J. Math. Anal. Appl. 349 (2009) 526–543

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Relaxation results for functions depending on polynomials changing sign on rank-one matrices

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ARTICLE INFO

Article history:
Received 11 January 2008
Available online 3 July 2008
Submitted by K.A. Lurie

Keywords:
Quasi-convex
Rank-one convex
Polyconvex envelopes
Polynomial of several variables
Calculus of variations

ABSTRACT

In this paper, we are interested in computing the different convex envelopes of functions depending on polynomials, especially those having it is main part change sign on rank-one matrices. Our main result applies to functions of the type $W(F)=\varphi(P(F))$, $W(F)=\varphi(P(F))+f(\det F)$ or $W(F)=\varphi(P(F))+g(\operatorname{adj}_n F)$ defined on the space of matrices, where φ, $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}^3\to\mathbb{R}$ are three continuous functions, and $P=P_0+P_1+\cdots+P_d$ is a polynomial such that P_d has the property of changing sign on rank-one matrices. Then the polyconvex, quasi-convex and rank-one convex envelopes of W are equal.

1. Introduction

The basic problem of the calculus of variations is to minimize functionals of the type:

$$\operatorname{Min} I(u)=\int_\Omega W\big(\nabla u(x)\big)\,dx$$

over a space of admissible functions, where $\Omega\subset\mathbb{R}^n$ is a smooth domain, $u:\Omega\to\mathbb{R}^m$ is the dependent variable and $\nabla u=(\frac{\partial u_i}{\partial x_j})$ denote the jacobien matrix of u. In nonlinear elasticity (see [2,8]), u stands for the displacement vector and ∇u is the deformation gradient.

To have existence of solutions to the minimization problem by using the direct method in the calculus of variations, one needs to have the weak lower semi-continuity of the functional I:

$$u_j\rightharpoonup u \quad \text{in } W^{1,p}\big(\Omega,\mathbb{R}^m\big) \quad\Rightarrow\quad \frac{1}{|\Omega|}\int_\Omega W\big(\nabla u(x)\big)\,dx\leqslant\frac{1}{|\Omega|}\int_\Omega W\big(\nabla u_j(x)\,dx\big).$$

This property is equivalent to the quasi-convexity of the integrand W (see [3]):

$$W(F)\leqslant\frac{1}{|\Omega|}\int_\Omega W\big(F+\nabla\psi(x)\big)\,dx$$

for any matrix F and any test function $\psi\in W_0^{1,\infty}(\Omega)$. The quasi-convexity was introduced by C.B. Morrey (see [12]), but it is very difficult to check it for a given function is quasi-convex. Two other notions were introduced, one necessary and the other sufficient. The sufficient one is polyconvexity, it was introduced by J. Ball [2] when dealing with problems in nonlinear

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.06.047

elasticity, the second one is known as rank-one-convexity. We say that a function is polyconvex if it can be written as a convex expression of the minors of the matrix and We say that a function W is rank-one convex if:

$$W\big(\lambda A + (1-\lambda)B\big) \leqslant \lambda W(A) + (1-\lambda)W(B)$$

provided that $(A-B)$ is a rank-one matrix.

If the quasi-convexity is not satisfied by the function W, the functional I do not satisfy the weak lower semi-continuity property, and the direct method of the calculus of variations can not be applied to the minimization problem. One way to overcome the situation is to consider the so-called relaxed problem:

$$\inf \int_{\Omega} QW\big(\nabla u(x)\big)\,dx$$

where QW stands for the quasi-convex envelope of the function W, which is the largest quasi-convex function less than W. For more details, see [10]. This process is like the quasi-saddlification used for relaxation of constained problems of optimal design in the absence of analytic description of a G-closure see [16]. The difficult part of the problem is to compute explicitly this envelope.

In the present work, we are interested in the relaxation of a class of integrands defined on the space of matrices depending on polynomial functions having the property of changing sign on the cone of rank-one matrices. We would like to compute explicitly the quasi-convex envelope of the associated functional. In Section 2 we deal with functions depending only on polynomials. Our main result in this situation is the following.

Theorem 1.1. *Let $P(X) = P_0(X) + P_1(X) + \cdots + P_d(X)$ be a polynomial function defined on the space of $n \times m$ real matrices, where each P_i is the part of the polynomial of degree i, such that there exist two rank-one matrices E_1, E_2 verifying*

$$P_d(E_1) > 0, \qquad P_d(-E_1) > 0, \qquad P_d(E_2) < 0, \qquad P_d(-E_2) < 0,$$

and let φ be a real-valued function bounded from bellow, $\mu = \inf \varphi > -\infty$. Then if we consider the function W defined on the space $\mathbb{R}^{n,m}$ of $n \times m$ matrices by the expression:

$$W(F) = \varphi\big(P(F)\big),$$

we have

$$CW(F) = PW(F) = QW(F) = RW(F) = \mu, \quad \forall F \in \mathbb{R}^{n,m}.$$

If the integrand is a sum of two terms, one depending on a polynomial and the other on the determinant of the matrix, then we prove in Section 3 that

Theorem 1.2. *Let $P(X): \mathbb{R}^{n,n} \to \mathbb{R}$, $P = P_0 + P_1 + \cdots + P_d$ be a polynomial function of degree d such that there exists a vector $T \in \mathbb{R}^n$ satisfying*:

$$P_d(0, T, 0, \ldots, 0) < 0, \qquad P_d(0, -T, 0, \ldots, 0) < 0, \qquad P_d(T, 0, \ldots, 0) > 0, \qquad P_d(-T, 0, \ldots, 0) > 0,$$

and let f, φ be two real-valued functions such that f is continuous and φ is bounded from below, $\mu = \inf \varphi$. If we set

$$W(F) = \varphi\big(P(F)\big) + f(\det F),$$

then

$$PW(F) = QW(F) = RW(F) = \mu + Cf(\det F).$$

Finally, in Section 4, we take, instead of the determinant, the function $\operatorname{adj}_n F$ and we prove that

Theorem 1.3. *Let $P = P_0 + P_1 + \cdots + P_d$ be a polynomial of any degree defined on the space $\mathbb{R}^{2,3}$ such that there exists a vector $L \in \mathbb{R}^2$ satisfying the following condition*

$$P_d(L, \alpha L, 0)^T > 0, \qquad P_d(-L, -\alpha L, 0)^T > 0, \qquad P_d(0, \alpha L, L)^T < 0, \qquad P_d(0, -\alpha L, -L)^T < 0, \quad \forall \alpha \in \mathbb{R},$$

and let $\varphi: \mathbb{R} \to \mathbb{R}$, $f: \mathbb{R}^3 \to \mathbb{R}$ be two continuous functions such that φ is bounded from below, $\mu = \inf \varphi$, and

$$W(F) = \varphi\big(P(F)\big) + f\big(\operatorname{adj}_2(F)\big).$$

Then

$$PW(F) = QW(F) = RW(F) = \mu + Cf\big(\operatorname{adj}_2(F)\big).$$

In the different cases, the general method consist to show that the polyconvex and the rank-one convex envelopes are equals.

Several works deal with the topic of computing these relaxed integrands.

(1) The optimal disigh problem: The quasi-convexification of the optimal disigh problem is given in [13] and [14]. For f defined by

$$F \in \mathbb{M}^{m,n} \mapsto f(F) = \begin{cases} 1 + \|F\|^2 & \text{if } F \neq 0, \\ 0 & \text{if } F = 0, \end{cases}$$

we have

(a) If $n = 1$, then

$$Cf(F) = Pf(F) = Qf(F) = Rf(F) = \begin{cases} 1 + \|F\|_2^2 & \text{if } \|F\|_2^2 \geqslant 1, \\ 2\|F\|_2 & \text{if } \|F\|_2^2 \leqslant 1. \end{cases}$$

(b) If $n > 1$, then $Pf(F) = Qf(F) = Rf(F) = h(F)$, where

$$h(F) = \begin{cases} 1 + \|F\|_2^2 & \text{if } \|F\|_2^2 + 2\|\operatorname{adj}_2 F\|_2 \geqslant 1, \\ 2(\|F\|_2^2 + 2\|\operatorname{adj}_2 F\|_2)^{\frac{1}{2}} - 2\|\operatorname{adj}_2 F\|_2 & \text{if } \|F\|_2^2 + 2\|\operatorname{adj}_2 F\|_2 \leqslant 1. \end{cases}$$

(2) In [9] and [10], B. Dacorogna has relaxed some funcionals depending either on determinant, euclidean norme or the function $\operatorname{adj}_n F$.

(a) For $f : \mathbb{M}^{m,n} \to \mathbb{R}$ such that $f(F) = g(\Phi(F))$, where $\Phi : \mathbb{M}^{m,n} \to \mathbb{R}$ is a quasi-affine function (i.e. Φ quasi-convex and $-\Phi$ is quasi-convex), g a real function. Then

$$Pf = Qf = Rf = Cg,$$

and in general

$$Qf > Cf.$$

(b) Let $f : \mathbb{M}^{n+1,n} \to \mathbb{R}$ and $g : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $f(F) = g(\operatorname{adj}_n F)$. Then

$$Pf = Qf = Rf = Cg,$$

and in general

$$Qf > Cf.$$

(c) Let $f : \mathbb{M}^{2,2} \to \mathbb{R}$, g and $h : \mathbb{R} \to \mathbb{R}$ where h is a convex function such that

$$F = (F_{ij})_{1 \leqslant i,j \leqslant 2}, \qquad f(F) = g(F_{11}) + h(\det F)$$

then

$$Pf = Qf = Rf = Cg + h.$$

(d) Let $g : \mathbb{R}_+ \to \mathbb{R}$ such that

$$g(0) = \inf\{g(x): \ x \geqslant 0\}$$

and $f : \mathbb{M}^{m,n} \to \mathbb{R}$ is such that

$$f(F) = g(\|F\|_2),$$

then in general

$$Pf > Cf = Cg.$$

If there exists a real $\alpha \geqslant 0$, verifying:

$$g(\alpha) = g(0) \quad \text{and} \quad Cg(x) = g(x), \quad \forall x \geqslant \alpha,$$

then

$$Cf = Pf.$$

(3) The Saint Venant–Kirchhoff energy is given by:

$$F \in \mathbb{M}^{n,n} \mapsto W(F) = \frac{\lambda}{2}(\operatorname{tr} \bar{E})^2 + \mu \operatorname{tr}(\bar{E})^2,$$

In [15], H. Ledret and A. Raoult have computed QW in terms of the singular values of the matrix:

$$0 \leqslant v_1(F) \leqslant v_2(F) \leqslant v_3(F).$$

For all $x \in \mathbb{R}$ let $[x]_+^2 = x^2$ if $x \geqslant 0$ and $[x]_+^2 = 0$ if $x \leqslant 0$, we define the function Ψ over the set:

$$\Sigma = \{v = (v_1, v_2, v_3) \in \mathbb{R}^3 \colon\ 0 \leqslant v_1 \leqslant v_2 \leqslant v_3\}$$

by:

$$\Psi(v) = \frac{E}{8}\left[v_3^2 - 1\right]_+^2 + \frac{E}{8(1-\upsilon^2)}\left[v_2^2 + \upsilon v_3^2 - (1+\upsilon)\right]_+^2 + \frac{E}{8(1-\upsilon^2)(1-2\upsilon)}\left[(1-\upsilon)v_1^2 + \upsilon\left(v_2^2 + v_3^2\right) - (1+\upsilon)\right]_+^2,$$

then the quasi-convex envelope of W is given by:

$$QW(F) = \Psi\left(v_1(F), v_2(F), v_3(F)\right).$$

In the case $n = 2$, we have:

$$QW(F) = \frac{E}{8(1-\upsilon^2)}\left[\left(v_2(F)\right)^2 - 1\right]_+^2 + \frac{E(1-\upsilon)}{8(1+\upsilon)(1-2\upsilon)}\left[\left(v_1(F)\right)^2 + \frac{\nu}{1-\nu}\left(v_2(F)\right)^2 - \frac{1}{1-\nu}\right]_+^2.$$

(4) The James–Eriksen energy can be writen as: $F = (F_{ij})_{1 \leqslant i,j \leqslant 2} \in \mathbb{M}^{2,2}$

$$\varphi(F) = k_1\left(F_{11}^2 + F_{12}^2 + F_{21}^2 + F_{22}^2 - 2\right)^2 + k_2(F_{11}F_{12} + F_{21}F_{22})^2 + k_3\left(\frac{F_{11}^2 + F_{21}^2 - F_{12}^2 - F_{22}^2}{2} - \varepsilon^2\right)^2,$$

where $k_1, k_2, k_3 \geqslant 0$. In [6] we have:
– if $k_1 = 0$, then

$$C\varphi = P\varphi = Q\varphi = R\varphi = 0;$$

– if $k_3 = 0$, then

$$C\varphi = P\varphi = Q\varphi = R\varphi.$$

If we let for $F \in \mathbb{M}^{2,2}$ $C = F^T F$, then:

$$\begin{cases} R\varphi(F) = 0 & \text{if } \mathrm{Tr}(C) \leqslant 2 \text{ and } 2|C_{12}| \leqslant 2 - \mathrm{Tr}(C), \\ R\varphi(F) = k_1(\mathrm{Tr}\,C - 2)^2 + k_2 C_{12}^2 & \text{if } \mathrm{Tr}\,C \geqslant 2 \text{ and } k_2|C_{12}| \leqslant 2k_1(\mathrm{Tr}\,C - 2), \\ R\varphi(F) = k_1(\mathrm{Tr}\,C - 2)^2 + k_2 C_{12}^2 - \dfrac{(2k_1(\mathrm{Tr}\,C - 2) - k_2|C_{12}|)}{4k_1 + k_2} & \text{if } \begin{cases} \mathrm{Tr}\,C \geqslant 2 \text{ and } k_2|C_{12}| \geqslant 2k_1(\mathrm{Tr}\,C - 2), \\ \text{where} \\ \mathrm{Tr}\,C \leqslant 2 \text{ and } 2|C_{12}| \geqslant 2 - \mathrm{Tr}\,C. \end{cases} \end{cases}$$

(5) In [5] we have considered functions of the type:

$$W(F) = f\left(F^1, F^2, \ldots, F^{n-1}\right) + \varphi(\mathrm{adj}_n F)$$

where $f : (\mathbb{R}^n)^{n-1} \to \mathbb{R}$ is a convex function and $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}$ is continuous, then

$$PW(F) = QW(F) = RW(F) = f\left(F^1, F^2, \ldots, F^{n-1}\right) + C\varphi(\mathrm{adj}_n F).$$

More relaxation results can be found in [4–7,11,17].

2. Case of functions depending only on polynomials

Let us denote by $P(X)$ a polynomial function defined on the space $\mathbb{R}^{n,m}$ of $n \times m$ matrices. $P(X)$ can be expressed as follow:

$$P(X) = P_0(X) + P_1(X) + \cdots + P_d(X)$$

such that $P_0(X)$ is of degree zero (a real constant), P_1 is of degree one, and $P_d(X)$ is of degree d.

Lemma 2.1. *Let $P(X) \in \mathbb{R}[X_1, X_2, \dots X_{nm}]$ such that*

$$P(X) = P_0(X) + P_1(X) + \cdots + P_d(X).$$

Suppose that there exists a rank-one matrix $E \in \mathbb{R}^{n,m}$ such that

$$P_d(E) > 0, \qquad P_d(-E) > 0.$$

Let $\alpha \in \mathbb{R}$ and $F \in \mathbb{R}^{n,m}$ be such that $P(F) \leqslant \alpha$. Then there exist two matrices $B, C \in \mathbb{R}^{n,m}$, a real $\lambda \in [0, 1]$ such that

$$F = \lambda B + (1-\lambda)C,$$

$$\operatorname{rank}(B - C) \leqslant 0,$$

$$P(B) = P(C) = \alpha.$$

Proof. We first choose $B_t = F + tE$. Then for $t = 0$, we have $P(B_0) = P(F) \leqslant \alpha$ and when t tends to $+\infty$, we get:

$$\begin{aligned}
\lim_{t\to+\infty} P(B_t) &= \lim_{t\to+\infty}\big[P_0(B_t) + P_1(B_t) + \cdots + P_d(B_t)\big] \\
&= \lim_{t\to+\infty}\big[P_0(F+tE) + P_1(F+tE) + \cdots + P_d(F+tE)\big] \\
&= \lim_{t\to+\infty}\left[t^d\left(\frac{1}{t^d}P_0 + \frac{1}{t^{d-1}}P_1\left(\frac{1}{t}F + E\right) + \cdots + P_d\left(\frac{1}{t}F + E\right)\right)\right] \\
&= \lim_{t\to+\infty} t^d P^d(E) = +\infty.
\end{aligned}$$

By continuity, there exists a real $t_0 \in]0, +\infty]$ such that $P(B_{t_0}) = \alpha$. For this value of t, we let for $\lambda \in [0, 1[$,

$$C_\lambda = F - \frac{\lambda}{1-\lambda} t_0 E.$$

Clearly, with this choice of C_λ, we have for all $\lambda \in [0, 1[$,

$$\lambda B_{t_0} + (1-\lambda)C_\lambda = F,$$

$$\operatorname{rank}(B_{t_0} - C_\lambda) = \operatorname{rank} E \leqslant 1.$$

We just have to choose λ such that $P(C_\lambda) = \alpha$. For $\lambda = 0$, one has

$$P(C_0) = P(F) \leqslant \alpha$$

and

$$\begin{aligned}
\lim_{t\to 1^-} P(C_\lambda) &= \lim_{t\to 1^-}\big[P_0(C_\lambda) + P_1(C_\lambda) + \cdots + P_d(C_\lambda)\big] \\
&= \lim_{t\to 1^- y}\left[P_0 + P_1\left(F - \frac{\lambda}{1-\lambda}t_0 E\right) + \cdots + P_d\left(F - \frac{\lambda}{1-\lambda}t_0 E\right)\right] \\
&= \lim_{t\to 1^-}\left(\frac{\lambda t_0}{1-\lambda}\right)^d \left[\left(\frac{1-\lambda}{\lambda t_0}\right)^d P_0 + \left(\frac{1-\lambda}{\lambda t_0}\right)^{d-1} P_1\left(\frac{1-\lambda}{\lambda t_0}F - E\right) + \cdots \right. \\
&\quad \left. + \left(\frac{1-\lambda}{\lambda t_0}\right) P_{d-1}\left(\frac{1-\lambda}{\lambda t_0}F - E\right) + P_d\left(\frac{1-\lambda}{\lambda t_0}F - E\right)\right] \\
&= \lim_{t\to 1^-}\left(\frac{\lambda t_0}{1-\lambda}\right)^d \big(P_d(-E)\big) = +\infty.
\end{aligned}$$

By continuity again, there exists a real $\lambda_0 \in [0, 1[$ such that $P(C_{\lambda_0}) = \alpha$ which concludes the proof of Lemma 2.1. □

Theorem 2.1. *Let $P(X) = P_0(X) + P_1(X) + \cdots + P_d(X)$ a polynomial function defined on the space of $n \times m$ real matrices such that there exist two rank-one matrices E_1, E_2 verifying*

$$P_d(E_1) > 0, \qquad P_d(-E_1) > 0, \qquad P_d(E_2) < 0, \qquad P_d(-E_2) < 0,$$

and let φ be a real-valued function bounded from bellow, $\mu = \inf \varphi > -\infty$. Then if we consider the function W defined on the space $\mathbb{R}^{n,m}$ of $n \times m$ matrices by the expression:

$$W(F) = \varphi\big(P(F)\big),$$

we have

$$CW(F) = PW(F) = QW(F) = RW(F) = \mu, \quad \forall F \in \mathbb{R}^{n,m}.$$

Proof. For $\varepsilon > 0$ we take α_ε such that

$$\mu + \varepsilon \geqslant \varphi(\alpha_\varepsilon).$$

Let $F \in \mathbb{R}^{n,m}$.

- Suppose first that $P(F) \leqslant \alpha_\varepsilon$. Thanks to Lemma 2.1, we have the existence of two matrices B_ε and C_ε and a real $\lambda_\varepsilon \in [0,1]$ such that

  $$F = \lambda_\varepsilon B_\varepsilon + (1-\lambda_\varepsilon)C_\varepsilon,$$
  $$\operatorname{rank}(B_\varepsilon - C_\varepsilon) \leqslant 0,$$
  $$P(B_\varepsilon) = P(C_\varepsilon) = \alpha_\varepsilon.$$

  By the rank one convexity of RW, we have:

  $$\begin{aligned} RW(F) &\leqslant \lambda_\varepsilon RW(B_\varepsilon) + (1-\lambda_\varepsilon)RW(C_\varepsilon) \\ &\leqslant \lambda_\varepsilon \varphi\big(P(B_\varepsilon)\big) + (1-\lambda_\varepsilon)\varphi\big(P(C_\varepsilon)\big) \\ &= \varphi(\alpha_\varepsilon) \leqslant \mu + \varepsilon. \end{aligned}$$

  As ε is arbitrary, we conclude that:

  $$P(F) \leqslant \alpha_\varepsilon \quad \Rightarrow \quad RW(F) \leqslant \mu.$$

- Suppose now that $P(F) \geqslant \alpha_\varepsilon$. Then $-P(F) \leqslant -\alpha_\varepsilon$. By Lemma 2.1 applied to $-P$ instead of P and E_2 instead of E_1, we can find two matrices B_ε and C_ε, a real $\lambda_\varepsilon \in [0,1]$ such that

  $$F = \lambda_\varepsilon B_\varepsilon + (1-\lambda_\varepsilon)C_\varepsilon,$$
  $$\operatorname{rank}(B_\varepsilon - C_\varepsilon) \leqslant 0,$$
  $$P(B_\varepsilon) = P(C_\varepsilon) = \alpha_\varepsilon,$$

  and we conclude as in the first case. Therefore we have

  $$\forall F \in \mathbb{R}^{n,m}: \quad \mu \leqslant CW(F) \leqslant PW(F) \leqslant QW(F) \leqslant RW(F) \leqslant \mu.$$

  Then

  $$\forall F \in \mathbb{R}^{n,m}: \quad CW(F) = PW(F) = QW(F) = RW(F) = \mu. \qquad \square$$

Examples.

(1) As an application of the previous relaxation result, we take

$$P(F) = \sum_{i=1}^{n} \big(|F_1|^i - |F_2|^i\big), \quad F = (F_1, F_2) \in R^{2,2}.$$

If we take $\varphi(x) = (x-\alpha)^2$, then $\min \varphi = 0$. Let $W(F) = \varphi(P(F))$. As a consequence of Theorem 2.1, we have

$$PW(F) = QW(f) = RW(f) = 0.$$

(2) In a more general situation we take

$$P(F) = \sum_{i,j \in K} a_{ij}(F_i.F_j)^2 - \sum_{i,j \notin K} a_{ij}(F_i.F_j)^2,$$

where F_i are the entries of the matrix F, $F_i.F_j$ stands for the scalar product, K is any nonempty subset of $\{1,2,\dots,n\} \times \{1,2,\dots,n\}$ and a_{ij} are positive constants. We take $\varphi(x) = (x-\alpha)^2$. Let $W(F) = \varphi(P(F))$ then we have

$$PW(F) = QW(f) = RW(f) = 0.$$

3. Cases including determinant

Now we consider the case of functions having a term depending on determinant, of the form:

$$W(F) = \varphi\big(P(F)\big) + f(\det F).$$

In this case, we prove that the polyconvex, quasi-convex and rank-one convex envelopes are equal to the function $\inf\varphi + Cf(\det F)$, where $\inf\varphi > -\infty$ and Cf is the convexification of the function f. Before proving this, we give some preliminary results.

Lemma 3.1. *For $n \in N$, $F \in \mathbb{R}^{n,n}$, we denote $F = (F_1, F_2, \dots, F_n)$, where $F_1, F_2, \dots, F_n$ are the column vectors of F. Let $P(X)$: $\mathbb{R}^{n,n} \to \mathbb{R}$ be a polynomial function*

$$P(X) = P_0(X) + P_1(X) + \dots + P_d(X),$$

$\alpha \in \mathbb{R}$ is a real number, F is an $n \times n$ matrix such that $F_2 \neq 0$, $P(F) \leqslant \alpha$, and

$$P_d(F_2, 0, 0, \dots, 0) > 0; \qquad P_d(-F_2, 0, 0, \dots, 0) > 0.$$

Then, there exist two matrices B and C such that

$$F = \lambda B + (1-\lambda)C,$$
$$\operatorname{rank}(B - C) \leqslant 0,$$
$$\det B = \det C = \det F,$$
$$P(B) = P(C) = \alpha.$$

Proof. We choose

$$B_t = (F_1 + tF_2, F_2, \dots, F_n), \qquad C_t = \left(F_1 - \frac{\lambda}{(1-\lambda)} tF_2, F_2, \dots, F_n\right).$$

Clearly, we have $F = \lambda B + (1 - \lambda C)$, $\det B_t = \det C_t = \det F$ and $\operatorname{rank}(B_t - C_t) = 1$, $\forall t \in \mathbb{R}$. For $t \in [0, +\infty[$ we let

$$\eta(t) = P\big(B_t\big).$$

η is a continuous function of the variable t on the interval $[0, +\infty[$, $\eta(0) = P(F) \leqslant \alpha$ and

$$\begin{aligned}
\lim_{t\to+\infty} \mu(t) &= \lim_{t\to+\infty} \big[P_0 + P_1(B_t) + \dots + P_d(B_t)\big] \\
&= \lim_{t\to+\infty} t^d \left[\frac{1}{t^d} P_0 + \frac{1}{t^{d-1}} P_1\left(\frac{F_1}{t} + F_2, \frac{F_2}{t}, \dots, \frac{F_n}{t}\right) + \dots + P_d\left(\frac{F_1}{t} + F_2, \frac{F_2}{t}, \dots, \frac{F_n}{t}\right)\right] \\
&= \lim_{t\to+\infty} t^d P_d(F_2, 0, \dots, 0) = +\infty.
\end{aligned}$$

By continuity, there exists a real t_0 such that $\eta(t_0) = \alpha$. For this value t_0, we consider the second equation and we let for $\lambda \in [0, 1[$:

$$\xi(\lambda) = P\left(F_1 - \frac{\lambda}{1-\lambda} t_0 F_2, F_2, \dots, F_n\right).$$

The function ξ is continuous and such that $\xi(0) = P(F) \leqslant \alpha$, Moreover

$$\begin{aligned}
\lim_{\lambda\to1^-} \xi(t) &= \lim_{t\to+\infty} \big[P_0 + P_1(C_{t_0}) + \dots + P_d(C_{t_0})\big] \\
&= \lim_{\lambda\to1^-} \left(\frac{1}{\varepsilon}\right)^d \big[\varepsilon^d P_0 + \varepsilon^{d-1} P_1(\varepsilon F_1 - F_2 \varepsilon F_2, \dots, \varepsilon F_n) + \dots + P_d(\varepsilon F_1 - F_2, \varepsilon F_2, \dots, \varepsilon F_n)\big] \\
&= \lim_{\lambda\to1^-} \varepsilon^d P_d(-F_2, 0, \dots, 0) = +\infty,
\end{aligned}$$

where $\varepsilon = \frac{1-\lambda}{\lambda t_0}$ verifies $\lim_{\lambda\to1^-} \varepsilon = 0$.

By continuity again, there exists a real $\lambda \in [0, 1[$ such that $\xi(\lambda) = \alpha$, and hence

$$P(B) = P(C) = \alpha.$$

The proof of the lemma is finished. □

Lemma 3.2. *Let $P(X) : \mathbb{R}^{n,n} \to \mathbb{R}$ be a polynomial function*

$$P(X) = P_0(X) + P_1(X) + \cdots + P_d(X),$$

$\alpha \in \mathbb{R}$ is a real number, F is an $n \times n$ matrix such that $F_1 \neq 0$, $P(F) \geqslant \alpha$, and

$$P_d(0, F_1, 0, \dots, 0) < 0, \qquad P_d(0, -F_1, 0, \dots, 0) < 0.$$

Then, there exist two matrices B and C such that

$$F = \lambda B + (1-\lambda)C,$$
$$\operatorname{rank}(B - C) \leqslant 1,$$
$$\det B = \det C = \det F,$$
$$P(B) = P(C) = \alpha.$$

The proof of the lemma is the same as in the above case. We have just to take $-P$ instead of P, $-\alpha$ instead of α, and then choose B_t and C_t as follows

$$B_t = (F_1, F_2 + tF_1, F_3, \dots, F_n), \qquad C_t = \left(F_1, F_2 - \frac{\lambda}{(1-\lambda)} tF_1, F_3, \dots, F_n\right).$$

Lemma 3.3. *Let $P(X) : \mathbb{R}^{n,n} \to \mathbb{R}$, $P = P_0 + P_1 + \cdots + P_d$ be a polynomial function of degree d such that there exists a vector $T \in \mathbb{R}^n$ satisfying*:

$$P_d(0, T, 0, \dots, 0) < 0, \qquad P_d(0, -T, 0, \dots, 0) < 0, \qquad P_d(T, 0, \dots, 0) < 0, \qquad P_d(-T, 0, \dots, 0) < 0.$$

Then, for every $n \times n$ matrix F such that $\det(F_1, T, F_3, \dots, F_n) \neq 0$ *there exist two real matrices $B, C \in \mathbb{R}^{n,n}$, and a real $\lambda \in [0, 1]$ such that*

$$F = \lambda B + (1-\lambda)C,$$
$$\operatorname{rank}(B - C) \leqslant 1,$$
$$\det B = \det C = \det F,$$
$$P_d(B_2, 0, \dots, 0) > 0, \qquad P_d(0, B_1, 0, \dots, 0) < 0, \qquad P_d(-B_2, 0, \dots, 0) > 0, \qquad P_d(0, -B_1, 0, \dots, 0) < 0,$$
$$P_d(C_2, 0, \dots, 0) > 0, \qquad P_d(0, C_1, 0, \dots, 0) < 0, \qquad P_d(-C_2, 0, \dots, 0) > 0, \qquad P_d(0, -C_1, 0, \dots, 0) < 0,$$
$$B_1(x) \neq 0, \qquad B_2(x) \neq 0, \qquad C_1(x) \neq 0, \qquad C_2(x) \neq 0.$$

Proof. We choose $B(x)$ and $C(x)$ as follows

$$B(x) = \big(F_1 - (1-\lambda)xT, F_2 - (1-\lambda)\alpha xT, F_3, \dots, F_n\big),$$
$$C(x) = (F_1 + \lambda xT, F_2 + \lambda\alpha xT, F_3, \dots, F_n),$$

with

$$\alpha = -\det \frac{(T, F_2, F_3, \dots, F_n)}{\det(F_1, T, F_3, \dots, F_n)}.$$

Clearly, with this choice we have $F = \lambda B(x) + (1-\lambda)C(x)$, $\operatorname{rank}(B(x) - C(x)) \leqslant 0$, $\det B(x) = \det C(x) = \det F$, and

$$\lim_{x\to+\infty} P_d\big(0, F_1 - (1-\lambda)xT, 0, \dots, 0\big) = \lim_{x\to+\infty} (1-\lambda)^d x^d P_d(0, -T, 0, \dots, 0) = -\infty,$$
$$\lim_{x\to+\infty} P_d\big(0, -\big(F_1 - (1-\lambda)xT\big), 0, \dots, 0\big) = \lim_{x\to+\infty} (1-\lambda)^d x^d P_d(0, T, 0, \dots, 0) = -\infty,$$
$$\lim_{x\to+\infty} P_d\big(F_2 - (1-\lambda)\alpha xT, 0, \dots, 0\big) = \lim_{x\to+\infty} (1-\lambda)^d \alpha^d x^d P_d(-T, 0, \dots, 0) = +\infty,$$
$$\lim_{x\to+\infty} P_d\big(-\big(F_2 - (1-\lambda)\alpha xT\big), 0, \dots, 0\big) = \lim_{x\to+\infty} (1-\lambda)^d \alpha^d x^d P_d(T, 0, \dots, 0) = +\infty.$$

If $\alpha \leqslant 0$ and d is odd we consider the following equations:

$$\lim_{x\to+\infty} P_d\big(F_2 - (1-\lambda)\alpha xT, 0, \dots, 0\big) = \lim_{x\to+\infty} (1-\lambda)^d (-1)^d \alpha^d x^d P_d(T0, \dots, 0) = +\infty,$$
$$\lim_{x\to+\infty} P_d\big(-(F_2 - (1-\lambda)\alpha xT), 0, \dots, 0\big) = \lim_{x\to+\infty} (1-\lambda)^d (-1)^d \alpha^d x^d P_d(-T, 0, \dots, 0) = +\infty.$$

Hence, we can choose a real x_1 such that for all $x \geqslant x_1$, we get $B_1(x) \neq 0$, $B_2(x) \neq 0$ and:

$$P_d\big(B_2(x), 0, \dots, 0\big) > 0, \qquad P_d\big(-B_2(x), 0, \dots, 0\big) > 0,$$
$$P_d\big(0, B_1(x), 0, \dots, 0\big) < 0, \qquad P_d\big(0, -B_1(x), 0, \dots, 0\big) < 0.$$

In the same way, we can choose another real x_2 such that $\forall x \geqslant x_2$, we get $C_1(x) \neq 0$, $C_2(x) \neq 0$ and:

$$P_d\big(C_2(x), 0, \dots, 0\big) > 0, \qquad P_d\big(-C_2(x), 0, \dots, 0\big) > 0,$$
$$P_d\big(0, C_1(x), 0, \dots, 0\big) < 0, \qquad P_d\big(0, -C_1(x), 0, \dots, 0\big) < 0.$$

To finish the proof, we have to choose $x \geqslant \max(x_1, x_2)$. $\square$

The main result of this section is the following:

Theorem 3.1. *Let $P(X) : \mathbb{R}^{n,n} \to \mathbb{R}$, $P = P_0 + P_1 + \cdots + P_d$ be a polynomial function of degree d such that there exists a vector $T \in \mathbb{R}^n$ satisfying*:

$$P_d(0, T, 0, \dots, 0) < 0, \qquad P_d(0, -T, 0, \dots, 0) < 0, \qquad P_d(T, 0, \dots, 0) > 0, \qquad P_d(-T, 0, \dots, 0) > 0,$$

and let f, φ be two real-valued functions such that f is continuous and φ is bounded from below, $\mu = \inf \varphi$. If we set

$$W(F) = \varphi\big(P(F)\big) + f(\det F),$$

then

$$PW(F) = QW(F) = RW(F) = \mu + Cf(\det F).$$

Proof. The function $F \mapsto \mu + Cf(\det F)$ is a polyconvex function less than W, then

$$\mu + Cf(\det F) \leqslant PW(F), \quad \forall F \in \mathbb{R}^{n,n}.$$

Let $F \in \mathbb{R}^{n,n}$, $\varepsilon > 0$ be fixed.

(1) *Step one*. We show that if

$$P_d(0, F_1, 0, \dots, 0) < 0, \qquad P_d(0, -F_1, 0, \dots, 0) > 0, \qquad P_d(F_2, 0, \dots, 0) < 0, \qquad P_d(-F_2, 0, \dots, 0) > 0,$$

then

$$RW(F) \leqslant \mu + f(\det F).$$

- We suppose first that $F_1 \neq 0$ and $F_2 \neq 0$.
  There exists a real constant α such that

  $$\mu + \varepsilon \geqslant \varphi(\alpha).$$

  Then, using Lemma 3.1 if $P(F) \leqslant \alpha$, or Lemma 3.2 if $P(F) \geqslant \alpha$, there exist two matrices B and C so that

  $$F = \lambda B + (1 - \lambda)C,$$
  $$\operatorname{rank}(B - C) \leqslant 1,$$
  $$\det B = \det C = \det F,$$
  $$P(B) = P(C) = \alpha.$$

  Therefore we have

  $$\begin{aligned} RW(F) &\leqslant \lambda RW(B) + (1 - \lambda) RW(C) \\ &\leqslant \lambda \varphi\big(P(B)\big) + \lambda f(\det B) + (1 - \lambda)\big(\varphi\big(P(C)\big) + f(\det C)\big) \\ &= \varphi(\alpha) + \lambda f(\det F) + (1 - \lambda) f(\det F) \leqslant \mu + f(\det F) + \varepsilon. \end{aligned}$$

  As ε is arbitrary we conclude that

  $$RW(F) \leqslant \mu + f(\det F).$$

- We assume now that $F_1 = 0$ or $F_2 = 0$. Let F^ε be a sequence of matrices such that $F^\varepsilon \to F$ and $F_i^\varepsilon \neq 0$, $i = 1, 2$. This choice is possible since we can choose $F^\varepsilon = (\varepsilon T, F_2, \ldots, F_n)$ if $F_1 = 0$, $F_2 \neq 0$, $F^\varepsilon = (F_1, \varepsilon T, F_3, \ldots, F_n)$ if $F_2 = 0$, $F_1 \neq 0$, and $F^\varepsilon = (\varepsilon T, \varepsilon T, F_3, \ldots, F_n)$ if $F_1 = F_2 = 0$. where T is the same vector as in Theorem 3.1.
  By hypothesis and continuity of P_d, the part of P of degree d, we can have for ε sufficiently small
  $$P_d\big(F_2^\varepsilon, 0, \ldots, 0\big) > 0, \qquad P_d\big(0, F_1^\varepsilon, 0, \ldots, 0\big) < 0,$$
  $$P_d\big(-F_2^\varepsilon, 0, \ldots, 0\big) > 0, \qquad P_d\big(0, -F_1^\varepsilon, 0, \ldots, 0\big) < 0.$$
  We then conclude as in the first case that
  $$RW\big(F^\varepsilon\big) \leqslant \mu + f\big(\det F^\varepsilon\big).$$
  Letting ε go to 0, we get:
  $$RW(F) \leqslant \mu + f(\det F).$$

(2) *Step two.* In this case we suppose that the hypothesis of the first step is not satisfied.
- If $\det(F_1, T, F_3, \ldots, F_n) \neq 0$, thanks to Lemma 3.3, there exist two matrices B, C, and a real $\lambda \in [0, 1]$ such that
  $$\begin{aligned} &F = \lambda B + (1-\lambda)C, \\ &\operatorname{rank}(B - C) \leqslant 1, \\ &\det B = \det C = \det F, \\ &P_d(B_2, 0, \ldots, 0) > 0, \qquad P_d(-B_2, 0, \ldots, 0) > 0, \\ &P_d(0, B_1, 0, \ldots, 0) < 0, \qquad P_d(0, -B_1, 0, \ldots, 0) < 0, \\ &P_d(C_2, 0, \ldots, 0) > 0, \qquad P_d(-C_2, 0, \ldots, 0) > 0, \\ &P_d(0, C_1, 0, \ldots, 0) < 0, \qquad P_d(0, -C_1, 0, \ldots, 0) < 0, \\ &B_1 \neq 0, \qquad B_2 \neq 0, \qquad C_1 \neq 0, \qquad C_2 \neq 0. \end{aligned}$$
  Then by using the first step for the matrices B and C, we have
  $$\begin{aligned} RW(F) &\leqslant \lambda RW(B) + (1-\lambda) RW(C) \\ &\leqslant \lambda\big[\mu + f(\det B)\big] + (1-\lambda)\big[\mu + f(\det C)\big] \\ &= \lambda\big[\mu + f(\det F)\big] + (1-\lambda)\big[\mu + f(\det F)\big] \\ &= \mu + f(\det F). \end{aligned}$$
- Let $\det(F_1, T, F_3, \ldots, F_n) = 0$. Consider the sequence F^ε such that $\lim_{\varepsilon \to 0} F^\varepsilon = F$, and
  $$\det\big(F_1^\varepsilon, T, F_3^\varepsilon, \ldots, F_n^\varepsilon\big) \neq 0.$$
  By making use of the first step, we have
  $$RW\big(F^\varepsilon\big) \leqslant \mu + f\big(\det F^\varepsilon\big).$$
  Letting ε go to 0, we get:
  $$RW(F) \leqslant \mu + f(\det F).$$

Finally, we have proved that
$$\forall F \in \mathbb{R}^{n,n}: \quad RW(F) \leqslant \mu + f(\det F).$$
The function $F \to RW(F) - \mu$ is rank-one convex and bounded from above by the function $f(\det F)$, so that
$$RW(F) - \mu = R(RW(F) - \mu) \leqslant R\big(f(\det F)\big) = Cf(\det F),$$
then
$$RW(F) = Cf(\det F) + \mu.$$
Which end the proof. $\square$

In the general case $n \geqslant 3$, by using the same techniques, we prove the following corollary:

Corollary 3.1. *Let $P(X) : \mathbb{R}^{n,n} \to \mathbb{R}$, $P = P_0 + P_1 + \cdots + P_d$ be a polynomial function of degree d such that there exists a vector $T \in \mathbb{R}^n$ satisfying*:

$$P_d(0, T, 0, \dots, 0) < 0, \qquad P_d(0, -T, 0, \dots, 0) < 0,$$

and

$$P_d(T, 0, 0, \dots, 0) > 0, \qquad P_d(-T, 0, 0, \dots, 0) < 0,$$

and let f, φ be two real-valued functions such that f is continuous and φ is bounded from below, $\mu = \inf \varphi$, and $h : \mathbb{R}^{n,n-2} \to \mathbb{R}$ be a convex function. If we set

$$W(F) = \varphi\big(P(F)\big) + f(\det F) + h(F_3, F_4, \dots, F_n),$$

then

$$PW(F) = QW(F) = RW(F) = \mu + Cf(\det F) + h(F_3, F_4, \dots, F_n).$$

The proof of the corollary is the same as the proof of Theorem 3.1 by using in addition the following result:

Theorem 3.2. *Let f be a real-valued continuous function and $h : \mathbb{R}^{n,n-1} \to \mathbb{R}$ a convex function. $W : \mathbb{R}^{n,n} \to \mathbb{R}$ a function defined by*

$$W(F) = f(\det F) + h(F_2, F_3, \dots, F_n),$$

then

$$PW(F) = QW(F) = RW(F) = Cf(\det F) + h(F_2, F_3, \dots, F_n).$$

The proof of Theorem 3.2 is essentially based on the following decomposition lemma.

Lemma 3.4. *Let b, c be real constants, $\lambda \in]0, 1[$, and $F \in \mathbb{R}^{n,n}$ such that*

$$\operatorname{rank}(F_2, F_3, \dots, F_n) = n - 1,$$

and

$$\det F = \lambda b + (1 - \lambda)c.$$

Then, there exist two real matrices B and C in $\mathbb{R}^{n,n}$ such that

$$F = \lambda B + (1 - \lambda)C, \qquad \operatorname{rank}(B - C) \leqslant 1, \qquad \det B = b, \qquad \det C = c,$$

and

$$(B_2, B_3, \dots, B_n) = (C_2, C_3, \dots, C_n) = (F_2, F_3, \dots, F_n).$$

Proof. Let T be a vector in $\mathbb{R}^n$ such that $T \notin \langle F_2, F_3, \dots, F_n \rangle$, where $\langle F_2, F_3, \dots, F_n \rangle$ denote the vector space generated by the vectors $F_2, F_3, \dots, F_n$. We choose

$$B = (F_1 + \alpha T, F_2, F_3, \dots, F_n)^t, \qquad C = \left(F_1 - \frac{\lambda}{1-\lambda}\alpha T, F_2, F_3, \dots, F_n\right)^t,$$

then, it is sufficient to choose $\alpha = \frac{b - \det F}{\det(T, F_2, F_3, \dots, F_n)^t}$ to have the result. $\square$

Proof of Theorem 3.2.

- Suppose first that $\operatorname{rank}(F_2, F_3, \dots, F_n) = n - 1$.

Let $\varepsilon > 0$, $F \in \mathbb{R}^{n,n}$ and $b, c \in \mathbb{R}$, $\lambda \in [0, 1]$ be such that

$$\det F = \lambda b + (1 - \lambda c),$$

and

$$Cf(\det F) + \varepsilon \geqslant \lambda f(b) + (1 - \lambda) f(c).$$

Let B and C two matrices in $\mathbb{R}^{n,n}$ as in Lemma 3.4, then

$$RW(F) \leqslant \lambda RW(B) + (1 - \lambda) RW(C) \leqslant Cf(\det F) + \varepsilon + h(F_2, F_3, \dots, F_n).$$

Since ε is arbitrary we conclude that

$$RW(F) \leqslant Cf(\det F) + h(F_2, F_3, \dots, F_n).$$

- Suppose now that $\operatorname{rank}(F_2, F_3, \dots, F_n) < n-1$, let $F_2, F_3, \dots, F_m$ such that

$$\operatorname{rank}(F_2, F_3, \dots, F_m) = m-1,$$

and

$$\forall i \in \{m+1, m+2, \dots, n\}: \quad F_i \in \langle F_2, F_3, \dots, F_m\rangle.$$

Let $X_1, X_2, \dots, X_{n-m}$, $n-m$ vectors in $\mathbb{R}^n$ such that

$$\operatorname{rank}(F_2, F_3, \dots, F_m, X_{m+1}, X_{m+2}, \dots, X_n) = n-1.$$

Consider the sequence of matrices F^ε defined by

$$F^\varepsilon = (F_1, \dots, F_m, F_{m+1} + \varepsilon X_1, F_{m+2} + \varepsilon X_2, \dots, F_n + \varepsilon X_{n-m}),$$

the matrices F^ε satisfy, $\operatorname{rank}(F_2^\varepsilon, F_3^\varepsilon, \dots, F_n^\varepsilon) = n-1$.

By using the first step we get

$$RW\big(F^\varepsilon\big) \leqslant Cf\big(\det F^\varepsilon\big) + h\big(F_2^\varepsilon, F_3^\varepsilon, F_n^\varepsilon\big),$$

letting ε go to 0 we obtain

$$RW(F) \leqslant Cf(\det F) + h(F_2, F_3, \dots, F_n).$$

The function $F \to Cf(\det F) + h(F_2, F_3, \dots, F_n)$ is polyconvex and less than W, then

$$Cf(\det F) + h(F_2, F_3, \dots, F_n) \leqslant PW(F),$$

which finishes the proof. □

Examples. As a nontrivial application of Theorem 3.1 we take the following example:

$$W(F) = \big(F_{11} + F_{12} - F_{21} - F_{22} + |F|^p - 1\big)^2 + \det F,$$

where $p > 1$ is any integer. Then,

$$PW(F) = QW(F) = RW(F) = \det F.$$

4. Situations including the function $\operatorname{adj}_n F$

In this situation, we consider functions of the form:

$$W(F) = \varphi\big(P(F)\big) + f\big(\operatorname{adj}_n(F)\big),$$

where $F \in \mathbb{R}^{n,n+1}$ and $\operatorname{adj}_n(F)$ is the vector of all n-minors of F. In the sequel we will use the following notation.

For $F \in \mathbb{R}^{n,n+1}$, $F = (F_1, F_2, \dots, F_{n+1})^T$ where $F_1, F_2, \dots F_{n+1} \in \mathbb{R}^n$ are the rows of the matrix F, we denote

$$\det \hat{F}_i = \det(F_1, F_2, \dots F_{i-1}, F_{i+1} \dots F_{n+1}).$$

For example, in the case $n = 2$, we have $F = (F_1, F_2, F_3)^T$ and then the vector $\operatorname{adj}_2(F)$ has the following expression:

$$(\det \hat{F}_1, \det \hat{F}_2, \det \hat{F}_3) = \big(\det(F_2, F_3)^T, \det(F_1, F_3)^T, \det(F_2, F_3)^T\big).$$

Before proceeding, let us give some preliminary results.

Lemma 4.1. *Let $P = P_0 + P_1 + \cdots + P_d$ be a polynomial defined on the space $\mathbb{R}^{2,3}$ and $F \in \mathbb{R}^{2,3}$ is a real matrix, $\det \hat{F}_2 \neq 0$, such that*

$$P_d\left(-F_3, -\frac{\det \hat{F}_1}{\det \hat{F}_2} F_3, 0t\right)^T > 0, \qquad P_d\left(F_3, \frac{\det \hat{F}_1}{\det \hat{F}_2} F_3, 0\right)^T > 0.$$

If α is a real number such that $P(F) \leqslant \alpha$, then there exist two real matrices B, C, and a real $\lambda \in [0,1]$ such that

$$F = \lambda B + (1-\lambda)C,$$

$$\operatorname{rank}(B - C) \leqslant 1,$$

$$\operatorname{adj}_2 B = \operatorname{adj}_2 C = \operatorname{adj}_2 F,$$

$$P(B) = P(C) = \alpha.$$

Proof. We choose B and C as follows

$$B_a = \left(F_1 + aF_3,\, F_2 + a\frac{\det \hat{F}_1}{\det \hat{F}_2}F_3,\, F_3\right)^T,$$

$$C_\lambda = \left(F_1 - \frac{\lambda}{1-\lambda}aF_3,\, F_2 - \frac{\lambda}{1-\lambda}a\frac{\det \hat{F}_1}{\det \hat{F}_2}F_3,\, F_3\right)^T.$$

The matrices B_a and C_λ verifies $F = \lambda B_a + (1-\lambda)C_\lambda$, $\operatorname{adj}_2 B_a = \operatorname{adj}_2 C_\lambda = \operatorname{adj}_2 F$, and $\operatorname{rank}(B_a - C_\lambda) \leqslant 1$, for each value of a and $\lambda \in]0,1[$. For $a = 0$, we have $P(B_0) = P(F) \leqslant \alpha$ and

$$\lim_{a\to+\infty} P(B_a) = \lim_{a\to+\infty} a^d P_d\left(F_3,\, \frac{\det \hat{F}_1}{\det \hat{F}_2}F_3,\, 0\right)^T = +\infty.$$

Then, there exists a real a_0 such that $P(B_{a_0}) = \alpha$. We consider the matrix

$$C_\lambda = \left(F_1 - \frac{\lambda}{1-\lambda}a_0F_3,\, F_2 - \frac{\lambda}{1-\lambda}a_0\frac{\det \hat{F}_1}{\det \hat{F}_2}F_3,\, F_3\right)^T.$$

For $\lambda = 0$, $P(C_0) = P(F) \leqslant \alpha$ and

$$\lim_{\lambda\to1^-} P(C_\lambda) = \lim_{\lambda\to1^-}\left(\frac{\lambda}{1-\lambda}\right)^d a_0^d P_d\left(-F_3,\, -\frac{\det \hat{F}_1}{\det \hat{F}_2}F_3,\, 0\right)^T = +\infty.$$

By continuity, there exist a real $\lambda \in [0,1[$ such that $P(C_\lambda) = \alpha$. $\square$

Lemma 4.2. *Let $F \in \mathbb{R}^{2,3}$ be a real matrix, $\det \hat{F}_2 \neq 0$, $P = P_0 + P_1 + \cdots + P_d$ a polynomial defined on the space $\mathbb{R}^{2,3}$ such that*

$$P_d\left(0,\, \frac{\det \hat{F}_3}{\det \hat{F}_2}F_1,\, F_1\right)^T < 0, \qquad P_d\left(0,\, -\frac{\det \hat{F}_3}{\det \hat{F}_2}F_1,\, -F_1\right)^T < 0.$$

If α is a real number such that $P(F) \geqslant \alpha$, then there exist two real matrices B, C, and a real $\lambda \in [0,1]$ so that

$$F = \lambda B + (1-\lambda)C,$$
$$\operatorname{rank}(B - C) \leqslant 1,$$
$$\operatorname{adj}_2 B = \operatorname{adj}_2 C = \operatorname{adj}_2 F,$$
$$P(B) = P(C) = \alpha.$$

The proof is the same as in the previous lemma. We have just to take $-P$ instead of P, $-\alpha$ instead of α, and to choose the matrices B and C as follows

$$B = \left(F_1,\, F_2 + a\frac{\det \hat{F}_3}{\det \hat{F}_2}F_1,\, F_3 + aF_1\right)^T,$$

$$C = \left(F_1,\, F_2 - \frac{\lambda}{1-\lambda}a\frac{\det \hat{F}_3}{\det \hat{F}_2}F_1,\, F_3 - \frac{\lambda}{1-\lambda}aF_1\right)^T.$$

Lemma 4.3. *Let $P(X)$ be a polynomial of degree d, defined as above such that there exists a vector $L \in \mathbb{R}^2$ satisfying for each $\alpha \in \mathbb{R}$:*

$$P_d(L, \alpha L, 0)^T > 0, \qquad P_d(0, \alpha L, L)^T < 0, \qquad P_d(-L, -\alpha L, 0)^T > 0, \qquad P_d(0, -\alpha L, -L)^T < 0.$$

If $F \in \mathbb{R}^{2,3}$ is such that $\det\binom{F_1}{L} \neq 0$, then there exist two matrices $B, C \in \mathbb{R}^{2,3}$, and a real $\lambda \in [0,1]$ such that

$$F = \lambda B + (1-\lambda)C,$$
$$\operatorname{rank}(B - C) \leqslant 1,$$
$$\operatorname{adj}_2 B = \operatorname{adj}_2 C = \operatorname{adj}_2 F,$$
$$P_d\left(B_3,\, \frac{\det \hat{B}_1}{\det \hat{B}_2}B_3,\, 0\right)^T > 0, \qquad P_d\left(C_3,\, \frac{\det \hat{C}_1}{\det \hat{C}_2}C_3,\, 0\right)^T > 0,$$
$$P_d\left(-B_3,\, -\frac{\det \hat{B}_1}{\det \hat{B}_2}B_3,\, 0\right)^T > 0, \qquad P_d\left(-C_3,\, -\frac{\det \hat{C}_1}{\det \hat{C}_2}C_3,\, 0\right)^T > 0,$$

$$P_d\left(0, \frac{\det \hat{B}_3}{\det \hat{B}_2} B_1, B_1\right)^T < 0, \qquad P_d\left(0, \frac{\det \hat{C}_3}{\det \hat{C}_2} C_1, C_1\right)^T < 0,$$
$$P_d\left(0, -\frac{\det \hat{B}_3}{\det \hat{B}_2} B_1, -B_1\right)^T < 0, \qquad P_d\left(0, -\frac{\det \hat{C}_3}{\det \hat{C}_2} C_1, -C_1\right)^T < 0.$$

Proof. We choose B and C as

$$B(a) = \left(F_1 + aL, F_2 + a\frac{\det(F_2, L)^T}{\det(F_1, L)^T} L, F_3 + a\frac{\det(F_3, L)^T}{\det(F_1, L)^T} L\right)^T,$$
$$C(a) = \left(F_1 - aL, F_2 - a\frac{\det(F_2, L)^T}{\det(F_1, L)^T} L, F_3 - a\frac{\det(F_3, L)^T}{\det(F_1, L)^T} L\right)^T.$$

Then

$$F = \frac{1}{2}B + \frac{1}{2}C, \qquad \operatorname{rank}(B - C) \leqslant 1,$$

and for all $a \in \mathbb{R}$

$$\operatorname{adj}_2 B = \operatorname{adj}_2 C = \operatorname{adj}_2 F.$$

If we denote by L_1^3 the number $\frac{\det(F_3,L)^T}{\det(F_1,L)^T}$, then

$$P_d\left(B_3(a), \frac{\det \hat{B}_1(a)}{\det \hat{B}_2(a)} B_3(a), 0\right)^T = P_d\left(F_3 + aL_1^3 L, \frac{\det \hat{F}_1}{\det \hat{F}_2}(F_3 + aL_1^3 L), 0\right)^T.$$

We define the function μ_{i+}, $i = 1, 2$, μ_{i-}, $i = 1, 2$ and η_{i+}, $i = 1, 2$, η_{i-}, $i = 1, 2$ as follows:

$$\mu_{1+}(a) = P_d\left(B_3(a), \frac{\det \hat{B}_1(a)}{\det \hat{B}_2(a)} B_3(a), 0\right)^T, \qquad \eta_{1+}(a) = P_d\left(0, \frac{\det \hat{B}_3(a)}{\det \hat{B}_2(a)} B_1(a), B_1(a)\right)^T,$$
$$\mu_{2+}(a) = P_d\left(C_3(a), \frac{\det \hat{C}_1(a)}{\det \hat{C}_2(a)} C_3(a), 0\right)^T, \qquad \eta_{2+}(a) = P_d\left(0, \frac{\det \hat{C}_3(a)}{\det \hat{C}_2(a)} C_1(a), C_1(a)\right)^T,$$
$$\mu_{1-}(a) = P_d\left(-B_3(a), -\frac{\det \hat{B}_1(a)}{\det \hat{B}_2(a)} B_3(a), 0\right)^T, \qquad \eta_{1-}(a) = P_d\left(0, -\frac{\det \hat{B}_3(a)}{\det \hat{B}_2(a)} B_1(a), -B_1(a)\right)^T,$$
$$\mu_{2-}(a) = P_d\left(-C_3(a), -\frac{\det \hat{C}_1(a)}{\det \hat{C}_2(a)} C_3(a), 0\right)^T, \qquad \eta_{2-}(a) = P_d\left(0, -\frac{\det \hat{C}_3(a)}{\det \hat{C}_2(a)} C_1(a), -C_1(a)\right)^T.$$

Then we have:

$$\mu_{1+}(a) = P_d\left(F_3 + a\frac{\det(F_3, L)^T}{\det(F_1, L)^T} L, \frac{\det \hat{F}_1}{\det \hat{F}_2}(F_3 + a\frac{\det(F_3, L)^T}{\det(F_1, L)^T} L), 0\right),$$
$$\mu_{2+}(a) = P_d\left(F_3 - a\frac{\det(F_3, L)^T}{\det(F_1, L)^T} L, \frac{\det \hat{F}_1}{\det \hat{F}_2}(F_3 - a\frac{\det(F_3, L)^T}{\det(F_1, L)^T} L), 0\right),$$
$$\eta_{1+}(a) = P_d\left(0, \frac{\det \hat{F}_1}{\det \hat{F}_2}(F_1 + aL), F_1 + aL\right)^T,$$
$$\eta_{2+}(a) = P_d\left(0, \frac{\det \hat{F}_1}{\det \hat{F}_2}(F_1 - aL), F_1 - aL\right)^T.$$

Hence

$$\mu_{1+}(a) = \left(L_1^3\right)^d a^d P_d\left(\frac{1}{aL_1^3} F_3 + L, \frac{\det \hat{F}_1}{\det \hat{F}_2}(\frac{1}{aL_1^3} F_3 + L), 0\right),$$

and

$$\lim_{a\to+\infty} \mu_{1+}(a) = \lim_{a\to\infty} \left(\frac{\det(F_3, L)^T}{\det(F_1, L)^T}\right)^d a^d P_d\left(L, \frac{\det \hat{F}_1}{\det \hat{F}_2} L, 0\right) = +\infty.$$

If $\frac{\det(F_3,L)^T}{\det(F_1,L)^T} < 0$, and the degree d of P_d is odd, then:

$$\lim_{a\to+\infty} \mu_{1+}(a) = \lim_{a\to\infty} \left(-\frac{\det(F_3, L)^T}{\det(F_1, L)^T}\right)^d a^d P_d\left(-L, -\frac{\det \hat{F}_1}{\det \hat{F}_2} L, 0\right) = +\infty.$$

In the same way, we have

$$\lim_{a\to+\infty}\mu_{2+}(a)=\lim_{a\to\infty}\left(\frac{\det(F_3,L)^T}{\det(F_1,L)^T}\right)^d a^d P_d\left(-L,-\frac{\det\hat F_1}{\det\hat F_2}L,0\right)=+\infty,$$

$$\lim_{a\to+\infty}\eta_{1+}(a)=\lim_{a\to+\infty}a^d P_d\left(0,\frac{\det\hat F_1}{\det\hat F_2}L,L\right)^T=-\infty,$$

$$\lim_{a\to+\infty}\eta_{2+}(a)=\lim_{a\to+\infty}a^d P_d\left(0,-\frac{\det\hat F_1}{\det\hat F_2}L,-L\right)^T=-\infty.$$

Similarly, we can have

$$\lim_{a\to+\infty}\mu_{1-}(a)=\lim_{a\to+\infty}\mu_{2-}(a)=+\infty \quad\text{and}\quad \lim_{a\to+\infty}\eta_{1-}(a)=\lim_{a\to+\infty}\eta_{2-}(a)=-\infty.$$

So, we can choose a constant a_1 such that $\forall a\geqslant a_1$: $\mu_{1+}(a)>0$, $\mu_{2+}(a)>0$, $\eta_{1+}(a)<0$, $\eta_{2+}(a)<0$, $\mu_{1-}(a)>0$, $\mu_{2-}(a)>0$, $\eta_{1-}(a)<0$ and $\eta_{2-}(a)<0$, and this leeds to the result. □

Theorem 4.1. *Let $P=P_0+P_1+\cdots+P_d$ be a polynomial of any degree defined on the space $\mathbb{R}^{2,3}$ such that there exists a vector $L\in\mathbb{R}^2$ satisfying the following condition*

$$P_d(L,\alpha L,0)^T>0,\qquad P_d(-L,-\alpha L,0)^T>0,\qquad P_d(0,\alpha L,L)^T<0,\qquad P_d(0,-\alpha L,-L)^T<0,\quad \forall\alpha\in\mathbb{R},$$

and let $\varphi:\mathbb{R}\to\mathbb{R}$, $f:\mathbb{R}^3\to\mathbb{R}$ be two continuous functions such that φ is bounded from below, $\mu=\inf\varphi$, and

$$W(F)=\varphi\big(P(F)\big)+f\big(\mathrm{adj}_2(F)\big).$$

Then

$$PW(F)=QW(F)=RW(F)=\mu+Cf\big(\mathrm{adj}_2(F)\big).$$

Remark 4.1. The result remains true in the general case of $n\times(n+1)$ matrices. The hypothesis on the polynomial is:

$$P_d(0,\alpha L,L,0,\dots,0)^t<0,\qquad P_d(0,-\alpha L,-L,0,\dots,0)^t<0,$$

and

$$P_d(L,\alpha L,0,\dots,0)^t>0,\qquad P_d(-L,-\alpha L,0,\dots,0)^t>0.$$

For the decomposition lemmas, we take the same form of the matrices B, C, we just keep the rows $F_4,F_5,\dots,F_n$ of the matrices.

Proof. • Let is begin by proving that if $\det\hat F_2\neq 0$, then $RW(F)\leqslant\mu+f(\mathrm{adj}_2(F))$.

(1) Suppose first that

$$P_d\left(F_3,\frac{\det\hat F_1}{\det\hat F_2}F_3,0\right)^T>0,\qquad P_d\left(-F_3,-\frac{\det\hat F_1}{\det\hat F_2}F_3,0\right)^T>0,$$

$$P_d\left(0,\frac{\det\hat F_3}{\det\hat F_2}F_1,F_1\right)^T<0,\qquad P_d\left(0,-\frac{\det\hat F_3}{\det\hat F_2}F_1,-F_1\right)^T<0.$$

Let $\varepsilon>0$. There exists $\alpha_\varepsilon\in\mathbb{R}$ such that

$$\mu+\varepsilon\geqslant\varphi\big(P(F)\big).$$

Thanks to Lemma 4.1 (if $P(F)\leqslant\alpha_\varepsilon$) or Lemma 4.2 (if $P(F)\geqslant\alpha_\varepsilon$), there are two real matrices B,C, and a real $\lambda\in[0,1]$ so that

$$F=\lambda B+(1-\lambda)C,$$

$$\mathrm{rank}(B-C)\leqslant 1,$$

$$\mathrm{adj}_2 B=\mathrm{adj}_2 C=\mathrm{adj}_2 F,$$

$$P(B)=P(C)=\alpha_\varepsilon.$$

Then

$$\begin{aligned} RW(F) &\leqslant \lambda RW(B) + (1-\lambda)RW(C) \leqslant \lambda W(B) + (1-\lambda)W(C) \\ &= \lambda\big[\varphi\big(P(B)\big) + f\big(\mathrm{adj}_2(B)\big)\big] + (1-\lambda)\big[\varphi\big(P(C)\big) + f\big(\mathrm{adj}_2(C)\big)\big] \\ &= \lambda\big[\varphi(\alpha_\varepsilon) + f\big(\mathrm{adj}_2(F)\big)\big] + (1-\lambda)\big[\varphi(\alpha_\varepsilon) + f\big(\mathrm{adj}_2(F)\big)\big] \\ &= \varphi(\alpha_\varepsilon) + f\big(\mathrm{adj}_2(F)\big) \leqslant \mu + \varepsilon + f\big(\mathrm{adj}_2(F)\big), \end{aligned}$$

as ε is arbitrary, we conclude that $RW(F) \leqslant \mu + f(\mathrm{adj}_2(F))$.

(2) Suppose that the hypothesis in the first case is not satisfied. Then, thanks to Lemma 4.3, we have the existence of two matrices B, C, and a real $\lambda \in [0,1]$ such that

$$\begin{aligned} &F = \lambda B + (1-\lambda)C, \\ &\mathrm{rank}(B - C) \leqslant 1, \\ &\mathrm{adj}_2 B = \mathrm{adj}_2 C = \mathrm{adj}_2 F, \\ &P_d\left(B_3, \frac{\det \hat{B}_1}{\det \hat{B}_2} B_3, 0\right)^T > 0, \qquad P_d\left(C_3, \frac{\det \hat{C}_1}{\det \hat{C}_2} C_3, 0\right)^T > 0, \\ &P_d\left(-B_3, -\frac{\det \hat{B}_1}{\det \hat{B}_2} B_3, 0\right)^T > 0, \qquad P_d\left(-C_3, -\frac{\det \hat{C}_1}{\det \hat{C}_2} C_3, 0\right)^T > 0, \\ &P_d\left(0, \frac{\det \hat{B}_3}{\det \hat{B}_2} B_1, B_1\right)^T < 0, \qquad P_d\left(0, \frac{\det \hat{C}_3}{\det \hat{C}_2} C_1, C_1\right)^T < 0, \\ &P_d\left(0, -\frac{\det \hat{B}_3}{\det \hat{B}_2} B_1, -B_1\right)^T < 0, \qquad P_d\left(0, -\frac{\det \hat{C}_3}{\det \hat{C}_2} C_1, -C_1\right)^T < 0. \end{aligned}$$

Then, by applying the first step to the matrices B and C, we get:

$$\begin{aligned} RW(F) &\leqslant \lambda RW(B) + (1-\lambda)RW(C) \\ &\leqslant \lambda\big(\mu + f\big(\mathrm{adj}_2(B)\big)\big) + (1-\lambda)\big(\mu + f\big(\mathrm{adj}_2(C)\big)\big) \\ &= \mu + f\big(\mathrm{adj}_2(F)\big). \end{aligned}$$

- Now, we suppose that $\det \hat{F}_2 = 0$, That means that the vectors F_1 and F_3 are collinear. Let $\alpha \in \mathbb{R}$ be such that $F_3 = \alpha F_1$, and let T be a vector not collinear with them. Let F^ε a sequence defined by

  $$F_1^\varepsilon = F_1 + \varepsilon\beta T, \qquad F_2^\varepsilon = F_2, \qquad F_3^\varepsilon = F_3 + \varepsilon T.$$

  If we choose β such that $1 - \alpha\beta \neq 0$, then

  $$\det \hat{F}_2^\varepsilon = \varepsilon(1 - \alpha\beta)\det(F_1, T) \neq 0.$$

  By using the first step, we have

  $$RW\big(F^\varepsilon\big) \leqslant \mu + f\big(\mathrm{adj}_n\big(F^\varepsilon\big)\big).$$

  Letting ε go to 0 we get

  $$RW(F) \leqslant \mu + f\big(\mathrm{adj}_n(F)\big).$$

So we have proved that

$$\forall F \in \mathbb{R}^{2,3}\colon \quad RW(F) \leqslant \mu + f\big(\mathrm{adj}_2(F)\big).$$

Then $\forall F \in \mathbb{R}^{2,3}$

$$RW(F) - \mu = R\big(RW(F) - \mu\big) \leqslant Rf\big(\mathrm{adj}_2(F)\big) = Cf\big(\mathrm{adj}_2(F)\big).$$

In other words, the function $F \to \mu + Cf(\mathrm{adj}_2(F))$ is polyconvex and less than W. Therefore $\forall F \in \mathbb{R}^{2,3}$

$$\mu + Cf\big(\mathrm{adj}_2(F)\big) \leqslant PW(F) \leqslant QW(F) \leqslant RW(F) \leqslant \mu + Cf\big(\mathrm{adj}_2(F)\big). \qquad \square$$

Examples.

(1) As examples we take first a function $W:\mathbb{R}^{2,3}\to\mathbb{R}$ defined by:

$$W(F)=\left(F_{11}-F_{12}+F_{21}-F_{12}+F_{31}-F_{13}+(F_1.F_2)^2+(F_1.F_3)^2-(F_2.F_3)^2-1\right)^2+\left(\left\|\operatorname{adj}_3(F)\right\|^2\right),$$

where $F=(F_1,F_2,F_3)^T$ and $F_i, i=1,2,3$, are the rows of the matrix F then our result asserts that:

$$PW(F)=Q(F)=RW(F)=\left(\left\|\operatorname{adj}_3(F)\right\|^2\right).$$

(2) We take now a second example

$$W(F)=\left(\|F_1+F_2\|^2-\|F_2+F_3\|^2-1\right)^2+\left(\|\operatorname{adj}_3 F\|\right)^2.$$

Then we obtain:

$$PW(F)=QW(F)=RW(F)=\left(\|\operatorname{adj}_3 F\|\right)^2.$$

In the general case $n\geqslant 3$, by using the same techniques, we prove the following corollary:

Corollary 4.1. *Let $P(X):\mathbb{R}^{n,n+1}\to\mathbb{R}$, $P=P_0+P_1+\cdots+P_d$ be a polynomial function of degree d such that there exists a vector $T\in\mathbb{R}^n$ satisfying for each $\alpha\in\mathbb{R}$:*

$$P_d(0,\alpha L,L,0,\ldots,0)^t<0,\qquad P_d(0,-\alpha L,-L,0,\ldots,0)^t<0,$$

and

$$P_d(L,\alpha L,0,\ldots,0)^t>0,\qquad P_d(-L,-\alpha L,0,\ldots,0)^t>0,$$

and let $f:\mathbb{R}\to\mathbb{R}$, $\varphi:\mathbb{R}^{n+1}\to\mathbb{R}$ be two real functions such that f is continuous and φ is bounded from below, $\mu=\inf\varphi$, and $h:\mathbb{R}^{n,n-2}\to\mathbb{R}$ be a convex function. If we set

$$W(F)=\varphi\big(P(F)\big)+f(\operatorname{adj}_n F)+h(F_4,F_5,\ldots,F_{n+1}),$$

then

$$PW(F)=QW(F)=RW(F)=\mu+Cf(\operatorname{adj}_n F)+h(F_3,F_4,\ldots,F_n).$$

The proof of the corollary is the same as the proof of Theorem 4.1 by using in addition the following result:

Theorem 4.2. *Let $f:\mathbb{R}^{n+1}\to\mathbb{R}$ be a continuous function and $h:\mathbb{R}^{n,n-1}\to\mathbb{R}$ a convex function. $W:\mathbb{R}^{n,n+1}\to\mathbb{R}$ a function defined by*

$$W(F)=f(\operatorname{adj}_n F)+h(F_3,F_4,\ldots,F_{n+1}),$$

then

$$PW(F)=QW(F)=RW(F)=Cf(\operatorname{adj}_n F)+h(F_3,F_4,\ldots,F_{n+1}).$$

For the proof of Theorem 4.2 we refer the reader to [5].

Acknowledgment

The author would like to thanks Professor Pablo Pedregal for fruitful discussion and remarks, following the progress of this paper.

Appendix A

We recall now some basics notions of convexity.
Let φ be a real-valued Borel measurable function defined on the space $\mathbb{R}^{n\times m}$ of $n\times m$ matrices.

- We say that φ is convex if:

$$\varphi\big(\lambda F_1+(1-\lambda)F_2\big)\leqslant\lambda\varphi(F_1)+(1-\lambda)\varphi(F_2)$$

for every real $\lambda\in[0,1]$ and all matrices $F_1,F_2\in\mathbb{R}^{n\times m}$.

- We say that φ is quasi-convex if:

$$\frac{1}{|\Omega|}\int_\Omega\varphi\big(F+\nabla u(x)\big)\,dx\geqslant\varphi(F)$$

for every F and every test function $u\in W_0^{1,\infty}(\Omega,\mathbb{R}^m)$.

- φ is said to be polyconvex if there exists a convex function ψ such that
$$\varphi(F) = \psi\big(T(F)\big)$$
for every matix $F \in \mathbb{R}^{n\times m}$. $T(F)$ stands for the vector of all minors of F.
- φ is said to be rank-one convex if:
$$\varphi\big(\lambda F_1 + (1-\lambda)F_2\big) \leqslant \lambda\varphi(F_1) + (1-\lambda)\varphi(F_2)$$
for every $\lambda \in [0,1]$ and $F_1, F_2 \in \mathbb{R}^{n,m}$ such that $\operatorname{rank}(F_1 - F_2) \leqslant 1$.

For more details on theses notions one can see [10].

It is well known that
$$\varphi \text{ convex } \Rightarrow \varphi \text{ polyconvex } \Rightarrow \varphi \text{ quasi-convex } \Rightarrow \varphi \text{ rank-one convex.}$$

However the converses are false in general (see [1,10]). The last one has been established by V. Sverak in the case $m \geqslant 3$, [18], but the case $m = 2$ and $n \geqslant 2$ is still open. In the scalar case $n = 1$ or $m = 1$ all theses notions are equivalents.

We define now the different envelopes associated with different notions of convexity by setting:
$$C\varphi(F) = \big\{\sup f(F):\ f \leqslant \varphi,\ f \text{ convex}\big\},$$
$$P\varphi(F) = \big\{\sup f(F):\ f \leqslant \varphi,\ f \text{ polyconvex}\big\},$$
$$Q\varphi(F) = \big\{\sup f(F):\ f \leqslant \varphi,\ f \text{ quasi-convex}\big\},$$
$$R\varphi(F) = \big\{\sup f(F):\ f \leqslant \varphi,\ f \text{ rank-one convex}\big\}.$$
As a direct result of the implications above, we have
$$C\varphi \leqslant P\varphi \leqslant Q\varphi \leqslant R\varphi$$
and all the envelopes coincide in the case $n = 1$ or $m = 1$. For a characterizations of the different envelopes see [10]. Let us recall that:
$$Cf(F) = \inf\left\{\sum_{i=1}^{n.m+1} \lambda_i f(F_i):\ \sum_{i=1}^{n.m+1} \lambda_i F_i = F\right\},$$
$$Pf(F) = \inf\left\{\sum_{i=1}^{\tau(n,m)+1} \lambda_i f(F_i):\ \sum_{i=1}^{\tau(n,m)+1} \lambda_i T(F_i) = T(F)\right\},$$
$$Rf(F) = \inf\left\{\sum_{i=1}^{I} \lambda_i f(F_i):\ \sum_{i=1}^{I} \lambda_i F_i = F \text{ and } (\lambda_i, F_i)_{1\leqslant i\leqslant I} \text{ satisfy } (H_I)\right\},$$
$$Qf(F) = \inf_{\varphi\in W_0^{1,\infty}(\Omega;\mathbb{R}^m)} \frac{1}{|\Omega|}\int_\Omega f\big(F + \nabla\varphi(x)\big)\,dx,$$
$\Omega \subset \mathbb{R}^n$ is a bounded regular domain.

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