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Riesz products on the ring of dyadic integers

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ABSTRACT

A class of Riesz products on the ring of dyadic integers is introduced. Using almost everywhere convergence of certain series, Hausdorff dimension of these Riesz products is determined. Other properties, such as mutually absolute continuity, quasi-invariance and quasi-Bernoulli property, are also discussed.

1. Introduction and main results

For a prime $p \geqslant 3$, the Riesz products on the ring $\mathbb{Z}_p$ of p-adic integers were studied in [7] by Fan and Zhang. This paper is devoted to the Riesz products on the ring $\mathbb{Z}_2$ of dyadic integers. The main difference between $p = 2$ and $p \geqslant 3$ lies in the lack of dissociate property in the former case. Without this property, the proofs for many results in the case of $p = 2$ are rather lengthy although they lead to similar results as $p \geqslant 3$.

Let $p \geqslant 2$ be a prime number and let $\mathbb{Q}_p$ be the field of p-adic numbers (see [12,18–20] for more information about p-adic numbers). Every p-adic number $x \in \mathbb{Q}_p$ admits a unique expansion $x = \sum_{j=-n}^{\infty} x_j p^j$, $x_j \in \{0, 1, \dots, p-1\}$. We denote by $\{x\}$ the p-adic fraction part of x, i.e., the rational number $\sum_{j=-n}^{-1} x_j p^j$. We further denote by $|\cdot|_p$ the absolute value on $\mathbb{Q}_p$, which is non-Archimedean. The unit ball $\mathbb{Z}_p = \{x \in \mathbb{Q}_p\colon |x|_p \leqslant 1\}$ is called the ring of p-adic integers.

Consider the ring $\mathbb{Z}_p$ as an additive group. The dual group of $\mathbb{Z}_p$ is denoted by

$$\widehat{\mathbb{Z}}_p = \{1\} \cup \{\gamma_{n,k}\colon n \geqslant 1,\ 1 \leqslant k < p^n \text{ and } p \nmid k\}$$

where $\gamma_{n,k}(x) := \exp(2\pi i\{p^{-n}kx\})$ (see [19,20]). We shall consider the subset of characters

$$\Gamma = \{\gamma_{n,1}\colon n \geqslant 1\} \subset \widehat{\mathbb{Z}}_p.$$

For simplicity, we write $\gamma_{n,1}$ as γ_n. Denote by $W(\Gamma)$ the set of all characters $\gamma \in \widehat{\mathbb{Z}}_p$ of the form

$$\gamma = \gamma_1^{\epsilon_1}\gamma_2^{\epsilon_2}\cdots\gamma_n^{\epsilon_n}, \quad \gamma_j \in \Gamma \tag{1}$$

where $\epsilon_j = 0, 1$ or -1 for any $1 \leqslant j \leqslant n$. Γ is called *dissociate* (in the sense of Hewitt–Zuckerman [8,9]) if each element of $W(\Gamma)$ has a unique representation of the form (1).

Denote by dx the normalized Haar measure on the additive group $\mathbb{Z}_p$. Let $a = (a_n)_{n\geqslant 1}$ be a sequence of complex numbers with $|a_n| \leqslant 1$. For $n \geqslant 1$, we define

$$P_{a,n}(x) = \prod_{k=1}^{n}\bigl(1 + \mathbf{Re}\, a_k\gamma_k(x)\bigr). \tag{2}$$

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If $p \geqslant 3$, one can show that Γ is dissociate. By this dissociate property, it is proved [7] that $P_{a,n}(x)\,dx$ converge in the weak* sense to μ_a, a measure which is called *Riesz product* on $\mathbb{Z}_p$ and is written formally as

$$\mu_a = \prod_{n=1}^{\infty}\bigl(1 + \mathbf{Re}\, a_n \gamma_n(x)\bigr). \tag{3}$$

However, Γ is not dissociate in the case of $p = 2$. This can be easily seen, for example, from the fact that γ_2 has at least two different representations γ_2 and $\gamma_1\gamma_2^{-1}$. Though without the dissociate property, we managed to prove the convergence of the sequence of measures $\{P_{a,n}(x)\,dx\}$ directly (Section 2). The limit measure of the form (3) for $p = 2$ is also called a Riesz product.

Let us state our main results. The first result concerns the almost everywhere convergence of certain lacunary series.

Let $\{f_k\}_{k\geqslant 1}$ be a sequence of analytic functions defined in some complex domain containing the unit disc $\{z \in \mathbb{C}\colon\ |z| \leqslant 1\}$. Let $\{\alpha_k\}_{k\geqslant 1}$ be any sequence of complex numbers. We consider the following lacunary series

$$\sum_{k=1}^{\infty} \alpha_k\bigl[f_k \circ \gamma_k(x) - \mathbb{E}_{\mu_a} f_k \circ \gamma_k\bigr]. \tag{4}$$

Theorem 1.1. *Let $\{c_j^{(k)}\}$ be the Taylor coefficients of f_k at the point zero. Suppose*

$$\sum_{j=1}^{\infty} \sqrt{1 + \log j}\, \sup_{k\geqslant 1}\bigl|c_j^{(k)}\bigr| < \infty.$$

Then for any sequence $\{\alpha_k\}_{k\geqslant 1} \in l^2$, the series (4) *converges for μ_a-a.e. x.*

Equipping $\mathbb{Z}_2$ with the dyadic norm $|\cdot|_2$, one can talk about Hausdorff dimension of any subset $E \subset \mathbb{Z}_2$ (see [13]). The Hausdorff dimension of μ_a, denoted by $\dim_{\mathrm{H}} \mu_a$, is defined as the infimum of $\dim_{\mathrm{H}} E$'s such that $\mu_a(E) = 1$ (see [5] for more details).

From Theorem 1.1, we deduce the Hausdorff dimension of the Riesz product.

Theorem 1.2. *The Hausdorff dimension of the Riesz products μ_a is equal to*

$$\dim_{\mathrm{H}} \mu_a = 1 - \frac{1}{\log 2} \varlimsup_{n\to\infty} \frac{\mathbb{E}_{\mu_a} \log P_{a,n}}{n}.$$

The shift transformation T on $\mathbb{Z}_2$, which takes $\sum_{n=0}^{\infty} a_n 2^n$ to $\sum_{n=0}^{\infty} a_{n+1} 2^n$, has the following analytic expression

$$Tx = \frac{x}{2} - \left\{\frac{x}{2}\right\}.$$

Recall that a probability measure μ is T*-invariant* (resp. T*-quasi-invariant*) if $\mu = \mu \circ T^{-1}$ (resp. $\mu \ll \mu \circ T^{-1}$ and $\mu \circ T^{-1} \ll \mu$).

We recall that in [7] we have proved that for the case $p \geqslant 3$, none of Riesz product is T-invariant, except the trivial case of Haar measure which corresponds to $a_n = 0$ for all n. For the case $p = 2$, we have the following theorem.

Theorem 1.3. *Let μ_a be a Riesz product defined by* (3)*. Assume that $|a_k| < 1$ for all $k \geqslant 1$. If $a_1, a_2, a_3 \in \mathbb{R}$, then μ_a is T-invariant if and only if $a_k = 0$ for any $k \geqslant 1$.*

The condition on a_1, a_2, a_3 is technical. If one of a_1, a_2, a_3 is not in $\mathbb{R}$, we do not know whether there exists a non-trivial sequence $\{a_n\}_{n\geqslant 1}$ such that the Riesz product associated to the coefficients a_n's is T-invariant or not.

In this paper, we will also discuss other properties, such as the mutually absolute continuity of two Riesz products, the invariance and the quasi-invariance with respect to the shift transformation, and the quasi-Bernoulli property of the Riesz products (Section 5).

We organize the paper as follows. In Section 2, we prove the existence of μ_a. In Section 3, we study the convergence of lacunary series (4) and prove Theorems 1.1 and 1.2. The invariance of μ_a will be given in Section 4. Some other properties of μ_a are stated in the last section.

2. Construction of Riesz products

In this section, we prove the convergence of the sequence $\{P_{a,n}(x)\,dx\}$ of measures. We begin with two lemmas.

For $m \geqslant 1$, let

$$\Gamma_m = \bigl\{\gamma_1^{\epsilon_1} \cdots \gamma_m^{\epsilon_m}\colon\ \epsilon_1, \dots, \epsilon_{m-1} = -1, 0, 1;\ \epsilon_m = -1, 1\bigr\}.$$

Lemma 2.1. *If $\gamma_{n,k} \in \Gamma_m$, then $n = m$.*

Proof. For any $\gamma_{n,k} \in \Gamma_m$ and for any sequence $(\epsilon_1, \dots, \epsilon_m)$ with $\gamma_{n,k} = \gamma_1^{\epsilon_1} \cdots \gamma_m^{\epsilon_m}$, we have

$$\exp\left(2\pi i\left\{\frac{k}{2^n}x\right\}\right) = \exp\left(2\pi i\left\{\left(\frac{\epsilon_1}{2} + \frac{\epsilon_2}{2^2} + \cdots + \frac{\epsilon_m}{2^m}\right)x\right\}\right), \quad \forall x \in \mathbb{Z}_2.$$

If $n \neq m$, then by taking the value at $x = 1$ of both sides, we get

$$\frac{k}{2^n} - \left(\frac{\epsilon_1}{2} + \frac{\epsilon_2}{2^2} + \cdots + \frac{\epsilon_m}{2^m}\right) = \frac{k}{2^n} - \frac{\epsilon_m + \cdots + \epsilon_1 \cdot 2^{m-1}}{2^m} \in \mathbb{Z}_2.$$

But this is impossible because the numerator of the fraction reduced to a common denominator, is an odd number. So, we must have $n = m$. $\square$

Lemma 2.2. *Let $n \geqslant 1$. Let F be a function on $\mathbb{Z}_2$ depending only on the first $n-1$ coordinates. Then for any odd integer k,*

$$\int_{\mathbb{Z}_2} F(x)\gamma_n^k(x)\,dx = 0.$$

Proof. Since the Haar measure is a probability measure, the integral in question is equal to

$$\int_{(\mathbb{Z}/2\mathbb{Z})^n} F(x_0, x_1, \dots, x_{n-2})\gamma_n^k(x_0, x_1, \dots, x_{n-1})\,dx_0\,dx_1 \cdots dx_{n-1}.$$

Write

$$\gamma_n^k(x) = \gamma(x_0, x_1, \dots, x_{n-2})\exp\left(2\pi ki\frac{x_{n-1}}{2}\right) = (-1)^{x_{n-1}}\gamma(x_0, x_1, \dots, x_{n-2}),$$

where γ depends only on the first $n-1$ coordinates. Thus by Fubini Theorem,

$$\int_{\mathbb{Z}_2} F(x)\gamma_n^k(x)\,dx = \int_{(\mathbb{Z}/2\mathbb{Z})^{n-1}} F\gamma\,dx_0\,dx_1 \cdots dx_{n-2} \int_{\mathbb{Z}/2\mathbb{Z}} (-1)^{x_{n-1}}\,dx_{n-1}.$$

Then the lemma follows by using

$$\int_{\mathbb{Z}/2\mathbb{Z}} (-1)^{x_{n-1}}\,dx_{n-1} = 1 \cdot \frac{1}{2} + (-1) \cdot \frac{1}{2} = 0. \qquad \square$$

Now we prove that the sequence $\{P_{a,n}(x)\,dx\}$ of measures admits a weak* limit.
For any $\gamma \in \widehat{\mathbb{Z}}_2$, let

$$\widehat{P}_{a,n}(\gamma) = \int_{\mathbb{Z}_2} P_{a,n}(x)\gamma(x)\,dx.$$

To prove the convergence of $\{P_{a,n}(x)\,dx\}$, it suffices to prove that for any character $\gamma \in \widehat{\mathbb{Z}}_2$, $\widehat{P}_{a,n}(\gamma)$ admits a limit as n tends to $+\infty$. First notice that $\widehat{P}_{a,n}(1) \equiv 1$. For any $\gamma_{n,k} \in \widehat{\mathbb{Z}}_2$ and for any $N > n$, by Lemma 2.2,

$$\begin{aligned}
\widehat{P}_{a,N}(\gamma_{n,k}) &= \int_{\mathbb{Z}_2} P_{a,N}(x)\gamma_{n,k}(x)\,dx \\
&= \int_{\mathbb{Z}_2} \gamma_{n,k}(x)P_{a,n}(x)\,dx \\
&= \int_{\mathbb{Z}_2} \gamma_{n,k}(x) \sum_{\epsilon_1, \epsilon_2, \dots, \epsilon_n} a_1^{(\epsilon_1)} \cdots a_n^{(\epsilon_n)}\gamma_1^{\epsilon_1} \cdots \gamma_n^{\epsilon_n}\,dx
\end{aligned}$$

where $a^{(\epsilon)}$ stands for 1, $\frac{\bar{a}}{2}$ or $\frac{a}{2}$ according to $\epsilon = 0, 1$ or -1. We have the following three facts:

- If $\epsilon_n = 0$, then by Lemma 2.1, there does not exist any sequence $(\epsilon_1, \epsilon_2, \dots, \epsilon_n)$ satisfying $\gamma_{n,k} = \gamma_1^{-\epsilon_1} \cdots \gamma_n^{-\epsilon_n}$.

- For any sequence $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ with $\gamma_{n,k} \neq \gamma_1^{-\epsilon_1} \cdots \gamma_n^{-\epsilon_n}$, the integral

$$\int_{\mathbb{Z}_2} \gamma_{n,k}(x)\gamma_1^{\epsilon_1} \cdots \gamma_n^{\epsilon_n} dx = 0.$$

- Let $E_{n,k} := \{(\epsilon_1, \ldots, \epsilon_n)\colon\ \epsilon_n \neq 0;\ \epsilon_n + \cdots + \epsilon_1 \cdot 2^{n-1} = k \text{ or } 2^n - k\}$ where $\epsilon_j \in \{-1, 0, 1\}$ for $1 \leqslant j \leqslant n$. For any sequence $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ with $\gamma_{n,k} = \gamma_1^{-\epsilon_1} \cdots \gamma_n^{-\epsilon_n}$, by Lemma 2.1 again, we have

$$\frac{k + \epsilon_n + \cdots + \epsilon_1 \cdot 2^{n-1}}{2^n} \in \mathbb{Z}_2,$$

which implies $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in E_{n,k}$.

By the above three facts, we have

$$\widehat{P}_{a,N}(\gamma_{n,k}) = \sum_{(\epsilon_1,\ldots,\epsilon_n)\in E_{n,k}} \prod_{j=1}^{n} a_j^{(\epsilon_j)}.$$

Hence $\widehat{P}_{a,N}(\gamma_{n,k})$ admits a limit as N tends to $+\infty$, which proves the existence and uniqueness of μ_a. Therefore, the following proposition follows.

Proposition 2.3. *The Riesz product μ_a of the form* (3) *is well defined as the weak* limit of $P_{a,n}(x)\,dx$.*

Let $B_n(y) = \{x \in \mathbb{Z}_2\colon\ |x - y|_2 \leqslant 2^{-n}\}$. As an application of Lemma 2.2, we also have the following proposition.

Proposition 2.4. *For any ball $B_n(y)$, we have*

$$\mu_a\big(B_n(y)\big) = 2^{-n} P_{a,n}(y). \tag{5}$$

Proof. Let F be the characteristic function of the ball $B_n(y)$ which is a function depending only on the first n coordinates. Applying Lemma 2.2 to this function F shows that for any odd integer k and for any $m > n$,

$$\int_{B_n(y)} \gamma_m^k(x)\,dx = 0.$$

So,

$$\int_{B_n(y)} d\mu_a(x) = \lim_{N\to\infty} \int_{B_n(y)} P_{a,N}(x)\,dx = \int_{B_n(y)} P_{a,n}(x)\,dx.$$

Then the proposition follows immediately since the integrand at the right side is constant on the ball $B_n(y)$. $\square$

We can consider the equality (5) as the definition of the Riesz product. Actually it is easy to see that the functions $2^{-n}P_{a,n}(x)$ are consistent in the sense of Kolmogorov. These two ways of defining Riesz products lead to the same Riesz products. We point out that both methods can be generalized to produce measures on the product space $\prod_{n=1}^{\infty} \mathbb{Z}/m_n\mathbb{Z}$ for any sequence of integers $(m_n)_{n\geqslant 1}$ with $m_n \geqslant 2$.

3. Almost everywhere convergence and Hausdorff dimension

In this section, we will prove Theorem 1.1. For the case $p \geqslant 3$, the series (4) can be decomposed into a sum of a finite number of martingales. But for the present case $p = 2$, the decomposition will be more complicated. The idea here is inspired by Peyrière's work [17] on the circle Riesz products. However, the contexts are different now. We prove Theorem 1.1 for the special case $f_k(z) = z$ first, then we will give the proof for the general case.

3.1. Case: $f_k(z) = z$

Let $\mathfrak{B}_n = \sigma(\gamma_1, \gamma_2, \ldots, \gamma_n)$ be the σ-algebra generated by the characters $\gamma_1, \ldots, \gamma_n$ (convention: $\mathfrak{B}_0 = \{\emptyset, \mathbb{Z}_2\}$).

Lemma 3.1. *For any $n \geqslant 1$, we have*

$$\mathbb{E}_{\mu_a}\big(\gamma_{n+1}(x)|\mathfrak{B}_n\big) = \frac{1}{2}\big(\bar{a}_{n+1} + a_{n+1}\gamma_n(x)\big).$$

Proof. It suffices to prove that for any ball $B_n(y)$,

$$\int_{B_n(y)} \gamma_{n+1}(x)\,d\mu_a(x) = \frac{1}{2}\int_{B_n(y)} \big(\bar{a}_{n+1} + a_{n+1}\gamma_n(x)\big)\,d\mu_a(x).$$

In fact, by Lemma 2.2, we have

$$\begin{aligned}\int_{B_n(y)} \gamma_{n+1}(x)\,d\mu_a(x) &= \int_{B_n(y)} \gamma_{n+1}(x)P_{a,n+1}(x)\,dx\\ &= \int_{B_n(y)} P_{a,n}(x)\gamma_{n+1}(x)\big(\operatorname{Re} a_{n+1}\gamma_{n+1}(x)\big)\\ &= \frac{1}{2}\int_{B_n(y)} P_{a,n}(x)\big(\bar{a}_{n+1} + a_{n+1}\gamma_n(x)\big)\,dx\\ &= \frac{1}{2}\int_{B_n(y)} \big(\bar{a}_{n+1} + a_{n+1}\gamma_n(x)\big)\,d\mu_a(x),\end{aligned}$$

where the third equality is because $\gamma_{n+1}^2(x) = \gamma_n(x)$. $\square$

By Lemma 3.1,

$$\mathbb{E}_{\mu_a}\gamma_n = \mathbb{E}_{\mu_a}\big(\mathbb{E}_{\mu_a}(\gamma_n|\mathfrak{B}_{n-1})\big) = \frac{1}{2}\bar{a}_n + \frac{1}{2}a_n\mathbb{E}_{\mu_a}\gamma_{n-1}.$$

By induction, we obtain

$$\mathbb{E}_{\mu_a}\gamma_n = \frac{1}{2}\bar{a}_n + \frac{1}{2^2}a_n\bar{a}_{n-1} + \cdots + \frac{1}{2^n}a_n a_{n-1}\cdots a_2\bar{a}_1, \tag{6}$$

which is useful in what follows. We also remark that the equality (6) implies

$$\gamma_n = \gamma_{n-1}\gamma_n^{-1} = \cdots = \gamma_1\gamma_2^{-1}\cdots\gamma_n^{-1}.$$

Proposition 3.2. *There exists a constant C such that for any $\alpha = \{\alpha_j\}_{j\geqslant 1} \in l^2$, we have*

$$\left[\int \sup_{n\geqslant 1}\left|\sum_{j=1}^{n}\alpha_j\big(\gamma_j(x) - \mathbb{E}_{\mu_a}\gamma_j\big)\right|^2 d\mu_a(x)\right]^{\frac{1}{2}} \leqslant C\left(\sum_{j\geqslant 1}|\alpha_j|^2\right)^{\frac{1}{2}}.$$

Proof. Put $\gamma_0(x) = 0$ by convention. From Lemma 3.1, we see that $\{\gamma_n(x) - \frac{1}{2}(\bar{a}_n + a_n\gamma_{n-1}(x))\}_{n\geqslant 1}$ is a sequence of martingale-differences. Furthermore, we observe that $\gamma_j(x) - \mathbb{E}_{\mu_a}\gamma_j$ can be decomposed into the following sum:

$$\begin{aligned}\gamma_j(x) - \mathbb{E}_{\mu_a}\gamma_j &= \gamma_j(x) - \frac{1}{2}\bar{a}_j - \frac{1}{2}a_j\gamma_{j-1}(x)\\ &\quad + \frac{1}{2}a_j\left(\gamma_{j-1}(x) - \frac{1}{2}\bar{a}_{j-1} - \frac{1}{2}a_{j-1}\gamma_{j-2}(x)\right)\\ &\quad \vdots\\ &\quad + \frac{1}{2^{j-2}}a_j a_{j-1}\cdots a_3\left(\gamma_2(x) - \frac{1}{2}\bar{a}_2 - \frac{1}{2}a_2\gamma_1(x)\right)\\ &\quad + \frac{1}{2^{j-1}}a_j a_{j-1}\cdots a_2\left(\gamma_1(x) - \frac{1}{2}\bar{a}_1\right).\end{aligned}$$

Now we define two sequences of numbers $u_{j,k}$ and $v_{j,k}$. For any $j \geqslant 1$, $u_{j,0} = v_{j,0} := 1$. For $j \geqslant 1$ and $1 \leqslant k \leqslant j-1$,

$$\begin{aligned}v_{j,k} &:= \frac{1}{2^k}a_j a_{j-1}\cdots a_{j-k+1},\\ u_{j,k} &:= \frac{1}{2^k}a_j a_{j-1}\cdots a_{j-k+2}\bar{a}_{j-k+1}.\end{aligned}$$

Then for any $j \geqslant 1$ and any $0 \leqslant k \leqslant j-1$, we have

$$|u_{j,k+1}| \leqslant \frac{1}{2}|v_{j,k}| \leqslant \frac{1}{2^{k+1}}, \qquad |v_{j,k+1}| \leqslant \frac{1}{2}|v_{j,k}| \leqslant \frac{1}{2^{k+1}},$$

and

$$\mathbb{E}_{\mu_a}\big(v_{j,k}\gamma_{j-k}(x)|\mathfrak{B}_{j-k-1}\big) = u_{j,k+1} + v_{j,k+1}\gamma_{j-k-1}(x),$$

$$\gamma_j(x) - \mathbb{E}_{\mu_a}\gamma_j = \sum_{k=0}^{j-1}\big(v_{j,k}\gamma_{j-k}(x) - u_{j,k+1} - v_{j,k+1}\gamma_{j-k-1}(x)\big),$$

$$\big\|v_{j,k}\gamma_{j-k}(x) - u_{j,k+1} - v_{j,k+1}\gamma_{j-k-1}(x)\big\|_{L^2(\mu_a)} \leqslant 2^{-(k-1)}. \tag{7}$$

Hence

$$\sup_{n\geqslant 1}\left|\sum_{j=1}^{n}\alpha_j\big(\gamma_j(x) - \mathbb{E}_{\mu_a}\gamma_j\big)\right| \leqslant \sum_{k=0}^{\infty}\sup_{n\geqslant 1}\left|\sum_{j=1}^{n}\alpha_j\big(v_{j,k}\gamma_{j-k}(x) - u_{j,k+1} - v_{j,k+1}\gamma_{j-k-1}(x)\big)\right|$$

where $\gamma_{j-k}(x) = 0$ $(j \leqslant k)$, and $u_{j,k} = v_{j,k} = 0$ $(j < k)$.

By Doob's inequality and (7), we obtain

$$\left\|\sup_{n\geqslant 1}\left|\sum_{j=1}^{n}\alpha_j\big(\gamma_j(x) - \hat{\mu}_a(\gamma_j)\big)\right|\right\|_{L^2(\mu_a)} \leqslant C\sum_{k=0}^{\infty}2^{-(k-1)}\sqrt{\sum_{j=1}^{\infty}|\alpha_j|^2} \leqslant C\sqrt{\sum_{j=1}^{\infty}|\alpha_j|^2}. \qquad \square$$

From Proposition 3.2, we get the following theorem which is nothing but the special case $f_k(z) = z$ of Theorem 1.1.

Theorem 3.3. *Assume that $\alpha = \{\alpha_j\}_{j\geqslant 1} \in l^2$. Then the series*

$$\sum_{j=1}^{\infty}\alpha_j\big(\gamma_j(x) - \mathbb{E}_{\mu_a}\gamma_j\big)$$

converges for μ_a-a.e. x.

Now we prove the convergence of the lacunary series (4) in the general case.

3.2. Proof of Theorem 1.1

Lemma 3.4. *For any integer n,*

$$\mathbb{E}_{\mu_a}\big(\gamma_{k+1}^n(x)|\mathfrak{B}_k\big) = \begin{cases} \gamma_{k+1}^n(x), & n \text{ even},\\ \frac{1}{2}\big(\bar{a}_{k+1}\gamma_k^{\frac{n-1}{2}}(x) + a_{k+1}\gamma_k^{\frac{n+1}{2}}(x)\big), & n \text{ odd}. \end{cases}$$

Proof. It is trivial if n is even. If n is odd, write $n = 2m+1$. We have

$$\mathbb{E}_{\mu_a}\big(\gamma_{k+1}^n(x)|\mathfrak{B}_k\big) = \gamma_k^m\mathbb{E}_{\mu_a}\big(\gamma_{k+1}(x)|\mathfrak{B}_k\big).$$

Then the desired result follows by Lemma 3.1. $\square$

Lemma 3.5. *Let $m \geqslant 1$ be an integer. For any integer n, if $n \equiv 0 \bmod 2^m$, then*

$$\mathbb{E}_{\mu_a}\big(\gamma_{k+m}^n(x)|\mathfrak{B}_k\big) = \gamma_k^{\frac{n}{2^m}}(x) = \gamma_{k+m}^n(x); \tag{8}$$

if $n \equiv j \bmod 2^m$ for some $1 \leqslant j \leqslant 2^m - 1$, then

$$\mathbb{E}_{\mu_a}\big(\gamma_{k+m}^n(x)|\mathfrak{B}_k\big) = u_k\gamma_k^{\frac{n-j}{2^m}}(x) + v_k\gamma_k^{\frac{n-j}{2^m}+1}(x) \tag{9}$$

where $|u_k| + |v_k| \leqslant 1$.

Proof. The first assertion is trivial. We will prove the second assertion by induction. The case $m = 1$ is nothing but Lemma 3.4. Suppose the lemma is established for $m \geqslant 1$, let us consider the case $m+1$ and $2^{m+1} \nmid n$. Notice that

$$\mathbb{E}_{\mu_a}\big(\gamma_{k+m+1}^n(x)|\mathfrak{B}_k\big) = \mathbb{E}_{\mu_a}\big(\mathbb{E}_{\mu_a}\big(\gamma_{k+m+1}^n(x)|\mathfrak{B}_{k+m}\big)|\mathfrak{B}_k\big).$$

If $n \equiv 2j \bmod 2^{m+1}$ for some $1 \leqslant j \leqslant 2^m - 1$, then $\frac{n}{2} \equiv j \bmod 2^m$. Hence by (9),

$$\begin{aligned}\mathbb{E}_{\mu_a}\big(\gamma_{k+m+1}^n(x)|\mathfrak{B}_k\big) &= \mathbb{E}_{\mu_a}\big(\gamma_{k+m}^{\frac{n}{2}}(x)|\mathfrak{B}_k\big)\\ &= u_k'\gamma_k^{\frac{\frac{n}{2}-j}{2^m}}(x) + v_k'\gamma_k^{\frac{\frac{n}{2}-j}{2^m}+1}(x)\\ &= u_k'\gamma_k^{\frac{n-2j}{2^{m+1}}}(x) + v_k'\gamma_k^{\frac{n-2j}{2^{m+1}}+1}(x)\end{aligned}$$

where $|u_k'| + |v_k'| \leqslant 1$.

If $n \equiv 2j+1 \bmod 2^{m+1}$ for some $0 \leqslant j \leqslant 2^m - 2$, then $\frac{n-1}{2} \equiv j \bmod 2^m$ and $\frac{n+1}{2} \equiv j+1 \bmod 2^m$. Thus by Lemma 3.4 and (9),

$$\begin{aligned}\mathbb{E}_{\mu_a}\big(\gamma_{k+m+1}^n(x)|\mathfrak{B}_k\big) &= \mathbb{E}_{\mu_a}\left(\frac{1}{2}\bar{a}_{k+m+1}\gamma_{k+m}^{\frac{n-1}{2}}(x) + \frac{1}{2}a_{k+m+1}\gamma_{k+m}^{\frac{n+1}{2}}(x)\middle|\mathfrak{B}_k\right)\\ &= \frac{1}{2}\bar{a}_{k+m+1}\big(u_k^{(1)}\gamma_k^{\frac{\frac{n-1}{2}-j}{2^m}}(x) + v_k^{(1)}\gamma_k^{\frac{\frac{n-1}{2}-j}{2^m}+1}(x)\big)\\ &\quad + \frac{1}{2}a_{k+m+1}\big(u_k^{(2)}\gamma_k^{\frac{\frac{n+1}{2}-(j+1)}{2^m}}(x) + v_k^{(2)}\gamma_k^{\frac{\frac{n+1}{2}-(j+1)}{2^m}+1}(x)\big)\\ &= \frac{1}{2}\bar{a}_{k+m+1}\big(u_k^{(1)}\gamma_k^{\frac{n-(2j+1)}{2^{m+1}}}(x) + v_k^{(1)}\gamma_k^{\frac{n-(2j+1)}{2^{m+1}}+1}(x)\big)\\ &\quad + \frac{1}{2}a_{k+m+1}\big(u_k^{(2)}\gamma_k^{\frac{n-(2j+1)}{2^{m+1}}}(x) + v_k^{(2)}\gamma_k^{\frac{n-(2j+1)}{2^{m+1}}+1}(x)\big)\\ &= u_k''\gamma_k^{\frac{n-(2j+1)}{2^{m+1}}}(x) + v_k''\gamma_k^{\frac{n-(2j+1)}{2^{m+1}}+1}(x)\end{aligned}$$

where $|u_k^{(1)}| + |v_k^{(1)}| \leqslant 1$, $|u_k^{(2)}| + |v_k^{(2)}| \leqslant 1$ and $u_k'' = \frac{1}{2}\bar{a}_{k+m+1}u_k^{(1)} + \frac{1}{2}a_{k+m+1}u_k^{(2)}$, $v_k'' = \frac{1}{2}\bar{a}_{k+m+1}v_k^{(1)} + \frac{1}{2}a_{k+m+1}v_k^{(2)}$. It is clear that $|u_k''| + |v_k''| \leqslant 1$.

Now, if $n \equiv 2^{m+1} - 1 \bmod 2^{m+1}$, then $\frac{n-1}{2} \equiv 2^m - 1 \bmod 2^m$, $\frac{n+1}{2} \equiv 0 \bmod 2^m$. Hence by Lemma 3.4, (8) and (9),

$$\begin{aligned}\mathbb{E}_{\mu_a}\big(\gamma_{k+m+1}^n(x)|\mathfrak{B}_k\big) &= \mathbb{E}_{\mu_a}\left(\frac{1}{2}\bar{a}_{k+m+1}\gamma_{k+m}^{\frac{n-1}{2}}(x) + \frac{1}{2}a_{k+m+1}\gamma_{k+m}^{\frac{n+1}{2}}(x)\middle|\mathfrak{B}_k\right)\\ &= \frac{1}{2}\bar{a}_{k+m+1}\big(u_k^{(3)}\gamma_k^{\frac{\frac{n-1}{2}-(2^m-1)}{2^m}}(x) + v_k^{(3)}\gamma_k^{\frac{\frac{n-1}{2}-(2^m-1)}{2^m}+1}(x)\big)\\ &\quad + \frac{1}{2}a_{k+m+1}\gamma_k^{\frac{\frac{n+1}{2}}{2^m}}(x)\\ &= \frac{1}{2}\bar{a}_{k+m+1}\big(u_k^{(3)}\gamma_k^{\frac{n-(2^{m+1}-1)}{2^{m+1}}}(x) + v_k^{(3)}\gamma_k^{\frac{n-(2^{m+1}-1)}{2^{m+1}}+1}(x)\big)\\ &\quad + \frac{1}{2}a_{k+m+1}\gamma_k^{\frac{n-(2^{m+1}-1)}{2^{m+1}}+1}(x)\\ &= u_k'''\gamma_k^{\frac{n-(2^{m+1}-1)}{2^{m+1}}}(x) + v_k'''\gamma_k^{\frac{n-(2^{m+1}-1)}{2^{m+1}}+1}(x)\end{aligned}$$

where $|u_k^{(3)}| + |v_k^{(3)}| \leqslant 1$ and $u_k''' = \frac{1}{2}\bar{a}_{k+m+1}u_k^{(3)}$, $v_k''' = \frac{1}{2}\bar{a}_{k+m+1}v_k^{(3)} + \frac{1}{2}a_{k+m+1}$. We also have $|u_k'''| + |v_k'''| \leqslant 1$. $\square$

Proposition 3.6. *There exists a constant C such that for any positive integer n and $\alpha = \{\alpha_j\}_{j\geqslant 1} \in l^2$,*

$$\left[\int \sup_{m\geqslant 1}\left|\sum_{j=1}^m \alpha_j\big(\gamma_j^n(x) - \mathbb{E}_{\mu_a}(\gamma_j^n)\big)\right|^2 d\mu_a(x)\right]^{\frac{1}{2}} \leqslant C(\sqrt{1+\log n})\left(\sum_{j\geqslant 1}|\alpha_j|^2\right)^{\frac{1}{2}}.$$

Proof. Let d be the smallest integer such that $d > \log_2 n$. Then $n = j$ for some $1 \leqslant j \leqslant 2^d - 1$ (the case $n = 0$ is trivial). Hence $\frac{n-j}{2^d} = 0$. By Lemma 3.5, we have

$$\mathbb{E}_{\mu_a}\big(\gamma_{j+d}^n(x)|\mathfrak{B}_j\big) = u_j + v_j\gamma_j(x)$$

where $|u_j| + |v_j| \leqslant 1$. Thus

$$\mathbb{E}_{\mu_a}\big(\gamma_{j+d}^n(x)|\mathfrak{B}_j\big) - \mathbb{E}_{\mu_a}\big(\gamma_{j+d}^n\big) = v_j\big(\gamma_j(x) - \mathbb{E}_{\mu_a}\gamma_j\big).$$

Hence by Proposition 3.2, we obtain

$$\left\| \sup_{m\geqslant 1} \left| \sum_{j=1}^{m} \alpha_{j+d} \big(\mathbb{E}_{\mu_a} \big(\gamma_{j+d}^{n}(x) | \mathfrak{B}_j \big) - \mathbb{E}_{\mu_a} \big(\gamma_{j+d}^{n} \big) \big) \right| \right\|_{L^2(\mu_a)} \leqslant C \sqrt{\sum_{j\geqslant 1} |\alpha_j|^2}. \tag{10}$$

Notice that $\{\gamma_{j+ld}^{n}(x) - \mathbb{E}_{\mu_a}(\gamma_{j+ld}^{n}(x)|\mathfrak{B}_{j+(l-1)d})\}_{l\geqslant 1}$ is also a sequence of martingale-differences for any $1 \leqslant j \leqslant d-1$. Thus by the same method in the proof of Proposition 6.5 in [7], we have

$$\left\| \sup_{m\geqslant 1} \left| \sum_{j=1}^{m} \alpha_{j+d} \big(\gamma_{j+d}^{n}(x) - \mathbb{E}_{\mu_a} \big(\gamma_{j+d}^{n}(x) | \mathfrak{B}_j \big) \big) \right| \right\|_{L^2(\mu_a)} \leqslant C \sqrt{d \sum_{j\geqslant 1} |\alpha_j|^2}. \tag{11}$$

Combining (10) and (11), we get the desired result. □

Proof of Theorem 1.1. Using the Taylor expansion of f_k, we write

$$f_k\big(\gamma_k(x)\big) - \mathbb{E}_{\mu_a} f_k\big(\gamma_k(x)\big) = \sum_{j=1}^{\infty} c_j^{(k)} \varphi_{k,j}(x)$$

where

$$\varphi_{k,j}(x) = \gamma_k^{j}(x) - \mathbb{E}_{\mu_a} \gamma_k^{j}(x).$$

Denote by

$$F_N(x) = \sum_{k=1}^{N} \alpha_k \big[f_k\big(\gamma_k(x)\big) - \mathbb{E}_{\mu_a} f_k\big(\gamma_k(x)\big) \big]$$

the partial sum of the series in question, and

$$F^*(x) = \sup_{N\geqslant 1} \big| F_N(x) \big|.$$

Observe that

$$\big| F_N(x) \big| \leqslant \left| \sum_{k=1}^{N} \alpha_k \sum_{j=1}^{\infty} c_j^{(k)} \varphi_{k,j}(x) \right| \leqslant \sum_{j=1}^{\infty} \left| \sum_{k=1}^{N} \alpha_k c_j^{(k)} \varphi_{k,j}(x) \right|.$$

Then we have

$$F^*(x) \leqslant \sum_{j\geqslant 1} \sup_{N\geqslant 1} \left| \sum_{k=1}^{N} \alpha_k c_j^{(k)} \varphi_{k,j}(x) \right|.$$

Thus, by Proposition 3.6,

$$\begin{aligned} \big\| F^*(x) \big\|_{L^2(\mu_a)} &\leqslant C \sum_{j\geqslant 1} \sqrt{1+\log j} \sqrt{\sum_{k=1}^{\infty} \big| \alpha_k c_j^{(k)} \big|^2} \\ &\leqslant C \left(\sum_{j=1}^{\infty} \sqrt{1+\log j} \sup_{k\geqslant 1} \big| c_j^{(k)} \big| \right) \sqrt{\sum_{k=1}^{\infty} |\alpha_k|^2} < \infty. \end{aligned}$$

Hence we obtain the μ_a-a.e. convergence of the series (4). □

3.3. Hausdorff dimension

We apply Theorem 1.1 to calculate the Hausdorff dimension of μ_a. Recall that the Hausdorff dimension of a measure μ is defined by $\dim_{\mathrm{H}} \mu = \inf\{\dim_{\mathrm{H}} E\colon\ E$ Borel set and $\mu(E^c) = 0\}$ (see [5]). The dimension $\dim_{\mathrm{H}} \mu$ is equal to the essential supremum of the lower local density

$$\underline{D}(\mu, x) = \liminf_{n\to\infty} \frac{\log \mu(B_n(x))}{\log |B_n(x)|}$$

where $|B_n(x)|$ denotes the Haar measure of $B_n(x)$ (see [5]).

For the Riesz product μ_a, we have

$$\underline{D}(\mu_a, x) = 1 - \frac{1}{\log 2} \limsup_{n\to\infty} \frac{\log P_{a,n}(x)}{n}.$$

By Theorem 1.1 and Kronecker Lemma,

$$\frac{1}{n}\sum_{k=1}^{n}\big[\log\big(1+\mathbf{Re}\, a_k\gamma_k(x)\big) - \mathbb{E}_{\mu_a}\log\big(1+\mathbf{Re}\, a_k\gamma_k(x)\big)\big]$$

tends to zero for μ_a-a.e x. It follows that

$$\dim_{\mathrm{H}} \mu_a = 1 - \frac{1}{\log 2} \limsup_{n\to\infty} \frac{\mathbb{E}_{\mu_a}\log P_{a,n}}{n}.$$

4. Proof of invariance

In this section we will prove Theorem 1.3.

Recall that the measure μ_a is said to be T-invariant if for any $\gamma \in \widehat{\mathbb{Z}}_2$,

$$\hat{\mu}_a(\gamma) = \widehat{\mu_a \circ T^{-1}}(\gamma).$$

We are going to express this condition in terms of the coefficients a_n's.

On one hand, by Lemma 2.2,

$$\hat{\mu}_a(\gamma_{n,k}) = \int_{\mathbb{Z}_2} \gamma_{n,k} P_{a,n}(x)\, dx. \tag{12}$$

On the other hand, for any $\gamma_{n,k} \in \widehat{\mathbb{Z}}_2 \setminus \{1\}$, by Lemma 2.2, we have

$$\widehat{\mu_a \circ T^{-1}}(\gamma_{n,k}) = \int_{\mathbb{Z}_2} \gamma_{n,k}(Tx) P_{a,n+1}(x)\, dx. \tag{13}$$

For $j = 0$ or 1, denote

$$b_n^{(j)} := a_{n+1} e^{2\pi i \frac{j}{2^{n+1}}}, \qquad c_j := \frac{1+\mathbf{Re}\, a_1\gamma_1(j)}{2}.$$

Let $\mu_b^{(j)}$ be the Riesz products associated with the coefficients $\{b_n^{(j)}\}_{n\geqslant 1}$. Then the right side of (13) can be written as

$$2\sum_{j=0}^{1} \int_{j+2\mathbb{Z}_2} c_j \gamma_{n,k}(Tx) P_{b^{(j)},n}(Tx)\, dx = \sum_{j=0}^{1} c_j \int_{\mathbb{Z}_2} \gamma_{n,k}(x) P_{b^{(j)},n}(x)\, dx. \tag{14}$$

Now we compute the integrals on the right hand sides of (12) and (14). Denote

$$E_n = \big\{(\epsilon_1,\ldots,\epsilon_n):\ \epsilon_1,\ldots,\epsilon_{n-1} = -1,0,1;\ \epsilon_n = -1,1\big\},$$
$$\Gamma_n = \big\{\gamma_1^{\epsilon_1}\cdots\gamma_n^{\epsilon_n}:\ (\epsilon_1,\ldots,\epsilon_n) \in E_n\big\},$$
$$E'_{n,k} = \big\{(\epsilon_1,\epsilon_2,\ldots,\epsilon_n) \in E_n:\ \epsilon_n + \cdots + \epsilon_1 \cdot 2^{n-1} = 2^n - k \text{ or } -k\big\}.$$

From the construction of Riesz products in Section 2 we see that if $\gamma_{n,k} \notin \Gamma_n$, then the right sides of (12) and (14) are both equal to zero. If $\gamma_{n,k} \in \Gamma_n$, then for any sequence $(\epsilon_1,\epsilon_2,\ldots,\epsilon_n) \in E_n$ with $\gamma_{n,k}(x) = \gamma_1^{-\epsilon_1}\cdots\gamma_n^{-\epsilon_n}$, we have $(\epsilon_1,\epsilon_2,\ldots,\epsilon_n) \in E'_{n,k}$. Hence the right side of (12) is equal to

$$\sum_{(\epsilon_1,\ldots,\epsilon_n)\in E'_{n,k}} a_1^{(\epsilon_1)}\cdots a_n^{(\epsilon_n)},$$

and the right side of (14) is equal to

$$\sum_{j=0}^{1} c_j \sum_{(\epsilon_1,\ldots,\epsilon_n)\in E'_{n,k}} b_1^{(j)(\epsilon_1)}\cdots b_n^{(j)(\epsilon_n)}.$$

Thus the Riesz products μ_a is T-invariant if and only if for any $n \in \mathbb{N}$ and for any odd $k \in \mathbb{N}$ with $1 \leqslant k < 2^n$,

$$\sum_{(\epsilon_1,\ldots,\epsilon_n)\in E'_{n,k}} a_1^{(\epsilon_1)}\cdots a_n^{(\epsilon_n)} = \sum_{j=0}^{1} c_j \sum_{(\epsilon_1,\ldots,\epsilon_n)\in E'_{n,k}} b_1^{(j)(\epsilon_1)}\cdots b_n^{(j)(\epsilon_n)},$$

i.e.,

$$\sum_{(\epsilon_1,\dots,\epsilon_n)\in E'_{n,k}} a_1^{(\epsilon_1)}\cdots a_n^{(\epsilon_n)}=\sum_{j=0}^{1}c_j\sum_{(\epsilon_1,\dots,\epsilon_n)\in E'_{n,k}} a_2^{(\epsilon_1)}\cdots a_{n+1}^{(\epsilon_n)}e^{2\pi i(\frac{\epsilon_1}{2^2}+\cdots+\frac{\epsilon_n}{2^{n+1}})j}. \tag{15}$$

Lemma 4.1. *The equation $\epsilon_n+\epsilon_{n-1}\cdot 2+\cdots+\epsilon_1\cdot 2^{n-1}=2^n-1$ or -1 admits only $n+1$ solutions*:

$$\begin{aligned}
&(1,1,\dots,1,1,1),\\
&(0,0,\dots,0,0,-1),\\
&(0,0,\dots,0,-1,1),\\
&\ \vdots\\
&(-1,1,\dots,1,1,1).
\end{aligned}$$

Proof. It is easy to see that there is only one sequence $(1,1,\dots,1)$ satisfying the relation $\epsilon_n+\epsilon_{n-1}\cdot 2+\cdots+\epsilon_1\cdot 2^{n-1}=2^n-1$. Now we consider the second relation.

If $\epsilon_n=-1$, then $\epsilon_{n-1}\cdot 2+\epsilon_{n-2}\cdot 2^2+\cdots+\epsilon_1\cdot 2^{n-1}=0$, i.e., $\epsilon_{n-1}+\epsilon_{n-2}\cdot 2+\cdots+\epsilon_1\cdot 2^{n-2}=0$, which implies that $\epsilon_{n-1}=0$. By induction, we have $\epsilon_{n-2}=\cdots=\epsilon_2=\epsilon_1=0$. Thus we obtain the second sequence $(0,0,\dots,0,0,-1)$.

If $\epsilon_n=1$, then $\epsilon_{n-1}\cdot 2+\epsilon_{n-2}\cdot 2^2+\cdots+\epsilon_1\cdot 2^{n-1}=-2$, i.e., $\epsilon_{n-1}+\epsilon_{n-2}\cdot 2+\cdots+\epsilon_1\cdot 2^{n-2}=-1$, which implies that $\epsilon_{n-1}\neq 0$. We continue to distinguish two cases: $\epsilon_{n-1}=-1$ or 1. If $\epsilon_{n-1}=-1$, then we obtain the third sequence. If $\epsilon_{n-1}=1$, we continue the above process. Finally we will obtain all other sequences. □

Lemma 4.2. *If μ_a is T-invariant, the for each $n\geqslant 1$, $a_{n+1}=0$ if and only if $a_n=0$.*

Proof. Suppose that μ_a is T-invariant. For $k=1$, by Lemma 4.1, (15) reads as

$$\begin{aligned}
&\sum_{j=0}^{1}e^{2\pi i\frac{-j}{2^{n+1}}}\left(\frac{a_{n+1}}{2}+\frac{\bar a_{n+1}}{2}\cdot\frac{a_n}{2}+\cdots+\frac{\bar a_{n+1}}{2}\cdots\frac{\bar a_3}{2}\cdot\frac{a_2}{2}-\frac{\bar a_{n+1}}{2}\cdots\frac{\bar a_2}{2}\right)\\
&\quad=\frac{a_n}{2}+\frac{\bar a_n}{2}\cdot\frac{a_{n-1}}{2}+\cdots+\frac{\bar a_n}{2}\cdots\frac{\bar a_2}{2}\cdot\frac{a_1}{2}+\frac{\bar a_n}{2}\cdots\frac{\bar a_1}{2}.
\end{aligned}$$

If $a_{n+1}=0$, then

$$\frac{a_n}{2}+\frac{\bar a_n}{2}\cdot\frac{a_{n-1}}{2}+\cdots+\frac{\bar a_n}{2}\cdots\frac{\bar a_2}{2}\cdot\frac{a_1}{2}+\frac{\bar a_n}{2}\cdots\frac{\bar a_1}{2}=0.$$

By absurdity, we suppose that $a_n\neq 0$. Then

$$\frac{a_n}{\bar a_n}=-\frac{a_{n-1}}{2}-\frac{\bar a_{n-1}}{2}\cdot\frac{a_{n-2}}{2}-\cdots-\frac{\bar a_{n-1}}{2}\cdots\frac{\bar a_2}{2}\cdot\frac{a_1}{2}-\frac{\bar a_{n-1}}{2}\cdots\frac{\bar a_1}{2}.$$

Notice that the modulus of the left side is equal to 1, but the modulus of the right side is smaller than 1. This contradiction implies $a_n=0$.

Conversely, if $a_n=0$, then

$$\sum_{j=0}^{1}c_je^{2\pi i\frac{-j}{2^{n+1}}}\cdot\frac{a_{n+1}}{2}=0.$$

But

$$\sum_{j=0}^{1}c_je^{2\pi i\frac{-j}{2^{n+1}}}=\frac{1+\operatorname{Re}a_1}{2}+\frac{1-\operatorname{Re}a_1}{2}\cdot e^{\frac{-\pi ij}{2^n}}\neq 0.$$

Hence $a_{n+1}=0$. □

Lemma 4.3. *If $a_1,a_2,a_3\in\mathbb{R}$, and μ_a is T-invariant, then $a_3=0$.*

Proof. Suppose that μ_a is T-invariant. Then

$$\begin{aligned}
&\mu_a\big(B_3(6)\big)+\mu_a\big(B_3(7)\big)=\mu_a\big(B_2(3)\big),\\
&\mu_a\big(B_3(2)\big)+\mu_a\big(B_3(3)\big)=\mu_a\big(B_2(1)\big),
\end{aligned}$$

since $T^{-1}B_2(3) = B_3(6) \sqcup B_3(7)$ and $T^{-1}B_2(1) = B_3(2) \sqcup B_3(3)$. If $a_1, a_2, a_3 \in \mathbb{R}$, then by the fact that $\mu_a(B_n(x)) = 2^{-n}P_{a,n}(x)$, we have

$$\frac{1+a_1}{2}\cdot\frac{1-a_2}{2}\cdot\frac{1}{2}+\frac{1-a_1}{2}\cdot\frac{1}{2}\cdot\frac{1+\frac{\sqrt{2}}{2}a_3}{2}=\frac{1-a_1}{2}\cdot\frac{1}{2},$$
$$\frac{1+a_1}{2}\cdot\frac{1-a_2}{2}\cdot\frac{1}{2}+\frac{1-a_1}{2}\cdot\frac{1}{2}\cdot\frac{1-\frac{\sqrt{2}}{2}a_3}{2}=\frac{1-a_1}{2}\cdot\frac{1}{2}.$$

So, $a_3 = 0$. □

Theorem 1.3 is a direct consequence of Lemmas 4.2 and 4.3.

5. Other properties

In this section, we give some other properties of Riesz products on the ring of dyadic integers. Since the proofs are similar to the case $p \geqslant 3$, we state the results without proofs.

5.1. Quasi-Bernoulli property

Recall that a probability measure μ on $\mathbb{Z}_2$ is *T-quasi-Bernoulli* if there exists a constant $C > 0$ such that

$$C^{-1} \leqslant \frac{\mu_a(B_n(x) \cap T^{-n}B_m(y))}{\mu_a(B_n(x))\mu_a(B_m(y))} \leqslant C$$

holds for all $x, y \in \mathbb{Z}_2$ and all integers $n, m \geqslant 1$ (the measure is Bernoulli if $C = 1$). For Riesz product μ_a, we have the following theorem.

Theorem 5.1. *Assume that $|a_k| < 1$ for any $k \geqslant 1$. The Riesz product μ_a is T-quasi-Bernoulli if there exists a complex number a with $|a| < 1$ such that*

$$\sum_{k=1}^{\infty} |a_k - a| < +\infty.$$

The notion of quasi-Bernoulli which was introduced by Brown, Michon and Peyrière [2], plays an important role in multifractal analysis. It was proved in [2] that the multifractal formalism holds for all quasi-Bernoulli measures.

Are the conditions in Theorem 5.1 necessary for μ_a being T-quasi-Bernoulli? We point out that the above conditions are sufficient and necessary for the case $p \geqslant 3$. Recall that the T-ergodicity of a measure μ means $\mu(A) = 0$ or 1 for all T-invariant set A (i.e., $A = T^{-1}A$). It is easy to see that if μ is a T-quasi-Bernoulli measure, then μ is ergodic and μ is equivalent to a T-invariant measure which is a limit of $\frac{1}{n}\sum_{k=0}^{n-1} \mu \circ T^{-k}$.

5.2. Mutually absolute continuity

Given two Riesz products μ_a and μ_b defined by two different coefficients $\{a_n\}$ and $\{b_n\}$, under what conditions, μ_a and μ_b are mutually absolutely continuous (mutually singular)?

Theorem 5.2. *Assume that $|a_n| < 1$, $|b_n| < 1$ for all $n \geqslant 1$. We have $\mu_a \sim \mu_b$ if*

$$\sum_{k=1}^{\infty} |a_k - b_k|^2 \left(1 + \frac{\cos^2(s_n - t_n)}{2 - |a_n + b_n|}\right) < \infty$$

where $s_n = \arg(a_n + b_n)$ and $t_n = \arg(a_n - b_n)$. Furthermore, if $\sup_{n \geqslant 1} |a_n| < 1$ and $\sup_{n \geqslant 1} |b_n| < 1$, then we have $\mu_a \perp \mu_b$ if

$$\sum_{k=1}^{\infty} |a_k - b_k|^2 = +\infty.$$

In contrast to the case $p \geqslant 3$, we need to add the supremum conditions for the mutually singular part. The second part of Theorem 5.2, which holds for Riesz products on any compact abelian group, is due to Peyrière (see [16,17]). There are many works on this topic of Riesz products defined on the circle $\mathbb{R}/\mathbb{Z}$ (for example, [3,4,8,11,14,16,17,21]). But it is still an open problem to find the exact condition for mutually absolute continuity of two Riesz products [6,10] both in the circle case and in the p-adic case.

5.3. Quasi-invariance

Recall that a measure μ is T-quasi-invariant if $\mu \circ T^{-1} \sim \mu$.

Theorem 5.3. *Assume* $\sup_{n\geqslant 1} |a_n| < 1$. *Then* μ_a *is* T*-quasi-invariant if and only if*

$$\sum_{n=1}^{\infty} |a_n - a_{n+1}|^2 < \infty.$$

Each quasi-invariant Riesz product produces a non-singular measure-theoretic dynamics (see [1,15] for general theory of non-singular measure-theoretic dynamics). It would be interesting to know if there exists a non-trivial T-invariant measure which is absolutely continuous with respect to a quasi-invariant Riesz product measure.

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