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The coexistence of quasi-periodic and blow-up solutions in a class of Hamiltonian systems

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ABSTRACT

The coexistence of quasi-periodic solutions and blow-up phenomena in a class of *higher* dimensional Duffing-type equations is proved in this paper. Moreover, we show that the initial point sets for both kinds of solutions are of infinite Lebesgue measure in the phase space. For the part of quasi-periodic solutions, the tool we used is the small twist theorem for higher dimensional cases.

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1. Introduction and results

The stability problem of the Hamiltonian systems can be traced back to the time of Newton. Since then a lot of results on this question have been obtained. But whether or not regular orbits are exceptional has been unknown until the establishment of the famous KAM theory in 1960s, which states that the measure of the initial point set for quasi-periodic solutions of a nearly-integrable Hamiltonian system is positive in the phase space. Recently, many nearly-integrable Hamiltonian systems have also been found to possess unstable solutions, see [2,3,15,22] and references therein. Thus it is interesting to study the coexistence of stable and unstable solutions for the Hamiltonian systems.

In this paper, we will study the equations:

$$x_l'' + x_l^{2n_l+1} + \frac{\partial}{\partial x_l} G(X,t) = 0, \quad l = 1, 2, \dots, m \tag{1.1}$$

where $n_l \in \mathbb{N}^+$, $X = (x_1, \dots, x_m) \subset \mathbb{R}^m$, x' denotes $\frac{dx}{dt}$, G is periodic on t and polynomial on X.

If $m = 1$, (1.1) becomes

$$x'' + x^{2n+1} + \sum_{i=0}^{k} x^i p_i(t) = 0, \quad p_i(t+1) = p_i(t), \tag{1.2}$$

which is actually a planar Duffing-type equation. The Lagrangian stability study of the Duffing equations was initiated by Littlewood [10,11] in 1960s. In [18], Moser commended that even for the equation

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$$x'' + x^{2n+1} + ax^3 + bx = p(t), \quad p(t+1) = p(t) \tag{1.3}$$

with $a, b > 0$ two constants and p small enough, it is very delicate to decide whether all solutions are bounded.

The first result on the boundedness of solutions of (1.2) was established by Morris [16] for the equation

$$x'' + x^3 = p(t)$$

with $p(t+1) = p(t)$ piecewise continuous. Then Dieckerhoff and Zehnder [4] extended Morris' result to the polynomial system (1.2) with $k < 2n+1$ and $p_j(t)$ $(j = 0, 1, \ldots, \ell)$ smooth 1-periodic functions.

For more results along this line, see [4,7,12,13,16,20,23–25] and references therein.

The idea for proving the boundedness of solutions for a planar Duffing equation is as follows. By means of transformation theory the original system outside of a large disc $D = \{(x, x') \in \mathbb{R}^2 \colon x^2 + (x')^2 \leqslant r^2\}$ in (x, x')-plane is transformed into a nearly integrable Hamiltonian system. The Poincaré map of the transformed system is closed to a so-called twist map in $\mathbb{R}^2 \backslash D$. Then Moser's twist theorem [17] guarantees the existence of arbitrarily large invariant curves diffeomorphic to circles and surrounding the origin in the (x, x')-plane. Every such curve is the base of a time-periodic and flow-invariant cylinder in the extended phase space $(x, x', t) \in \mathbb{R}^2 \times \mathbb{R}$, which confines the solutions in the interior and which leads to a bound of these solutions.

On the other hand, many results on the existence of unbounded solutions have been established for planar superlinear Duffing equations, see [8,9,11,14,21,27].

In the study of planar semilinear Duffing equations, some results on the coexistence of periodic and unbounded solutions have been obtained, see [1,5] and references therein.

A natural question then arises on the stability and instability of the coupled Duffing equations (1.1). As the authors know, the only two results on the existence of quasi-periodic solutions for this system were obtained by [6,26] for Eqs. (1.1) under the assumptions that $n_l \in \mathbb{N}^+$, G is periodic on t and polynomial on $X = (x_1, \ldots, x_m)$ with some suitable restrictions on its degrees. In particular, the degree of each monomial on x_l in G is assumed to be smaller than $2n_l + 2$.

In this paper, we will consider the case for which the assumption on degrees of G in [6,26] stated as above is not satisfied. More precisely, we consider (1.1) with the following assumptions:

(**G**) G is a monomial of X of the form

$$G(X, t) = x_1^{i_1} \cdots x_m^{i_m} p(t), \quad p(t) \in \mathbb{C}^1\big(S^1\big)$$

with i_1 odd and satisfying $0 < i_1 < \frac{n_1}{2} - 2$, $n_1 > 6$.

Remark 1.1. The condition (**G**) imposes a strong restriction on i_1. However, the degree of G on x_l, $l \geqslant 2$ can be arbitrarily large. For example, consider the case $m = 2$, $n_1 = 100$ and $n_2 = 100$, $G(X, t) = x_1^{47} \cdot x_2^{10\,000} \cdot sin\, t$, which obviously satisfies the condition (**G**). The degree of G on x_2 is 10 000, which is much larger than n_2. Thus in this case the conditions in [6,26] are not satisfied.

We will prove the following result:

Theorem 1. *Assume $G(X, t)$ satisfy the condition* (**G**)*. Then for any given*

$$D_0 = [\underline{\omega}, \overline{\omega}] \times \cdots \times [\underline{\omega}, \overline{\omega}] \subset \mathbb{R}^m,$$

there exists (*large*) $A^* > 0$ *such that for* $A > A^*$ *and* $(\omega_1, \ldots, \omega_m) \in D_0$ *satisfying*

$$\big|e^{i\langle K, \Omega\rangle} - 1\big| \geqslant \frac{\alpha}{|K|^\tau}, \quad \forall 0 \neq K \in \mathbb{Z}^m,\ n \in \mathbb{N} \tag{1.4}$$

where $\tau > m$, $\alpha = \gamma \cdot A^{-2/(n_1-1)} > 0$ *with* $\gamma > 0$ *a constant,* $\Omega = (A\omega_1, \omega_2, \ldots, \omega_m)$*, there is an analytic vector function* $f(\theta_1, \ldots, \theta_m, t)$ *periodic in every variable with period 1 such that for any* $(\theta_1, \ldots, \theta_m, t_0) = (\Theta, t_0) \in \mathbb{T}^{m+1}$, $X(t) := f(\Theta + \Omega t, t_0 + t)$ *is a quasi-periodic solution of* (1.1)*. Moreover for the set* $\widetilde{D}_0$ *of* $(\omega_1, \ldots, \omega_m)$ *satisfying* (1.4)*, we have*

$$meas(D_0 \setminus \widetilde{D}_0) = O(\gamma).$$

Remark 1.2. (1.4) is a variant of the Diophantine condition. It is well known that for every bounded region D in $\mathbb{R}^m$, the measure of the subset of it in which every point $\omega = (\omega_1, \ldots, \omega_m)$ satisfies the classical Diophantine condition

$$\big|e^{i\langle K, \omega\rangle} - 1\big| \geqslant \frac{\gamma}{|K|^\tau}, \quad \forall 0 \neq K \in \mathbb{Z}^m,\ n \in \mathbb{N} \tag{1.5}$$

is $(1 - C\gamma)\,\mathrm{mes}(D)$, where the constant $C > 0$ is independent of D. Similarly, it holds that the subset of D_0 in which every point satisfies (1.4) has a measure $(1 - C'\gamma)\,\mathrm{mes}(D_0)$ with $C' > 0$ independent of A, see Section 4. Thus for sufficiently small γ, we have a positive measure subset of D_0 such that every point in it satisfies (1.4). One can easily see that the union of all quasi-periodic solutions obtained in Theorem 1 is of infinite measure in the phase space $\mathbb{R}^{2m} \times S^1$.

Remark 1.3. Theorem 1 can be easily extended to the case in which G is of the form: $G = x^{i_1} \cdot G_1(x_2, \dots, x_n, t)$ with G_1 periodic on t and polynomial on $x_2, \dots, x_n$.

Theorem 1 states the stable aspect of (1.1). On the other hand, we will show the unstable aspect of it by proving that with some further conditions besides those in Theorem 1, (1.1) possesses also infinitely many blow-up solutions.

Theorem 2. *Consider the system* (1.1)*, where $G(X,t) = x_1^{i_1} \cdots x_m^{i_m} p(t)$ satisfies the assumptions that $p \in \mathbb{C}(S^1)$ and $p(t_0) < 0$ for some point $t_0 \in [0,1]$. Suppose $\sum_{l=1}^m i_l > \max_{1 \leqslant l \leqslant m}(2n_l + 2)$. Then we can find a constant $c_1 > 0$ which depends only on $p(t)$ and m such that if $|\frac{i_j}{i_k} - 1| < c_1$, $j, k = 1, \dots, m$, then there is an open set with an infinite measure in the phase space $\mathbb{R}^{2m} \times S^1$ such that each solution of the system* (1.1) *starting from this set will blow up.*

Remark 1.4. In Theorem 2, if we also suppose that i_1 is an odd number satisfying the inequality $i_1 < \frac{n_1}{2} - 2$, $n_1 > 6$ and $p(t) \in \mathbb{C}^1$, then the function $G(X,t)$ meets the assumptions of Theorem 1 but fails to satisfy the conditions in [26] or [6].

Remark 1.5. The invariant tori we obtained lie in the following strip region of the action-variable vector (see Section 3),

$$(\lambda_1, \dots, \lambda_m) \in [A, \infty] \times \underbrace{\left[\underline{\rho}^{(0)}, \overline{\rho}^{(0)}\right] \times \cdots \times \left[\underline{\rho}^{(0)}, \overline{\rho}^{(0)}\right]}_{m-1} \tag{1.6}$$

with $A \gg \overline{\rho}^{(0)} \gg 1$. That is, the scale of the first action-variable is much larger than the others. On the other hand, the blow-up solutions we obtained lie in the following strip region of the action-variable vector:

$$(\lambda_1, \dots, \lambda_m) \in \underbrace{\left[\underline{\rho}^{(0)}, \overline{\rho}^{(0)}\right] \times \cdots \times \left[\underline{\rho}^{(0)}, \overline{\rho}^{(0)}\right]}_{m}, \tag{1.7}$$

that is, all the action-variables are of the same scale.

It is worth pointing out that it is not clear whether or not the coexistence of stable and unstable solutions is generic in the higher dimensional Duffing equations.

The remaining part of the paper is organized as follows. We introduce the action-angle variables in Section 2. In Section 3, we construct a canonical transformation to transform the original system to a nearly integrable one. Theorem 1 will be proved in Section 4 by a variant of the small Moser's twist theorem for higher dimensional cases in [19]. The proof of Theorem 2 is given in Section 5.

2. Action-angle variables

If $G = 0$, (1.1) is of the form

$$x_l'' + x_l^{2n_l+1} = 0, \quad l = 1, \dots, m,$$

which is m uncoupled one degree of freedom Hamiltonian systems:

$$x_l' = \frac{\partial}{\partial y_l} h_l(x_l, y_l), \qquad y_l' = -\frac{\partial}{\partial x_l} h_l(x_l, y_l) \tag{2.1}$$

with $h_l(x_l, y_l) = \frac{1}{2} y_l^2 + \frac{1}{2(n_l+1)} x_l^{2(n_l+1)}$, $l = 1, 2, \dots, m$.

With the notation $(X, Y) = (x_1, \dots, x_m, y_1, \dots, y_m)$ and $h(X, Y) = \sum_{l=1}^m h_l(x_l, y_l)$, we have that (1.1) is equivalent to the Hamiltonian system

$$x_l' = \frac{\partial H}{\partial y_l}, \qquad y_l' = -\frac{\partial H}{\partial x_l}, \quad l = 1, 2, \dots, m, \tag{2.2}$$

where the Hamiltonian function is

$$H(X, Y, t) = h(X, Y) + G(X, t). \tag{2.3}$$

Denote by $(C_l(t), S_l(t))$ the periodic solution of the Hamiltonian system (2.1) satisfying $(C_l(0), S_l(0)) = (1, 0)$, $l = 1, \dots, m$. Then $h_l(C_l(t), S_l(t)) \equiv \frac{1}{2(n_l+1)}$. Let $T_l > 0$ be the minimal period of $(C_l(t), S_l(t))$, then C_l, S_l, $l = 1, 2, \dots, m$ satisfy the following propositions:

(1) $C_l(t) = C_l(t + T_l)$, $S_l(t) = S_l(t + T_l)$;
(2) $C_l'(t) = S_l(t)$ and $S_l'(t) = -C_l(t)^{2n_l+1}$;

(3) $(n_l+1)S_l(t)^2 + C_l(t)^{2(n_l+1)} = 1$;
(4) $C_l(-t) = C_l(t)$ and $S_l(-t) = -S_l(t)$;
(5) $C_l(-t+1/2) = -C_l(t)$ and $S_l(-t+1/2) = -S_l(t)$.

Define the action-angle variables by the symplectic transformation

$$x_l = c_l^{\alpha_l}\lambda_l^{\alpha_l} C_l(\theta_l T_l), \qquad y_l = c_l^{\beta_l}\lambda_l^{\beta_l} S_l(\theta_l T_l),$$

where

$$\alpha_l = \frac{1}{n_l+2}, \qquad \beta_l = 1-\alpha_l, \qquad c_l = \frac{1}{\alpha_l T_l}, \quad l = 1,2,\dots,m.$$

Then (2.2) is transformed into another Hamiltonian system

$$\begin{cases} \theta_l' = 2d_l\beta_l\lambda_l^{2\beta_l-1} + \dfrac{\partial R_1}{\partial \lambda_l}, \\ \lambda_l' = -\dfrac{\partial R_1}{\partial \theta_l}, \end{cases} \quad l = 1,2,\dots,m \tag{2.4}$$

with the Hamiltonian function

$$H_1(\Lambda,\Theta,t) = \sum_{l=1}^m d_l\lambda_l^{2\beta_l} + R_1(\Lambda,\Theta,t), \tag{2.5}$$

where $(\Lambda,\Theta) = (\lambda_1,\dots,\lambda_m,\theta_1,\dots,\theta_m)$, $d_l = \frac{c_l^{2\beta_l}}{2(n_l+1)}$, $l = 1,\dots,m$ and

$$R_1(\Lambda,\Theta,t) = (c_1\lambda_1)^{i_1\alpha_1} C_1^{i_1}(\theta_1 T_1)\cdots(c_m\lambda_m)^{i_m\alpha_m} C_m^{i_m}(\theta_m T_m)p(t) \tag{2.6}$$
$$\equiv R_1^*(\Lambda^*,\Theta^*,t)\lambda_1^{i_1\alpha_1} C_1^{i_1}(\theta_1 T_1), \tag{2.7}$$

where $\Lambda^* = (\lambda_2,\dots,\lambda_m)$, $\Theta^* = (\theta_2,\dots,\theta_m)$.

Remark 2.1. Remind that i_1 is assumed to be odd in (**G**). Thus with the symmetric properties of $C_1(t)$ stated in (4) and (5), we have

$$\int_{\mathbb{T}} R_1(\Lambda,\Theta,t)\,d\theta_1 = R_1^*(\Lambda^*,\Theta^*,t)\lambda_1^{i_1\alpha_1}\int_{\mathbb{T}} C_1^{i_1}(\theta_1 T_1)\,d\theta_1 = 0. \tag{2.8}$$

This fact is crucial in the proof of Theorem 1.

3. More canonical transformations

In the following, for any m-dimensional vector $Z = (z_1, z_2, \dots, z_m)$, we denote $(z_2,\dots,z_m)$ by Z^*, e.g., $f(Z) \equiv f(z_1, Z^*)$. Let D_{m-1} be any domain in $\mathbb{R}^{m-1}$. Next we introduce a space of functions $\mathcal{F}_1(r)$.

Definition 3.1. For $r \in \mathbb{R}$, we call $f(\Lambda,\Theta,t) \in \mathcal{F}_1(r)$ if $f(\cdot,t) \in C^\infty([1,\infty] \times D_{m-1} \times \mathbb{T}^m)$ and for all nonnegative integer vectors J, L and nonnegative integer j, it holds that

$$\sup_{(\lambda_1,\Lambda^*,\Theta,t)\in[1,\infty]\times D_{m-1}\times\mathbb{T}^{m+1}} \lambda_1^{j-r}\left|(D_{\lambda_1})^j (D_{\Lambda^*})^{J^*}(D_\Theta)^L f\big(\lambda_1,\Lambda^*,\Theta,t\big)\right| < \infty.$$

We also call a vector function

$$G(\Lambda,\Theta,t) = \big(g_1(\Lambda,\Theta,t), g_2(\Lambda,\Theta,t),\dots,g_m(\Lambda,\Theta,t)\big) \in \mathcal{F}_1(r),$$

if $g_l(\Lambda,\Theta,t) \in \mathcal{F}_1(r)$, $l = 1,2,\dots,m$.

From the definition of $\mathcal{F}_1(r)$, we can easily verify the following properties:

Lemma 3.1.

(1) *If* $r_1 < r_2$, *then* $\mathcal{F}_1(r_1) \subset \mathcal{F}_1(r_2)$.
(2) *If* $f \in \mathcal{F}_1(r)$, *then* $(D_{\Lambda^*})^{J^*}(D_\Theta)^L f \in \mathcal{F}_1(r)$.

(3) *If* $f \in \mathcal{F}_1(r_1)$, $g \in \mathcal{F}_1(r_2)$, *then* $f \cdot g \in \mathcal{F}_1(r_1 + r_2)$.
(4) *If* $f \in \mathcal{F}_1(r)$ *satisfies* $|f| \geqslant c\lambda_1^r$, *then* $\frac{1}{f} \in \mathcal{F}_1(-r)$.

Without loss of generality, consider the Hamiltonian

$$H_1(\Lambda, \Theta, t) = d_1\lambda_1^{2\beta_1} + \sum_{l=2}^{m} d_l\lambda_l^{2\beta_l} + R_1\bigl(\lambda_1, \Lambda^*, \Theta, t\bigr) \tag{3.1}$$

defined in $D_m^{(0)} \times \mathbb{T}^{m+1}$ with

$$D_m^{(0)} := [1, \infty] \times D_{m-1}^{(0)}, \tag{3.2}$$

where $D_{m-1}^{(0)} = \underbrace{[\underline{\rho}^{(0)}, \overline{\rho}^{(0)}] \times \cdots \times [\underline{\rho}^{(0)}, \overline{\rho}^{(0)}]}_{m-1}$ is a bounded domain. Thus, from (2.6) we have $R_1 \in \mathcal{F}_1(b_1)$ with $b_1 = i_1\alpha_1$.

Next we show that there exists a canonical transformation with which the term $R_1 \in \mathcal{F}_1(b_1)$ in (3.1) is transformed into another one in $\mathcal{F}_1(b_2)$ with $b_2 < 0$. More precisely, we have

Proposition 3.1. *For the Hamiltonian* (3.1) *in* $D_m^{(0)}$, *there exist* $A^{(1)} \gg 1$ *and a canonical diffeomorphism* Φ *defined in* $D_m^{(1)} = [A^{(1)}, \infty) \times D_{m-1}^{(1)} \subset D_m^{(0)}$ *depending periodically on t of the form*:

$$\Phi\colon \quad \begin{cases} \Lambda = U + \Phi_1(U, V, t), \\ \Theta = V + \Phi_2(U, V, t) \end{cases}$$

with $\Phi_1, \Phi_2 \in \mathcal{F}_1(b_1 - 2\beta_1 + 1)$ *such that for* $u_1 > A^{(1)}$, $\Phi(D_m^{(1)}) \subset D_m^{(0)}$ *and the Hamiltonian is transformed into* $\Phi^*(X_{H_1}) = X_{H_2}$ *with*

$$H_2 = d_1u_1^{2\beta_1} + \sum_{l=2}^{m} d_lu_l^{2\beta_l} + R_2\bigl(u_1, U^*, V, t\bigr), \tag{3.3}$$

where $R_2 \in \mathcal{F}_1(b_2)$ *with* $b_2 = 2b_1 - 2\beta_1 + 1 < 0$.

Proof. We will construct the canonical transformation Φ by means of the generating function:

$$\Phi\colon \quad \begin{cases} \Lambda = U + \frac{\partial}{\partial \Theta} S(U, \Theta, t), \\ V = \Theta + \frac{\partial}{\partial U} S(U, \Theta, t). \end{cases}$$

The transformed Hamiltonian function expressed in the variables (U, Θ) instead of (U, V) is of the form:

$$\hat{H}_1(U, \Theta, t) = d_1u_1^{2\beta_1} + \sum_{l=2}^{m} d_lu_l^{2\beta_l} + [R_1]_1\bigl(U, \Theta^*, t\bigr) + \hat{R}_2 \tag{3.4}$$

with

$$[R_1]_1\bigl(U, \Theta^*, t\bigr) = \int_{\mathbb{T}} R_1(U, \Theta, t)\, d\theta_1,$$

and $\hat{R}_2 = \hat{R}_{21} + \hat{R}_{22} + \cdots + \hat{R}_{25}$, where

$$\hat{R}_{21} = 2d_1\beta_1u_1^{2\beta_1 - 1}\frac{\partial S}{\partial \theta_1} + R_1(U, \Theta, t) - [R_1]_1, \tag{3.5}$$

$$\hat{R}_{22} = d_1\bigl(\lambda_1^{2\beta_1} - u_1^{2\beta_1}\bigr) - d_1 2\beta_1u_1^{2\beta_1 - 1}\frac{\partial S}{\partial \theta_1}, \tag{3.6}$$

$$\hat{R}_{23} = \sum_{l=2}^{m} d_l\lambda_l^{2\beta_l} - \sum_{l=2}^{m} d_lu_l^{2\beta_l}, \tag{3.7}$$

$$\hat{R}_{24} = R_1\left(U + \frac{\partial S}{\partial \Theta}, \Theta, t\right) - R_1(U, \Theta, t), \tag{3.8}$$

$$\hat{R}_{25} = \frac{\partial S}{\partial t}. \tag{3.9}$$

It follows from (2.8) that $[R_1]_1 = 0$.

We define S by the equation

$$\hat{R}_{21} = 0, \tag{3.10}$$

i.e.

$$\frac{\partial S}{\partial \theta_1} = -\frac{u_1^{1-2\beta_1}}{2d_1\beta_1} R_1(U, \Theta, t).$$

Thus S is defined by

$$S = -\frac{1}{2d_1\beta_1}\int_0^{\theta_1} u_1^{1-2\beta_1} R_1(U, \Theta, t)\, d\theta_1. \tag{3.11}$$

Note that $R_1 \in \mathcal{F}_1(b_1)$. Thus we can prove that $\exists A^{(1)} \gg 1$, s.t.

$$\sup_{(u_1, U^*, \Theta, t)\in[A^{(1)},\infty]\times D_{m-1}^{(0)}\times\mathbb{T}^{m+1}} \left|u_1^{2\beta_1-1-b_1+j}(D_{u_1})^j (D_{U^*})^{J^*}(D_\Theta)^L\big(S\big(u_1, U^*, \Theta, t\big)\big)\right| < \infty,$$

i.e. $S \in \mathcal{F}_1(b_1 - 2\beta_1 + 1)$, where $b_1 - 2\beta_1 + 1 = \frac{i_1}{n_1+2} - \frac{n_1}{n_1+2} < 0$.

Let $\Phi_1(U, V, t), \Phi_2(U, V, t)$ be determined implicitly by

$$\Phi_2 + \frac{\partial}{\partial U} S(U, V + \Phi_2, t) = 0, \qquad \Phi_1(U, V, t) = \frac{\partial}{\partial \Theta} S(U, V + \Phi_2, t).$$

Similar to [4], we have $\Phi_1(U, V, t), \Phi_2(U, V, t) \in \mathcal{F}_1(b_1 - 2\beta_1 + 1)$ with $b_1 - 2\beta_1 + 1 < 0$. Thus shrinking the domain $D_{m-1}^{(0)}$ a little, we can easily find $D_{m-1}^{(1)} := [\underline{\rho}^{(1)}, \overline{\rho}^{(1)}] \times \cdots \times [\underline{\rho}^{(1)}, \overline{\rho}^{(1)}] \subset D_{m-1}^{(0)}$ such that

$$\Phi\big(D_m^{(1)}\big) = \Phi\big(\big[A^{(1)}, \infty\big] \times D_{m-1}^{(1)}\big) \subset D_m^{(0)}.$$

Moreover the following functions, expressed in U, Θ, t, possess the properties:

$$\begin{aligned}
&\hat{R}_{22} \in \mathcal{F}_1(2b_1 - 2\beta_1),\\
&\hat{R}_{23} \in \mathcal{F}_1(b_1 - 2\beta_1 + 1),\\
&\hat{R}_{24} \in \mathcal{F}_1(2b_1 - 2\beta_1 + 1),\\
&\hat{R}_{25} \in \mathcal{F}_1(b_1 - 2\beta_1 + 1).
\end{aligned}$$

Thus we have $\hat{R}_2 \in \mathcal{F}_1(2b_1 - 2\beta_1 + 1)$. And by definition, $R_2(U, V, t) = \hat{R}_2(U, V + \Phi_2, t)$ which implies that

$$R_2 \in \mathcal{F}_1(2b_1 - 2\beta_1 + 1).$$

The proof is completed by setting $H_2(U, V, t) = \hat{H}_1(U, \Theta(U, V, t), t)$. □

4. The proof of Theorem 1

In this section, we will prove the existence of quasi-periodic solutions of (1.1) via the following theorem [6], which is a variant of the small twist theorem for higher dimensional cases in [19].

For any two vectors $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$, we define $x * y = (x_1 y_1, \dots, x_n y_n)$.

Theorem 3. *Let $\gamma > 0$, $\tau > n + 1$, $a = (a_1, a_2, \dots, a_n)^T$ be a constant vector with $0 < a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n \leqslant 1$ and $b = (b_1, b_2, \dots, b_n)^T$ be any constant vector. Consider a family of exact symplectic mappings $S_a : (p, q) \to (\hat{p}, \hat{q})$ defined in phase space $D \times \mathbb{T}^n$ by*

$$\begin{aligned}
&\hat{p} = p - \partial_2 h(\hat{p}, q),\\
&\hat{q} = q + \widetilde{\omega}(\hat{p}) + \partial_1 h(\hat{p}, q),
\end{aligned} \tag{4.1}$$

where D is a bounded open set in $\mathbb{R}^n$, h possesses the same regularity as in Theorem 4 [6] and $\widetilde{\omega}(\hat{p})$ is of the form: $\widetilde{\omega}(\hat{p}) = a^T * \omega(\hat{p}) + b$, where $\omega(\hat{p})$ is analytic and satisfies the non-degenerate condition. Then there exists a constant $\delta_0 > 0$ such that if $\|h\|_{D\times\mathbb{T}^n} \leqslant \delta_0\gamma^2 a_1^2$, there is a Cantor set $\widetilde{D}_{a,\gamma} \subset D$ such that for each $\nu_0 \in \widetilde{D}_{a,\gamma}$, $\omega(\nu_0)$ is in the set

$$\widetilde{\Omega}_{a,\gamma} = \left\{\omega\colon \left|e^{i\langle k,\widetilde{\omega}(\omega)\rangle} - 1\right| \geqslant \frac{a_1\gamma}{|k|^\tau},\ \textit{for all } 0 \neq k \in \mathbb{Z}^n\right\},$$

and S_a has an invariant torus diffeomorphic to $\{\widetilde{\omega}(\omega(\nu_0))\} \times T^n$. Moreover, the measure of $\widetilde{D}_{a,\gamma}$ satisfies

$$meas(\widetilde{D}_{a,\gamma}) > \bigl(1 - O(\gamma)\bigr)meas(D). \tag{4.2}$$

Remark 4.1. The proof of this theorem can be found in [6].

Consider the Hamiltonian system given by (3.3):

$$\begin{cases} v_l' = 2d_l\beta_l u_l^{2\beta_l-1} + \partial_{u_l}R_2(U,V,t), \\ u_l' = -\partial_{v_l}R_2(U,V,t), \end{cases} \quad l = 1,\dots,m.$$

Similar to Lemma 4 of [4], the time 1 map $\mathcal{P}^1$ of the flow $\mathcal{P}^t$ of the vector field X_{H_2} defined in

$$D_m^{(1)} = \bigl[A^{(1)},\infty\bigr) \times D_{m-1}^{(1)} \times \mathbb{T}^m$$

is of the form

$$\mathcal{P}^1\colon \quad \begin{cases} V_1 = V + r(U) + f(U,V), \\ U_1 = U + g(U,V) \end{cases}$$

with

$$r(U) = \bigl(r_1(u_1),\dots,r_m(u_m)\bigr),$$
$$f = (f_1,\dots,f_m), \qquad g = (g_1,\dots,g_m),$$

where

$$r_l(u_l) = 2d_l\beta_l u_l^{2\beta_l-1}, \tag{4.3}$$

$$f_l(U,V) = \int_0^1 \partial_{u_l}R_2\bigl(U(t),V(t),t\bigr)\,dt + \int_0^1 2d_l\beta_l u_l(t)^{2\beta_l-1}\,dt - r_l(u_l), \tag{4.4}$$

$$g_l(U,V) = -\int_0^1 \partial_{v_l}R_2\bigl(U(t),V(t),t\bigr)\,dt \tag{4.5}$$

with $l = 1,\dots,m$.

Moreover for every pair (J,L):

$$\bigl|(D_U)^J(D_V)^L f_l(U,V)\bigr|, \bigl|(D_U)^J(D_V)^L g_l(U,V)\bigr| < c \cdot u_1^{b_2}, \quad l = 1,\dots,m$$

with c some positive constant in $D_m^{(1)} \times \mathbb{T}^m$.

Given a sufficiently large u_1^0, let $u_1 = u_1^0 + \mu_1$, $u_2 = \mu_2,\dots,u_m = \mu_m$, $\mu_1 \in [\underline{\omega},\overline{\omega}]$, some bounded interval determined later, then

$$u_1^{2\beta_1-1} = \bigl(u_1^0\bigr)^{2\beta_1-1} + \bigl(u_1^0\bigr)^{2\beta_1-2}\mu + O\bigl(\bigl(u_1^0\bigr)^{2\beta_1-3}\bigr).$$

For simplicity, we still denote by (U,V) the coordinates of the transformed symplectic map, which is of the form

$$\widetilde{\mathcal{P}}^1\colon \quad \begin{cases} V_1 = V + \widetilde{r}(U) + \widetilde{f}(U,V), \\ U_1 = U + g(U,V) \end{cases}$$

with

$$\widetilde{r}(U) = \bigl(2d_1\beta_1\bigl(u_1^0\bigr)^{2\beta_1-1} + 2d_1\beta_1\bigl(u_1^0\bigr)^{2\beta_1-2}\mu_1, 2d_2\beta_2\mu_2^{2\beta_2-1},\dots,2d_m\beta_m\mu_m^{2\beta_m-1}\bigr),$$
$$f = \bigl(f_1 + O\bigl(\bigl(u_1^0\bigr)^{2\beta_1-3}\bigr),\dots,f_m\bigr), \qquad g = (g_1,\dots,g_m),$$

where

$$\left|(D_U)^J(D_V)^L f_l(U,V)\right|, \left|(D_U)^J(D_V)^L g_l(U,V)\right| < c\cdot\left(u_1^0\right)^{b_2}, \quad l=1,\dots,m, \tag{4.6}$$

in $D_m^{(2)}\times\mathbb{T}^m$ with $D_m^{(2)}=[\underline{\omega},\overline{\omega}]\times D_{m-1}^{(1)}$.

Denote $\widetilde{r}(U)=a*\omega^T(U)+b$ with

$$\omega(U)=\left(\mu_1, 2d_2\beta_2\mu_2^{2\beta_2-1},\dots,2d_m\beta_m\mu_m^{2\beta_m-1}\right),$$
$$a=\left(2d_1\beta_1\left(u_1^0\right)^{2\beta_1-2},1,\dots,1\right), \qquad b=\left(2d_1\beta_1\left(u_1^0\right)^{2\beta_1-1},0,\dots,0\right).$$

Note that $(u_1^0)^{2\beta_1-2}=o(1)$ since $u_1^0\gg 1$ and $2\beta_1-2<0$. Thus the twists for all the action-variables of U are in different scales.

Then the map $\widetilde{\mathcal{P}}^1$ is, with its derivatives, closed to a generalized small twist map. Moreover it is an exact symplectic map. On the other hand, since the twists of action-variables are not of the same scale in our case, which are defined by the vector a as above, we cannot use the small twist theorem for higher dimensional cases in [19] directly. Instead, we will apply Theorem 3 to our case.

Note that $|b_2|>|4\beta_1-4|$ provided $i_1<\frac{n_1}{2}-2$, thus $(u_1^0)^{b_2}<((u_1^0)^{2\beta_1-2})^2$. Combining this with (4.6), we have that the map $\widetilde{\mathcal{P}}^1$ meets the small assumption on the perturbation in Theorem 3. It follows that if μ_1^0 is sufficiently large, then there is an embedding $\phi:\mathbb{T}^m\to D_m^{(2)}\times\mathbb{T}$ of an m-torus, which is invariant under the map $\widetilde{\mathcal{P}}^1$. Moreover, $\widetilde{\mathcal{P}}^1\circ\phi(S)=\phi(S+\Omega)$ with

$$\Omega=\left(2d_1\beta_1\left(u_1^0\right)^{2\beta_1-1}+2d_1\beta_1\left(u_1^0\right)^{2\beta_1-2}\omega_1^*,\omega_2^*,\dots,\omega_m^*\right),$$

where

$$\omega^*=\left(\omega_1^*,\omega_2^*,\dots,\omega_m^*\right)\in[\underline{\omega},\overline{\omega}]\times\cdots\times[\underline{\omega},\overline{\omega}]$$

lies in the set $\widetilde{\Omega}_{a,\gamma}$ in Theorem 3 and $[\underline{\omega},\overline{\omega}]\subset[2d_l\beta_l(\underline{\rho}^{(1)})^{2\beta_l-1},2d_l\beta_l(\overline{\rho}^{(1)})^{2\beta_l-1}]$, $l=2,\dots,m$. By setting $A=2d_1\beta_1(u_1^0)^{2\beta_1-2}$, we have that Ω satisfies the Diophantine condition ($\overline{1}.4$).

Thus by Theorem 3, the solutions of the Hamiltonian equation starting at time $t=0$ on this invariant torus determine a 1-periodic "hypercylinder" in the set $\{(U,V,t)\mid(U,V,t)\in D_m^{(2)}\times T^m\times\mathbb{R}\}$. Since the Hamiltonian vector field X_{H_2} is time periodic, the phase space is $D_m^{(2)}\times\mathbb{T}^{m+1}$. Let Ψ^t with $\Psi^0=\mathrm{Id}$ be the flow of the time-independent vector field $(X_{H_2},1)$ on $D_m^{(2)}\times\mathbb{T}^{m+1}$ and define the embedded torus $\psi:\mathbb{T}^{m+1}\to D_m^{(2)}\times\mathbb{T}^{m+1}$ by setting

$$\psi(S,\tau)=\left(\phi(S),\tau\right).$$

In view of the rigid rotation, we have

$$\psi(\dots,s_l+1,\dots,\tau)=\psi(\dots,s_l,\dots,\tau+1)=\psi(S,\tau).$$

Moreover $\Psi^t\circ\psi(S,\tau)=\psi(S+\Omega t,\tau+t)$. So the torus $\psi(\mathbb{T}^{m+1})$ is quasi-periodic with the frequencies $(\Omega,1)$.

5. Blow up

We have already obtained Theorem 1 about the existence of infinitely many invariant tori in the region (1.6). It means that in this region the orbits are stable in the sense of possibility. But there still remains a large region in the phase space where it is not clear whether or not the orbits are also stable. In this section, we will prove the unstable aspect of (1.1) described in Theorem 2.

We first consider the special system:

$$x_l''+x_l^{2n+1}+\frac{i}{x_l}x_1^i\cdots x_m^i p(t)=0, \quad p\in\mathbb{C}\left(S^1\right),\ l=1,\dots,m \tag{5.1}$$

where i, m, n are positive integers satisfying the inequality $\frac{2n+2}{m}<i$. Moreover, we assume $p(t_0)<0$ for some point $t_0\in[0,1]$. Under these assumptions, we have the following result:

Theorem 4. *There is an open set with an infinite measure in the phase space $\mathbb{R}^{2m}\times S^1$ of the system* (5.1) *such that each solution of the system starting from this set will blow up.*

Next we consider a special situation of (5.1), which is helpful for us to understand the general case.

Lemma 5.1. *Under the assumptions of Theorem* 3*, each solution of* (5.1) *with an initial condition satisfying $x_1(t_0)=\cdots=x_m(t_0)\gg 1$, $x_1'(t_0)=\cdots=x_m'(t_0)\gg 1$ will blow up.*

Proof. Consider the auxiliary equation:

$$x'' + x^{2n+1} + ix^{mi-1}p(t) = 0. \tag{5.2}$$

The relation between (5.1) and (5.2) is that if $x(t)$ is a solution of (5.2), then $(x(t), x(t), \dots, x(t))$ is a solution of (5.1); conversely, if $X(t) = (x_1(t), \dots, x_m(t))$ is a solution of (5.1) satisfying the conditions of Lemma 5.1, then $x_1(t) = \dots = x_m(t)$ and $x = x_1(t)$ is a solution of (5.2). Thus the proof of Lemma 5.1 is reduce to prove the existence of blow-up solutions for (5.2).

Since $p(t_0) < 0$ and p is continuous, there exist $\epsilon_0 > 0$ and $t_0 < t_1 < 1$ such that $p(t) < -2\epsilon_0$ for $t \in [t_0, t_1]$. From the condition $\frac{2n+2}{m} < i$, we have $mi - 1 > 2n + 1$. Thus it follows from (5.2) that for $x(t) \gg 1$ for $t \in [t_0, t_1]$ with some $t_0 < t_1 < 1$, it holds that

$$x'' = -x^{2n+1} - ix^{mi-1}p(t) > 2i\epsilon_0 x^{mi-1} - x^{2n+1} > i\epsilon_0 x^{mi-1}. \tag{5.3}$$

Consider the equation

$$x'' = i\epsilon_0 x^{mi-1}. \tag{5.4}$$

It is easy to prove that for any initial condition satisfying $x(t_0), x'(t_0) \gg 1$, the corresponding solution $x(t)$ will blow up on the interval $[t_0, t_2)$ with $t_2 = \frac{t_0+t_1}{2}$.

Comparing the solutions of (5.3) and (5.4), we have that each solution of (5.2) with the initial condition $x(t_0), x'(t_0) \gg 1$ will blow up on the interval $[t_0, t_2)$. $\square$

Proof of Theorem 4. Denote $G_1(X) = x_1^i \cdots x_m^i$. Then for any small $\delta > 0$, we have that

$$(1-\delta)\big(-p(t)\big)\partial G_1/\partial x_i < x_i'' < (1+\delta)\big(-p(t)\big)\partial G_1/\partial x_i \tag{5.5}$$

for sufficiently large r.

Fix $0 < c < 1$. Let $0 < \delta_0, \eta < 1$, $r > 1$ and define D_r be a set in the phase space $\mathbb{R}^{2m}$ satisfying the following conditions for $k, j = 1, \dots, m$:

(i) $\quad x_k, x_k' > r$;

(ii) $\quad c < \dfrac{x_j'}{x_k'} < c^{-1}, \qquad \eta c < \dfrac{x_k'^2}{G_1(X)} < (\eta c)^{-1}$;

(iii) $\quad c < \dfrac{x_j}{x_k} < c^{-1} + \dfrac{\delta_0}{1-c}\left(\dfrac{x_j'}{x_k'} - 1\right) \quad \text{for } c < \dfrac{x_j'}{x_k'} \leqslant 1$;

(iv) $\quad c + \dfrac{\delta_0}{c^{-1}-1}\left(\dfrac{x_j'}{x_k'} - 1\right) < \dfrac{x_j}{x_k} < c^{-1} \quad \text{for } 1 < \dfrac{x_j'}{x_k'} < c^{-1}$.

Obviously, D_r is of infinite measure. The proof of Theorem 4 can be reduced to the following proposition:

Proposition 5.1. *There exist* $0 < \delta_0, \eta < 1$ *and* $r > 1$ *such that* D_r *is an invariant set of the flow defined by Eq.* (5.1)*.*

Proof. It is equivalent to prove that every vector at the boundary of D_r points inward.

To analyze the situation on the boundary of D_r, it is sufficient to deal with the cases $x_k, x_k' = r$ as well as the following cases:

$$\frac{x_k'^2}{G_1(X)} = \eta c \quad \text{or} \quad \frac{x_k'^2}{G_1(X)} = (\eta c)^{-1};$$

$$\frac{x_j'}{x_k'} = c \quad \text{or} \quad \frac{x_j'}{x_k'} = c^{-1};$$

$$\frac{x_j}{x_k} = c \quad \text{or} \quad \frac{x_j}{x_k} = c^{-1} + \frac{\delta_0}{1-c}\left(\frac{x_j'}{x_k'} - 1\right), \quad c < \frac{x_j'}{x_k'} \leqslant 1;$$

$$\frac{x_j}{x_k} = c^{-1} \quad \text{or} \quad \frac{x_j}{x_k} = c + \frac{\delta_0}{c^{-1}-1}\left(\frac{x_j'}{x_k'} - 1\right), \quad 1 < \frac{x_j'}{x_k'} < c^{-1}.$$

For the case $\frac{x_k'^2}{G_1(X)} = \eta c$, we have

$$\begin{aligned}\left(\frac{x_k'^2}{G_1(X)}\right)' &= \frac{2x_k'x_k''G_1 - x_k'^2\sum_{l=1}^m \partial G_1/\partial x_l\cdot x_l'}{G_1^2}\\ &> \frac{2i(1-\delta_0)x_k'G_1^2(-p(t))/x_k - ix_k'^2\sum_{l=1}^m G_1/x_l\cdot x_l'}{G_1^2}\quad \text{from (5.5)}\\ &= \left(4\epsilon_0 i(1-\delta_0)G_1 - imc^{-2}x_k'^2\right)\frac{x_k'}{G_1x_k}\quad \text{since } c<\frac{x_j}{x_k},\frac{x_j'}{x_k'}<c^{-1}\\ &> 0\quad \text{with } \eta < m^{-1}c\epsilon_0,\ \delta_0<\frac{1}{2}.\end{aligned}$$

We can analysis the case $\frac{x_k'^2}{G_1(X)} = (\eta c)^{-1}$ in a similar way as above.

For the part of the boundary of D_r in the hyperplane $\frac{x_j'}{x_k'} = c$, from the condition (iii), we have that every point in this part also satisfies $c < \frac{x_j}{x_k} < c^{-1} - \delta_0$. It implies that for sufficiently large r and sufficiently small δ_0, it holds that

$$\left(\frac{x_j'}{x_k'}\right)' = \frac{x_j''x_k' - x_j'x_k''}{x_k'^2} > \frac{iG_1(X)(-p(t))((1-\delta_1)x_k'/x_j - (1+\delta_1)x_j'/x_k)}{x_k'^2} > 0,$$

where $\delta_1 = ([2c^{-1}]+1)^{-1}\delta_0$. Thus each vector at this part of the boundary of D_r points inward.

Similarly, we can prove the same conclusion for the case $\frac{x_j'}{x_k'} = c^{-1}$.

For the part of the boundary of D_r satisfying $\frac{x_j}{x_k} = c$, $c < \frac{x_j'}{x_k'} \leqslant 1$, we can easily have that

$$\left(\frac{x_j}{x_k}\right)' = \frac{x_j'x_k - x_jx_k'}{x_j^2} > 0.$$

The situation for the case $\frac{x_j}{x_k} = c^{-1}$, $1 < \frac{x_j'}{x_k'} < c^{-1}$ is similar.

For the subset of the boundary of D_r satisfying $\frac{x_j}{x_k} = c^{-1} + \frac{\delta_0}{1-c}(\frac{x_j'}{x_k'} - 1)$, $c < \frac{x_j'}{x_k'} \leqslant 1$, we have that

$$\begin{aligned}\left(\frac{x_j}{x_k} - \frac{\delta_0}{1-c}\left(\frac{x_j'}{x_k'}-1\right)\right)' &= \frac{x_j'x_k - x_jx_k'}{x_k^2} - \frac{\delta_0}{1-c}\frac{x_j''x_k' - x_j'x_k''}{x_k'^2}\\ &< \frac{(\delta_0+1-c^{-1})x_kx_k'}{x_k^2} - \frac{i\delta_0G_1(X)(-p(t))}{(1-c)x_k'^2}\cdot(1+\delta_1)\left(\frac{x_k'}{x_j}+\frac{x_j'}{x_k}\right)\\ &< \frac{(\delta_0+1-c^{-1})x_kx_k'}{x_k^2} + \frac{i\|p\|\delta_0}{(1-c)c\eta}\cdot\left(\frac{x_k'}{x_j}+\frac{x_j'}{x_k}\right)\quad \text{from (ii)},\ \|p\| = \max_{t\in S^1}\bigl|p(t)\bigr|\\ &< \left(\delta_0 + 1 - c^{-1} + \frac{2i\|p\|\delta_0}{(1-c)c^2\eta}\right)\frac{x_k'}{x_k}\quad \text{since } c<\frac{x_j}{x_k},\frac{x_j'}{x_k'}<c^{-1}\\ &< 0\quad \text{for } \delta_0 \ll 1.\end{aligned}$$

We can deal with the subset of the boundary of D_r satisfying $\frac{x_j}{x_k} = c + \frac{\delta_0}{c^{-1}-1}(\frac{x_j'}{x_k'} - 1)$, $1 < \frac{x_j'}{x_k'} < c^{-1}$ in a similar way.

In conclusion, a flow $X(t) = (\dots, x_k(t), \dots)$ of (5.1) starting from D_r always satisfies the conditions (ii)–(iv). Especially, we have $c < \frac{x_k(t)}{x_j(t)} < c^{-1}$, $1 \leqslant k, j \leqslant m$. Hence similar to the argument in Lemma 5.1, we can find $\epsilon_1 > 0$ such that on the time interval $t \in [t_0, t_1]$ the following inequality holds true for $X(t)$:

$$x_k''(t) > \epsilon_1 x_k(t)^{mi-1},\quad k = 1, 2, \dots, m. \tag{5.6}$$

Then we have that the flow is inward on the boundary $x_k = r$ or $x_k' = r$ for $r \gg 1$. Thus we complete the proof of the proposition. $\square$

From (5.6) and the same argument as in Lemma 5.1, we can prove that a flow $X(t) = (\dots, x_k(t), \dots)$ of (5.1) starting from D_r with $r \gg 1$ will blow up during $t \in [t_0, t_1)$. Thus we complete the proof of Theorem 4. $\square$

Proof of Theorem 2. Theorem 4 is the special situation of Theorem 2 with $i_1 = \dots = i_m$. For the general case, we observe that in the proof of Proposition 5.1, all the inequalities hold strictly. Thus we can find $c_1 > 0$ dependent only on $p(t)$ and m

such that with the assumptions $|\frac{i_j}{i_k} - 1| \leqslant c_1$ for $j, k = 1, \ldots, m$, we can prove that Proposition 5.1 holds true for (1.1) instead of (5.1). Remember the condition that $\sum_{l=1}^m i_l > \max_{1 \leqslant l \leqslant m}(2n_l + 2)$. Thus we obtain that a flow $X(t) = (\ldots, x_k(t), \ldots)$ of (1.1) starting from D_r always satisfies

$$x_k''(t) > \epsilon_1 x_k(t)^{(\sum_{l=1}^m i_l - 1)}, \quad k = 1, 2, \ldots, m, \ t \in [t_0, t_1] \tag{5.7}$$

for some $\epsilon_1 > 0$. Thus we can prove Theorem 2 with the same argument as in Lemma 5.1. □

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References

[1] J.M. Alonso, R. Ortega, Roots of unity and unbounded motions of an asymmetric oscillator, J. Differential Equations 143 (1998) 201–220.
[2] V.I. Arnold, Instability of dynamical systems with several degrees of freedom, Soviet Math. Dokl. 5 (1964) 581–585.
[3] C. Cheng, J. Yan, Existence of diffusion orbits in a priori unstable Hamiltonian systems, J. Differential Geom. 67 (3) (2004) 457–517.
[4] R. Dieckerhoff, E. Zehnder, Boundedness of solutions via the Twist Theorem, Ann. Sc. Norm. Super. Pisa 14 (1) (1987) 79–85.
[5] C. Fabry, J. Mawhin, Oscillations of a forced asymmetric oscillator at resonance, Nonlinearity 13 (2000) 493–505.
[6] H. Jin, B. Liu, Y. Wang, The existence of quasiperiodic solutions for coupled Duffing-type equations, J. Math. Anal. Appl. 374 (2011) 411–429.
[7] S. Laederich, M. Levi, Invariant curves and time-dependent potentials, Ergodic Theory Dynam. Systems 11 (2) (1991) 365–378.
[8] M. Levi, On Littlewood's counterexample of unbounded motions in superquadratic potentials, in: C.K.R.T. Jones, U. Kirchgraber, H.O. Walther (Eds.), Forschungsinstitut für Mathematik, ETH Zürich, Dynamics Reported I, Springer, Berlin, 1992, pp. 113–124.
[9] M. Levi, J. You, Oscillatory escape in a Duffing equation with polynomial potential, J. Differential Equations 140 (2) (1997) 415–426.
[10] J. Littlewood, Some Problems in Real and Complex Analysis, Heath Lexington, Massachusetts, 1968.
[11] J. Littlewood, Unbounded solutions of $y'' + g(y) = p(t)$, J. London Math. Soc. 41 (1966) 491–496.
[12] B. Liu, Boundedness for solutions of nonlinear Hill's equations with periodic forcing terms via Moser's twist theorem, J. Differential Equations 79 (1989) 304–315.
[13] B. Liu, Boundedness for solutions of nonlinear periodic differential equations via Moser's twist theorem, Acta Math. Sin. (N.S.) 8 (1992) 91–98.
[14] Y. Long, An unbounded solution of a superlinear Duffing's equation, Acta Math. Sin. (N.S.) 7 (1991) 360–369.
[15] J. Mather, Arnold diffusion, I: Announcement of results, J. Math. Sci. 124 (2004) 5275–5289.
[16] G. Morris, A case of boundedness in Littlewood's problem on oscillatory differential equations, Bull. Austral. Math. Soc. 14 (1976) 71–93.
[17] J. Moser, On invariant curves of area-preserving mappings of an annulus, Nachr. Akad. Wiss. Gottingen Math.-Phys. Kl. II (1962) 1–20.
[18] J. Moser, Stable and Random Motions in Dynamical Systems: with Special Emphasis on Celestial Mechanics, Ann. of Math. Stud., vol. 77, Princeton University Press, 1973.
[19] Z. Shang, A note on the KAM theorem for symplectic mappings, J. Dynam. Differential Equations 12 (2) (2000) 357–383.
[20] Y. Wang, J. You, Boundedness of solutions in polynomial potentials with C^2 coefficients, Z. Angew. Math. Phys. 47 (1996) 943–952.
[21] Y. Wang, Unboundedness in a Duffing equation with polynomial potentials, J. Differential Equations 160 (2) (2000) 467–479.
[22] Z. Xia, Arnold diffusion: a variational construction, Doc. Math. J. DMV Extra Volume ICM II (1998) 867–877.
[23] J. You, Invariant tori and Lagrange stability of pendulum-type equations, J. Differential Equations 85 (1990) 54–65.
[24] X. Yuan, Lagrange stability for Duffing-type equations, J. Differential Equations 160 (1) (2000) 94–117.
[25] X. Yuan, Invariant tori of Duffing-type equations, J. Differential Equations 142 (1998) 231–262.
[26] F. Zhang, Quasiperiodic solutions of higher dimensional Duffing's equations via the KAM theorem, Sci. China Ser. A 44 (5) (2001) 631–644.
[27] V. Zharnitsky, Breakdown of stability of motion in superquadratic potentials, Comm. Math. Phys. 189 (1) (1997) 165–204.