\right)(0)\right\|\leqslant c\left\|q_{\widetilde{J}^\delta}(u)\frac{\partial}{\partial z}u\right\|_{\mathcal{C}^r(\overline{\Delta})}\leqslant c'''\delta^s\|u\|_{\mathcal{C}^{1,r}(\overline{\Delta})}, \tag{3.44}$$

for some positive constant c and c'''. Moreover, according to (3.42) we have

$$\|u\|_{\mathcal{C}^{1,r}(\overline{\Delta})}=\big\|\Phi^{-1}_{\widetilde{J}^\delta}(h_w)\big\|_{\mathcal{C}^{1,r}(\overline{\Delta})}\leqslant\big(1+c''\delta^s\big)\|h_w\|_{\mathcal{C}^{1,r}(\overline{\Delta})}\leqslant 2\|w\|. \tag{3.45}$$

Finally (3.43), (3.44) and (3.45) lead to

$$\big(1-2c'''\delta^s\big)\|w\|\leqslant\left\|\frac{\partial}{\partial x}\big(\Phi^{-1}_{\widetilde{J}^\delta}(h_w)\big)(0)\right\|\leqslant\big(1+2c'''\delta^s\big)\|w\|. \tag{3.46}$$

This implies that the map $w\mapsto\frac{\partial}{\partial x}(\Phi^{-1}_{\widetilde{J}^\delta}h_w)(0)$ is a small continuously differentiable perturbation of the identity. More precisely, using (3.46), there exists a positive constant c_4 such that for every vector $v\in\mathbb{R}^4\setminus\{0\}$ and for $r=1-c_4\delta^s$, there is a vector $w\in\mathbb{R}^4$ satisfying $\|w\|\leqslant 1+c_3\delta^s$ and such that $\frac{\partial}{\partial x}(\Phi^{-1}_{\widetilde{J}^\delta}h_w)(0)=rv/\|v\|$ (see Fig. 3).

Hence the $\widetilde{J}^\delta$-holomorphic disc $\Phi^{-1}_{\widetilde{J}^\delta}h_w:\Delta\to\mathbb{B}$ satisfies

$$\begin{cases}\Phi^{-1}_{\widetilde{J}^\delta}h_w(0)=0,\\[4pt] \dfrac{\partial}{\partial x}\Phi^{-1}_{\widetilde{J}^\delta}h_w(0)=r\dfrac{v}{\|v\|}.\end{cases}$$

This proves estimate (3.41), giving the upper estimate of Theorem A.

The lower estimate (3.40) and the upper estimate (3.41) imply estimate (0.1) of Theorem A. □

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