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Classes of weights related to Schrödinger operators ☆

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ABSTRACT

In this work we obtain boundedness on weighted Lebesgue spaces on $\mathbb{R}^d$ of the semi-group maximal function, Riesz transforms, fractional integrals and g-function associated to the Schrödinger operator $-\Delta + V$, where V satisfies a reverse Hölder inequality with exponent greater than $d/2$. We consider new classes of weights that locally behave as Muckenhoupt's weights and actually include them. The notion of locality is defined by means of the critical radius function of the potential V given in Shen (1995) [8].

1. Introduction

Let us consider the Schrödinger operator on $\mathbb{R}^d$ with $d \geqslant 3$,

$$\mathcal{L} = -\Delta + V,$$

where the potential V is non-negative, non-identically zero, and for some $q > d/2$ satisfies the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B V(y)^q\,dy\right)^{1/q} \leqslant \frac{C}{|B|}\int_B V(y)\,dy, \tag{1}$$

for every ball $B \subset \mathbb{R}^d$. The set of functions with the last property is usually denoted by RH_q.

Since $V \in RH_q$ implies $V \in RH_{q+\epsilon}$ for some $\epsilon > 0$, notice that the assumption $q > d/2$ is equivalent to $q \geqslant d/2$.

We will be interested in weighted L^p inequalities for the following operators associated to $\mathcal{L}$:

- Maximal operator of the diffusion semi-group

$$\mathcal{T}^* f(x) = \sup_{t>0} e^{-t\mathcal{L}} f(x). \tag{2}$$

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- $\mathcal{L}$-Riesz potential or $\mathcal{L}$-Fractional Integral

$$\mathcal{I}_\alpha f(x) = \mathcal{L}^{-\alpha/2} f(x) = \int_0^\infty e^{-t\mathcal{L}} f(x) t^{\alpha/2} \frac{dt}{t}, \quad 0 < \alpha < d. \tag{3}$$

- $\mathcal{L}$-Riesz transforms

$$\mathcal{R} = \nabla \mathcal{L}^{-1/2}, \tag{4}$$

and their adjoints

$$\mathcal{R}^* = \mathcal{L}^{-1/2} \nabla.$$

- $\mathcal{L}$-Square Function

$$\mathfrak{g}(f)(x) = \left(\int_0^\infty \left| \frac{d}{dt} e^{-t\mathcal{L}}(f)(x) \right|^2 t\, dt \right)^{1/2}. \tag{5}$$

It is well known that for the classical case of $\mathcal{L} = -\Delta$, the maximal operator of the heat diffusion semi-group, the Riesz transforms and the g-function are bounded on $L^p(w)$, $1 < p < \infty$, for w in the Muckenhoupt A_p classes defined by the inequality

$$\left(\int_B w \right)^{1/p} \left(\int_B w^{-\frac{1}{p-1}} \right)^{1/p'} \leqslant C|B|, \tag{6}$$

for every ball $B \subset \mathbb{R}^d$, and of weak type $(1,1)$ for weights satisfying the A_1 condition

$$w(B) \sup_B w^{-1} \leqslant C|B|, \tag{7}$$

for every ball $B \subset \mathbb{R}^d$.

Also, the classical Fractional Integral of order $0 < \alpha < d$, is bounded from $L^p(w)$ into $L^\nu(w^{\nu/p})$, $\frac{1}{\nu} = \frac{1}{p} - \frac{\alpha}{d}$, when $w^{\nu/p} \in A_{1+\frac{\nu}{p'}}$, and of weak type $(1, \frac{d}{d-\alpha})$ for weights such that $w^{\frac{d}{d-\alpha}} \in A_1$.

As we shall see the classes of weights for Schrödinger operators under the stated hypothesis on V will be in general larger than Muckenhoupt's. It is well known that the operators derived from $\mathcal{L}$ behave "locally" quite similar to those corresponding to the Laplacian (see for example [3] or [8]). This notion of locality is given by the critical radius function

$$\rho(x) = \sup \left\{ r > 0\colon \frac{1}{r^{d-2}} \int_{B(x,r)} V \leqslant 1 \right\}, \quad x \in \mathbb{R}^d, \tag{8}$$

which, under our assumptions, is easy to check that $0 < \rho(x) < \infty$ (see [8]).

Our new classes of weights are given in terms of this critical radius function. More precisely, given $p \geqslant 1$ we introduce the class $A_p^{\rho,\infty} = \bigcup_{\theta \geqslant 0} A_p^{\rho,\theta}$, where $A_p^{\rho,\theta}$ is defined as those weights w such that

$$\left(\int_B w \right)^{1/p} \left(\int_B w^{-\frac{1}{p-1}} \right)^{1/p'} \leqslant C|B| \left(1 + \frac{r}{\rho(x)} \right)^\theta,$$

for every ball $B = B(x,r)$.

Clearly, the classes $A_p^{\rho,\theta}$ are increasing with θ, and for $\theta = 0$ they are the Muckenhoupt classes A_p. Moreover, the inclusions are proper. Take for instance, $\rho \equiv 1$ and $w(x) = 1 + |x|^\gamma$. Now, for $\gamma > d(p-1)$, the weight w belongs to $A_p^{\rho,\infty}$, but it is not in A_p.

In this work we will show that the operators (2) and (5) defined above are bounded on $L^p(w)$ for w in $A_p^{\rho,\infty}$, $1 < p < \infty$, and of weak type $(1,1)$ for w in $A_1^{\rho,\infty}$ (see Theorems 2 and 5). As for the Riesz transforms (4) we get the same results when the potential satisfies (1) with $q \geqslant d$. However, when $d/2 < q < d$, the range of p must be restricted and also the classes of weights shrink when q get closer to $d/2$ (see Theorem 3 and Corollary 2). Indeed, in [8] it is shown that for $w = 1$ the given range is optimal.

Regarding the Fractional Integral (3) we prove that it maps $L^p(w)$ into $L^\nu(w^{\nu/p})$ for weights such that $w^{\nu/p} \in A^{\rho,\infty}_{1+\frac{\nu}{p'}}$, $1 < p \leqslant d/\alpha$, $\frac{1}{\nu} = \frac{1}{p} - \frac{\alpha}{d}$, and of weak type $(1, \frac{d}{d-\alpha})$ for weights such that $w^{\frac{d}{d-\alpha}} \in A^{\rho,\infty}_1$.

We achieve these results through the comparison of the new operators with the classical ones when restricted to local regions given by ρ.

In this way we are led to study weighted L^p inequalities for the ρ-localized classical operators. In this setting larger classes of weights appear, namely the classes $A^{\rho,\mathrm{loc}}_p$ defined as those weights that satisfy (6) for balls $B(x,r)$ with $r \leqslant \rho(x)$. We summarized all of these results in Section 2, since we believe they are interesting by themselves.

The four remaining sections are devoted to state and prove weighted L^p inequalities for the maximal operator of the semi-group, the Riesz transforms, the Fractional Integral and the Square Function, respectively.

2. Localized classical operators and weights

Let us denote by T^*, I_α, R, and $\mathbf{g}$, the classical versions of the operators considered in the introduction, that means, those defined by (2), (3), (4) and (5), when $\mathcal{L} = -\Delta$. Let us also consider M, the Hardy–Littlewood maximal function.

If S stands for any of the above operators we denote by S_{loc}, the ρ-localization of S,

$$S_{\mathrm{loc}}(f)(x) = S(f\chi_{B(x,\rho(x))})(x). \tag{9}$$

Our aim is to show weighted inequalities for S_{loc}, but first we give some properties of ρ that we will use later.

Proposition 1. *(See [8].) If $V \in RH_{d/2}$, there exist c_0 and $N_0 \geqslant 1$ such that*

$$c_0^{-1}\rho(x)\left(1 + \frac{|x-y|}{\rho(x)}\right)^{-N_0} \leqslant \rho(y) \leqslant c_0\,\rho(x)\left(1 + \frac{|x-y|}{\rho(x)}\right)^{\frac{N_0}{N_0+1}}, \tag{10}$$

for all $x, y \in \mathbb{R}^d$.

A ball of the form $B(x,\rho(x))$ is called critical. Inequality (10) implies that if $\sigma > 0$ and $x, y \in \sigma Q$, where Q is a critical ball, then

$$\rho(x) \leqslant C_\sigma\,\rho(y), \tag{11}$$

where $C_\sigma = c_0^2(1+\sigma)^{\frac{2N_0+1}{N_0+1}}$, and c_0 is the constant appearing in (10).

In what follows we will call critical radius function to any positive continuous function ρ that satisfies (10), not necessarily coming from a potential V. Clearly, if ρ is such a function, so it is $\beta\rho$ for any $\beta > 0$.

As a consequence of (11) we derive the following result.

Proposition 2. *(See [4].) There exists a sequence of points x_j, $j \geqslant 1$, in $\mathbb{R}^d$, so that the family $Q_j = B(x_j, \rho(x_j))$, $j \geqslant 1$, satisfies*

(i) $\bigcup_j Q_j = \mathbb{R}^d$.
(ii) *For every $\sigma \geqslant 1$ there exist constants C and N_1 such that $\sum_j \chi_{\sigma Q_j} \leqslant C\sigma^{N_1}$.*

Now we turn to present some properties of weights in the class $A^{\rho,\mathrm{loc}}_p$ defined in the previous section, i.e., those weights that satisfy (6) for balls $B(x,r)$ with $r \leqslant \rho(x)$.

We define D_ρ as the set of weights w for which there exists a constant C such that for any pair of balls $B_0 = B(x_0,r)$ and $B_1 = B(x_1,r/2)$, with $B_1 \subset B_0$ and $r \leqslant \rho(x_0)$,

$$w(B_0) \leqslant C\,w(B_1).$$

Clearly for any $1 \leqslant p < \infty$, $w \in A^{\rho,\mathrm{loc}}_p$ implies $w \in D_\rho$.

Proposition 3. *For every $\beta > 1$, $D_\rho = D_{\beta\rho}$.*

Proof. We only have to prove $D_\rho \subset D_{\beta\rho}$. Let $w \in D_\rho$, and consider two balls $B_0 = B(x_0,r)$ and $B_1 = B(x_1,r/2)$, with $B_1 \subset B_0$ and $\rho(x_0) \leqslant r \leqslant \beta\rho(x_0)$; we must prove that there exists a constant C_β such that

$$w(B_0) \leqslant C_\beta\,w(B_1). \tag{12}$$

First observe that for $x \in 2B_0$, by property (10) we have

$$\rho(x) \geqslant \frac{1}{c_0}\rho(x_0)\left(1+\frac{2r}{\rho(x_0)}\right)^{-N_0} = \rho_{\min}.$$

Now let us consider a covering of B_0 by balls $P_j = B(y_j, \rho_{\min}/4)$ such that

(i) $P_j \cap B_0 \neq \emptyset$, for every j.
(ii) $\frac{1}{2}P_j \cap \frac{1}{2}P_k = \emptyset$, for $k \neq j$.

Therefore, for any j and k such that $P_j \cap P_k \neq \emptyset$, we have

$$w(P_j) \leqslant w(4P_k) \leqslant C^2 w(P_k),$$

where in the last inequality we have used that $w \in D_\rho$ since the radius of $4P_k$ is $\rho_{\min} \leqslant \rho(y_k)$, which in turn holds because $y_k \in 2B_0$.

Also, property (ii) implies that the number of balls $\{P_j\}$ is bounded by $C_1\beta^{d(N_0+1)}$, with C_1 depending only on the dimension d and c_0. Finally, if P_{j_1} denotes a member of the family such that $x_1 \in P_{j_1}$,

$$w(B_0) \leqslant \sum_j w(P_j) \leqslant \sum_j C^{2C_1\beta^{d(N_0+1)}} w(P_{j_1}) \leqslant C_1\beta^{d(N_0+1)} C^{2C_1\beta^{d(N_0+1)}} w(B_1),$$

since $P_{j_1} \subset B_1$. $\square$

Corollary 1. *For $1 \leqslant p < \infty$ and $\beta > 1$, $A_p^{\rho,\mathrm{loc}} = A_p^{\beta\rho,\mathrm{loc}}$.*

Proof. We first prove the case $1 < p < \infty$. Assuming w in $A_p^{\rho,\mathrm{loc}}$, it is easy to check that both w and $w^{-\frac{1}{p-1}}$ belong to D_ρ and hence also to $D_{\beta\rho}$, by Proposition 3. Now, it is easy to derive $w \in A_p^{\beta\rho,\mathrm{loc}}$. In fact, if $B = B(x,r)$ with $r \leqslant \beta\rho(x)$, and N is the least integer such that $2^N > \beta$ we have

$$w(B)\left[w^{-\frac{1}{p-1}}(B)\right]^{p-1} \lesssim w\left(\frac{1}{2^N}B\right)\left[w^{-\frac{1}{p-1}}\left(\frac{1}{2^N}B\right)\right]^{p-1} \lesssim |B|^p,$$

where the last inequality holds since the radius of $\frac{1}{2^N}B$ is less than $\rho(x)$.

Finally, we deal with the case $p = 1$. For B and N as above, we choose a ball $B_N \subset B$ with radius $\frac{r}{2^N}$ such that $\sup_{B_N} w^{-1} = \sup_B w^{-1}$. Using that $w \in A_1^{\rho,\mathrm{loc}}$ implies $w \in D_\rho$, and hence $w \in D_{\beta\rho}$, we have

$$w(B)\sup_B w^{-1} \lesssim w(B_N)\sup_{B_N} w^{-1} \lesssim |B_N| \lesssim |B|. \qquad \square$$

Notice that, even all the classes $A_p^{\beta\rho,\mathrm{loc}}$ are the same, the membership constant may increase with β, otherwise the weight would be in A_p.

Next we give a general result that can be applied to prove the boundedness of the localized classical operators. To this end, we consider a covering of balls $\{Q_j\}$ such that the family of a fixed dilation of them, $\{\tilde{Q}_j\}$, has bounded overlapping (for instance, a covering associated to ρ like in Proposition 2).

For a given operator $\mathcal{S}$, we define

$$\mathcal{S}_0(f) = \sum_j \chi_{Q_j}\left|\mathcal{S}(f\chi_{\tilde{Q}_j})\right|. \tag{13}$$

Proposition 4. *Let $1 \leqslant p \leqslant \nu < \infty$, and a weight w on $\mathbb{R}^d$ with the following property: for each j, $w|_{\tilde{Q}_j}$ admits an extension w_j to $\mathbb{R}^d$ such that*

$$\mathcal{S} : L^p(w_j) \mapsto L^\nu\left(w_j^{\nu/p}\right) \tag{14}$$

boundedly with a constant independent of j. Then

$$\mathcal{S}_0 : L^p(w_j) \mapsto L^\nu\left(w_j^{\nu/p}\right)$$

continuously. If for $p = 1$ the assumption (14) is changed by weak type $(1,\nu)$, the corresponding weak type can be concluded for $\mathcal{S}_0$.

Proof. Let $I_j = \{k\colon\ Q_k \cap Q_j \neq \emptyset\}$. Due to the bounded overlapping property of the family Q_j, and the assumptions on S and w_j,

$$\begin{aligned}
\|\mathcal{S}_0 f\|^{\nu}_{L^{\nu}(w^{\nu/p})} &\lesssim \sum_j \sum_{k\in I_j} \int_{Q_k\cap Q_j} \left|\mathcal{S}(\chi_{\tilde{Q}_k} f)\right|^{\nu} w^{\nu/p} \\
&\lesssim \sum_k \sum_{j\in I_k} \int_{Q_k} \left|\mathcal{S}(\chi_{\tilde{Q}_k} f)\right|^{\nu} w^{\nu/p} \\
&\lesssim \sum_k \int_{\mathbb{R}^d} \left|\mathcal{S}(\chi_{\tilde{Q}_k} f)\right|^{\nu} w_k^{\nu/p} \\
&\lesssim \sum_k \left(\int_{\tilde{Q}_k} |f|^p w_k\right)^{\nu/p} \\
&\lesssim \left(\int_{\mathbb{R}^d} |f|^p w\right)^{\nu/p},
\end{aligned}$$

where in the last inequality we have used that $p \leqslant \nu$ and the bounded overlapping property of the family $\{\tilde{Q}_j\}$.

To prove the weak type statement, we proceed in a similar way to get

$$\begin{aligned}
\left\{\left|\mathcal{S}_0(f)\right| > \lambda\right\} &\subset \bigcup_j \bigcup_{k\in I_j} Q_j \cap Q_k \cap \left\{\left|S(f\chi_{\tilde{Q}_k})\right| > \lambda/M\right\} \\
&\subset \bigcup_k Q_k \cap \left\{\left|S(f\chi_{\tilde{Q}_k})\right| > \lambda/M\right\},
\end{aligned}$$

where M is such that $\sum_k \chi_{Q_k} \leqslant M$. Hence,

$$\begin{aligned}
w^{\nu}\left(\left\{\left|\mathcal{S}_0(f)\right| > \lambda\right\}\right) &\lesssim \sum_k w_k^{\nu}\left(\left\{\left|S(f\chi_{\tilde{Q}_k})\right| > \lambda/M\right\}\right) \\
&\lesssim \lambda^{-\nu} \sum_k \left(\int_{\tilde{Q}_k} |f| w_k\right)^{\nu} \\
&\lesssim \lambda^{-\nu} \left(\int_{\mathbb{R}^d} |f| w\right)^{\nu}. \qquad \square
\end{aligned}$$

We say that a weight w defined on a ball B_0, belongs to $A_p(B_0)$ if the inequality (6) is satisfied for every ball $B \subset B_0$.

We will use the following fact concerning extension of weights. For related results see also [5].

Lemma 1. *Given a ball B_0 and a weight $w \in A_p(B_0)$, $1 \leqslant p < \infty$, there exists an extension $w_0 \in A_p(\mathbb{R}^d)$ with the same constant.*

Proof. By means of a conformal mapping it is equivalent to work with cubes. For that case a construction process was given in [6], namely proceeding by reflecting the weight to neighboring cubes of the same size. $\square$

Now, we are ready to prove the main result of this section.

Theorem 1. *Let ρ be a function satisfying* (10)*, then*:

(a) *M_{loc}, T^*_{loc}, R_{loc} and $\mathbf{g}_{\mathrm{loc}}$ are bounded on $L^p(w)$, $1 < p < \infty$, for $w \in A_p^{\rho,\mathrm{loc}}$, and they are of weak type $(1,1)$ for $w \in A_1^{\rho,\mathrm{loc}}$.*

(b) *For $0 < \alpha < d$, $(I_\alpha)_{\mathrm{loc}}$ is bounded from $L^p(w)$ into $L^s(w^{s/p})$, $1 < p < d/\alpha$, $\frac{1}{s} = \frac{1}{p} - \frac{\alpha}{d}$, for $w^{s/p} \in A^{\rho,\mathrm{loc}}_{1+\frac{s}{p'}}$ and it is of weak type $(1, \frac{d}{d-\alpha})$ for $w^{\frac{d}{d-\alpha}} \in A_1^{\rho,\mathrm{loc}}$.*

Proof. Let $\sigma = c_0 2^{\frac{N_0}{N_0+1}}$, with N_0 and c_0 as in Proposition 1. Let $\{Q_j\}$ be the family given by Proposition 2 and set $\tilde{Q}_j = \sigma Q_j$. Clearly, we have

$$\bigcup_{x \in Q_j} B\big(x, \rho(x)\big) \subset \tilde{Q}_j. \tag{15}$$

Let $w \in A_p^{\rho,\mathrm{loc}}$, $1 \leqslant p < \infty$. Due to Corollary 1, $w \in A_p^{\sigma\rho,\mathrm{loc}}$, and then, for any j, $w|_{\tilde{Q}_j} \in A_p(\tilde{Q}_j)$, with a constant independent of j. Applying Lemma 1, we obtain for each j an extension w_j such that $w_j \in A_p(\mathbb{R}^d)$, and uniformly in j.

It is well known that M, T^*, R and g are bounded on $L^p(w)$, $1 < p < \infty$, for $w \in A_p$, and weak type $(1,1)$ for $w \in A_1$. Hence, we get that M_0, T_0^*, R_0 and g_0, associated to the covering $\{Q_j\}$ as in (13), are bounded in $L^p(w)$, $1 < p < \infty$, for $w \in A_p^{\rho,\mathrm{loc}}$, and weak type $(1,1)$ for $w \in A_1^{\rho,\mathrm{loc}}$, in view of Proposition 4.

The conclusion of (a) follows immediately for M_{loc} and T_{loc}^* since

$$M_{\mathrm{loc}} f(x) \leqslant M_0 f(x) \quad \text{and} \quad T_{\mathrm{loc}}^* f(x) \leqslant T_0^* |f|(x),$$

as a consequence of (15).

Regarding R and $\mathbf{g}$, since they are not positive operators, we have to be more careful.

First, for $x \in Q_j$, by (15),

$$\big|R_{\mathrm{loc}} f(x) - R(\chi_{\tilde{Q}_j} f)(x)\big| \lesssim \int_{\tilde{Q}_j \setminus B(x,\rho(x))} \frac{|f(y)|}{|x-y|^d}\,dy \lesssim \frac{1}{|\tilde{Q}_j|} \int_{\tilde{Q}_j} |f|, \tag{16}$$

since $\rho(x) \simeq \rho(x_j)$ when $x \in Q_j$. Therefore, for $1 < p < \infty$,

$$\|R_{\mathrm{loc}} f\|_{L^p(w)}^p \lesssim \sum_j \int_{Q_j} \left(\frac{1}{|\tilde{Q}_j|} \int_{\tilde{Q}_j} \big|f(y)\big|\,dy \right)^p w + \|R_0 f\|_{L^p(w)}^p,$$

and the first term can be bounded by

$$\begin{aligned} \sum_j w(\tilde{Q}_j)\big[w^{-p'/p}(\tilde{Q}_j)\big]^{p/p'} \frac{1}{|\tilde{Q}_j|^p} \int_{\tilde{Q}_j} |f|^p w &\lesssim \sum_j \int_{\tilde{Q}_j} |f|^p w \\ &\lesssim \int_{\mathbb{R}^d} |f|^p w, \end{aligned}$$

where we have used again Corollary 1. Therefore, from Proposition 4 we are done for $1 < p < \infty$.

For the case $p = 1$, again from (16) and $w \in A_1^{\rho,\mathrm{loc}}$ we have for each j,

$$w\big(\big\{x \in Q_j\colon \big|R_{\mathrm{loc}} f(x) - R(\chi_{\tilde{Q}_j} f)(x)\big| > \lambda\big\}\big) \lesssim \frac{1}{\lambda} \int_{\tilde{Q}_j} |f| w.$$

Besides, from Proposition 4,

$$w\big(\big\{x \in Q_j\colon \big|R_0 \chi_{\tilde{Q}_j} f(x)\big| > \lambda\big\}\big) \lesssim \frac{1}{\lambda} \int_{\tilde{Q}_j} |f| w.$$

Therefore, summing over j we get the weak type $(1,1)$.

Let us remind that for the heat kernel $h_t(z) = (4\pi t)^{-\frac{d}{2}} \exp(-\frac{|z|^2}{4t})$, we have

$$\left|\frac{d}{dt} h_t(z)\right| \lesssim t^{-\frac{d}{2}-1} \exp\left(-\frac{|z|^2}{5t}\right). \tag{17}$$

Consequently, for $x \in Q_j$,

$$\big|\mathbf{g}_{\mathrm{loc}} f(x) - \mathbf{g}(\chi_{\tilde{Q}_j} f)(x)\big|^2 \lesssim \int_0^\infty \left(\int_{\tilde{Q}_j \setminus B(x,\rho(x))} \frac{e^{-\frac{|x-y|^2}{5t}}}{t^{d/2+1}} \big|f(y)\big|\,dy \right)^2 t\,dt.$$

Using that $|x-y| \geqslant \rho(x) \simeq \rho(x_j)$ when $x \in Q_j$ and $y \notin B(x,\rho(x))$, the last expression is bounded by

$$\int_0^\infty \frac{e^{-2\frac{\rho(x)^2}{5t}}}{t^{d+1}}\,dt \left(\int_{\tilde{Q}_j} |f|\right)^2 \lesssim \left(\frac{1}{|\tilde{Q}_j|}\int_{\tilde{Q}_j} |f|\right)^2.$$

Taking square root we obtain the same point-wise bound as in (16), and the proof follows in the same way as above for R.

Finally, (b) can be derived easily from Proposition 4 as for M_{loc} and T^*_{loc}, since $(I_\alpha)_{\mathrm{loc}} f(x) \leqslant (I_\alpha)_0 f(x)$, for $f \geqslant 0$. $\square$

Remark 1. Let us note that, following standard arguments, the condition $A_p^{\rho,\mathrm{loc}}$ is also necessary for the boundedness of M_{loc} in $L^p(w)$.

3. The maximal of the semi-group

We begin this section with an estimate of the kernel of the operator $e^{-t\mathcal{L}}$, $t>0$, that will be denoted by k_t.

As it is well known

$$k_t(x,y) \leqslant h_t(x-y), \tag{18}$$

where h_t is the classical heat kernel. Moreover, we have the following result.

Lemma 2. *(See [7].) Given $N>0$, there exists a constant C_N such that for all x and y in $\mathbb{R}^d$,*

$$k_t(x,y) \leqslant C_N t^{-d/2} e^{-\frac{|x-y|^2}{5t}} \left(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}.$$

Now we present the main result of this section.

Theorem 2. *For $1<p<\infty$ the operator $\mathcal{T}^*$ is bounded on $L^p(w)$ when $w \in A_p^{\rho,\infty}$, and of weak type $(1,1)$ when $w \in A_1^{\rho,\infty}$.*

Proof. First, for $x \in \mathbb{R}^d$ we denote $B_x = B(x,\rho(x))$ and thus

$$\mathcal{T}^* f(x) \leqslant \mathcal{T}^*_{\mathrm{loc}} f(x) + \mathcal{T}^*_{\mathrm{glob}} f(x), \tag{19}$$

for all f in $L^p(w)$, where $\mathcal{T}^*_{\mathrm{loc}} f(x) = \mathcal{T}^* f\chi_{B_x}(x)$ and $\mathcal{T}^*_{\mathrm{glob}} f(x) = \mathcal{T}^* f\chi_{B_x^c}(x)$.

To deal with the first term of (19) we use (18) to obtain

$$\mathcal{T}^*_{\mathrm{loc}} f(x) \lesssim T^*_{\mathrm{loc}} f(x).$$

Hence, the $L^p(w)$ boundedness and the weak type $(1,1)$ for $\mathcal{T}^*_{\mathrm{loc}}$ follow from Theorem 1 since $A_p^{\rho,\infty} \subset A_p^{\rho,\mathrm{loc}}$.

For the second term of (19) we use again Lemma 2 and the estimate $e^{-s} \lesssim \frac{1}{s^{M/2}}$ with $d \leqslant M < N$. Splitting into annuli, we obtain

$$\begin{aligned}
\mathcal{T}^*_{\mathrm{glob}} f(x) &\lesssim \sup_{t>0} t^{-d/2}\left(1+\frac{\sqrt{t}}{\rho(x)}\right)^{-N} \int_{B_x^c} e^{-\frac{|x-y|^2}{5t}} \left|f(y)\right| dy \\
&\lesssim \sup_{t>0} t^{-d/2}\left(\frac{\sqrt{t}}{\rho(x)}\right)^{M}\left(1+\frac{\sqrt{t}}{\rho(x)}\right)^{-N} \sum_{k=1}^{\infty} 2^{-kM} \int_{2^k B_x \setminus 2^{k-1} B_x} \left|f(y)\right| dy \\
&\lesssim g_1(x),
\end{aligned}$$

with

$$g_1(x) = \rho(x)^{-d} \sum_{k=0}^{\infty} 2^{-kM} \int_{2^k B_x} |f|, \tag{20}$$

and the last inequality is easily obtained considering the cases $0<t<\rho(x)^2$ and $t \geqslant \rho(x)^2$.

Let $\{Q_j\}$ be a covering of critical balls given by Proposition 1 and we set $\tilde{Q}_j$ like in the proof of Theorem 1 satisfying (15). Denoting $\tilde{Q}_j^k = 2^k \tilde{Q}_j$, then $2^k B_x \subset \tilde{Q}_j^k$ and $\rho(x) \simeq \rho(x_j)$, whenever $x \in Q_j$. Therefore, for $1<p<\infty$, by Hölder's inequality

$$\|g_1\|_{L^p(w)} \lesssim \sum_{k=0}^{\infty} 2^{-kM}\left(\sum_j \int_{Q_j} \rho(x)^{-dp}\left(\int_{2^k B_x} |f|\right)^p w(x)\,dx\right)^{1/p}$$
$$\lesssim \sum_{k=0}^{\infty} 2^{-kM}\left(\sum_j \rho(x_j)^{-dp}\left(\int_{\tilde{Q}_j^k} w^{-p'/p}\right)^{p/p'} w(Q_j)\int_{\tilde{Q}_j^k} |f|^p w\right)^{1/p}.$$

Since $Q_j \subset \tilde{Q}_j^k$ and $w \in A_p^{\rho,\theta}$ for some $\theta > 0$, the last expression is bounded by

$$\sum_{k=0}^{\infty} 2^{-k(M-d-\theta)}\left(\sum_j \int_{\tilde{Q}_j^k} |f|^p w\right)^{1/p} \lesssim \|f\|_{L^p(w)} \sum_{k=0}^{\infty} 2^{-k(M-d-\theta-N_1)},$$

where the last inequality is due to the control on the overlapping of the family $\{\tilde{Q}_j^k\}$ given by Proposition 2. For M large enough we are done in the case $1 < p < \infty$. For the case $p = 1$ the same proof, with the obvious changes, gives strong boundedness on $L^1(w)$ for $w \in A_1^{\rho,\infty}$. □

4. Riesz transforms

The operators $\mathcal{R}$ and $\mathcal{R}^*$ were studied by Shen in [8]. There he proved that when $q > d$, $\mathcal{R}$ and $\mathcal{R}^*$ are bounded on $L^p(dx)$, $1 < p < \infty$. Moreover, he showed that they are in fact Calderón–Zygmund operators and thus of weak type $(1,1)$. However, if we only know that $q > d/2$, he obtains L^p boundedness on a smaller range of p, depending on q, that he proves to be optimal.

These operators have singular kernels with values in $\mathbb{R}^d$ that will be denoted by $\mathcal{K}$ and $\mathcal{K}^*$ respectively. For such kernels, we have the following estimates that are basically proved in [8].

Lemma 3. *Let $V \in RH_q$ with $q > d/2$. Then we have*:

(i) *For every N there exists a constant C_N such that*

$$\left|\mathcal{K}^*(x,y)\right| \leqslant \frac{C_N\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}}{|x-y|^{d-1}}\left(\int_{B(y,|x-y|/4)} \frac{V(u)}{|u-y|^{d-1}}\,du + \frac{1}{|x-y|}\right). \tag{21}$$

Moreover, the last inequality also holds with $\rho(x)$ replaced by $\rho(y)$.

(ii) *If $\mathbf{K}^*$ denotes the $\mathbb{R}^d$ vector-valued kernel of the adjoint of the classical Riesz operator, then*

$$\left|\mathcal{K}^*(x,z) - \mathbf{K}^*(x,z)\right| \leqslant \frac{C}{|x-z|^{d-1}}\left(\int_{B(z,|x-z|/4)} \frac{V(u)}{|u-z|^{d-1}}\,du + \frac{1}{|x-z|}\left(\frac{|x-z|}{\rho(x)}\right)^{2-\frac{d}{q}}\right), \tag{22}$$

whenever $|x-z| \leqslant \rho(x)$.

(iii) *When $q > d$, the term involving V can be dropped from inequalities* (21) *and* (22)*.*

For the proof of the following lemma, see [1].

Lemma 4. *Let $V \in RH_q$ with $q > d/2$ and $\epsilon > \frac{d}{q}$. Then for any constant C_1 there exists a constant C_2 such that*

$$\int_{B(x,C_1 r)} \frac{V(u)}{|u-x|^{d-\epsilon}}\,du \leqslant C_2\, r^{\epsilon-2}\left(\frac{r}{\rho(x)}\right)^{2-d/q},$$

if $0 < r \leqslant \rho(x)$.

Theorem 3. *Let $V \in RH_q$.*

(i) *If $q \geqslant d$, the operators $\mathcal{R}$ and $\mathcal{R}^*$ are bounded on $L^p(w)$, $1 < p < \infty$, for $w \in A_p^{\rho,\infty}$, and are of weak type $(1,1)$ for $w \in A_1^{\rho,\infty}$.*

(ii) *If $d/2 < q < d$, and s is such that $\frac{1}{s} = \frac{1}{q} - \frac{1}{d}$, the operator $\mathcal{R}^*$ is bounded on $L^p(w)$, for $s' < p < \infty$ and $w \in A_{p/s'}^{\rho,\infty}$ and hence by duality $\mathcal{R}$ is bounded on $L^p(w)$, for $1 < p < s$, with w satisfying $w^{-\frac{1}{p-1}} \in A_{p'/s'}^{\rho,\infty}$. Moreover, $\mathcal{R}$ is of weak type $(1,1)$ for $w^{s'} \in A_1^{\rho,\infty}$.*

Proof. First of all, notice that there is no need to consider $q = d$ since in that case there exists an $\epsilon > 0$ such that $V \in RH_{d+\epsilon}$.
We begin giving estimates for $\mathcal{R}^*$.

The *local* and *global* operators associated with $\mathcal{R}^*$ are

$$\mathcal{R}^*_{\mathrm{loc}} f(x) = \int_{B(x,\rho(x))} \mathcal{K}^*(x,y) f(y)\,dy \tag{23}$$

and

$$\mathcal{R}^*_{\mathrm{glob}} f(x) = \int_{B(x,\rho(x))^c} \mathcal{K}^*(x,y) f(y)\,dy$$

respectively, where the first integral should be understood in the sense of principal value.

Now with the notation of Theorem 1 we write

$$\mathcal{R}^* f = R^*_{\mathrm{loc}} f + \mathcal{R}^*_{\mathrm{glob}} f + \big(\mathcal{R}^*_{\mathrm{loc}} - R^*_{\mathrm{loc}}\big) f. \tag{24}$$

As a consequence of Theorem 1 the first term is bounded on $L^p(w)$ for $w \in A_p^{\rho,\mathrm{loc}}$, $1 < p < \infty$, and of weak type $(1,1)$ for $w \in A_1^{\rho,\mathrm{loc}}$. Since $w \in A_p^{\rho,\infty} \subset A_p^{\rho,\mathrm{loc}}$, $1 \leqslant p < \infty$, and $w \in A_{p/s'}^{\rho,\infty} \subset A_p^{\rho,\mathrm{loc}}$, $s' < p < \infty$, all the conclusions for R^*_{loc} hold.

For the term $\mathcal{R}^*_{\mathrm{glob}} f$ of (24) we use (21) to obtain

$$\big|\mathcal{R}^*_{\mathrm{glob}} f(x)\big| \leqslant \int_{B(x,\rho(x))^c} \big|\mathcal{K}^*(x,y)\big|\big|f(y)\big|\,dy \lesssim g_1(x) + g_2(x),$$

where

$$g_1(x) = \sum_{k=0}^{\infty} \frac{2^{-kN}}{(2^k\rho(x))^d} \int_{B(x,2^k\rho(x))} \big|f(y)\big|\,dy,$$

and

$$g_2(x) = \sum_{k=0}^{\infty} \frac{2^{-kN}}{(2^k\rho(x))^{d-1}} \int_{B(x,2^k\rho(x))} \bigg(\int_{B(x,2^k\rho(x))} \frac{V(u)}{|u-y|^{d-1}}\,du \bigg) \big|f(y)\big|\,dy.$$

Notice that g_1 is the same as (20) for $M = N + d$. Then its boundedness, for N large enough, follows from the arguments given there.

Regarding g_2, according to Lemma 3, we only have to consider $\frac{d}{2} < q < d$.

We consider a covering by critical balls as in the proof of Theorem 2. With the notation there, for $x \in Q_j$ we have $B(x, 2^k\rho(x)) \subset \tilde{Q}_j^k$, and so

$$\int_{B(x,2^k\rho(x))} \frac{V(u)}{|u-y|^{d-1}}\,du \lesssim I_1(\chi_{\tilde{Q}_j^k} V)(y),$$

where I_1 is the classical Fractional Integral of order 1. Therefore,

$$\|g_2\|_{L^p(w)} \lesssim \sum_{k=0}^{\infty} 2^{-kN} \bigg(\sum_j \frac{w(Q_j)}{(2^k\rho(x_j))^{p(d-1)}} \bigg(\int_{\tilde{Q}_j^k} I_1(\chi_{\tilde{Q}_j^k} V) f \bigg)^p \bigg)^{1/p}.$$

If we choose γ such that $\frac{1}{\gamma} + \frac{1}{s} + \frac{1}{p} = 1$, then by Hölder's inequality,

$$\int_{\tilde{Q}_j^k} I_1(\chi_{\tilde{Q}_j^k} V) f \leqslant \big\|I_1(\chi_{\tilde{Q}_j^k} V)\big\|_s \|\chi_{\tilde{Q}_j^k} f\|_{L^p(w)} \bigg(\int_{\tilde{Q}_j^k} w^{-\gamma/p} \bigg)^{1/\gamma}.$$

Recall that $V \in RH_q$ for some $q > 1$ implies that V satisfies the doubling condition, i.e., there exist constants $\mu \geqslant 1$ and C such that

$$\int_{tB} V \leqslant C t^{d\mu} \int_B V$$

holds for every ball B and $t>1$. Therefore, due to the boundedness of I_1 from L^q into L^s, and the assumptions on V,

$$\left\|I_1(\chi_{\tilde{Q}_j^k}V)\right\|_s \lesssim \|\chi_{\tilde{Q}_j^k}V\|_q \lesssim \left|\tilde{Q}_j^k\right|^{-1/q'}\int_{\tilde{Q}_j^k} V$$

$$\lesssim 2^{kd\mu}\left|\tilde{Q}_j^k\right|^{-1/q'}\int_{\tilde{Q}_j} V \lesssim 2^{kd(\mu-\frac{1}{q'})}\rho(x_j)^{\frac{d}{q}-2}$$

where the last inequality follows from the definition of ρ (see (8)). Hence,

$$\|g_2\|_{L^p(w)} \lesssim \sum_{k=0}^{\infty} 2^{-k(N-d\mu+d/q')}\left(\sum_j \frac{w(Q_j)}{\rho(x_j)^{p(\frac{d}{q'}+1)}}\left(\int_{\tilde{Q}_j^k} w^{-\gamma/p}\right)^{p/\gamma}\int_{\tilde{Q}_j^k}|f|^p w\right)^{1/p}.$$

Since $w\in A_{p/s'}^{\rho,\theta}$, for some $\theta>0$, and $\frac{p}{\gamma}=\frac{p}{s'}-1$ the last expression is bounded by

$$\sum_{k=0}^{\infty} 2^{-k(N-d\mu-1-\frac{\theta}{s'})}\left(\sum_j\int_{\tilde{Q}_j^k}|f|^p w\right)^{1/p} \lesssim \|f\|_{L^p(w)}\sum_{k=0}^{\infty}2^{-k(N-d\mu-1-\frac{\theta}{s'}-N_1)},$$

where the last inequality is due to Proposition 2. Finally, the series converges if we choose N large enough.

Now we have to deal with the term $(\mathcal{R}^*_{\mathrm{loc}}-R^*_{\mathrm{loc}})f$ of (24). By using estimate (22), we have

$$\left|\left(\mathcal{R}^*_{\mathrm{loc}}-R^*_{\mathrm{loc}}\right)f(x)\right| \lesssim h_1(x)+h_2(x)$$

where

$$h_1(x)=\rho(x)^{-2+d/q}\int_{B(x,\rho(x))}\frac{|f(z)|}{|x-z|^{d-2+d/q}}\,dz$$

and

$$h_2(x)=\int_{B(x,\rho(x))}\frac{|f(z)|}{|x-z|^{d-1}}\left(\int_{B(z,|x-z|/4)}\frac{V(u)}{|u-z|^{d-1}}\,du\right)dz.$$

For h_1 we have

$$h_1(x)\leqslant \rho(x)^{-2+d/q}\sum_{k=1}^{\infty}\left(2^{-k}\rho(x)\right)^{-d+2-d/q}\int_{B(x,2^{-k}\rho(x))}|f|$$

$$\lesssim M_{\mathrm{loc}}f(x).$$

Hence, as a consequence of Theorem 1, we obtain all the needed results for h_1.

To deal with h_2, according to Lemma 3, we only have to consider $\frac{d}{2}<q<d$.

Let us take a covering $\{Q_j\}$ as before. For each j and k there exist 2^{dk} balls of radio $2^{-k}\rho(x_j)$, $B_l^{j,k}=B(x_l^{j,k},2^{-k}\rho(x_j))$ such that $Q_j\subset\bigcup_{l=1}^{2^{dk}}B_l^{j,k}\subset 2Q_j$ and $\sum_{l=1}^{2^{dk}}\chi_{B_l^{j,k}}\leqslant 2^d$. Moreover, this construction can be done in a way that for each k the family of a fixed dilation $\{\tilde{B}_l^{j,k}\}_{j,l}$ is a covering of $\mathbb{R}^d$ with

$$\sum_j\sum_{l=1}^{2^{dk}}\chi_{\tilde{B}_l^{j,k}}\leqslant C \tag{25}$$

with the constant C independent of k. To our purpose we take the dilation $\tilde{B}_l^{j,k}=5c_0B_l^{j,k}$.

Splitting into annuli,

$$h_2(x)\lesssim\sum_{k=0}^{\infty}2^{k(d-1)}h_{2,k}(x),$$

where

$$h_{2,k}(x) = \rho(x)^{-d+1} \int\limits_{B(x,2^{-k}\rho(x))} \left|f(z)\right| \left(\int\limits_{B(z,|x-z|/4)} \frac{V(u)}{|u-z|^{d-1}}\, du \right) dz.$$

Observe that if $x \in B_l^{j,k}$,

$$h_{2,k}(x) \lesssim \rho(x_j)^{-d+1} \int\limits_{\tilde{B}_l^{j,k}} |f| I_1(\chi_{\tilde{B}_l^{j,k}} V),$$

but

$$\int\limits_{\tilde{B}_l^{j,k}} |f| I_1(\chi_{\tilde{B}_l^{j,k}} V) \leqslant \left\| I_1(\chi_{\tilde{B}_l^{j,k}} V) \right\|_s \| \chi_{\tilde{B}_l^{j,k}} f \|_{L^p(w)} \left(\int\limits_{\tilde{B}_l^{j,k}} w^{-\gamma/p} \right)^{1/\gamma}.$$

Using that I_1 is bounded from L^q into L^s, and Lemma 4,

$$\begin{aligned} \left\| I_1(\chi_{\tilde{B}_l^{j,k}} V) \right\|_s &\lesssim \| \chi_{\tilde{B}_l^{j,k}} V \|_q \\ &\lesssim \left| \tilde{B}_l^{j,k} \right|^{-1+1/q} \int\limits_{\tilde{B}_l^{j,k}} V \\ &\lesssim \rho(x_j)^{-2+d/q}. \end{aligned} \tag{26}$$

Therefore, from the above estimate, using that $w \in A_{p/s'}^{\rho,\infty}$ and $\frac{p}{\gamma} = \frac{p}{s'} - 1$,

$$\begin{aligned} \|h_{2,k}\|_{L^p(w)} &\lesssim \sum_{j,l} \frac{w(B_l^{j,k})}{\rho(x_j)^{p(1+d/q')}} \left(\int\limits_{\tilde{B}_l^{j,k}} w^{-\gamma/p} \right)^{p/\gamma} \| \chi_{B_l^{j,k}} f \|_{L^p(w)}^p \\ &\lesssim 2^{-k(1+\frac{d}{q'})} \left(\sum_{j,l} \| \chi_{B_l^{j,k}} f \|_{L^p(w)}^p \right)^{1/p} \\ &\lesssim 2^{-k(1+\frac{d}{q'})} \| f \|_{L^p(w)}, \end{aligned}$$

where in the last inequality we have used the finite overlapping property (25).

Now,

$$\|h_2\|_{L^p(w)} \lesssim \sum_{k=0}^{\infty} 2^{k(d-1)} \|h_{2,k}\|_{L^p(w)} \lesssim \|f\|_{L^p(w)},$$

and the summability of the series is due to $\frac{d}{q} - 2 < 0$.

In this way we have proved all the stated boundedness for $\mathcal{R}^*$. By duality we obtain the results for $\mathcal{R}$, except the weak type $(1,1)$ for both (i) and (ii).

To take care of that we decompose $\mathcal{R}$ in a slightly different way.

We write

$$\mathcal{R} f = R_{\mathrm{loc}^*} f + \mathcal{R}_{\mathrm{glob}^*} f + (\mathcal{R}_{\mathrm{loc}^*} - R_{\mathrm{loc}^*}) f, \tag{27}$$

where given an operator S, $S_{\mathrm{loc}^*} f(x) = S(f \chi_{E_x^\rho})(x)$, with $E_x^\rho = \{y\colon\ |x-y| < \rho(y)\}$, and $S_{\mathrm{glob}^*} = S - S_{\mathrm{loc}^*}$.

With this notation we have

$$(\mathcal{R}_{\mathrm{glob}^*})^* = \mathcal{R}_{\mathrm{glob}}^*, \tag{28}$$

and

$$(\mathcal{R}_{\mathrm{loc}^*} - R_{\mathrm{loc}^*})^* = \mathcal{R}_{\mathrm{loc}}^* - R_{\mathrm{loc}}^*. \tag{29}$$

We claim that the first term of (27), R_{loc^*}, is of weak type $(1,1)$ for $w \in A_1^{\rho,\mathrm{loc}}$. This can be proved as for R_{loc} in Theorem 1. A careful look reveals that the clue fact is that $B(x,\rho(x)) \subset \tilde{Q}_j$ for $x \in Q_j$. So the proof of the weak type $(1,1)$ for R_{loc^*} follows as before changing $\tilde{Q}_j$ by $8c_0^2 Q_j$, since in our situation it is easy to prove that $E_x^\rho \subset 8c_0^2 Q_j$ for $x \in Q_j$.

For the other two terms we proceed by duality. In view of (28) and (29) it is enough to show that $\mathcal{R}^*_{\mathrm{glob}}$ and $\mathcal{R}^*_{\mathrm{loc}} - R^*_{\mathrm{loc}}$ are bounded on $L^\infty_{w^{-1}} = \{f\colon\ fw^{-1} \in L^\infty\}$ for w such that $w \in A_1^{\rho,\infty}$ under the assumption $q > d$, or $w^{s'} \in A_1^{\rho,\infty}$ for $\frac{d}{2} < q < d$, $\frac{1}{s} = \frac{1}{q} - \frac{1}{d}$. We shall do that using the same estimates already obtained in the case $p < \infty$.

First, let us check that $\|g_1 w^{-1}\|_\infty$ and $\|g_2 w^{-1}\|_\infty$ are bounded by $\|fw^{-1}\|_\infty$. In fact, since as before, for $x \in Q_j$, $B(x, 2^k\rho(x)) \subset \tilde{Q}_j^k$ and $\rho(x) \approx \rho(x_j)$, we get for $w \in A_1^{\rho,\infty}$,

$$\|g_1 w^{-1}\|_\infty \lesssim \sum_{k=0}^{\infty} 2^{-k(N+d)} \sup_j \left(\rho(x_j)^{-d} \Big(\sup_{\tilde{Q}_j^k} w^{-1} \Big) \int_{\tilde{Q}_j^k} |f| \right)$$

$$\lesssim \|fw^{-1}\|_\infty \sum_{k=0}^{\infty} 2^{-k(N-\theta)},$$

and the last sum converges taking N large enough.

As for g_2 we only have to consider the case $\frac{d}{2} < q < d$. Using the same estimates as before

$$\|g_2 w^{-1}\|_\infty \lesssim \sum_{k=0}^{\infty} 2^{-kN} \left(\sup_j \frac{\sup_{\tilde{Q}_j^k} w^{-1}}{(2^k\rho(x_j))^{(d-1)}} \left(\int_{\tilde{Q}_j^k} I_1(\chi_{\tilde{Q}_j^k} V) f \right) \right),$$

but for each k and j,

$$\int_{\tilde{Q}_j^k} I_1(\chi_{\tilde{Q}_j^k} V) f \leqslant \|fw^{-1}\|_\infty \|I_1(\chi_{\tilde{Q}_j^k} V)\|_s \left(\int_{\tilde{Q}_j^k} w^{s'} \right)^{1/s'}.$$

Using (26) and that $w^{s'}$ belongs to $A_1^{\rho,\infty}$ the conclusion follows taking N large enough.

With similar techniques the boundedness of h_1 and h_2 on $L^\infty_{w^{-1}}$ can be achieved and the proof is finished. □

Now, we present an important property of the classes of weights in $A_p^{\rho,\infty}$ that will allow us to improve Theorem 3, part (ii).

It is well known that a weight in A_p, also belongs to $A_{p-\epsilon}$ for some $\epsilon > 0$. Following the proof in [2], it becomes clear that the classes $A_p^{\rho,\mathrm{loc}}$ share this property. In the next proposition, we show that it also holds for the intermediate classes $A_p^{\rho,\infty}$.

Proposition 5. *If $w \in A_p^{\rho,\infty}$, $1 < p < \infty$, then there exists $\epsilon > 0$ such that $w \in A_{p-\epsilon}^{\rho,\infty}$.*

As in the classical theory of weights, the previous proposition is a consequence of the following property that resembles a reverse Hölder property.

Lemma 5. *If $w \in A_p^{\rho,\infty}$, $1 \leqslant p < \infty$, then there exist positive constants δ, η and C such that*

$$\left(\frac{1}{|B|} \int_B w^{1+\delta} \right)^{\frac{1}{1+\delta}} \leqslant C \left(\frac{1}{|B|} \int_B w \right) \left(1 + \frac{r}{\rho(x)} \right)^{\eta},$$

for every ball $B = B(x,r)$.

Proof. Since $w \in A_p^{\rho,\infty}$, also $w \in A_p^{\rho,\mathrm{loc}}$ and thus there exist constants δ and C, independent of B, such that

$$\left(\frac{1}{|B|} \int_B w^{1+\delta} \right)^{\frac{1}{1+\delta}} \leqslant C \frac{w(B)}{|B|}, \tag{30}$$

for every ball $B = B(x,r)$ with $r \leqslant \rho(x)$. (This can be seen following the classical proof for A_p in [2].)

On the other hand, let $B = B(x,r)$, with $r > \rho(x)$ and $\mathcal{F} = \{j\colon\ Q_j \cap B \neq \emptyset\}$, where $Q_j = B(x_j, \rho(x_j))$ and $\{x_j\}_j$ is the sequence of Proposition 2.

For $j \in \mathcal{F}$, using (10) and $Q_j \cap B \neq \emptyset$ we have

$$\rho(x_j) \leqslant 2c_0^2\left(1 + \frac{r}{\rho(x)}\right)^{\frac{N_0}{N_0+1}} r.$$

Therefore

$$\bigcup_{j\in\mathcal{F}} Q_j \subset c_r B, \tag{31}$$

with $c_r = 4c_0^2(1 + \frac{r}{\rho(x)})^{\frac{N_0}{N_0+1}}$. Also from (10),

$$\rho(x_j) \geqslant \frac{1}{c_0}\rho(x)\left(1 + \frac{r}{\rho(x)}\right)^{-N_0}, \tag{32}$$

for every $j \in \mathcal{F}$.

Now let δ be the constant in (30). By using Proposition 2,

$$\begin{aligned}\left(\int_B w^{1+\delta}\right)^{\frac{1}{1+\delta}} &\leqslant \sum_{j\in\mathcal{F}}\left(\int_{Q_j} w^{1+\delta}\right)^{\frac{1}{1+\delta}}\\ &\lesssim \sum_{j\in\mathcal{F}} w(Q_j)|Q_j|^{-\frac{\delta}{1+\delta}}\\ &\lesssim w(c_r B)\rho(x)^{-\frac{d\delta}{1+\delta}}\left(1 + \frac{r}{\rho(x)}\right)^{\frac{d\delta N_0}{1+\delta}},\end{aligned} \tag{33}$$

where in the last inequality we have used (32), the finite overlapping property (ii) in Proposition 2 and then (31).

Finally, since $w \in A_p^\theta$ for some θ, we have

$$\begin{aligned} w(c_r B) &\lesssim c_r^{dp}|B|^p\left(\int_{c_r B} w^{-\frac{1}{p-1}}\right)^{-p+1}\left(1 + \frac{c_r r}{\rho(x)}\right)^{\theta p}\\ &\lesssim w(B)\left(1 + \frac{r}{\rho(x)}\right)^{\theta p + \frac{\theta p N_0}{N_0+1} + \frac{dpN_0}{N_0+1}}.\end{aligned} \tag{34}$$

Combining (33) and (34) we obtain the desired inequality with

$$\eta = \theta p + (\theta + d)\frac{pN_0}{N_0+1} + (N_0+1)\frac{d\delta}{1+\delta}. \qquad \square$$

Corollary 2. *Let $q_0 = \sup\{q\colon\ w \in RH_q\}$ and s_0 such that $\frac{1}{s_0} = \frac{1}{q_0} - \frac{1}{d}$. If $w \in A^{\rho,\infty}_{p/s_0'}$, then the operator $\mathcal{R}^*$ is bounded on $L^p(w)$, for $s_0' < p < \infty$. By duality, if $w^{-\frac{1}{p-1}} \in A^{\rho,\infty}_{p'/s_0'}$, then $\mathcal{R}$ is bounded on $L^p(w)$, for $1 < p < s_0$. Moreover, $\mathcal{R}$ is of weak type $(1,1)$ for $w^{s_0'} \in A_1^{\rho,\infty}$.*

Proof. In terms of s_0 the part (ii) of Theorem 3 for $\mathcal{R}^*$ can be re-written as: $\mathcal{R}^*$ is bounded on $L^p(w)$, whenever $s_0 < p < \infty$ and $w \in \bigcup_{s<s_0} A^{\rho,\infty}_{p/s'}$. But, Proposition 5 implies $A^{\rho,\infty}_{p/s_0'} = \bigcup_{s<s_0} A^{\rho,\infty}_{p/s'}$. Regarding the weak type of $\mathcal{R}$ we notice that if $w^{s_0'} \in A_1^{\rho,\infty}$ then $w^{s_0'+\epsilon} \in A_1^{\rho,\infty}$ for some $\epsilon > 0$ due to Proposition 5. Therefore we are in the hypothesis of Theorem 3. $\square$

5. The Fractional Integral

Let us remind that the $\mathcal{L}$-Fractional Integral of order $0 < \alpha < d$ can be written as

$$\mathcal{I}_\alpha f(x) = \int_{\mathbb{R}^d}\int_0^\infty k_t(x,y)t^{\alpha/2}\frac{dt}{t} f(y)\,dy.$$

Theorem 4. *For $1 < p < d/\alpha$, the operator $\mathcal{I}_\alpha$ is bounded from $L^p(w)$ into $L^\nu(w^{\nu/p})$ for $w^{\nu/p} \in A^{\rho,\infty}_{1+\frac{\nu}{p'}}$, with $\frac{1}{\nu} = \frac{1}{p} - \frac{\alpha}{d}$, and of weak type $(1, \frac{d}{d-\alpha})$ when $w^{\frac{d}{d-\alpha}} \in A_1^{\rho,\infty}$.*

Proof. As before, for each $x \in \mathbb{R}^d$ we split $\mathcal{I}_\alpha$ as

$$\mathcal{I}_\alpha f = (\mathcal{I}_\alpha)_{\mathrm{loc}} f + (\mathcal{I}_\alpha)_{\mathrm{glob}} f, \tag{35}$$

where $(\mathcal{I}_\alpha)_{\mathrm{loc}} f(x) = \mathcal{I}_\alpha f \chi_{B(x,\rho(x))}(x)$ and $(\mathcal{I}_\alpha)_{\mathrm{glob}} f(x) = \mathcal{I}_\alpha f \chi_{B(x,\rho(x))^c}(x)$.

It follows from (18) that the first term is bounded by $(I_\alpha)_{\mathrm{loc}} f(x)$, for f non-negative. Therefore, the conclusion for the first operator is a consequence of Theorem 1, and $A^{\rho,\infty}_{1+\frac{\nu}{p'}} \subset A^{\rho,\mathrm{loc}}_{1+\frac{\nu}{p'}}$.

To deal with the second term of (35) we obtain a point-wise estimate by an expression like (20). Indeed, using Lemma 2, for any M and N,

$$\begin{aligned}
\left|(\mathcal{I}_\alpha)_{\mathrm{glob}} f(x)\right| &\leqslant \int_0^\infty \int_{B(x,\rho(x))^c} k_t(x,y)\left|f(y)\right| dy\, t^{\alpha/2} \frac{dt}{t} \\
&\lesssim \int_{B(x,\rho(x))^c} \frac{|f(y)|}{|x-y|^M} dy \int_0^\infty \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N} t^{(M-d+\alpha)/2} \frac{dt}{t}.
\end{aligned}$$

Now, choosing $N \geqslant M > d - \alpha$,

$$\begin{aligned}
\int_0^\infty \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{-N} t^{(M-d+\alpha)/2} \frac{dt}{t} &\leqslant \int_0^{\rho(x)^2} t^{(M-d+\alpha)/2} \frac{dt}{t} + \rho(x)^N \int_{\rho(x)^2}^\infty t^{(-N+M-d+\alpha)/2} \frac{dt}{t} \\
&\lesssim \rho(x)^{M-d+\alpha}.
\end{aligned}$$

Therefore, splitting into annuli,

$$\left|(\mathcal{I}_\alpha)_{\mathrm{glob}} f(x)\right| \lesssim \rho(x)^{-d+\alpha} \sum_{k=1}^\infty 2^{-kM} \int_{2^k B_x} |f|\, dy.$$

Now we argue as we did with g_1 in the proof of Theorem 2. Denoting $\tilde{Q}_j^k = 2^k \tilde{Q}_j$, since $w \in A^{\rho,\theta}_{1+\frac{\nu}{p'}}$ for some $\theta > 0$, we have for $p > 1$,

$$\begin{aligned}
\left\|(\mathcal{I}_\alpha)_{\mathrm{glob}} f\right\|_{L^\nu(w^{\nu/p})} &\lesssim \sum_{k=0}^\infty 2^{-kM} \left(\sum_j \int_{Q_j} \rho(x)^{\nu(\alpha-d)} \left(\int_{2^k B_x} |f| \right)^\nu w^{\nu/p}(x)\, dx \right)^{1/\nu} \\
&\lesssim \sum_{k=0}^\infty 2^{-kM} \left(\sum_j \rho(x_j)^{\nu(\alpha-d)} \left(\int_{\tilde{Q}_j^k} w^{-p'/p} \right)^{\nu/p'} w^{\nu/p}(Q_j) \left(\int_{\tilde{Q}_j^k} |f|^p w \right)^{\nu/p} \right)^{1/\nu} \\
&\lesssim \sum_{k=0}^\infty 2^{-k(M-d+\alpha-\theta)} \left(\sum_j \left(\int_{\tilde{Q}_j^k} |f|^p w \right)^{\nu/p} \right)^{1/\nu}.
\end{aligned}$$

Finally, since $\nu \geqslant p$ and using Proposition 2, the last expression is bounded by

$$\sum_{k=0}^\infty 2^{-k(M-d-\theta+\alpha+N_1)} \|f\|_{L^p(w)}.$$

Choosing M large enough we are done.

For $p = 1$, $(I_\alpha)_{\mathrm{glob}}$ is also strong type $(1, \frac{d}{d-\alpha})$ and it follows in the same way as in the case $p > 1$ with the obvious changes. $\square$

6. The associated Square Function

Let us point out that in [3] the authors introduce a Square Function associated to $\mathcal{L}$, that after a change of variables can be written as (5).

For this operator we have the following result.

Theorem 5. *For $1 < p < \infty$ the operator $\mathfrak{g}$ is bounded on $L^p(w)$, when $w \in A_p^{\rho,\infty}$, and of weak type $(1,1)$ when $w \in A_1^{\rho,\infty}$.*

Proof. As before, we define

$$\mathfrak{g}_{\mathrm{loc}}(f)(x) = \mathfrak{g}(f\chi_{B(x,\rho(x))})(x) \quad \text{and} \quad \mathfrak{g}_{\mathrm{glob}}(f)(x) = \mathfrak{g}(f\chi_{B^c(x,\rho(x))})(x),$$

with $B_x = B(x,\rho(x))$, and thus

$$\big\|\mathfrak{g}(f)\big\|_{L^p(w)} \leqslant \big\|\mathfrak{g}_{\mathrm{loc}}(f)\big\|_{L^p(w)} + \big\|\mathfrak{g}_{\mathrm{glob}}(f)\big\|_{L^p(w)}.$$

We start with $\mathfrak{g}_{\mathrm{glob}}$. Denoting by q_t the kernel of $\frac{d}{dt}e^{-t\mathcal{L}}$, from (2.7) of [3], for any positive integer N we have

$$\big|q_t(x,y)\big| \leqslant \frac{C_N}{t^{d/2+1}}\left(1 + \frac{t}{\rho(x)^2} + \frac{t}{\rho(x)^2}\right)^{-N} e^{-\frac{|x-y|^2}{Ct}}. \tag{36}$$

Therefore, for any $M > 0$, and $B_x = B(x,\rho(x))$,

$$\begin{aligned}
\left|\int_{|x-y|>\rho(x)} q_t(x,y)f(y)\,dy\right| &\lesssim t^{-d/2-1}\left(1+\frac{t}{\rho(x)^2}\right)^{-N} \int_{|x-y|>\rho(x)} e^{-\frac{|x-y|^2}{Ct}}\big|f(y)\big|\,dy \\
&\lesssim t^{\frac{M-d}{2}-1}\left(1+\frac{t}{\rho(x)^2}\right)^{-N} \int_{|x-y|>\rho(x)} \frac{|f(y)|}{|x-y|^M}\,dy \\
&\lesssim \frac{t^{\frac{M-d}{2}-1}}{\rho(x)^M}\left(1+\frac{t}{\rho(x)^2}\right)^{-N} \sum_{k=0}^{\infty} 2^{-kM} \int_{2^k B_x} |f| \\
&\lesssim \frac{t^{\frac{M-d}{2}-1}}{\rho(x)^{M-d}}\left(1+\frac{t}{\rho(x)^2}\right)^{-N} g_1(x),
\end{aligned}$$

where g_1 is defined in (20).

Hence,

$$\begin{aligned}
\mathfrak{g}_{\mathrm{glob}}(f)(x) &\lesssim g_1(x)\left(\int_0^{\infty}\left(\frac{t}{\rho(x)^2}\right)^{M-d}\left(1+\frac{t}{\rho(x)^2}\right)^{-2N}\frac{dt}{t}\right)^{1/2} \\
&\lesssim g_1(x),
\end{aligned}$$

choosing M and N such that $M-d > 0$ and $2N > M-d$. Hence, the estimates for $\mathfrak{g}_{\mathrm{glob}}$ follow from those for g_1.

To deal with $\mathfrak{g}_{\mathrm{loc}}$ we write

$$\mathfrak{g}_{\mathrm{loc}}(f)(x) \lesssim I(x) + \mathbf{g}_{\mathrm{loc}}(x) + II(x), \tag{37}$$

where $\mathbf{g}_{\mathrm{loc}}$ is the localization of the classical square function as in Theorem 1,

$$I(x) = \left(\int_0^{\rho(x)^2}\left|\int_{|x-y|<\rho(x)} \big[q_t(x,y) - \tilde{q}_t(x,y)\big]f(y)\,dy\right|^2 t\,dt\right)^{1/2},$$

where $\tilde{q}_t$ is the kernel of $\frac{d}{dt}e^{t\Delta}$, and

$$II(x) = \left(\int_{\rho(x)^2}^{\infty}\left|\int_{|x-y|<\rho(x)} q_t(x,y)f(y)\,dy\right|^2 t\,dt\right)^{1/2}.$$

By (36) with $N = 1/2$,

$$II(x) \lesssim \left(\int_{\rho(x)^2}^{\infty} \left(\frac{\rho(x)}{t} \right)^2 \left(\int_{|x-y|<\rho(x)} \frac{e^{-\frac{|x-y|^2}{Ct}}}{t^{d/2}} \left| f(y) \right| dy \right)^2 dt \right)^{1/2}$$

$$\lesssim T^*_{\mathrm{loc}}(f)(x) \left(\int_{\rho(x)^2}^{\infty} \left(\frac{\rho(x)}{t} \right)^2 dt \right)^{1/2}$$

$$\lesssim T^*_{\mathrm{loc}}(f)(x).$$

As we have already seen the last operator is bounded in $L^p(w)$ for $w \in A_p^{\rho,\mathrm{loc}}$.

For $I(x)$, as in [3] (Eq. (5.25)) we use the following consequence of the perturbation formula

$$\frac{d}{dt} e^{-t\mathcal{L}} - \frac{d}{dt} e^{t\Delta} = e^{\frac{t}{2}\Delta} V e^{-\frac{t}{2}\mathcal{L}} + \int_0^{t/2} \frac{d}{dt} e^{(t-s)\Delta} V e^{-s\mathcal{L}} \, ds + \int_{t/2}^{t} e^{(t-s)\Delta} V \frac{d}{ds} e^{-s\mathcal{L}} \, ds.$$

Let us call K_1, K_2, K_3, the kernels of the first, second and third term respectively.

Our arguments will be based on the inequality

$$e^{-a|\alpha|^2} e^{-b|\beta|^2} \leqslant e^{-\frac{b}{2}|\beta|^2} e^{-\frac{c}{2}|\alpha+\beta|^2}, \quad \alpha, \beta \in \mathbb{R}^d, \ c = \min\left\{a, \frac{b}{2}\right\}, \tag{38}$$

and the estimate

$$\int_{\mathbb{R}^d} \phi_t(x-z) V(z) \, dz \lesssim \frac{1}{t} \left(\frac{\sqrt{t}}{\rho(x)} \right)^{\delta}, \tag{39}$$

for some $\delta > 0$ and $t < \rho(x)^2$, where $\phi_t(z) = \frac{1}{t^{d/2}} \phi(\frac{z}{\sqrt{t}})$, with ϕ any rapidly decreasing function (see (2.8) in [3]).

For K_1, by using (18), (38) (with $a = b = \frac{1}{2t}$, $\alpha = x - z$ and $\beta = z - y$) and (39), we get

$$K_1(x, y, t) \lesssim \frac{1}{t^d} \int_{\mathbb{R}^d} e^{-\frac{|x-z|^2}{2t}} V(z) e^{-\frac{|z-y|^2}{2t}} \, dz$$

$$\lesssim \frac{1}{t^d} e^{-\frac{|x-y|^2}{8t}} \int_{\mathbb{R}^d} V(z) e^{-\frac{|z-y|^2}{4t}} \, dz$$

$$\lesssim \frac{1}{t^{\frac{d}{2}+1}} \left(\frac{\sqrt{t}}{\rho(x)} \right)^{\delta} e^{-\frac{|x-y|^2}{8t}}.$$

For K_2 we use first (17), (18), together with $t/2 < t - s < t$ for $0 < s < t/2$; then we apply inequality (38) with $a = \frac{1}{5t}$, $b = \frac{1}{4s}$, $\alpha = x - z$ and $\beta = z - y$, to get

$$\left| K_2(x, y, t) \right| \lesssim \frac{1}{t^{\frac{d}{2}+1}} \int_0^{t/2} \frac{1}{s^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|x-z|^2}{5t}} V(z) e^{-\frac{|y-z|^2}{4s}} \, dz \, ds$$

$$\lesssim \frac{1}{t^{\frac{d}{2}+1}} e^{-\frac{|x-y|^2}{10t}} \int_0^{t/2} \frac{1}{s^{d/2}} \int_{\mathbb{R}^d} V(z) \, e^{-\frac{|y-z|^2}{8s}} \, dz \, ds$$

$$\lesssim \frac{1}{t^{\frac{d}{2}+1}} \left(\frac{\sqrt{t}}{\rho(y)} \right)^{\delta} e^{-\frac{|x-y|^2}{10t}}, \tag{40}$$

where in the last inequality we have used (39). Noting that for $|x - y| \leqslant \rho(x)$ we have $\rho(x) \approx \rho(y)$, and we obtain a similar estimate as for K_1.

Finally, we proceed in a similar way for K_3. By (36) and performing a change of variables, we get

$$\left| K_3(x, y, t) \right| \lesssim \int_0^{t/2} \frac{1}{s^{d/2}(t-s)^{d/2+1}} \int_{\mathbb{R}^d} e^{-\frac{|x-z|^2}{2s}} V(z) e^{-\frac{|y-z|^2}{C(t-s)}} \, dz \, ds.$$

Since $t/2 < t - s < t$ for $0 < s < t/2$, we arrive to an expression similar to the first line of (40) exchanging the roles of x and y, and with a different constant in the exponential, and then we proceed as there.

From the above estimates, we have

$$\left|K_i(x, y, t)\right| \lesssim \frac{1}{t^{\frac{d}{2}+1}} \left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta} e^{-\epsilon \frac{|x-y|^2}{t}},$$

for $i = 1, 2, 3$, and some $\epsilon > 0$.

Therefore,

$$\begin{aligned}\left(\int_0^{\rho(x)^2} \left|\int_{|x-y|<\rho(x)} \left|K_i(x, y, t)\right|\left|f(y)\right| dy\right|^2 t\, dt\right)^{1/2} &\lesssim \left(\int_0^{\rho(x)^2} \left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta} \left|\int_{|x-y|<\rho(x)} \frac{e^{-\epsilon \frac{|x-y|^2}{t}}}{t^{d/2}} f(y)\, dy\right|^2 \frac{dt}{t}\right)^{1/2} \\ &\lesssim T^*_{\mathrm{loc}} f(x), \qquad (41)\end{aligned}$$

for $i = 1, 2, 3$, and thus

$$I(x) \lesssim T^*_{\mathrm{loc}} f(x).$$

Coming back to (37), from the previous estimates

$$\mathfrak{g}_{\mathrm{loc}}(x) \lesssim T^*_{\mathrm{loc}} f(x) + \mathbf{g}_{\mathrm{loc}}(x),$$

then the desired estimates follow from Theorem 1 since $A_p^{\rho,\infty} \subset A_p^{\rho,\mathrm{loc}}$. $\square$

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