J. Math. Anal. Appl. 356 (2009) 179–185

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Trace inequalities on a generalized Wigner–Yanase skew information

S. Furuichi [a,*,1], K. Yanagi [b,2], K. Kuriyama [b]

[a] Department of Computer Science and System Analysis, College of Humanities and Sciences, Nihon University, 3-25-40, Sakurajyousui, Setagaya-ku, Tokyo, 156-8550, Japan

[b] Division of Applied Mathematical Science, Graduate School of Science and Engineering, Yamaguchi University, Tokiwadai 2-16-1, Ube City, 755-0811, Japan

ARTICLE INFO

Article history:
Received 9 September 2008
Available online 4 March 2009
Submitted by Goong Chen

Keywords:
Trace inequality
Wigner–Yanase skew information
Wigner–Yanase–Dyson skew information
and uncertainty relation

ABSTRACT

We introduce a generalized Wigner–Yanase skew information and then derive the trace inequality related to the uncertainty relation. This inequality is a non-trivial generalization of the uncertainty relation derived by S. Luo for the quantum uncertainty quantity excluding the classical mixture. In addition, several trace inequalities on our generalized Wigner–Yanase skew information are argued.

1. Introduction

Wigner–Yanase skew information

$$I_\rho(H) \equiv \frac{1}{2}\mathrm{Tr}\left[\left(i\left[\rho^{1/2}, H\right]\right)^2\right] = \mathrm{Tr}\left[\rho H^2\right] - \mathrm{Tr}\left[\rho^{1/2} H \rho^{1/2} H\right] \tag{1.1}$$

was defined in [8]. This quantity can be considered as a kind of the degree for non-commutativity between a quantum state ρ and an observable H. Here we denote the commutator by $[X, Y] \equiv XY - YX$. This quantity was generalized by Dyson

$$I_{\rho,\alpha}(H) \equiv \frac{1}{2}\mathrm{Tr}\left[\left(i\left[\rho^\alpha, H\right]\right)\left(i\left[\rho^{1-\alpha}, H\right]\right)\right] = \mathrm{Tr}\left[\rho H^2\right] - \mathrm{Tr}\left[\rho^\alpha H \rho^{1-\alpha} H\right], \quad \alpha \in [0,1],$$

which is known as the Wigner–Yanase–Dyson skew information. It is famous that the convexity of $I_{\rho,\alpha}(H)$ with respect to ρ was successfully proven by E.H. Lieb in [5]. From the physical point of view, an observable H is generally considered to be an unbounded operator, however in the present paper, unless otherwise stated, we consider $H \in B(\mathcal{H})$, where $B(\mathcal{H})$ represents the set of all bounded linear operators on the Hilbert space $\mathcal{H}$, as a mathematical interest. We also denote the set of all self-adjoint operators (observables) by $\mathcal{L}_h(\mathcal{H})$ and the set of all density operators (quantum states) by $\mathfrak{S}(\mathcal{H})$ on the Hilbet space $\mathcal{H}$. The relation between the Wigner–Yanase skew information and the uncertainty relation was studied in [7]. Moreover the relation between the Wigner–Yanase–Dyson skew information and the uncertainty relation was studied in [4,9]. In our previous paper [9], we defined a generalized skew information and then derived a kind of an uncertainty relation. In Section 2, we introduce a new generalized Wigner–Yanase skew information. On a generalization of the original

* Corresponding author.
E-mail addresses: furuichi@chs.nihon-u.ac.jp (S. Furuichi), yanagi@yamaguchi-u.ac.jp (K. Yanagi), kuriyama@yamaguchi-u.ac.jp (K. Kuriyama).

1 Supported in part by the Japanese Ministry of Education, Science, Sports and Culture, Grant-in-Aid for Encouragement of Young Scientists (B), 20740067 and Grant-in-Aid for Scientific Research (B), 18300003.

2 Partially supported by the Ministry of Education, Science, Sports and Culture, Grant-in-Aid for Scientific Research (B), 18300003.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.02.043

Wigner–Yanase skew information, our generalization is different from the Wigner–Yanase–Dyson skew information and a generalized skew information defined in our previous paper [9]. Moreover we define a new quantity by our generalized Wigner–Yanase skew information and then we derive the trace inequality expressing a kind of the uncertainty relation.

2. Trace inequalities on a generalized Wigner–Yanase skew information

Firstly we review the relation between the Wigner–Yanase skew information and the uncertainty relation. In quantum mechanical system, the expectation value of an observable H in a quantum state ρ is expressed by $\mathrm{Tr}[\rho H]$. It is natural that the variance for a quantum state ρ and an observable H is defined by $V_\rho(H) \equiv \mathrm{Tr}[\rho(H - \mathrm{Tr}[\rho H]I)^2] = \mathrm{Tr}[\rho H^2] - \mathrm{Tr}[\rho H]^2$. It is famous that we have the Heisenberg's uncertainty relation:

$$V_\rho(A)V_\rho(B) \geqslant \frac{1}{4}\left|\mathrm{Tr}\left[\rho[A,B]\right]\right|^2 \tag{2.1}$$

for a quantum state ρ and two observables A and B. The further strong result was given by Schrödinger

$$V_\rho(A)V_\rho(B) - \left|\mathrm{Cov}_\rho(A,B)\right|^2 \geqslant \frac{1}{4}\left|\mathrm{Tr}\left[\rho[A,B]\right]\right|^2,$$

where the covariance is defined by $\mathrm{Cov}_\rho(A,B) \equiv \mathrm{Tr}[\rho(A - \mathrm{Tr}[\rho A]I)(B - \mathrm{Tr}[\rho B]I)]$. However, the uncertainty relation for the Wigner–Yanase skew information failed (see [4,7,9])

$$I_\rho(A)I_\rho(B) \geqslant \frac{1}{4}\left|\mathrm{Tr}\left[\rho[A,B]\right]\right|^2.$$

Recently, S. Luo introduced the quantity $U_\rho(H)$ representing a quantum uncertainty excluding the classical mixture:

$$U_\rho(H) \equiv \sqrt{V_\rho(H)^2 - \left(V_\rho(H) - I_\rho(H)\right)^2}, \tag{2.2}$$

then he derived the uncertainty relation on $U_\rho(H)$ in [6]:

$$U_\rho(A)U_\rho(B) \geqslant \frac{1}{4}\left|\mathrm{Tr}\left[\rho[A,B]\right]\right|^2. \tag{2.3}$$

Note that we have the following relation

$$0 \leqslant I_\rho(H) \leqslant U_\rho(H) \leqslant V_\rho(H). \tag{2.4}$$

The inequality (2.3) is a refinement of the inequality (2.1) in the sense of (2.4).

In this section, we study one-parameter extended inequality for the inequality (2.3).

Definition 2.1. For $0 \leqslant \alpha \leqslant 1$, a quantum state ρ and an observable H, we define the Wigner–Yanase–Dyson skew information

$$I_{\rho,\alpha}(H) \equiv \frac{1}{2}\mathrm{Tr}\left[\left(i\left[\rho^\alpha, H_0\right]\right)\left(i\left[\rho^{1-\alpha}, H_0\right]\right)\right] \tag{2.5}$$

and we also define

$$J_{\rho,\alpha}(H) \equiv \frac{1}{2}\mathrm{Tr}\left[\left\{\rho^\alpha, H_0\right\}\left\{\rho^{1-\alpha}, H_0\right\}\right],$$

where $H_0 \equiv H - \mathrm{Tr}[\rho H]I$ and we denote the anti-commutator by $\{X,Y\} = XY + YX$.

Note that we have

$$\frac{1}{2}\mathrm{Tr}\left[\left(i\left[\rho^\alpha, H_0\right]\right)\left(i\left[\rho^{1-\alpha}, H_0\right]\right)\right] = \frac{1}{2}\mathrm{Tr}\left[\left(i\left[\rho^\alpha, H\right]\right)\left(i\left[\rho^{1-\alpha}, H\right]\right)\right]$$

but we have

$$\frac{1}{2}\mathrm{Tr}\left[\left\{\rho^\alpha, H_0\right\}\left\{\rho^{1-\alpha}, H_0\right\}\right] \neq \frac{1}{2}\mathrm{Tr}\left[\left\{\rho^\alpha, H\right\}\left\{\rho^{1-\alpha}, H\right\}\right].$$

Then we have the following inequalities:

$$I_{\rho,\alpha}(H) \leqslant I_\rho(H) \leqslant J_\rho(H) \leqslant J_{\rho,\alpha}(H), \tag{2.6}$$

since we have $\mathrm{Tr}[\rho^{1/2}H\rho^{1/2}H] \leqslant \mathrm{Tr}[\rho^\alpha H\rho^{1-\alpha}H]$. (See [1,2] for example.) If we define

$$U_{\rho,\alpha}(H) \equiv \sqrt{V_\rho(H)^2 - \left(V_\rho(H) - I_{\rho,\alpha}(H)\right)^2}, \tag{2.7}$$

as a direct generalization of Eq. (2.2), then we have

$$0 \leqslant I_{\rho,\alpha}(H) \leqslant U_{\rho,\alpha}(H) \leqslant U_{\rho}(H) \tag{2.8}$$

due to the first inequality of (2.6). We also have

$$U_{\rho,\alpha}(H) = \sqrt{I_{\rho,\alpha}(H) J_{\rho,\alpha}(H)}. \tag{2.9}$$

Remark 2.2. From the inequalities (2.4), (2.6) and (2.8), our situation is that we have

$$0 \leqslant I_{\rho,\alpha}(H) \leqslant I_{\rho}(H) \leqslant U_{\rho}(H)$$

and

$$0 \leqslant I_{\rho,\alpha}(H) \leqslant U_{\rho,\alpha}(H) \leqslant U_{\rho}(H).$$

Therefore our first concern is the ordering between $I_\rho(H)$ and $U_{\rho,\alpha}(H)$. However we have no ordering between them. Because we have the following examples. We set the density matrix ρ and the observable H such as

$$\rho = \begin{pmatrix} 0.6 & 0.48 \\ 0.48 & 0.4 \end{pmatrix}, \qquad H = \begin{pmatrix} 1.0 & 0.5 \\ 0.5 & 5.0 \end{pmatrix}.$$

If $\alpha = 0.1$, then $U_{\rho,\alpha}(H) - I_\rho(H)$ approximately takes -0.14736. If $\alpha = 0.2$, then $U_{\rho,\alpha}(H) - I_\rho(H)$ approximately takes 0.4451.

Conjecture 2.3. *Our second concern is to show an uncertainty relation with respect to $U_{\rho,\alpha}(H)$ as a direct generalization of the inequality* (2.3) *such that*

$$U_{\rho,\alpha}(X) U_{\rho,\alpha}(Y) \geqslant \frac{1}{4} \left| \mathrm{Tr}\big[\rho[X,Y]\big] \right|^2. \tag{2.10}$$

However we have not found the proof of the above inequality (2.10)*. In addition, we have not found any counter-examples of the inequality* (2.10) *yet.*

In the present paper, we introduce a generalized Wigner–Yanase skew information which is a generalization of the Wigner–Yanase skew information defined in Eq. (1.1), but different from the Wigner–Yanase–Dyson skew information defined in Eq. (2.5).

Definition 2.4. For $0 \leqslant \alpha \leqslant 1$, a quantum state ρ and an observable H, we define a generalized Wigner–Yanase skew information by

$$K_{\rho,\alpha}(H) \equiv \frac{1}{2} \mathrm{Tr}\left[\left(i\left[\frac{\rho^\alpha + \rho^{1-\alpha}}{2}, H_0\right]\right)^2\right]$$

and we also define

$$L_{\rho,\alpha}(H) \equiv \frac{1}{2} \mathrm{Tr}\left[\left(\left\{\frac{\rho^\alpha + \rho^{1-\alpha}}{2}, H_0\right\}\right)^2\right].$$

Remark 2.5. For two generalized Wigner–Yanase skew informations $I_{\rho,\alpha}(H)$ and $K_{\rho,\alpha}(H)$, we have the relation:

$$I_{\rho,\alpha}(H) \leqslant K_{\rho,\alpha}(H).$$

Indeed, for a spectral decomposition of ρ such as $\rho = \sum_k \lambda_k |\phi_k\rangle\langle\phi_k|$, we have the following expressions:

$$I_{\rho,\alpha}(H) = \frac{1}{2} \sum_{m,n} \left(\lambda_m^\alpha - \lambda_n^\alpha\right)\left(\lambda_m^{1-\alpha} - \lambda_n^{1-\alpha}\right) \left|\langle\phi_m|H|\phi_n\rangle\right|^2$$

and

$$K_{\rho,\alpha}(H) = \frac{1}{2} \sum_{m,n} \left(\frac{\lambda_m^\alpha - \lambda_n^\alpha + \lambda_m^{1-\alpha} - \lambda_n^{1-\alpha}}{2}\right)^2 \left|\langle\phi_m|H|\phi_n\rangle\right|^2.$$

By simple calculations, we see

$$\left(\frac{\lambda_m^\alpha - \lambda_n^\alpha + \lambda_m^{1-\alpha} - \lambda_n^{1-\alpha}}{2}\right)^2 - \left(\lambda_m^\alpha - \lambda_n^\alpha\right)\left(\lambda_m^{1-\alpha} - \lambda_n^{1-\alpha}\right) \geqslant 0.$$

Throughout this section, we put $X_0 \equiv X - \mathrm{Tr}[\rho X]I$ and $Y_0 \equiv Y - \mathrm{Tr}[\rho Y]I$. Then we show the following trace inequality.

Theorem 2.6. *For a quantum state ρ and observables X, Y and $\alpha \in [0,1]$, we have*

$$W_{\rho,\alpha}(X)W_{\rho,\alpha}(Y) \geqslant \frac{1}{4}\left|\mathrm{Tr}\left[\left(\frac{\rho^\alpha+\rho^{1-\alpha}}{2}\right)^2[X,Y]\right]\right|^2 \tag{2.11}$$

where

$$W_{\rho,\alpha}(X) \equiv \sqrt{K_{\rho,\alpha}(X)L_{\rho,\alpha}(X)}.$$

Proof. Putting

$$M \equiv i\left[\frac{\rho^\alpha+\rho^{1-\alpha}}{2}, X_0\right]x + \left\{\frac{\rho^\alpha+\rho^{1-\alpha}}{2}, Y_0\right\} \tag{2.12}$$

for any $x \in \mathbb{R}$, then we have

$$\begin{aligned}0 \leqslant \mathrm{Tr}\big[M^*M\big] = {} & \left(\frac{1}{4}\mathrm{Tr}\big[\big(i\big[\rho^\alpha, X_0\big]\big)^2 + \big(i\big[\rho^{1-\alpha}, X_0\big]\big)^2\big] + I_{\rho,\alpha}(X)\right)x^2 \\ & + \frac{1}{2}\mathrm{Tr}\big[\big(i\big[\rho^\alpha, X_0\big] + i\big[\rho^{1-\alpha}, X_0\big]\big)\big(\big\{\rho^\alpha, Y_0\big\} + \big\{\rho^{1-\alpha}, Y_0\big\}\big)\big]x \\ & + \left(\frac{1}{4}\mathrm{Tr}\big[\big\{\rho^\alpha, Y_0\big\}^2 + \big\{\rho^{1-\alpha}, Y_0\big\}^2\big] + J_{\rho,\alpha}(Y)\right).\end{aligned}$$

Therefore we have

$$\begin{aligned}\frac{1}{4}\big|\mathrm{Tr}\big[\big(\rho^\alpha+\rho^{1-\alpha}\big)^2\big(i[X,Y]\big)\big]\big|^2 \leqslant {} & 4\left(\frac{1}{4}\mathrm{Tr}\big[\big(i\big[\rho^\alpha, X_0\big]\big)^2 + \big(i\big[\rho^{1-\alpha}, X_0\big]\big)^2\big] + I_{\rho,\alpha}(X)\right) \\ & \times \left(\frac{1}{4}\mathrm{Tr}\big[\big\{\rho^\alpha, Y_0\big\}^2 + \big\{\rho^{1-\alpha}, Y_0\big\}^2\big] + J_{\rho,\alpha}(Y)\right),\end{aligned}$$

since we have

$$\mathrm{Tr}\big[\big(i\big[\rho^\alpha, X_0\big] + i\big[\rho^{1-\alpha}, X_0\big]\big)\big(\big\{\rho^\alpha, Y_0\big\} + \big\{\rho^{1-\alpha}, Y_0\big\}\big)\big] = \mathrm{Tr}\big[\big(\rho^\alpha+\rho^{1-\alpha}\big)^2\big(i[X,Y]\big)\big].$$

As similar as we have

$$\begin{aligned}\frac{1}{4}\big|\mathrm{Tr}\big[\big(\rho^\alpha+\rho^{1-\alpha}\big)^2\big(i[X,Y]\big)\big]\big|^2 \leqslant {} & 4\left(\frac{1}{4}\mathrm{Tr}\big[\big(i\big[\rho^\alpha, Y_0\big]\big)^2 + \big(i\big[\rho^{1-\alpha}, Y_0\big]\big)^2\big] + I_{\rho,\alpha}(Y)\right) \\ & \times \left(\frac{1}{4}\mathrm{Tr}\big[\big\{\rho^\alpha, X_0\big\}^2 + \big\{\rho^{1-\alpha}, X_0\big\}^2\big] + J_{\rho,\alpha}(X)\right).\end{aligned}$$

By the above two inequalities, we have

$$W_{\rho,\alpha}(X)W_{\rho,\alpha}(Y) \geqslant \frac{1}{4}\left|\mathrm{Tr}\left[\left(\frac{\rho^\alpha+\rho^{1-\alpha}}{2}\right)^2[X,Y]\right]\right|^2. \qquad \square$$

Corollary 2.7. *For a quantum state ρ and observables (possibly unbounded operators) X, Y and $\alpha \in [0,1]$, if we have the relation $[X,Y] = \frac{1}{2\pi i}I$ on $\mathbf{dom}(XY) \cap \mathbf{dom}(YX)$ and ρ is expressed by $\rho = \sum_k \lambda_k |\phi_k\rangle\langle\phi_k|$, $|\phi_k\rangle \in \mathbf{dom}(XY) \cap \mathbf{dom}(YX)$, then*

$$W_{\rho,\alpha}(X)W_{\rho,\alpha}(Y) \geqslant \frac{1}{4}\big|\mathrm{Tr}\big[\rho[X,Y]\big]\big|^2.$$

Proof. It follows from Theorem 2.6 and the following inequality:

$$\frac{1}{4}\left|\mathrm{Tr}\left[\left(\frac{\rho^\alpha+\rho^{1-\alpha}}{2}\right)^2[X,Y]\right]\right|^2 \geqslant \frac{1}{4}\big|\mathrm{Tr}\big[\rho[X,Y]\big]\big|^2,$$

whenever we have the canonical commutation relation such as $[X,Y] = \frac{1}{2\pi i}I$. $\square$

Remark 2.8. Theorem 2.6 is not trivial one in the sense of the following (i) and (ii).

(i) Since the arithmetic mean is greater than the geometric mean, $\mathrm{Tr}[(i[\rho^\alpha, X_0])^2] \geqslant 0$ and $\mathrm{Tr}[(i[\rho^{1-\alpha}, X_0])^2] \geqslant 0$ imply $K_{\rho,\alpha}(X) \geqslant I_{\rho,\alpha}(X)$, by the use of Schwarz's inequality. Similarly, $\mathrm{Tr}[\{\rho^\alpha, Y_0\}^2] \geqslant 0$ and $\mathrm{Tr}[\{\rho^{1-\alpha}, Y_0\}^2] \geqslant 0$ imply $L_{\rho,\alpha}(Y) \geqslant J_{\rho,\alpha}(Y)$. We then have $W_{\rho,\alpha}(X) \geqslant U_{\rho,\alpha}(X)$.

From the inequality (2.8) and the above, our situation is that we have

$$U_{\rho,\alpha}(H) \leqslant U_\rho(H)$$

and

$$U_{\rho,\alpha}(H) \leqslant W_{\rho,\alpha}(H).$$

Our third concern is the ordering between $U_\rho(H)$ and $W_{\rho,\alpha}(H)$. However, we have no ordering between them. Because we have the following examples. We set

$$\rho = \begin{pmatrix} 0.8 & 0.0 \\ 0.0 & 0.2 \end{pmatrix}, \qquad H = \begin{pmatrix} 2.0 & 3.0 \\ 3.0 & 1.0 \end{pmatrix}.$$

If we take $\alpha = 0.8$, then $U_\rho(H) - W_{\rho,\alpha}(H)$ approximately takes -0.0241367. If we take $\alpha = 0.9$, then $U_\rho(H) - W_{\rho,\alpha}(H)$ approximately takes 0.404141. This example actually shows that there exists a triplet of α, ρ and H such that $W_{\rho,\alpha}(H) < V_\rho(H)$, since we have $U_\rho(H) \leqslant V_\rho(H)$ in general.

(ii) We have no ordering between $|\mathrm{Tr}[(\frac{\rho^\alpha+\rho^{1-\alpha}}{2})^2[X,Y]]|^2$ and $|\mathrm{Tr}[\rho[X,Y]]|^2$, by the following examples. If we take

$$\rho = \frac{1}{7}\begin{pmatrix} 2 & 2i & 1 \\ -2i & 3 & -2i \\ 1 & 2i & 2 \end{pmatrix}, \qquad X = \begin{pmatrix} 3 & 3 & -i \\ 3 & 1 & 0 \\ i & 0 & 1 \end{pmatrix}, \qquad Y = \begin{pmatrix} 1 & -i & 1-i \\ i & 1 & i \\ 1+i & -i & 3 \end{pmatrix},$$

then we have

$$\left|\mathrm{Tr}\left[\left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2}\right)^2 [X,Y]\right]\right|^2 \simeq 0.348097, \qquad \left|\mathrm{Tr}\big[\rho[X,Y]\big]\right|^2 \simeq 0.326531.$$

If we take

$$\rho = \frac{1}{7}\begin{pmatrix} 2 & 2i & 1 \\ -2i & 3 & -2i \\ 1 & 2i & 2 \end{pmatrix}, \qquad X = \begin{pmatrix} 3 & 3 & -i \\ 3 & 1 & 0 \\ i & 0 & 1 \end{pmatrix}, \qquad Y = \begin{pmatrix} 1 & -i & 0 \\ i & 1 & i \\ 0 & -i & 3 \end{pmatrix},$$

then we have

$$\left|\mathrm{Tr}\left[\left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2}\right)^2 [X,Y]\right]\right|^2 \simeq 0.304377, \qquad \left|\mathrm{Tr}\big[\rho[X,Y]\big]\right|^2 \simeq 0.326531.$$

Remark 2.9.

(i) If we take $M = \rho^{1/2}X_0 x + \rho^{1/2}Y_0$ for any $x \in \mathbb{R}$ presented in Eq. (2.12), we recover the Heisenberg's uncertainty relation (2.1) shown in [3].
(ii) If we take $\alpha = \frac{1}{2}$, then we recover the inequality (2.3) presented in [6].
(iii) We have another inequalities which are different from the inequality (2.11), by taking different self-adjoint operators M appeared in the proof of Theorem 2.6.

Conjecture 2.10. *Our fourth concern is whether the following inequality*:

$$U_{\rho,\alpha}(X)U_{\rho,\alpha}(Y) \geqslant \frac{1}{4}\left|\mathrm{Tr}\left[\left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2}\right)^2 [X,Y]\right]\right|^2 \tag{2.13}$$

holds or not. However we have not found its proof and any counter-examples yet.

$K_{\rho,\alpha}(H)$ and $L_{\rho,\alpha}(H)$ are respectively rewritten by

$$K_{\rho,\alpha}(H) = \mathrm{Tr}\left[\left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2}\right)^2 H_0^2 - \left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2}\right)H_0\left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2}\right)H_0\right]$$

and

$$L_{\rho,\alpha}(H) = \mathrm{Tr}\left[\left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2}\right)^2 H_0^2 + \left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2}\right)H_0\left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2}\right)H_0\right].$$

Thus we have

$$\frac{1}{2}\operatorname{Tr}\left[\left(i\left[\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2}, H_0\right]\right)^2\right] = \frac{1}{2}\operatorname{Tr}\left[\left(i\left[\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2}, H\right]\right)^2\right]$$

but we have

$$\frac{1}{2}\operatorname{Tr}\left[\left(\left\{\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2}, H_0\right\}\right)^2\right] \neq \frac{1}{2}\operatorname{Tr}\left[\left(\left\{\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2}, H\right\}\right)^2\right].$$

In addition, we have $L_{\rho,\alpha}(H) \geqslant K_{\rho,\alpha}(H)$ which implies

$$W_{\rho,\alpha}(H) \equiv \sqrt{K_{\rho,\alpha}(H)L_{\rho,\alpha}(H)} \geqslant \sqrt{K_{\rho,\alpha}(H)K_{\rho,\alpha}(H)} \geqslant K_{\rho,\alpha}(H).$$

Therefore our fifth concern is whether the following inequality for $\alpha \in [0,1]$ holds or not:

$$K_{\rho,\alpha}(X)K_{\rho,\alpha}(Y) \geqslant \frac{1}{4}\left|\operatorname{Tr}\left[\left(\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2}\right)^2 [X,Y]\right]\right|^2. \tag{2.14}$$

However this inequality fails, because we have a counter-example. If we set $\alpha = \frac{1}{2}$ and

$$\rho = \frac{1}{4}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then we have

$$K_{\rho,\alpha}(X)K_{\rho,\alpha}(Y) = I_{\rho}(X)I_{\rho}(Y) = \left(\frac{1-\sqrt{3}}{2}\right)^2$$

and

$$\frac{1}{4}\left|\operatorname{Tr}\left[\left(\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2}\right)^2 [X,Y]\right]\right|^2 = \frac{1}{4}\left|\operatorname{Tr}\left[\rho[X,Y]\right]\right|^2 = \frac{1}{4}.$$

Thus the inequality (2.14) does not hold in general.

Before closing this section, we reconsider the ordering $W_{\rho,\alpha}(H)$ and $V_{\rho}(H)$, although we have already stated an example of the triplet α, ρ and H satisfying $W_{\rho,\alpha}(H) < V_{\rho}(H)$ in the last line of (i) of Remark 2.8. If we set $\alpha = \frac{1}{5}$ and

$$\rho = \begin{pmatrix} 0.3 & 0.45 \\ 0.45 & 0.7 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}.$$

Then $V_{\rho}(H) - W_{\rho,\alpha}(H)$ approximately takes -0.3072. If we set $\alpha = \frac{1}{5}$ and

$$\rho = \begin{pmatrix} 0.3 & 0.4 \\ 0.4 & 0.7 \end{pmatrix}, \qquad H = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}.$$

Then $V_{\rho}(H) - W_{\rho,\alpha}(H)$ approximately takes 0.682011. Therefore we have no ordering between $W_{\rho,\alpha}(H)$ and $V_{\rho}(H)$. Thus it is natural for us to have an interest in the following conjecture, since we have $K_{\rho,\alpha}(H) \leqslant W_{\rho,\alpha}(H)$ in general.

Conjecture 2.11. *Our final concern is whether the following inequality*:

$$K_{\rho,\alpha}(H) \leqslant V_{\rho}(H), \quad \alpha \in [0,1], \tag{2.15}$$

holds or not. However we have not found its proof and any counter-examples yet.

3. Concluding remarks

As we have seen, we introduced a generalized Wigner–Yanase skew information $K_{\rho,\alpha}(H)$ and then defined a new quantity $W_{\rho,\alpha}(H)$. We note that our generalized Wigner–Yanase skew information $K_{\rho,\alpha}(H)$ is different type of the Wigner–Yanase–Dyson skew information $I_{\rho,\alpha}(H)$. For the quantity $K_{\rho,\alpha}(H)$, we do not have a trace inequality related to an uncertainty relation. However, we showed that we have a trace inequality related to an uncertainty relation for the quantity $W_{\rho,\alpha}(H)$. This inequality is a non-trivial one-parameter extension of the uncertainty relation (2.3) shown by S. Luo in [6]. In addition, we studied several trace inequalities on informational quantities.

Finally, we give another generalized trace inequality of the inequality (2.3). For a quantum state ρ an observable H and $\alpha \in [0,1]$, we define

$$Z_{\rho,\alpha}(H) \equiv \frac{1}{4}\sqrt{\mathrm{Tr}\big[\big(i\big[\rho^{\alpha}, H_0\big]\big)^2\big]\,\mathrm{Tr}\big[\big(i\big[\rho^{1-\alpha}, H_0\big]\big)^2\big]\,\mathrm{Tr}\big[\big\{\rho^{\alpha}, H_0\big\}^2\big]\,\mathrm{Tr}\big[\big\{\rho^{1-\alpha}, H_0\big\}^2\big]},$$

with $H_0 \equiv H - \mathrm{Tr}[\rho H]I$. Then we have the following inequality

$$\sqrt{Z_{\rho,\alpha}(X)Z_{\rho,\alpha}(Y)} \geqslant \frac{1}{4}\big|\mathrm{Tr}\big[\rho^{2\alpha}[X,Y]\big]\,\mathrm{Tr}\big[\rho^{2(1-\alpha)}[X,Y]\big]\big|, \tag{3.1}$$

for a quantum state ρ, two observables X, Y and $\alpha \in [0,1]$. We note that the inequality (3.1) recovers the inequality (2.3) by taking $\alpha = 1/2$ and we do not have any weak-strong relation between the inequality (2.11) and the inequality (3.1).

References

[1] J.-C. Bourin, Some inequalities for norms on matrices and operators, Linear Algebra Appl. 292 (1999) 139–154.
[2] J.I. Fujii, A trace inequality arising from quantum information theory, Linear Algebra Appl. 400 (2005) 141–146.
[3] W. Heisenberg, Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik, Z. Phys. 43 (1927) 172–198.
[4] H. Kosaki, Matrix trace inequality related to uncertainty principle, Internat. J. Math. 16 (2005) 629–646.
[5] E.H. Lieb, Convex trace functions and the Wigner–Yanase–Dyson conjecture, Adv. Math. 11 (1973) 267–288.
[6] S. Luo, Heisenberg uncertainty relation for mixed states, Phys. Rev. A 72 (2005) 042110.
[7] S. Luo, Q. Zhang, On skew information, IEEE Trans. Inform. Theory 50 (2004) 1778–1782; and Correction to "On skew information", IEEE Trans. Inform. Theory 51 (2005) 4432.
[8] E.P. Wigner, M.M. Yanase, Information content of distribution, Proc. Natl. Acad. Sci. USA 49 (1963) 910–918.
[9] K. Yanagi, S. Furuichi, K. Kuriyama, A generalized skew information and uncertainty relation, IEEE Trans. Inform. Theory 51 (2005) 4401–4404.