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Local C^1 stability versus global C^1 unstability for iterative roots ☆

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ABSTRACT

Stability of iterative roots is important in the numerical computation of iterative roots. Known results show that under some conditions iterative roots of strictly monotonic self-mappings are C^0 stable in both the local sense and the global sense. In this paper we discuss the C^1 stability for iterative roots of strictly increasing self-mappings on a compact interval between two fixed points. We prove that those iterative roots are locally C^1 stable but globally C^1 unstable.

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1. Introduction

Regarded as a weak version of the problem of embedding flows [5] for dynamical systems, iterative root [2,10,11,22,26] is interesting in both dynamical systems and functional equations. Let X be a Banach space, $I \subset X$ and $C^r(I,I)$ for $r \geqslant 0$ be the set of all C^r self-mappings defined on I. The n-th iterate F^n of $F \in C^r(I,I)$ is defined inductively by $F^0(x) = x$ and $F^n(x) = F(F^{n-1}(x))$ for all $n \in \mathbb{N}$ and $x \in I$. If the inverse F^{-1} of F (when it is invertible) is regarded as an extension of n from $\mathbb{N}$ to $\mathbb{Z}$ with $n = -1$, the general problem of iterative roots, which is to solve the functional equation

$$f^k(x) = F(x), \quad \forall x \in I, \tag{1.1}$$

where $k \in \mathbb{N}$ is given, for the unknown $f \in C^r(I,I)$, is an extension of n from $\mathbb{N}$ to $\mathbb{Q}$ with $n = 1/k$. If the mapping F can be embedded into a flow, i.e., F is a time one mapping of a flow, then the index n can be extended to the whole $\mathbb{R}$ [18].

The theory of iterative roots has a very long history and it seems that it was Ch. Babbage [1] who first, yet at the beginning of 19th century, wrote on iterative roots explicitly. Since iterative roots were discussed well for monotonic mappings [10,11], many advances had been made to non-monotonic cases [3,15,28], self-mappings on circles [4,21,25], set-valued functions [14,19,20] and high dimensional mappings [12,13,16]. This problem leads to a philosophical discussion on the concept of time, as indicated by Targonski in [23]: *If we ever find a physical process represented by a map which is not embeddable or does not have iterative roots of every order, this suggests a minimal time interval, the chronon.* Since the theory is applicable to information science [7,8] and graph theory [17], it is necessary to develop algorithms for their numerical computation. A strategy is to give algorithms for polygonal functions [9,27] at first and then consider their approximation to general continuous functions. The errors from the numerical computation and approximation highly affect the validity of the computation, which requires the stability of iterative roots. A result on C^0 stability was given in 2007 in [24], which substantially is a local result because the C^0 stability is extended from a small neighborhood of a fixed point and the domain of iterative

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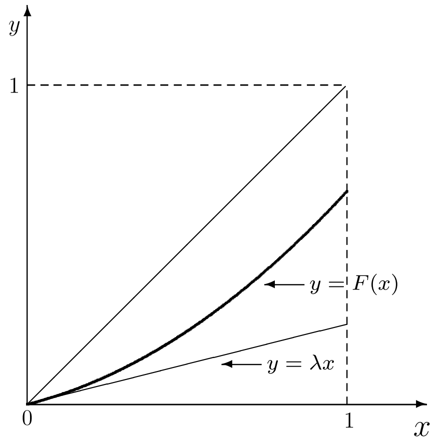

Fig. 1. $F \in \mathcal{H}^2_-(\lambda)$.

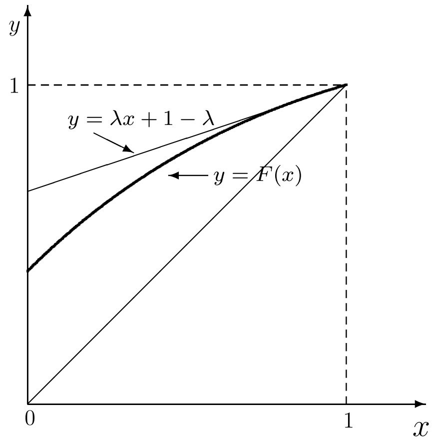

Fig. 2. $F \in \mathcal{H}^2_+(\lambda)$.

roots, being an open neighborhood of the fixed point, cannot be extended to include another fixed point. Recently, a result on globally C^0 stability was given in [29], which holds in an interval with more than one fixed point.

In this paper we discuss the C^1 stability for iterative roots of strictly increasing functions F's defined on the interval $I := [0, 1]$. We prove the C^1 stability in I which includes one fixed point but does not include the other by a reduction to Schröder's equation, giving the local C^1 stability for iterative roots. Moreover, we prove that F's are not C^1 stable for their iterative roots defined on the closed interval with the two endpoints being fixed points, showing the globally C^1 unstability for iterative roots.

2. Local C^1 stability

For each $\lambda \in (0, 1)$ let

$$\mathcal{H}^2_-(\lambda) := \{h \in C^2(I, I)\colon\ h(0) = 0,\ h'(0) = \lambda,\ h'(x) > 0 \text{ and } h(x) < x,\ \forall x \in (0, 1]\},$$
$$\mathcal{H}^2_+(\lambda) := \{h \in C^2(I, I)\colon\ h(1) = 1,\ h'(1) = \lambda,\ h'(x) > 0 \text{ and } h(x) > x,\ \forall x \in [0, 1)\}$$

(cf. Fig. 1 and Fig. 2). Let the norm $\|\cdot\|_r$ be defined by

$$\|f\|_r := \sup_{x \in I}|f(x)| + \cdots + \sup_{x \in I}|f^{(r)}(x)|$$

for all $r \in \mathbb{N} \cup \{0\}$ and $f \in C^r(I, I)$.

Given an arbitrary integer $k \geqslant 2$, by Theorem 11.4.2 in [11], a function F belonging to the class $\bigcup_{\lambda \in (0,1)} \mathcal{H}^2_-(\lambda)$ has a unique k-th order C^1 iterative root f defined on I, which is strictly increasing and is given by the formula

$$f(x) := \varphi^{-1}\big(\lambda^{1/k}\varphi(x)\big), \tag{2.1}$$

where $\varphi : I \to \mathbb{R}$ is the principle solution of Schröder's equation

$$\varphi\big(F(x)\big) = \lambda\varphi(x).$$

The principle solution is given by

$$\varphi(x) = \lim_{n \to \infty} \lambda^{-n} F^n(x), \tag{2.2}$$

which is C^1 differentiable in I with $\varphi'(0) = 1$ and is strictly increasing by Theorem 3.5.1 in [11]. A similar fact can be stated when $F \in \bigcup_{\lambda \in (0,1)} \mathcal{H}^2_+(\lambda)$.

Our aim of this section is to prove the following stability result.

Theorem 2.1. *Let $F \in \mathcal{H}^2_-(\lambda)$ [or $\mathcal{H}^2_+(\lambda)$] with some $\lambda \in (0, 1)$ and let (F_m) be a sequence of functions in $\mathcal{H}^2_-(\lambda)$ [or $\mathcal{H}^2_+(\lambda)$]. If*

$$\lim_{m \to \infty} \|F_m - F\|_2 = 0, \tag{2.3}$$

then

$$\lim_{m \to \infty} \|f_m - f\|_1 = 0,$$

where f and f_m are unique k-th order C^1 iterative roots of F and F_m, respectively, defined on I.

In what follows we only discuss the case that $F \in \mathcal{H}_-^2(\lambda)$ in details as the other one is similar. In order to prove Theorem 2.1 we need the following lemma.

Lemma 2.1. *Let $F \in \mathcal{H}_-^2(\lambda)$ with some $\lambda \in (0,1)$ and let (F_m) be a sequence of functions in $\mathcal{H}_-^2(\lambda)$ satisfying condition* (2.3)*. Then, for a given number $\mu_0 \in (\lambda, 1)$, there exist an $\varepsilon > 0$ and an $N_0 \in \mathbb{N}$ such that*

$$\left|F^n(x)\right| \leqslant \mu_0^n, \qquad \left|\left(F^n\right)'(x)\right| \leqslant \mu_0^n, \qquad \left|F_m^n(x)\right| \leqslant \mu_0^n, \qquad \left|\left(F_m^n\right)'(x)\right| \leqslant \mu_0^n \tag{2.4}$$

and

$$\left|F_m^n(x) - F^n(x)\right| \leqslant n\mu_0^{n-1}\|F_m - F\|_2 \tag{2.5}$$

for all $m \geqslant N_0$, $n \in \mathbb{N}$ and for all $x \in I_\varepsilon := [0, \varepsilon]$.

Proof. Since $F \in \mathcal{H}_-^2(\lambda)$ is C^2 differentiable in I, we have

$$\left|F'(x) - F'(y)\right| \leqslant L|x-y|, \quad \forall x, y \in I, \tag{2.6}$$

where $L := 2\|F\|_2 \geqslant 0$ is a constant independent of x and y. Choose a sufficiently small $\varepsilon > 0$ such that $\lambda + L\varepsilon \leqslant \mu_0$. Then by the chain rule, for all $n \in \mathbb{N}$,

$$\left|\left(F^n\right)'(x)\right| = \prod_{i=0}^{n-1}\left|F'\left(F^i(x)\right)\right| \leqslant \prod_{i=0}^{n-1}\left(\left|F'(0)\right| + L\left|F^i(x)\right|\right) \leqslant \mu_0^n, \quad \forall x \in I_\varepsilon, \tag{2.7}$$

because $|F^i(x)| \leqslant \varepsilon$ for all $i \in \mathbb{N}$ provided $|x| \leqslant \varepsilon$. It follows from (2.7) that

$$\left|F^n(x)\right| = \left|F^n(x) - F^n(0)\right| \leqslant \sup_{\xi \in I_\varepsilon}\left|\left(F^n\right)'(\xi)\right| \cdot |x| \leqslant \mu_0^n.$$

This proves the first two inequalities given in (2.4).

Let $\delta := \|F\|_2/2$. There is an $N_0 \in \mathbb{N}$ such that if $m \geqslant N_0$ then the inequality $\|F_m - F\|_2 \leqslant \delta$ holds by (2.3), implying that $\|F_m\|_2 \leqslant \|F\|_2 + \delta \leqslant L$. Thus, $|F_m'(x) - F_m'(y)| \leqslant L|x-y|$ for all $x, y \in I$. By the same procedure as before we can prove the last two inequalities given in (2.4).

It is clear that (2.5) holds when $n = 1$. Assume that (2.5) holds for $n = \ell \in \mathbb{N}$, i.e., $|F_m^\ell(x) - F^\ell(x)| \leqslant \ell\mu_0^{\ell-1}\|F_m - F\|_2$. Then, by (2.4), we get

$$\begin{aligned}
\left|F_m^{\ell+1}(x) - F^{\ell+1}(x)\right| &= \left|F_m\left(F_m^\ell(x)\right) - F\left(F^\ell(x)\right)\right| \\
&\leqslant \left|F_m\left(F_m^\ell(x)\right) - F_m\left(F^\ell(x)\right)\right| + \left|F_m\left(F^\ell(x)\right) - F\left(F^\ell(x)\right)\right| \\
&\leqslant \sup_{\xi \in I_\varepsilon}\left|F_m'(\xi)\right| \cdot \left|F_m^\ell(x) - F^\ell(x)\right| + \sup_{\xi \in I_\varepsilon}\left|(F_m - F)'(\xi)\right| \cdot \left|F^\ell(x)\right| \\
&\leqslant \mu_0\left|F_m^\ell(x) - F^\ell(x)\right| + \mu_0^\ell\|F_m - F\|_2 \\
&\leqslant (\ell+1)\mu_0^\ell\|F_m - F\|_2
\end{aligned}$$

for all $m \geqslant N_0$ and for all $x \in I_\varepsilon$. Thus we can obtain (2.5) by induction. This completes the proof. □

We also give the following lemma on C^1 stability of Schröder's equation.

Lemma 2.2. *Let $F \in \mathcal{H}_-^2(\lambda)$ with some $\lambda \in (0,1)$ and let (F_m) be a sequence of functions in $\mathcal{H}_-^2(\lambda)$ satisfying condition* (2.3)*. Then*

$$\lim_{m\to\infty}\|\varphi_m - \varphi\|_1 = 0, \tag{2.8}$$

where $\varphi : I \to \mathbb{R}$ and $\varphi_m : I \to \mathbb{R}$ are the principle solutions of Schröder's equations

$$\varphi\left(F(x)\right) = \lambda\varphi(x) \quad \textit{and} \quad \varphi_m\left(F_m(x)\right) = \lambda\varphi_m(x),$$

respectively.

Proof. First of all, by (2.2), φ and φ_m can be defined by

$$\varphi(x) = \lim_{n\to\infty}\lambda^{-n}F^n(x) \quad \text{and} \quad \varphi_m(x) = \lim_{n\to\infty}\lambda^{-n}F_m^n(x), \quad \forall x \in I, \tag{2.9}$$

respectively. In what follows we intend to discuss our results in a sufficiently small interval $I_\varepsilon = [0, \varepsilon]$ first and then extend them to the whole interval I.

In order to prove the convergence of the sequence (φ_m) in I_ε, note that

$$\left|F'\left(F^n(x)\right)\right| \leqslant \lambda + L\mu_0^n \quad \text{and} \quad \left|F'_m\left(F^n_m(x)\right)\right| \leqslant \lambda + L\mu_0^n, \quad \forall x \in I_\varepsilon, \tag{2.10}$$

for all $m \geqslant N_0$ and $n \in \mathbb{N}$ by (2.4). Let

$$M := \prod_{j=0}^{\infty}\left(1 + \lambda^{-1}L\mu_0^j\right)\sup_{x\in I_\varepsilon}\left|F'(x)\right|^{-1} < \infty.$$

It follows from (2.4), (2.5) and (2.10) that

$$\begin{aligned}
\lambda^{-n}\left|\left(F^n_m\right)'(x) - \left(F^n\right)'(x)\right| &= \lambda^{-n}\left|F'_m\left(F^{n-1}_m(x)\right)\cdots F'_m(x) - F'\left(F^{n-1}(x)\right)\cdots F'(x)\right| \\
&\leqslant \lambda^{-n}\sum_{i=0}^{n-1}\left|F'\left(F^{n-1}(x)\right)\right|\cdots\left|F'\left(F^{i+1}(x)\right)\right|\cdot\left|F'_m\left(F^i_m(x)\right) - F'\left(F^i(x)\right)\right| \\
&\quad\cdot\left|F'_m\left(F^{i-1}_m(x)\right)\right|\cdots\left|F'_m(x)\right| \\
&\leqslant \lambda^{-n}\prod_{j=0}^{n-1}\left(\lambda + L\mu_0^j\right)\sum_{i=0}^{n-1}\left|F'\left(F^i(x)\right)\right|^{-1}\cdot\left|F'_m\left(F^i_m(x)\right) - F'\left(F^i(x)\right)\right| \\
&\leqslant M\sum_{i=0}^{n-1}\left(\left|F'_m\left(F^i_m(x)\right) - F'\left(F^i_m(x)\right)\right| + \left|F'\left(F^i_m(x)\right) - F'\left(F^i(x)\right)\right|\right) \\
&\leqslant M\sum_{i=0}^{n-1}\Big(\sup_{\xi\in I_\varepsilon}\left|(F_m - F)''(\xi)\right|\cdot\left|F^i_m(x)\right| + L\left|F^i_m(x) - F^i(x)\right|\Big) \\
&\leqslant M\sum_{i=0}^{n-1}\left(\mu_0^i\|F_m - F\|_2 + Li\mu_0^{i-1}\|F_m - F\|_2\right) \\
&= M\left(\frac{L+1-\mu_0}{(1-\mu_0)^2} - \left(\frac{L-(1-\mu_0)(L-\mu_0)}{(1-\mu_0)^2} + \frac{L}{1-\mu_0}n\right)\mu_0^{n-1}\right)\|F_m - F\|_2 \\
&\leqslant M_1\|F_m - F\|_2,
\end{aligned} \tag{2.11}$$

where $M_1 := M(L+1-\mu_0)/(1-\mu_0)^2$ is a number independent of m, n and x. Thus, by (2.9) and (2.11), we get

$$\left|\varphi'_m(x) - \varphi'(x)\right| = \lim_{n\to\infty}\lambda^{-n}\left|\left(F^n_m\right)'(x) - \left(F^n\right)'(x)\right| \leqslant M_1\|F_m - F\|_2 \tag{2.12}$$

for all $m \geqslant N_0$ and for all $x \in I_\varepsilon$.

Next we extend the result (2.12) from I_ε to the whole interval I. Note that

$$\lim_{m\to\infty}\left\|F^n_m - F^n\right\|_2 = 0, \quad \forall n \in \mathbb{N}, \tag{2.13}$$

by (2.3) because we can see that the composition operator $T : \mathcal{H}^2_-(\lambda) \times \mathcal{H}^2_-(\lambda) \to \mathcal{H}^2_-(\lambda^2)$ such that $T(h_1, h_2) = h_1 \circ h_2$ for all $h_1, h_2 \in \mathcal{H}^2_-(\lambda)$ is continuous by Example 4.4.5 in [6]. Since 0 is a unique stable fixed point of F in I, by (2.13), there is an integer $N \in \mathbb{N}$ such that $F^N(x), F^N_m(x) \in I_\varepsilon$ for all $m \in \mathbb{N}$ and $x \in I$. Then, according to Schröder's equation, we can obtain the formulae

$$\varphi(x) = \lambda^{-N}\tilde{\varphi}\left(F^N(x)\right) \quad \text{and} \quad \varphi_m(x) = \lambda^{-N}\tilde{\varphi}_m\left(F^N_m(x)\right), \quad \forall x \in I, \tag{2.14}$$

where $\tilde{\varphi} := \varphi|_{I_\varepsilon}$ and $\tilde{\varphi}_m := \varphi_m|_{I_\varepsilon}$. It follows from (2.12) and (2.14) that

$$\begin{aligned}
\left|\varphi'_m(x) - \varphi'(x)\right| &= \lambda^{-N}\left|\tilde{\varphi}'_m\left(F^N_m(x)\right)\left(F^N_m\right)'(x) - \tilde{\varphi}'\left(F^N(x)\right)\left(F^N\right)'(x)\right| \\
&\leqslant \lambda^{-N}\left|\tilde{\varphi}'_m\left(F^N_m(x)\right) - \tilde{\varphi}'\left(F^N(x)\right)\right|\cdot\left|\left(F^N_m\right)'(x)\right| \\
&\quad + \lambda^{-N}\left|\tilde{\varphi}'\left(F^N(x)\right)\right|\cdot\left|\left(F^N_m\right)'(x) - \left(F^N\right)'(x)\right| \\
&\leqslant K_1\left(\left|\tilde{\varphi}'_m\left(F^N_m(x)\right) - \tilde{\varphi}'\left(F^N_m(x)\right)\right| + \left|\tilde{\varphi}'\left(F^N_m(x)\right) - \tilde{\varphi}'\left(F^N(x)\right)\right|\right) \\
&\quad + K_2\left|\left(F^N_m\right)'(x) - \left(F^N\right)'(x)\right| \\
&\leqslant K_1M_1\|F_m - F\|_2 + K_1\left|\tilde{\varphi}'\left(F^N_m(x)\right) - \tilde{\varphi}'\left(F^N(x)\right)\right| \\
&\quad + K_2\left\|F^N_m - F^N\right\|_2, \quad \forall x \in I,
\end{aligned} \tag{2.15}$$

for all $m \geqslant N_0$, where

$$K_1 := \lambda^{-N} \sup_{x\in I, m\in\mathbb{N}} \left|\left(F_m^N\right)'(x)\right| < \infty \quad \text{and} \quad K_2 := \lambda^{-N} \sup_{x\in I_\varepsilon} \left|\tilde{\varphi}'(x)\right| < \infty.$$

Hence, by (2.3), (2.13) and (2.15), by the uniform continuity of $\tilde{\varphi}'$ and by the fact that $\varphi(x) = \int_0^x \varphi'(t)\,dt$ and $\varphi_m(x) = \int_0^x \varphi_m'(t)\,dt$, we get

$$\lim_{m\to\infty} \left\|\varphi_m' - \varphi'\right\|_0 = 0 \quad \text{and} \quad \lim_{m\to\infty} \|\varphi_m - \varphi\|_0 = 0.$$

This proves equality (2.8) and the proof is completed. □

Having those preparations, we can give a proof to the main result of this section.

Proof of Theorem 2.1. By (2.1) the C^1 iterative roots f and f_m for each $m \in \mathbb{N}$ can be presented by

$$f(x) = \varphi^{-1}\left(\lambda^{1/k}\varphi(x)\right) \quad \text{and} \quad f_m(x) = \varphi_m^{-1}\left(\lambda^{1/k}\varphi_m(x)\right), \quad \forall x \in I, \tag{2.16}$$

respectively. In order to prove the convergence of (f_m) in I, note that, for sufficiently large $m \in \mathbb{N}$ such that $\lambda^{1/k}\varphi_m(x) \in \varphi(I)$ for all $x \in I$, we have

$$\left|\varphi_m'\left(\varphi_m^{-1}\left(\lambda^{1/k}\varphi_m(x)\right)\right) - \varphi'\left(\varphi^{-1}\left(\lambda^{1/k}\varphi(x)\right)\right)\right| \leqslant A_m(x) + B_m(x) + C_m(x), \tag{2.17}$$

where

$$\begin{aligned}
A_m(x) &:= \left|\varphi_m'\left(\varphi_m^{-1}\left(\lambda^{1/k}\varphi_m(x)\right)\right) - \varphi'\left(\varphi_m^{-1}\left(\lambda^{1/k}\varphi_m(x)\right)\right)\right| \leqslant \|\varphi_m - \varphi\|_1, \\
B_m(x) &:= \left|\varphi'\left(\varphi_m^{-1}\left(\lambda^{1/k}\varphi_m(x)\right)\right) - \varphi'\left(\varphi^{-1}\left(\lambda^{1/k}\varphi_m(x)\right)\right)\right| \\
&= \left|\varphi' \circ \varphi^{-1} \circ \varphi\left(\varphi_m^{-1}\left(\lambda^{1/k}\varphi_m(x)\right)\right) - \varphi' \circ \varphi^{-1} \circ \varphi_m\left(\varphi_m^{-1}\left(\lambda^{1/k}\varphi_m(x)\right)\right)\right|, \\
C_m(x) &:= \left|\varphi' \circ \varphi^{-1}\left(\lambda^{1/k}\varphi_m(x)\right) - \varphi' \circ \varphi^{-1}\left(\lambda^{1/k}\varphi(x)\right)\right|.
\end{aligned}$$

By (2.8) and by the fact that $\varphi' \circ \varphi^{-1}$ is uniformly continuous, we obtain

$$A_m(x) \to 0, \qquad B_m(x) \to 0, \qquad C_m(x) \to 0 \tag{2.18}$$

uniformly in I as $m \to \infty$. On the other hand, by the first equality in (2.14), we have $\inf_{x\in I} |\varphi'(x)| > 0$ because $\inf_{x\in I_\varepsilon} |\tilde{\varphi}'(x)| > 0$ and $\inf_{x\in I} |(F^N)'(x)| > 0$. Also we have

$$0 < \frac{1}{2}\inf_{x\in I}\left|\varphi'(x)\right| \leqslant \inf_{x\in I}\left|\varphi_m'(x)\right| \leqslant \sup_{x\in I}\left|\varphi_m'(x)\right| \leqslant 2\sup_{x\in I}\left|\varphi'(x)\right| < \infty$$

for all sufficiently large $m \in \mathbb{N}$ by (2.8). This implies that (φ_m') and $(1/\varphi_m')$ are uniformly bounded. Hence, by (2.16) and (2.17),

$$\begin{aligned}
\left|f_m'(x) - f'(x)\right| &= \left|\frac{\lambda^{1/k}\varphi_m'(x)}{\varphi_m'(\varphi_m^{-1}(\lambda^{1/k}\varphi_m(x)))} - \frac{\lambda^{1/k}\varphi'(x)}{\varphi'(\varphi^{-1}(\lambda^{1/k}\varphi(x)))}\right| \\
&\leqslant \left|\frac{\lambda^{1/k}\varphi_m'(x)}{\varphi_m'(\varphi_m^{-1}(\lambda^{1/k}\varphi_m(x)))} - \frac{\lambda^{1/k}\varphi_m'(x)}{\varphi'(\varphi^{-1}(\lambda^{1/k}\varphi(x)))}\right| \\
&\quad + \left|\frac{\lambda^{1/k}\varphi_m'(x)}{\varphi'(\varphi^{-1}(\lambda^{1/k}\varphi(x)))} - \frac{\lambda^{1/k}\varphi'(x)}{\varphi'(\varphi^{-1}(\lambda^{1/k}\varphi(x)))}\right| \\
&\leqslant \left|\frac{\lambda^{1/k}\varphi_m'(x)}{\varphi_m'(\varphi_m^{-1}(\lambda^{1/k}\varphi_m(x))) \cdot \varphi'(\varphi^{-1}(\lambda^{1/k}\varphi(x)))}\right| \\
&\quad \cdot \left|\varphi_m'\left(\varphi_m^{-1}\left(\lambda^{1/k}\varphi_m(x)\right)\right) - \varphi'\left(\varphi^{-1}\left(\lambda^{1/k}\varphi(x)\right)\right)\right| \\
&\quad + \frac{\lambda^{1/k}}{|\varphi'(\varphi^{-1}(\lambda^{1/k}\varphi(x)))|}\left|\varphi_m'(x) - \varphi'(x)\right| \\
&\leqslant M_2\left(A_m(x) + B_m(x) + C_m(x) + \|\varphi_m - \varphi\|_1\right), \quad \forall x \in I,
\end{aligned} \tag{2.19}$$

for all sufficiently large $m \in \mathbb{N}$, where $M_2 > 0$ is a number independent of m and x. Then, by (2.8), (2.18) and (2.19) and by the fact that $f_m(x) = \int_0^x f_m'(t)\,dt$ and $f(x) = \int_0^x f'(t)\,dt$, we get

$$\lim_{m\to\infty}\|f'_m - f'\|_0 = 0 \quad \text{and} \quad \lim_{m\to\infty}\|f_m - f\|_0 = 0.$$

This completes the proof. □

Remark that in Theorem 2.1 we proved the C^1 stability for iterative roots defined on I in the case that F has a unique fixed point 0. As mentioned in the Introduction, this is a local result because the interval includes only one fixed point.

3. Globally C^1 unstability

In this section we show the C^1 unstability for iterative roots defined on I in the case when F has exactly two fixed points 0 and 1. This is clearly a global result as mentioned in the Introduction. Contrary to the previous situation we prove here what follows.

Theorem 3.1. *For any $r \in \mathbb{N}$ and function $F \in C^r(I, I)$ satisfying*

$$\begin{aligned} &F(0) = 0, \qquad F'(0) \in \mathbb{R}\backslash\{0, 1\},\\ &F(1) = 1, \qquad F'(1) \in \mathbb{R}\backslash\{0, 1\},\\ &F(x) \neq x \quad \textit{and} \quad F'(x) > 0, \quad \forall x \in (0, 1) \end{aligned} \tag{3.1}$$

(*see Fig.* 3), *there is a sequence (F_m) of functions in $C^r(I, I)$ such that*

$$\lim_{m\to\infty}\|F_m - F\|_r = 0$$

and having no k-th order C^1 iterative roots for any integers $k \geqslant 2$.

Proof. If F does not have a C^1 iterative root defined on I, then it is enough to take $F_m = F$ for each $m \in \mathbb{N}$. Otherwise we construct a desired sequence using the following notion of δ-pulse.

Given a continuous function $h : I \to I$, a point $a \in (0, 1)$ and a positive δ such that $\delta \leqslant \min\{a, 1-a\}$, a continuous function $\tilde{h} : I \to I$ is said to be a *δ-pulse of h at a* if $\tilde{h}(x) \neq h(x)$ for each $x \in (a-\delta, a+\delta)$ and $\tilde{h}(x) = h(x)$ for each $x \in I\backslash(a-\delta, a+\delta)$ (see Fig. 4).

We only discuss the case that F satisfies (3.1) and $0 < F(x) < x$ for all $x \in (0, 1)$ in details as the other one is similar. Let f be the strictly increasing k-th order C^1 iterative root of F defined on I for an integer $k \geqslant 2$. Clearly, $0 < f(x) < x$ for all $x \in (0, 1)$; otherwise, the assumption staying at the end of (3.1) implies that $f(x) > x$ for each $x \in (0, 1)$, whence $F(x) = f^k(x) > x$ contrary to the assumption. Our idea is to find a sequence convergent to F such that their C^1 iterative roots do not exist. For this purpose let $\delta > 0$ be a sufficiently small number such that

$$f(a+\delta) < a-\delta \tag{3.2}$$

and let G be a δ-pulse of F satisfying

$$G(x) \neq x \quad \text{and} \quad G'(x) > 0, \quad \forall x \in (0, 1).$$

By the definition, without loss of generality, we may assume that

$$G(x)\begin{cases} > F(x), & \forall x \in (a-\delta, a+\delta),\\ = F(x), & \forall x \notin (a-\delta, a+\delta). \end{cases} \tag{3.3}$$

In what follows we will prove that the δ-pulse G does not have k-th order C^1 iterative roots defined on I. For reduction to absurdity, we assume that G has a strictly increasing k-th order C^1 iterative root g. Then

$$g(x) = f(x), \quad \forall x \in [0, a-\delta], \tag{3.4}$$

by (3.3) and the uniqueness of C^1 iterative roots of G and F in the interval $[0, a-\delta]$, as mentioned in the second paragraph of Section 2. We then assert that

$$F^{-n}\circ F(a) > G^{-n}\circ F(a), \qquad G^{-1}\circ g\circ F(a) = f(a), \qquad G^{-n}\circ f(a) = f\circ F^{-n}(a) \tag{3.5}$$

for all $n \in \mathbb{N}$. In fact, it follows from (3.3) that

$$G^{-1}(x)\begin{cases} < F^{-1}(x), & \forall x \in (F(a-\delta), F(a+\delta)),\\ = F^{-1}(x), & \forall x \notin (F(a-\delta), F(a+\delta)). \end{cases} \tag{3.6}$$

Then one can prove by induction that

$$F^{-n}(x) = G^{-n}(x), \quad \forall n \in \mathbb{N},\ \forall x > F(a+\delta). \tag{3.7}$$

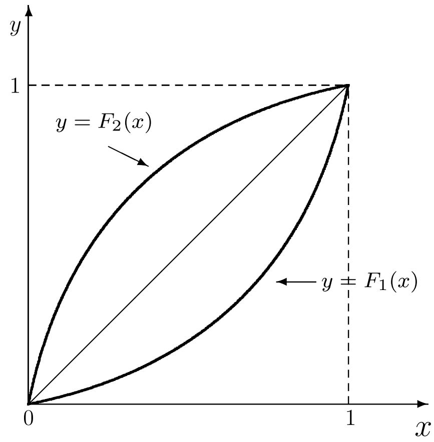

Fig. 3. F_1, F_2 satisfying (3.1).

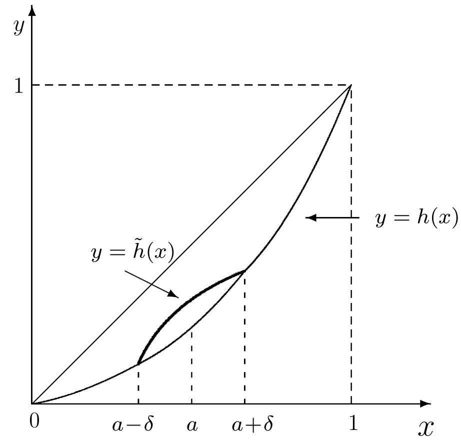

Fig. 4. $\tilde{h}$: a δ-pulse of h.

For the first formula of (3.5), noting that

$$F(a+\delta) = f^k(a+\delta) \leqslant f(a+\delta) < a-\delta = G^{-1} \circ G(a-\delta) = G^{-1} \circ F(a-\delta) < G^{-1} \circ F(a) < F^{-1} \circ F(a)$$

by (3.2), (3.3) and (3.6), we have

$$F^{-n} \circ F(a) = F^{-(n-1)} \circ F^{-1} \circ F(a) > F^{-(n-1)} \circ G^{-1} \circ F(a) = G^{-(n-1)} \circ G^{-1} \circ F(a) = G^{-n} \circ F(a)$$

for all $n \in \mathbb{N}$ by (3.6) and (3.7), which proves the first formula given in (3.5). For the second one, since

$$F(a) < F(a+\delta) < a-\delta$$

and

$$f \circ F(a) = F \circ f(a) < F \circ f(a+\delta) < F(a-\delta)$$

by (3.2), implying that $F(a) \in [0, a-\delta]$ and $f \circ F(a) \notin (F(a-\delta), F(a+\delta))$ respectively, we get

$$G^{-1} \circ g \circ F(a) = G^{-1} \circ f \circ F(a) = F^{-1} \circ f \circ F(a) = f(a)$$

by (3.4) and (3.6). This proves the second formula given in (3.5). By (3.2) again,

$$F(a+\delta) = f^{k-1} \circ f(a+\delta) < f^{k-1}(a-\delta) < f(a)$$

since $k \geqslant 2$, as indicated in the theorem, which implies that

$$F^{-n}(a) = f^{-1} \circ F^{-n} \circ f(a) = f^{-1} \circ G^{-n} \circ f(a)$$

for all $n \in \mathbb{N}$ by (3.7). This proves the third formula given in (3.5) and the assertion is proved.

Put $a_n := F^{-n}(a)$ for all $n \in \mathbb{N}$. Then, by (3.5),

$$\begin{aligned} g(a_n) &= g \circ F^{-n}(a) = g \circ F^{-(n+1)} \circ F(a) \\ &> g \circ G^{-(n+1)} \circ F(a) = G^{-n} \circ G^{-1} \circ g \circ F(a) = G^{-n} \circ f(a) \\ &= f \circ F^{-n}(a) = f(a_n). \end{aligned} \tag{3.8}$$

Since 1 is a stable fixed point of F^{-1}, we can see that the sequence (a_n) tends to 1. Thus f and g are not identical in any neighborhood of 1 by (3.8). This leads to a contradiction since, by the uniqueness of C^1 iterative roots and by (3.3), the C^1 iterative roots of F and G are identical near 1. Hence, G does not have k-th order C^1 iterative roots defined on I for any integers $k \geqslant 2$.

Choose a sequence (F_m) in the set of δ-pulses of F such that

$$\lim_{m\to\infty} \|F_m - F\|_r = 0. \tag{3.9}$$

Actually, choose a C^∞ bump function ϱ in I such that $\varrho(x) > 0$ for all $x \in (a-\delta, a+\delta)$ and $\varrho(x) = 0$ for all $x \notin (a-\delta, a+\delta)$. Then, for every $m \in \mathbb{N}$, the function F_m used in (3.9) can be defined by

$$F_m(x) := F(x) + \frac{1}{\gamma m}\varrho(x), \quad \forall x \in I,$$

where the constant $\gamma > 0$ is so large that $F_m(x) \neq x$ and $F_m'(x) > 0$ for all $x \in (0,1)$. One can see that none of those F_m's has a k-th order C^1 iterative root by our discussion before. This completes the proof. $\square$

4. Further remarks

Remark that the C^1 stability for strictly increasing functions F's defined on $[0,1)$ fixing 0 and satisfying $F(x) \to 1$ as $x \to 1$ is not contained in the above discussion. We guess that it is C^1 unstable but we are not able to give a proof yet.

The problem of iterative roots of self-mappings of higher dimensional space is more complicated. We also want to know whether it is stable or not in such a case. This is one of our further directions of investigations.

At the end we give an example to show the C^1 stability and C^1 unstability of iterative roots. Consider the mappings $\tilde{F}$ and F given by

$$\tilde{F}(x) := \frac{1}{2}x^2 + \frac{1}{3}x \quad \text{and} \quad F(x) := \frac{1}{27}x^4 + \frac{4}{27}x^3 + \frac{10}{27}x^2 + \frac{4}{9}x, \quad \forall x \in I.$$

It is easy to verify that $\tilde{F} \in \mathcal{H}_-^2(\frac{1}{3})$ and F satisfies (3.1), respectively. By our Theorem 2.1 the C^1 iterative root of $\tilde{F}$ is C^1 stable. Moreover, F has a second order C^1 iterative root defined on I by $f(x) := \frac{1}{3}x^2 + \frac{2}{3}x$. According to Theorem 3.1, it is C^1 unstable since we can always find a sequence (F_m) in the set of δ-pules of F such that $\lim_{m\to\infty} \|F_m - F\|_r = 0$ and each F_m does not have any C^1 iterative roots defined on I.

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