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Global existence and asymptotic behavior of the Cauchy problem for fourth-order Schrödinger equations with combined power-type nonlinearities

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ABSTRACT

We study the global existence of solutions in Sobolev spaces and the asymptotic behavior for the Cauchy problem of the following fourth-order Schrödinger equation with combined power-type nonlinearities $iu_t + \Delta^2 u + \lambda_1 |u|^p u + \lambda_2 |u|^{\frac{8}{n-4}} u = 0$, where $t \in R, x \in R^n$, $\frac{8}{n} \le p < \frac{8}{n-4}$, λ_1, λ_2 are nonzero real numbers.

1. Introduction

In this paper we study the Cauchy problem of the following energy-critical fourth-order Schrödinger equation with a subcritical perturbation in $R \times R^n$:

$$\begin{cases} iu_t + \Delta^2 u + \lambda_1 |u|^p u + \lambda_2 |u|^{\frac{8}{n-4}} u = 0, & \text{for } t \in R,\ x \in R^n, \\ u(0,x) = \varphi(x), \quad x \in R^n, \end{cases} \tag{1.1}$$

where λ_1, λ_2 are nonzero real numbers, and $\frac{8}{n} \le p < \frac{8}{n-4}$ is a positive constant.

Fourth-order Schrödinger equations are very important equations which arise in many physical application fields. They have been introduced by Karpman [1] and Karpman and Shagalov [2] to take into account the effects of small fourth-order dispersion terms on the solitons. They have also been studied from a mathematical viewpoint by many authors. For instance, the energy-critical focusing fourth-order Schrödinger equation ($\lambda_1 = 0, \lambda_2 < 0$) has been studied by Miao et al. [3]. They obtained that the solution is global and scatters in the radial case under some conditions. In [4] Miao et al. proved that any finite energy solution is global and scatters for defocusing case ($\lambda_1 = 0, \lambda_2 > 0$) and $n \ge 9$. Pausader [5] has obtained the global well-posedness and scattering in the defocusing case ($\lambda_1 = 0, \lambda_2 > 0$) for radially symmetrical initial data and $n \ge 5$. In [6] Pausader investigated the cubic defocusing fourth-order Schrödinger equation ($\lambda_1 = 0, \lambda_2 > 0, |u|^{\frac{8}{n-4}} u \to |u|^2 u$) and proved that the equation is globally well-posed for $n \le 8$ and the solution scatters for $5 \le n \le 8$. For the case $\lambda_1 \ne 0, \lambda_2 \ne 0$ and $n = 8$, Zhang and Zheng [7] proved that the solution is global for $1 < p < 2$, scattering will occur either $\lambda_2 > 0, 1 < p < 2$ or when the mass of solution is small enough, $1 \le p < 2$. But they only discuss the case $n = 8$ (obviously $|u|^{\frac{8}{n-4}} u = |u|^2 u$). As far as we know, there are few results about global existence and scattering for the energy-critical fourth-order Schrödinger equations

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with combined power-type nonlinearities in arbitrary dimensions. The aim of this paper is to study the global existence of solutions and scattering of the Cauchy problem of (1.1) for any dimensions. In [8] Tao et al. studied a Schrödinger equation with combined power-type nonlinearities. So we will utilize the ideas and techniques of [8]. The method of [7,8] is the "perturbative" method. The idea of this method is to obtain the global well-posedness by the good local well-posedness combining with the global kinetic energy control. The same method is used in [9]. The aim of this paper is to improve the global existence and scattering for fourth-order Schrödinger equations in arbitrary dimensions. As for the local well-posedness, we can obtain it by similar techniques in [10]. Here we omit it. We also refer the reader to [11–18] for other results about fourth-order Schrödinger equations.

The main results of this work are the following theorems:

Theorem 1.1 (*Global Solvability*)**.** *Assume that $4 < n \le 8$. Let $\varphi(x) \in H^2(R^n)$ and $\lambda_2 > 0$, then there exists a unique global solution $u(t,x)$ of (1.1) which satisfies*

$$\|u\|_{L^q_{\mathrm{loc}}(R;H^{2,r}(R^n))} \le C\big(\|\varphi\|_{H^2(R^n)}\big)$$

for all biharmonic admissible pairs (q,r) (biharmonic admissible pairs will be introduced in Section 2).

Theorem 1.2 (*Scattering Results*)**.** *Assume that $4 < n \le 8$. For any $\varphi(x) \in H^2(R^n)$, let u be the unique global solution of (1.1), there exist $u_\pm$ such that*

$$\lim_{t\to\pm\infty} \|u(t) - W(t)u_\pm\|_{H^2(R^n)} = 0,$$

provided

$$\lambda_1 > 0, \qquad \lambda_2 > 0, \qquad \frac{8}{n} < p < \frac{8}{n-4},$$

or

$$\lambda_2 > 0, \qquad \frac{8}{n} \le p < \frac{8}{n-4} \quad \text{and} \quad \|\varphi\|_{L^2(R^n)} \le C\big(\|\Delta\varphi\|_{L^2(R^n)}\big),$$

(where the operator $W(t)$ will be introduced in Section 2, $C\big(\|\Delta\varphi\|_{L^2(R^n)}\big)$ is a small enough parameter, see the proof of Theorem 1.2).

Obviously, (1.1) has two conservation laws:

$$M(u)(t) = \|u(t,\cdot)\|_{L^2(R^n)}, \tag{1.2}$$

and

$$E(u)(t) = \int_{R^n}\left[\frac{1}{2}|\Delta u(t,x)|^2 + \frac{\lambda_1}{p+2}|u(t,x)|^{p+2} + \frac{(n-4)\lambda_2}{2n}|u(t,x)|^{\frac{2n}{n-4}}\right]dx. \tag{1.3}$$

We recall the dispersive estimates for the linear equation related to the Eq. (1.1) in Section 2 and we will present the nonlinear estimates in Section 3. In Section 4 we present the proofs of the theorems.

2. Notations and fundamental solution operator estimates

Given $T > 0$ and a function space on R^n, we denote by $\|\cdot\|_{L^q([-T,T];X)}$ and $L^q([-T,T];X)$ respectively the following norm and the corresponding function space on $[-T,T]\times R^n$:

For $1 \le q < +\infty$,

$$\|f\|_{L^q([-T,T];X)} = \left(\int_{-T}^{T} \|f(\cdot,t)\|_X^q dt\right)^{\frac{1}{q}}.$$

And for $q = \infty$,

$$\|f\|_{L^\infty([-T,T];X)} = \mathrm{ess.\,sup}_{-T<t<T}\|f(\cdot,t)\|_X.$$

Later we shall particularly take $X = H^{s,r}(R^n)(s \in R, 1 \le r \le \infty)$. For simplicity of notation, we respectively abbreviate $\|\cdot\|_{L^q([-T,T];L^r)}$ and $\|\cdot\|_{L^q([-T,T];H^{s,r})}$ as respectively $\|\cdot\|_{L^q_T L^r_x}$ and $\|\cdot\|_{L^q_T H^{s,r}_x}$. We also abbreviate $H^{s,2}(R^n) = H^s(R^n)$. In the following, we will introduce our three working spaces.

For any time interval T, we denote

$$\dot{X}^0(T) = L_T^{\gamma_1}L_x^{\rho_1} \cap L_T^{\gamma_2}L_x^{\rho_2} \cap L_T^{\gamma_3}L_x^{\rho_3} \cap L_T^{\gamma_4}L_x^{\rho_4},$$

$$\dot{Y}^0(T) = \begin{cases} L_T^{\gamma_1}L_x^{\rho_1} \cap L_T^{\gamma_2}L_x^{\rho_2} \cap L_T^{\gamma_5}L_x^{\rho_5}, & \dfrac{8}{n} < p \le \dfrac{8(n-1)}{n^2-3n+8}, \\ L_T^{\gamma_1}L_x^{\rho_1} \cap L_T^{\gamma_2}L_x^{\rho_2} \cap L_T^{\gamma_6}L_x^{\rho_6}, & \dfrac{8(n-1)}{n^2-3n+8} < p < \dfrac{8}{n-4}, \end{cases}$$

$$\dot{Z}^0(T) = L_T^{\gamma_1}L_x^{\rho_1} \cap L_T^{\gamma_2}L_x^{\rho_2}$$

where

$$(\gamma_1, \rho_1) = \left(\frac{2(n+4)}{n}, \frac{2(n+4)}{n}\right), \qquad (\gamma_2, \rho_2) = \left(\frac{2(n+4)}{n-4}, \frac{2n(n+4)}{n^2+16}\right),$$

$$(\gamma_3, \rho_3) = \left(\frac{8}{(n-4)p-4}, \frac{2n}{n+4-(n-4)p}\right), \qquad (\gamma_4, \rho_4) = \left(\frac{8(p+2)}{(n-4)p}, \frac{n(p+2)}{n+2p}\right),$$

$$(\gamma_5, \rho_5) = \left(\frac{2(n+2)}{n-6+(n-4)p}, \frac{2n(n+2)}{n^2-2n+24-4(n-4)p}\right),$$

$$(\gamma_6, \rho_6) = \left(\frac{2(n-4)}{n+4-np}, \frac{2n(n-4)}{n^2-8n-16+4np}\right).$$

And

$$\dot{X}^1(T) = \{u : \Delta u \in \dot{X}^0(T)\}, \qquad X^1(T) = \dot{X}^0(T) \cap \dot{X}^1(T),$$
$$\dot{Y}^1(T) = \{u : \Delta u \in \dot{Y}^0(T)\}, \qquad Y^1(T) = \dot{Y}^0(T) \cap \dot{Y}^1(T),$$
$$\dot{Z}^1(T) = \{u : \Delta u \in \dot{Z}^0(T)\}, \qquad Z^1(T) = \dot{Z}^0(T) \cap \dot{Z}^1(T).$$

The fundamental solution of the linear equation related to (1.1) is given by the following oscillatory integral:

$$I(x,t) = (2\pi)^{-n} \int_{R^n} e^{ix\cdot\xi + it|\xi|^4} d\xi.$$

We denote by $W(t)(t \in R)$ the fundamental solution operator

$$W(t)\varphi = I(x,t) * \varphi(x), \quad \varphi(x) \in S'(R^n).$$

Definition 2.1. For two integers $2 \le q \le \infty$ and $2 \le r < \infty$, we say that (q, r) is a *Schrödinger admissible pair* if the following condition is satisfied:

$$\frac{2}{q} = n\left(\frac{1}{2} - \frac{1}{r}\right).$$

Definition 2.2. For two integers $2 \le \gamma \le \infty$ and $2 \le \rho < \infty$, we say that (γ, ρ) is an *biharmonic admissible pair* if the following condition is satisfied:

$$\frac{4}{\gamma} = n\left(\frac{1}{2} - \frac{1}{\rho}\right).$$

Lemma 2.1 (*See [19]*)**.** *For any biharmonic admissible pairs (q, r) and (γ, ρ), there hold the following estimates*

$$\|W(t)\varphi\|_{L_T^q H_x^{s,r}} \le C\|\varphi\|_{H_x^s},$$

$$\left\| \int_0^t W(t-\tau) f(\cdot,\tau)\, d\tau \right\|_{L_T^q H_x^{s,r}} \le C\|f\|_{L_T^{\gamma'} H_x^{s,\rho'}},$$

where $\varphi \in H^s(R^n)$, $f \in L^{\gamma'}([-T,T]; H^{s,\rho'}(R^n))$.

For any Schrödinger admissible pairs (a, b) and (c, d), there hold the following estimates

$$\|W(t)\varphi\|_{L_T^a \dot{H}_x^{s,b}} \le C\|\varphi\|_{\dot{H}_x^{s-\frac{2}{a}}},$$

$$\left\| \int_0^t W(t-\tau) f(\cdot,\tau)\, d\tau \right\|_{L_T^a \dot{H}_x^{s,b}} \le C\|f\|_{L_T^{c'} \dot{H}_x^{s-\frac{2}{a}-\frac{2}{c},\, d'}},$$

where $\varphi \in \dot{H}^{s-\frac{2}{a}}(R^n)$, $f \in L^{c'}([-T,T]; \dot{H}^{s-\frac{2}{a}-\frac{2}{c},\, d'}(R^n))$.

By Sobolev inequality and Lemma 2.1, we have

Corollary 2.2. *For any biharmonic admissible pair (q, r), there holds the following estimate*

$$\left\| \int_0^t W(t-\tau)f(\cdot,\tau)d\tau \right\|_{L_T^q \dot{H}_x^{2,r}} \le C\|f\|_{L_T^2 \dot{H}_x^{1,\frac{2n}{n+2}}},$$

where $f \in L^2([-T, T]; \dot{H}^{1,\frac{2n}{n+2}}(R^n))$.

Proof. By Sobolev inequality, we have

$$\left\| \int_0^t W(t-\tau)f(\cdot,\tau)d\tau \right\|_{L_T^q \dot{H}_x^{2,r}} \le \left\| \int_0^t W(t-\tau)f(\cdot,\tau)d\tau \right\|_{L_T^q \dot{H}_x^{s,r_1}},$$

where r_1 and s satisfy $\frac{2}{q} = n\left(\frac{1}{2} - \frac{1}{r_1}\right)$ and $\frac{1}{r_1} - \frac{1}{r} = \frac{s-2}{n}$.

Taking $a = q$, $b = r_1$ and $c = 2$, $d = \frac{2n}{n-2}$ in Lemma 2.1, the desired result is obtained. □

Using Lemma 2.1 and Corollary 2.2, by Duhamel formula, we get the following lemma:

Lemma 2.3. *If u is the solution of the problem (1.1), then for any biharmonic admissible pares (q, r), (γ_1, ρ_1), (γ_2, ρ_2), we have*

$$\|u\|_{L_T^q H_x^{2,r}} \le C_0\|u_0\|_{H^2} + C_1\| |u|^p u\|_{L_T^{\gamma_1'} L_x^{\rho_1'}} + C_1\| |u|^{\frac{8}{n-4}} u\|_{L_T^{\gamma_2'} L_x^{\rho_2'}} + C_1\|\nabla(|u|^p u)\|_{L_T^2 L_x^{\frac{2n}{n+2}}} + C_1\|\nabla(|u|^{\frac{8}{n-4}} u)\|_{L_T^2 L_x^{\frac{2n}{n+2}}}.$$

Lemma 2.4 (*See [4,20]*). *If u is a solution of the problem (1.1), then the following holds for $\lambda_1 > 0$, $\lambda_2 > 0$ and $n > 4$*

$$\| |\nabla|^{-\frac{n-5}{4}} u\|_{L_T^4 L_x^4}^4 \le C \sup_{[0,T]} \|u(t)\|_{\dot{H}_x^{\frac{1}{2}}}^2 \|u(t)\|_{L_x^2}^2.$$

Using Lemma 2.4 and interpolation theorem [21], we obtain

Lemma 2.5. *If u is a solution of the problem (1.1), then the following holds for $\lambda_1 > 0$, $\lambda_2 > 0$ and $n > 4$*

$$\|u\|_{L_T^{n-1} L_x^{\frac{2(n-1)}{n-3}}} \le C\| |\nabla|^{-\frac{n-5}{4}} u\|_{L_T^4 L_x^4}^{\frac{4}{n-1}} \|\nabla u\|_{L_T^\infty L_x^2}^{\frac{n-5}{n-1}}.$$

3. Nonlinear estimates

Lemma 3.1. *Assume that $4 < n \le 8$, then we have*

$$\| |u|^p u\|_{L_T^{\gamma_4'} L_x^{\rho_4'}} \le CT^{1-\frac{(n-4)p}{8}} \|u\|_{L_T^{\gamma_4} \dot{H}_x^{2,\rho_4}}^p \|u\|_{L_T^{\gamma_4} L_x^{\rho_4}}, \tag{3.1}$$

$$\|\nabla(|u|^p u)\|_{L_T^2 L_x^{\frac{2n}{n+2}}} \le CT^{1-\frac{(n-4)p}{8}} \|u\|_{L_T^\infty \dot{H}_x^2}^p \|\Delta u\|_{L_T^{\gamma_3} L_x^{\rho_3}}, \tag{3.2}$$

where (γ_i, ρ_i), $i = 3, 4$ are as in Section 2.

Proof. Using Hölder inequality and Sobolev inequality, we have

$$\begin{aligned} \| |u|^p u\|_{L_T^{\gamma_4'} L_x^{\rho_4'}} &\le CT^{1-\frac{(n-4)p}{8}} \|u\|_{L_T^{\gamma_4} L_x^{\rho^*}}^p \|u\|_{L_T^{\gamma_4} L_x^{\rho_4}} \left(\rho^* = \frac{n(p+2)}{n-4}\right) \\ &\le CT^{1-\frac{(n-4)p}{8}} \|u\|_{L_T^{\gamma_4} \dot{H}_x^{2,\rho_4}}^p \|u\|_{L_T^{\gamma_4} L_x^{\rho_4}}, \\ \|\nabla(|u|^p u)\|_{L_T^2 L_x^{\frac{2n}{n+2}}} &\le CT^{1-\frac{(n-4)p}{8}} \|u\|_{L_T^\infty L_x^{\frac{2n}{n-4}}}^p \|\nabla u\|_{L_T^{\gamma_3} L_x^{\frac{2n}{n+2-(n-4)p}}} \\ &\le CT^{1-\frac{(n-4)p}{8}} \|u\|_{L_T^\infty \dot{H}_x^2}^p \|\Delta u\|_{L_T^{\gamma_3} L_x^{\rho_3}}. \end{aligned}$$

□

Lemma 3.2. *Assume* $4 < n \le 8$, *we have*

$$\| |u|^{\frac{8}{n-4}} u \|_{L_T^{\frac{2(n+4)}{n+8}} L_x^{\frac{2(n+4)}{n+8}}} \le C \|u\|^{\frac{8}{n-4}}_{L_T^{\gamma_2} \dot{H}_x^{2,\rho_2}} \|u\|_{L_T^{\gamma_1} L_x^{\rho_1}}, \tag{3.3}$$

$$\|\nabla(|u|^{\frac{8}{n-4}} u)\|_{L_T^2 L_x^{\frac{2n}{n+2}}} \le C \|u\|^{\frac{n+4}{n-4}}_{L_T^{\gamma_2} \dot{H}_x^{2,\rho_2}} \tag{3.4}$$

where $(\gamma_i, \rho_i),\ i = 1,\ 2$ *are as in Section* 2.

Proof. Using Hölder inequality and Sobolev inequality, we have

$$\begin{aligned}
\| |u|^{\frac{8}{n-4}} u \|_{L_T^{\frac{2(n+4)}{n+8}} L_x^{\frac{2(n+4)}{n+8}}} &\le C \| |u|^{\frac{8}{n-4}} \|_{L_T^{\frac{n+4}{4}} L_x^{\frac{n+4}{4}}} \|u\|_{L_T^{\frac{2(n+4)}{n}} L_x^{\frac{2(n+4)}{n}}} \\
&\le C \|u\|^{\frac{8}{n-4}}_{L_T^{\frac{2(n+4)}{n-4}} L_x^{\frac{2(n+4)}{n-4}}} \|u\|_{L_T^{\frac{2(n+4)}{n}} L_x^{\frac{2(n+4)}{n}}} \\
&\le C \|u\|^{\frac{8}{n-4}}_{L_T^{\frac{2(n+4)}{n-4}} \dot{H}_x^{2,\frac{2n(n+4)}{n^2+16}}} \|u\|_{L_T^{\frac{2(n+4)}{n}} L_x^{\frac{2(n+4)}{n}}} \\
&= C \|u\|^{\frac{8}{n-4}}_{L_T^{\gamma_2} \dot{H}_x^{2,\rho_2}} \|u\|_{L_T^{\gamma_1} L_x^{\rho_1}}.
\end{aligned}$$

Similarly we can obtain

$$\begin{aligned}
\|\nabla(|u|^{\frac{8}{n-4}} u)\|_{L_T^2 L_x^{\frac{2n}{n+2}}} &\le C \|u\|^{\frac{8}{n-4}}_{L_T^{\frac{2(n+4)}{n-4}} L_x^{\frac{2(n+4)}{n-4}}} \|\nabla u\|_{L_T^{\frac{2(n+4)}{n-4}} L_x^{\frac{2n(n+4)}{n^2-2n+8}}} \\
&\le C \|u\|^{\frac{8}{n-4}}_{L_T^{\frac{2(n+4)}{n-4}} \dot{H}_x^{2,\frac{2n(n+4)}{n^2+16}}} \|u\|_{L_T^{\frac{2(n+4)}{n-4}} \dot{H}_x^{2,\frac{2n(n+4)}{n^2+16}}} \\
&= C \|u\|^{\frac{n+4}{n-4}}_{L_T^{\gamma_2} \dot{H}_x^{2,\rho_2}}. \quad \square
\end{aligned}$$

Lemma 3.3. *Assume that* $4 < n \le 8$. (i) *For the case* $\frac{8(n-1)}{n^2-3n+8} < p < \frac{8}{n-4}$, *we have the following results:*

$$\| |u|^p u \|_{L_T^2 L_x^{\frac{2n}{n+4}}} \le C \|u\|^{\frac{[8-(n-4)p](n-1)}{2(n+2)}}_{L_T^{n-1} L_x^{\frac{2(n-1)}{n-3}}} \|\Delta u\|^{\frac{(n^2-3n+8)p-8(n-1)}{2(n+2)}}_{L_T^\infty L_x^2} \|u\|_{L_T^{\gamma_5} L_x^{\rho_5}}, \tag{3.5}$$

$$\|\nabla(|u|^p u)\|_{L_T^2 L_x^{\frac{2n}{n+2}}} \le C \|u\|^{\frac{[8-(n-4)p](n-1)}{2(n+2)}}_{L_T^{n-1} L_x^{\frac{2(n-1)}{n-3}}} \|\Delta u\|^{\frac{(n^2-3n+8)p-8(n-1)}{2(n+2)}}_{L_T^\infty L_x^2} \|\Delta u\|_{L_T^{\gamma_5} L_x^{\rho_5}}. \tag{3.6}$$

(ii) *For the case* $\frac{8}{n} < p \le \frac{8(n-1)}{n^2-3n+8}$, *we have the following results:*

$$\| |u|^p u \|_{L_T^2 L_x^{\frac{2n}{n+4}}} \le C \|u\|^{\frac{(np-8)(n-1)}{2(n-4)}}_{L_T^{n-1} L_x^{\frac{2(n-1)}{n-3}}} \|u\|^{\frac{-(n^2-3n+8)p+8(n-1)}{2(n-4)}}_{L_T^\infty L_x^2} \|u\|_{L_T^{\gamma_6} L_x^{\rho_6}}, \tag{3.7}$$

$$\|\nabla(|u|^p u)\|_{L_T^2 L_x^{\frac{2n}{n+2}}} \le C \|u\|^{\frac{(np-8)(n-1)}{2(n-4)}}_{L_T^{n-1} L_x^{\frac{2(n-1)}{n-3}}} \|u\|^{\frac{-(n^2-3n+8)p+8(n-1)}{2(n-4)}}_{L_T^\infty L_x^2} \|\Delta u\|_{L_T^{\gamma_6} L_x^{\rho_6}}, \tag{3.8}$$

where $(\gamma_i, \rho_i),\ i = 5, 6$ *are as in Section* 2.

Proof. (i) For the case $\frac{8(n-1)}{n^2-3n+8} < p < \frac{8}{n-4}$, using Hölder inequality, we have

$$\begin{aligned}
\| |u|^p u \|_{L_T^2 L_x^{\frac{2n}{n+4}}} &\le C \|u\|^p_{L_T^{\frac{2(n+2)p}{8-(n-4)p}} L_x^{\frac{n(n+2)p}{4n-8+2(n-4)p}}} \|u\|_{L_T^{\frac{2(n+2)}{n-6+(n-4)p}} L_x^{\frac{2n(n+2)}{n^2-2n+24-4(n-4)p}}} \\
&\le C \|u\|^{\frac{[8-(n-4)p](n-1)}{2(n+2)}}_{L_T^{n-1} L_x^{\frac{2(n-1)}{n-3}}} \|u\|^{\frac{(n^2-3n+8)p-8(n-1)}{2(n+2)}}_{L_T^\infty L_x^{\frac{2n}{n-4}}} \|u\|_{L_T^{\frac{2(n+2)}{n-6+(n-4)p}} L_x^{\frac{2n(n+2)}{n^2-2n+24-4(n-4)p}}}
\end{aligned}$$

$$
\begin{aligned}
&\le C\|u\|^{\frac{[8-(n-4)p](n-1)}{2(n+2)}}_{L_T^{n-1}L_x^{\frac{2(n-1)}{n-3}}}\|\Delta u\|^{\frac{(n^2-3n+8)p-8(n-1)}{2(n+2)}}_{L_T^\infty L_x^2}\|u\|_{L_T^{\frac{2(n+2)}{n-6+(n-4)p}}L_x^{\frac{2n(n+2)}{n^2-2n+24-4(n-4)p}}}\\
&= C\|u\|^{\frac{[8-(n-4)p](n-1)}{2(n+2)}}_{L_T^{n-1}L_x^{\frac{2(n-1)}{n-3}}}\|\Delta u\|^{\frac{(n^2-3n+8)p-8(n-1)}{2(n+2)}}_{L_T^\infty L_x^2}\|u\|_{L_T^{\gamma_5}L_x^{\rho_5}}.
\end{aligned}
$$

Similarly we can obtain

$$
\begin{aligned}
\|\nabla(|u|^pu)\|_{L_T^2L_x^{\frac{2n}{n+2}}} &\le C\|u\|^p_{L_T^{\frac{2(n+2)p}{8-(n-4)p}}L_x^{\frac{n(n+2)p}{4n-8+2(n-4)p}}}\|\nabla u\|_{L_T^{\frac{2(n+2)}{n-6+(n-4)p}}L_x^{\frac{2n(n+2)}{n^2-4n+20-4(n-4)p}}}\\
&\le C\|u\|^{\frac{[8-(n-4)p](n-1)}{2(n+2)}}_{L_T^{n-1}L_x^{\frac{2(n-1)}{n-3}}}\|\Delta u\|^{\frac{(n^2-3n+8)p-8(n-1)}{2(n+2)}}_{L_T^\infty L_x^2}\|\Delta u\|_{L_T^{\frac{2(n+2)}{n-6+(n-4)p}}L_x^{\frac{2n(n+2)}{n^2-2n+24-4(n-4)p}}}\\
&= C\|u\|^{\frac{[8-(n-4)p](n-1)}{2(n+2)}}_{L_T^{n-1}L_x^{\frac{2(n-1)}{n-3}}}\|\Delta u\|^{\frac{(n^2-3n+8)p-8(n-1)}{2(n+2)}}_{L_T^\infty L_x^2}\|\Delta u\|_{L_T^{\gamma_5}L_x^{\rho_5}}.
\end{aligned}
$$

(ii) For the case $\frac{8}{n} < p \le \frac{8(n-1)}{n^2-3n+8}$, using Hölder inequality, we have

$$
\begin{aligned}
\||u|^pu\|_{L_T^2L_x^{\frac{2n}{n+4}}} &\le C\|u\|^p_{L_T^{\frac{2p(n-4)}{np-8}}L_x^{\frac{p(n-4)}{4-2p}}}\|u\|_{L_T^{\frac{2(n-4)}{4+n-np}}L_x^{\frac{2n(n-4)}{n^2-8n-16+4np}}}\\
&\le C\|u\|^{\frac{(np-8)(n-1)}{2(n-4)}}_{L_T^{n-1}L_x^{\frac{2(n-1)}{n-3}}}\|u\|^{\frac{-(n^2-3n+8)p+8(n-1)}{2(n-4)}}_{L_T^\infty L_x^2}\|u\|_{L_T^{\frac{2(n-4)}{4+n-np}}L_x^{\frac{2n(n-4)}{n^2-8n-16+4np}}}\\
&= C\|u\|^{\frac{(np-8)(n-1)}{2(n-4)}}_{L_T^{n-1}L_x^{\frac{2(n-1)}{n-3}}}\|u\|^{\frac{-(n^2-3n+8)p+8(n-1)}{2(n-4)}}_{L_T^\infty L_x^2}\|u\|_{L_T^{\gamma_6}L_x^{\rho_6}}.
\end{aligned}
$$

Similarly we can obtain

$$
\begin{aligned}
\|\nabla(|u|^pu)\|_{L_T^2L_x^{\frac{2n}{n+2}}} &\le C\|u\|^p_{L_T^{\frac{2p(n-4)}{np-8}}L_x^{\frac{p(n-4)}{4-2p}}}\|\nabla u\|_{L_T^{\frac{2(n-4)}{4+n-np}}L_x^{\frac{2n(n-4)}{n^2-10n-8+4np}}}\\
&\le C\|u\|^{\frac{(np-8)(n-1)}{2(n-4)}}_{L_T^{n-1}L_x^{\frac{2(n-1)}{n-3}}}\|u\|^{\frac{-(n^2-3n+8)p+8(n-1)}{2(n-4)}}_{L_T^\infty L_x^2}\|\Delta u\|_{L_T^{\frac{2(n-4)}{4+n-np}}L_x^{\frac{2n(n-4)}{n^2-8n-16+4np}}}\\
&= C\|u\|^{\frac{(np-8)(n-1)}{2(n-4)}}_{L_T^{n-1}L_x^{\frac{2(n-1)}{n-3}}}\|u\|^{\frac{-(n^2-3n+8)p+8(n-1)}{2(n-4)}}_{L_T^\infty L_x^2}\|\Delta u\|_{L_T^{\gamma_6}L_x^{\rho_6}}. \quad \square
\end{aligned}
$$

Lemma 3.4. *If $\frac{8}{n} \le p < \frac{8}{n-4}$, then*

$$
\||u|^{p+1}\|_{L_T^{\frac{2(n+4)}{n+8}}L_x^{\frac{2(n+4)}{n+8}}} \le C\|u\|^{\frac{12-(n-4)p}{4}}_{L_T^{\gamma_1}L_x^{\rho_1}}\|\Delta u\|^{\frac{np-8}{4}}_{L_T^{\gamma_2}L_x^{\rho_2}}, \tag{3.9}
$$

$$
\||u|^{p+1}\|_{L^2\dot H^{1,\frac{2n}{n+2}}} \le C\|u\|^{\frac{8-(n-4)p}{4}}_{L_T^{\gamma_1}L_x^{\rho_1}}\|\Delta u\|^{\frac{np-4}{4}}_{L_T^{\gamma_2}L_x^{\rho_2}}, \tag{3.10}
$$

where (γ_i,ρ_i), $i=1,2$ are as in Section 2.

Proof. Using Hölder inequality, interpolation inequality and Sobolev inequality, we have

$$
\begin{aligned}
\||u|^{p+1}\|_{L_T^{\frac{2(n+4)}{n+8}}L_x^{\frac{2(n+4)}{n+8}}} &\le C\||u|^p\|_{L_T^{\frac{n+4}{4}}L_x^{\frac{n+4}{4}}}\|u\|_{L_T^{\gamma_1}L_x^{\rho_1}}\\
&\le C\|u\|^p_{L_T^{\frac{(n+4)p}{4}}L_x^{\frac{(n+4)p}{4}}}\|u\|_{L_T^{\gamma_1}L_x^{\rho_1}}\\
&\le C\|u\|^{\frac{8-(n-4)p}{4}}_{L_T^{\gamma_1}L_x^{\rho_1}}\|u\|^{\frac{np-8}{4}}_{L_T^{\frac{2(n+4)}{n-4}}L_x^{\frac{2(n+4)}{n-4}}}\|u\|_{L_T^{\gamma_1}L_x^{\rho_1}}\\
&\le C\|u\|^{\frac{8-(n-4)p}{4}}_{L_T^{\gamma_1}L_x^{\rho_1}}\|\Delta u\|^{\frac{np-8}{4}}_{L_T^{\gamma_2}L_x^{\rho_2}}\|u\|_{L_T^{\gamma_1}L_x^{\rho_1}}\\
&= C\|u\|^{\frac{12-(n-4)p}{4}}_{L_T^{\gamma_1}L_x^{\rho_1}}\|\Delta u\|^{\frac{np-8}{4}}_{L_T^{\gamma_2}L_x^{\rho_2}}.
\end{aligned}
$$

Similarly we obtain

$$\begin{aligned}\| |u|^{p+1}\|_{L^2\dot{H}^{1,\frac{2n}{n+2}}} &\le C\| |u|^p\|_{L_T^{\frac{n+4}{4}} L_x^{\frac{n+4}{4}}} \|\nabla u\|_{L_T^{\frac{2(n+4)}{n-4}} L_x^{\frac{2n(n+4)}{n^2-2n+8}}} \\ &\le C\|u\|^p_{L_T^{\frac{(n+4)p}{4}} L_x^{\frac{(n+4)p}{4}}} \|\Delta u\|_{L_T^{\gamma_2} L_x^{\rho_2}} \\ &\le C\|u\|^{\frac{8-(n-4)p}{4}}_{L_T^{\gamma_1} L_x^{\rho_1}} \|\Delta u\|^{\frac{np-4}{4}}_{L_T^{\gamma_2} L_x^{\rho_2}}. \quad \square\end{aligned}$$

4. Proofs of Theorems

Let u be the solution of problem (1.1). We can decompose the solution u into

$$u = v + w,$$

where v stands for the solution of the initial value problem

$$\begin{cases} iv_t + \Delta^2 v + \lambda_2 |v|^{\frac{8}{n-4}} v = 0, \quad t \in R,\ x \in R^n, \\ v(0,x) = \varphi(x), \quad x \in R^n, \end{cases} \tag{4.1}$$

and w is the solution of the initial value problem

$$\begin{cases} iw_t + \Delta^2 w = -\lambda_1 |v+w|^p (v+w) - \lambda_2 |v+w|^{\frac{8}{n-4}} (v+w) + \lambda_2 |v|^{\frac{8}{n-4}} v, \quad t \in R,\ x \in R^n, \\ w(0,x) = 0, \quad x \in R^n. \end{cases} \tag{4.2}$$

By the remark of [6], we know that (4.1) is globally well-posed if $\lambda_2 > 0$. And we have

$$\|v\|_{L^q(R;\dot{H}^{2,r})} \le C(\|\varphi\|_{\dot{H}^2}), \qquad \|v\|_{L^q(R;L^r)} \le C(\|\varphi\|_{H^2}), \tag{4.3}$$

where (q,r) is a biharmonic admissible pair.

In the following, for any given time interval $[0,T]$, we prove the existence of a solution w for problem (4.2) and make estimates on some norms of it.

For any given ε (which will be specified later), by (4.3), we can divide R into subintervals $I_0, I_1, \dots, I_{J-1}$ such that on each I_j

$$\|v\|_{X^1(I_j)} \sim \varepsilon, \quad 0 \le j \le J-1.$$

So for any given time interval $[0,T]$, there exists $J' \le J$ such that (renumbering, if necessary)

$$[0,T] = \cup_{j=0}^{J'-1} (I_j \cap [0,T]), \quad I_j = [t_j, t_{j+1}], \qquad t_0 = 0, \qquad t_{J'} = T.$$

Lemma 4.1. *For any given time interval* $[0,T]$*, there is a unique solution* $w(t,x)$ *for the problem* (4.2)

$$\|w\|_{X^1[0,T]} \le C,$$

where C will be dependent on p.

Proof. First we prove that there exists a solution $w(t,x)$ for problem (4.2) on $I_0 = [0,t_1]$ (renumbering if necessary).

We shall prove it by Banach fixed point theorem. For this purpose we rewrite problem (4.2) into integral form, namely

$$w(t) = i\int_0^{t_1} W(t-\tau)\left(\lambda_1 |v+w|^p (v+w) + \lambda_2 |v+w|^{\frac{8}{n-4}} (v+w) - \lambda_2 |v|^{\frac{8}{n-4}} v\right)(\tau)d\tau.$$

We denote

$$B_0 = \left\{ w : \|w\|_{L^\infty(I_0;H^2(R^n))} + \|w\|_{X^1(I_0)} \le \eta_0 = |I_0|^{1-\frac{(n-4)p}{8}} \right\},$$

and define a mapping S as follows

$$Sw(t) = i\int_0^{t_1} W(t-\tau)\left(\lambda_1 |v+w|^p (v+w) + \lambda_2 |v+w|^{\frac{8}{n-4}} (v+w) - \lambda_2 |v|^{\frac{8}{n-4}} v\right)(\tau)d\tau.$$

In the sequel we prove that S is well-defined and it maps B_0 into B_0.

By Lemma 2.3 and Lemmas 3.1–3.2, we have

$$\begin{aligned}
\|Sw\|_{B_0} &\le C\left\| |v+w|^{\frac{8}{n-4}}(v+w) - |v|^{\frac{8}{n-4}}v \right\|_{L^{\frac{2(n+4)}{n+8}}\left(I_0;L^{\frac{2(n+4)}{n+8}}\right)\cap L^2\left(I_0;\dot H^{1,\frac{2n}{n+2}}\right)} \\
&\quad + C\| |v+w|^p(v+w)\|_{L^{\gamma_4'}(I_0;L^{\rho_4'})\cap L^2\left(I_0;\dot H^{1,\frac{2n}{n+2}}\right)} \\
&\le C_1\left(\|v\|^{\frac{8}{n-4}}_{X^1(I_0)} + \|w\|^{\frac{8}{n-4}}_{X^1(I_0)}\right)\|w\|_{X^1(I_0)} + C_1|I_0|^{1-\frac{(n-4)p}{8}}\left(\|v\|^p_{X^1(I_0)} + \|w\|^p_{X^1(I_0)}\right)\|w\|_{X^1(I_0)} \\
&\le C_1\left(\varepsilon^{\frac{8}{n-4}} + \eta_0^{\frac{8}{n-4}}\right)\eta_0 + C_1|I_0|^{1-\frac{(n-4)p}{8}}(\varepsilon^p + \eta_0^p)\eta_0,
\end{aligned}$$

and

$$\begin{aligned}
\|Sw_1 - Sw_2\|_{B_0} &\le C\left\| |v+w_1|^{\frac{8}{n-4}}(v+w_1) - |v+w_2|^{\frac{8}{n-4}}(v+w_2) \right\|_{L^{\frac{2(n+4)}{n+8}}\left(I_0;L^{\frac{2(n+4)}{n+8}}\right)\cap L^2\left(I_0;\dot H^{1,\frac{2n}{n+2}}\right)} \\
&\quad + C\| |v+w_1|^p(v+w_1) - |v+w_2|^p(v+w_2)\|_{L^{\gamma_4'}(I_0;L^{\rho_4'})\cap L^2\left(I_0;\dot H^{1,\frac{2n}{n+2}}\right)} \\
&\le C_1\left(2\|v\|^{\frac{8}{n-4}}_{X^1(I_0)} + \|w_1\|^{\frac{8}{n-4}}_{X^1(I_0)} + \|w_2\|^{\frac{8}{n-4}}_{X^1(I_0)}\right)\|w_1 - w_2\|_{X^1(I_0)} \\
&\quad + C_1|I_0|^{1-\frac{(n-4)p}{8}}\left(2\|v\|^p_{X^1(I_0)} + \|w_1\|^p_{X^1(I_0)} + \|w_2\|^p_{X^1(I_0)}\right)\|w_1 - w_2\|_{X^1(I_0)} \\
&\le C_1\left(2\varepsilon^{\frac{8}{n-4}} + 2\eta_0^{\frac{8}{n-4}}\right)\|w_1 - w_2\|_{X^1(I_0)} + C_1|I_0|^{1-\frac{(n-4)p}{8}}(2\varepsilon^p + 2\eta_0^p)\|w_1 - w_2\|_{X^1(I_0)}.
\end{aligned}$$

Hence if we first take $2C_1\varepsilon^{\frac{8}{n-4}} < \frac{1}{2}$, then take t_1 such that $2C_1(\varepsilon^p t_1^{1-\frac{n-4}{8}p} + t_1^{\frac{8}{n-4}(1-\frac{n-4}{8}p)} + t_1^{(p+1)(1-\frac{n-4}{8}p)}) < \frac{1}{2}$, then S is a contraction mapping of B_0 in itself. We see that there is a solution on $I_0 = [0, t_1]$ from Banach's fixed point theorem.

Secondly, we prove that there exists a solution $w(t,x)$ for problem (4.2) on $I_1 = [t_1, t_2]$.

Taking $w(t_1)$ as an initial value, we have

$$w(t) = W(t)w(t_1) + i\int_{t_1}^{t_2} W(t-\tau)\left(\lambda_1|v+w|^p(v+w) + \lambda_2|v+w|^{\frac{8}{n-4}}(v+w) - \lambda_2|v|^{\frac{8}{n-4}}v\right)(\tau)d\tau.$$

We denote

$$B_1 = \left\{w : \|w\|_{L^\infty(I_1;H^2(R^n))} + \|w\|_{X^1(I_1)} \le \eta_1 = 2C_0|I_1|^{1-\frac{(n-4)p}{8}}\right\},$$

and define a mapping S as follows

$$Sw(t) = W(t)w(t_1) + i\int_{t_1}^{t_2} W(t-\tau)\left(\lambda_1|v+w|^p(v+w) + \lambda_2|v+w|^{\frac{8}{n-4}}(v+w) - \lambda_2|v|^{\frac{8}{n-4}}v\right)(\tau)d\tau.$$

Similarly by Lemma 2.3 and Lemmas 3.1–3.2, we have

$$\begin{aligned}
\|Sw\|_{B_1} &\le C_0\|w(t_1)\|_{H^2} + C\left\| |v+w|^{\frac{8}{n-4}}(v+w) - |v|^{\frac{8}{n-4}}v \right\|_{L^{\frac{2(n+4)}{n+8}}\left(I_1;L^{\frac{2(n+4)}{n+8}}\right)\cap L^2\left(I_1;\dot H^{1,\frac{2n}{n+2}}\right)} \\
&\quad + C\| |v+w|^p(v+w)\|_{L^{\gamma_4'}(I_1;L^{\rho_4'})\cap L^2\left(I_1;\dot H^{1,\frac{2n}{n+2}}\right)} \\
&\le C_0\|w(t_1)\|_{H^2} + C_1\left(\|v\|^{\frac{8}{n-4}}_{X^1(I_1)} + \|w\|^{\frac{8}{n-4}}_{X^1(I_1)}\right)\|w\|_{X^1(I_1)} + C_1|I_1|^{1-\frac{(n-4)p}{8}}\left(\|v\|^p_{X^1(I_1)} + \|w\|^p_{X^1(I_1)}\right)\|w\|_{X^1(I_1)} \\
&\le C_0\|w(t_1)\|_{H^2} + C_1\left(\varepsilon^{\frac{8}{n-4}} + \eta_1^{\frac{8}{n-4}}\right)\eta_1 + C_1|I_1|^{1-\frac{(n-4)p}{8}}(\varepsilon^p + \eta_1^p)\eta_1,
\end{aligned}$$

and

$$\begin{aligned}
\|Sw_1 - Sw_2\|_{B_1} &\le C\left\| |v+w_1|^{\frac{8}{n-4}}(v+w_1) - |v+w_2|^{\frac{8}{n-4}}(v+w_2) \right\|_{L^{\frac{2(n+4)}{n+8}}\left(I_1;L^{\frac{2(n+4)}{n+8}}\right)\cap L^2\left(I_1;\dot H^{1,\frac{2n}{n+2}}\right)} \\
&\quad + C\| |v+w_1|^p(v+w_1) - |v+w_2|^p(v+w_2)\|_{L^{\gamma_4'}(I_1;L^{\rho_4'})\cap L^2\left(I_1;\dot H^{1,\frac{2n}{n+2}}\right)}
\end{aligned}$$

$$\le C_1\left(2\|v\|_{X^1(I_1)}^{\frac{8}{n-4}}+\|w_1\|_{X^1(I_1)}^{\frac{8}{n-4}}+\|w_2\|_{X^1(I_1)}^{\frac{8}{n-4}}\right)\|w_1-w_2\|_{X^1(I_1)}$$
$$+C_1|I_1|^{1-\frac{(n-4)p}{8}}(2\|v\|_{X^1(I_1)}^p+\|w_1\|_{X^1(I_1)}^p+\|w_2\|_{X^1(I_1)}^p)\|w_1-w_2\|_{X^1(I_1)}$$
$$\le C_1\left(2\varepsilon^{\frac{8}{n-4}}+2\eta_1^{\frac{8}{n-4}}\right)\|w_1-w_2\|_{X^1(I_1)}+C_1|I_1|^{1-\frac{(n-4)p}{8}}(2\varepsilon^p+2\eta_1^p)\|w_1-w_2\|_{X^1(I_1)}.$$

Hence if we take t_2 such that $2C_1(\varepsilon^p(t_2-t_1)^{1-\frac{n-4}{8}p}+(2C_0)^{\frac{8}{n-4}}(t_2-t_1)^{\frac{8}{n-4}(1-\frac{n-4}{8}p)}+(2C_0)^p(t_2-t_1)^{(p+1)(1-\frac{n-4}{8}p)})<\frac{1}{2}$, then S is a contraction mapping of B_1 in itself. We see that there is a solution on $I_1=[t_1,t_2]$ from Banach's fixed point theorem.

Using an induction argument, we take

$$B_j=\left\{w:\|w\|_{L^\infty(I_j;H^2(R^n))}+\|w\|_{X^1(I_j)}\le\eta_j=(2C_0)^j|I_j|^{1-\frac{(n-4)p}{8}}\right\},$$

and let

$$Sw(t)=W(t)w(t_j)+i\int_{t_j}^{t_{j+1}}W(t-\tau)\left(\lambda_1|v+w|^p(v+w)+\lambda_2|v+w|^{\frac{8}{n-4}}(v+w)-\lambda_2|v|^{\frac{8}{n-4}}v\right)(\tau)d\tau.$$

If we take t_{j+1} such that $2C_1(\varepsilon^p(t_{j+1}-t_j)^{1-\frac{n-4}{8}p}+(2C_0)^{\frac{8}{n-4}j}(t_{j+1}-t_j)^{\frac{8}{n-4}(1-\frac{n-4}{8}p)}+(2C_0)^{pj}(t_{j+1}-t_j)^{(p+1)(1-\frac{n-4}{8}p)})<\frac{1}{2}$, then S is a contraction mapping of B_j in itself. We see that there is a solution on $I_j=[t_j,t_{j+1}]$ from Banach's fixed point theorem.

Finally we get a unique solution of (4.2) on $[0,T]$ such that

$$\|w(t)\|_{X^1([0,T])}\le\sum_{j=0}^{J'-1}\|w(t)\|_{X^1(I_j)}\le\sum_{j=0}^{J'-1}(2C_0)^jI_j^{1-\frac{n-4}{8}p}\le J'(2C_0)^{J'}T^{1-\frac{n-4}{8}p}\le C.$$

Thus we have $\|u\|_{X^1([0,T])}\le\|v\|_{X^1([0,T])}+\|w\|_{X^1([0,T])}\le C(\|\varphi\|_{H^2})$. □

Lemma 4.2. *For the problem (1.1), if $\lambda_2>0$, then*

$$\|u(\cdot,t)\|_{H^2}\le C(E,M),$$

where E, M are as in (1.2) and (1.3).

Proof. It is obvious in the case $\lambda_1\ge0$. So we only discuss the case $\lambda_1<0$.

Using Young's inequality with ε, we have

$$|u|^{p+2}\le\varepsilon\cdot\frac{p(n-4)}{8}|u|^{\frac{2n}{n-4}}+\varepsilon^{-\frac{p(n-4)}{8-(n-4)p}}\frac{8-(n-4)p}{8}|u|^2,\quad \varepsilon=\frac{4(n-4)(p+2)\lambda_2}{n(n-4)p|\lambda_1|}.$$

So

$$|u|^{p+2}\ge-\frac{(n-4)\lambda_2}{2n}|u|^{\frac{2n}{n-4}}+C(\lambda_1,\lambda_2,n,p)|u|^2,$$

substituting it into (1.2) and (1.3), we obtain the desired result. □

Proof of Theorem 1.1. We obtain Theorem 1.1 by Lemmas 4.1 and 4.2. □

Proof of Theorem 1.2. Firstly, we prove that $\|u\|_{Y^1(R)}$ for the case $\lambda_1,\lambda_2>0$ and $\|u\|_{Z^1(R)}$ for the case $\lambda_2>0$, $\frac{8}{n}\le p<\frac{8}{n-4}$ are bounded.

Case 1: $\lambda_1,\lambda_2>0$.

By Theorem 1.1 and Lemma 2.5, we have

$$\|u\|_{L^{n-1}\left(R,L^{\frac{2(n+1)}{n-1}}(R^n)\right)}\le C\|u\|_{L^\infty(R,H^2(R^n))}\le C(E,M).$$

So we take ε (which will be specified later), divide R into subintervals $I_0,I_1,\ldots,I_{J-1}$ such that on each I_j

$$\|u\|_{L^{n-1}\left(I_j,L^{\frac{2(n+1)}{n-1}}(R^n)\right)}\sim\varepsilon. \tag{4.4}$$

On the other hand, by (4.3), we take η (which will also be specified later), divide R into subintervals $I'_0,I'_1,\ldots,I'_{K-1}$ such that on each I'_k

$$\|v\|_{Y(I'_k)}\sim\eta. \tag{4.5}$$

For convenience, we denote one of I_j by $[a,b]$. Without loss of generality, we may assume that

$$[a,b]=\cup_{k=0}^{K'-1} I'_k, \quad K'\le K.$$

Similar to Lemma 4.1, by inductive arguments, if we take the above ε and η small enough, using Lemma 3.3, we can obtain that for the case $\frac{8}{n}<p\le\frac{8(n-1)}{n^2-3n+8}$:

$$\|w\|_{Y^1(I'_k)}\le (2C)^k\varepsilon^{\frac{(np-8)(n-1)}{2(n+2)}}; \tag{4.6}$$

and for the case $\frac{8(n-1)}{n^2-3n+8}<p<\frac{8}{n-4}$:

$$\|w\|_{Y^1(I'_k)}\le (2C)^k\varepsilon^{\frac{[8-(n-4)p](n-1)}{2(n+2)}}. \tag{4.7}$$

In the following, we only prove the case $\frac{8}{n}<p\le\frac{8(n-1)}{n^2-3n+8}$, the case $\frac{8(n-1)}{n^2-3n+8}<p<\frac{8}{n-4}$ is similar, so here we omit it. Indeed, by (4.2) and Lemmas 2.3 and 3.3, we have

$$\begin{aligned}
\|w\|_{Y^1(I'_k)} &\le C\|w(t_k)\|_{H^2}+C\big\||u|^p u\big\|_{L^2\left(I'_k;L^{\frac{2n}{n+4}}\right)}+C\big\||u|^p u\big\|_{L^2\left(I'_k;\dot H^{1,\frac{2n}{n+2}}\right)}\\
&\quad +C\big\||v+w|^{\frac{8}{n-4}}(v+w)-|v|^{\frac{8}{n-4}}v\big\|_{L^{\frac{2(n+4)}{n+8}}\left(I'_k;L^{\frac{2(n+4)}{n+8}}\right)}\\
&\quad +C\big\||v+w|^{\frac{8}{n-4}}(v+w)-|v|^{\frac{8}{n-4}}v\big\|_{L^2\left(I'_k;\dot H^{1,\frac{2n}{n+2}}\right)}\\
&\le C\|w(t_k)\|_{H^2}+C\varepsilon^{\frac{(np-8)(n-1)}{2(n+2)}}\left(\eta+\|w\|_{Y^1(I'_k)}\right)+C\eta^{\frac{8}{n-4}}\|w\|_{Y^1(I'_k)}+C\eta\|w\|_{Y^1(I'_k)}^{\frac{8}{n-4}}+C\|w\|_{Y^1(I'_k)}^{\frac{n+4}{n-4}}.
\end{aligned}$$

So (4.6) holds by taking ε and η small enough.

Thus

$$\|w\|_{Y^1[a,b]}\le\sum_{k=0}^{K'-1}\|w\|_{Y^1(I'_k)}\le\begin{cases}\displaystyle\sum_{k=0}^{K'-1}(2C)^k\varepsilon^{\frac{(np-8)(n-1)}{2(n+2)}}, & \displaystyle\frac{8}{n}<p\le\frac{8(n-1)}{n^2-3n+8},\\ \displaystyle\sum_{k=0}^{K'-1}(2C)^k\varepsilon^{\frac{[8-(n-4)p](n-1)}{2(n+2)}}, & \displaystyle\frac{8(n-1)}{n^2-3n+8}<p<\frac{8}{n-4}.\end{cases}$$

Furthermore we have

$$\|u\|_{Y^1[a,b]}\le\|w\|_{Y^1[a,b]}+\|v\|_{Y^1[a,b]}\le C(\|\varphi\|_{H^2}).$$

Noting that $[a,b]$ is arbitrary, we have that

$$\|u\|_{Y^1(R)}\le\sum_{j=0}^{J-1}\|u\|_{Y^1(I_j)}\le JC(\|\varphi\|_{H^2}).$$

Using Lemma 2.3 and Lemmas 3.2–3.3, we obtain

$$\|u\|_{L^q(R;H^{2,r})}\le C(\|\varphi\|_{H^2})$$

for any biharmonic admissible pair (q,r).

Remark. Obviously, we have $\|u\|_{Z^1(R)}$ is bounded. We will use this fact later.

Case 2: $\lambda_2>0$, $\frac{8}{n}\le p<\frac{8}{n-4}$.

By (4.3), we take ε (which will also be specified later), divide R into subintervals $I_0,I_1,\dots,I_{K-1}$ such that on each I_k

$$\|v\|_{\dot Z^1(I_k)}\sim\varepsilon. \tag{4.8}$$

We claim that for small M and ε

$$\|v\|_{Z^1(I_k)}\le C\varepsilon. \tag{4.9}$$

Indeed, by Lemma 2.1 and Corollary 2.2, we have

$$\|v\|_{\dot Z^0(I_k)}\le C\|v\|_{L^2_x}+C\left\||v|^{\frac{8}{n-4}}v\right\|_{L_T^{\frac{2(n+4)}{n+8}}L_x^{\frac{2(n+4)}{n+8}}}\le CM+C\varepsilon^{\frac{8}{n-4}}\|v\|_{\dot Z^0(I_k)}.$$

Obviously if we take $C\varepsilon^{\frac{8}{n-4}} < \frac{1}{2}$, then we have $\|v\|_{\dot{Z}^0(I_k)} \le 2CM$. Furthermore, if we take M small enough, then we have

$$\|v\|_{Z^1(I_k)} = \|v\|_{\dot{Z}^0(I_k)} + \|v\|_{\dot{Z}^1(I_k)} \le C\varepsilon.$$

Similar to Lemma 4.1, by inductive arguments, using Lemma 3.4, if we take the above ε and M small enough, we have

$$\|w\|_{Z^1(I_k)} \le (2C)^k M^{\frac{8}{n-4}-p} \le C.$$

Thus

$$\|u\|_{Z^1(I_k)} \le \|w\|_{Z^1(I_k)} + \|v\|_{Z^1(I_k)} \le C(\|\varphi\|_{H^2}),$$
$$\|u\|_{L^q(R,H^{2,r})} \le C(\|\varphi\|_{H^2}),$$

for all biharmonic admissible pair (q,r).

Secondly, we prove the asymptotic state.

For $0 < t < +\infty$, we define

$$u_+(t) = \varphi + i\int_0^t W(-s)\left[\lambda_1|u|^p u + \lambda_2|u|^{\frac{8}{n-4}}u\right](s)ds,$$
$$u_+ = \varphi + i\int_0^{+\infty} W(-s)\left[\lambda_1|u|^p u + \lambda_2|u|^{\frac{8}{n-4}}u\right](s)ds,$$

and for $-\infty < t < 0$, we define

$$u_-(t) = \varphi + i\int_t^0 W(-s)\left[\lambda_1|u|^p u + \lambda_2|u|^{\frac{8}{n-4}}u\right](s)ds,$$
$$u_- = \varphi + i\int_{-\infty}^0 W(-s)\left[\lambda_1|u|^p u + \lambda_2|u|^{\frac{8}{n-4}}u\right](s)ds.$$

For $0 < t < \tau$, by Lemmas 2.3 and 3.4, we have

$$\begin{aligned}
\|u_+(t) - u_+(\tau)\|_{H_x^2} &= \left\|\int_\tau^t W(-s)\left(\lambda_1|u|^p u + \lambda_2|u|^{\frac{8}{n-4}}u\right)(s)ds\right\|_{H_x^2} \\
&\le \left\|\int_\tau^t W(-s)\left(\lambda_1|u|^p u + \lambda_2|u|^{\frac{8}{n-4}}u\right)(s)ds\right\|_{L_t^\infty([\tau,t];H_x^2(R^n))} \\
&\le C\left\|\lambda_1|u|^p u + \lambda_2|u|^{\frac{8}{n-4}}u\right\|_{L^{\frac{2(n+4)}{n+8}}\left([\tau,t];L^{\frac{2(n+4)}{n+8}}(R^n)\right)} + \left\|\lambda_1|u|^p u + \lambda_2|u|^{\frac{8}{n-4}}u\right\|_{L^2\left([\tau,t];\dot{H}^{1,\frac{2n}{n+2}}(R^n)\right)} \\
&\le C\left(\|u\|_{Z^1([\tau,t])}^{p+1} + \|u\|_{Z^1([\tau,t])}^{\frac{n+4}{n-4}}\right),
\end{aligned}$$

which combines the fact $\|u\|_{Z^1(R)}$ is bounded, implying $u_+(t)$ is a Cauchy sequence in $H^2(R^n)$, thus $u_+(t)$ will converge to some function in $H^2(R^n)$ as $t \to +\infty$. Obviously, this function must be function u^+.

Moreover, we have

$$\begin{aligned}
\|u(t) - W(t)u_+\|_{H^2(R^n)} &= \|W(-t)u(t) - u_+\|_{H^2(R^n)} \\
&\le \left\|\int_t^{+\infty} W(-s)\left(\lambda_1|u|^p u + \lambda_2|u|^{\frac{8}{n-4}}u\right)(s)ds\right\|_{L_t^\infty([t,+\infty);H_x^2(R^n))} \\
&\le C\left\|\lambda_1|u|^p u + \lambda_2|u|^{\frac{8}{n-4}}u\right\|_{L^{\frac{2(n+4)}{n+8}}\left([t,+\infty);L^{\frac{2(n+4)}{n+8}}(R^n)\right)} \\
&\quad + \left\|\lambda_1|u|^p u + \lambda_2|u|^{\frac{8}{n-4}}u\right\|_{L^2\left([t,+\infty);\dot{H}^{1,\frac{2n}{n+2}}(R^n)\right)} \\
&\le C\left(\|u\|_{Z^1([t,+\infty))}^{p+1} + \|u\|_{Z^1([t,+\infty))}^{\frac{n+4}{n-4}}\right) \to 0, \quad t \to +\infty,
\end{aligned}$$

so we have

$$\lim_{t\to+\infty}\|u(t) - W(t)u_+\|_{H^2(R^n)} = 0.$$

Using similar arguments, we can obtain

$$\lim_{t\to-\infty} \|u(t) - W(t)u_-\|_{H^2(R^n)} = 0. \quad \square$$

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References

[1] V.I. Karpman, Stabilization of soliton instabilities by higher-order dispersion: fourth-order nonlinear Schrödinger-type equations, Phys. Rev. E 53 (1996) 1336–1339.
[2] V.I. Karpman, A.G. Shagalov, Stability of solitons described by nonlinear Schrödinger-type equations with higher-order dispersion, Physica D 144 (2000) 194–210.
[3] C. Miao, G. Xu, L. Zhao, Global well-posedness and scattering for the focusing energy-critical nonlinear Schrödinger equations of fourth order in the radial case, J. Differential Equations 246 (2009) 3715–3749.
[4] C. Miao, G. Xu, L. Zhao, Global well-posedness and scattering for the focusing energy critical nonlinear Schrödinger equations of fourth-order in dimensions $d \geq 9$, J. Differential Equations 251 (2011) 3381–3402.
[5] B. Pausader, Global well-posedness for energy critical fourth-order Schrödinger equations in the radial case, Dyn. Partial Differ. Equ. 4 (2007) 197–225.
[6] B. Pausader, The cubic fourth-order Schrödinger equation, J. Funct. Anal. 256 (2009) 2473–2517.
[7] J.Y. Zhang, J.Q. Zheng, Energy critical fourth-order Schrödinger equations with subcritical perturbations, Nonlinear Anal. 73 (2010) 1004–1014.
[8] T. Tao, M. Visan, X.Y. Zhang, The nonlinear Schrödinger equation with combined power-type nonlinearities, Comm. Partial Differential Equations 32 (2007) 1281–1343.
[9] Xiaoyi Zhang, On the Cauchy problem of 3-D energy-critical Schrödinger equations with subcritical perturbations, J. Differential Equations 230 (2006) 422–445.
[10] T. Cazenave, Semilinear Schrödinger Equations, American Mathematical Society, 2003.
[11] M. Ben-Artzi, H. Koch, J.C. Saut, Dispersion estimates for fourth-order nonlinear Schrödinger equation, C. R. Math. Acad. Sci. Paris 330 (2000) 87–92.
[12] S.B. Cui, C.H. Guo, Well-posedness of higher-order nonlinear Schrödinger equations in Sobolev spaces $H^s(R^n)$ and applications, Nonlinear Anal. 67 (2007) 687–707.
[13] G. Fibich, B. Ilan, G. Papanicolaou, Self focusing with fourth-order dispersion, SIAM J. Appl. Math. 62 (2002) 1437–1462.
[14] C.H. Guo, S.B. Cui, Global existence of solutions for a fourth-order nonlinear Schrödinger equation, Appl. Math. Lett. 19 (2006) 706–711.
[15] C.H. Guo, Global existence of solutions for a fourth-order nonlinear Schrödinger equation in $n+1$ dimensions, Nonlinear Anal. 73 (2010) 555–563.
[16] C. Hao, L. Hsiao, B.X. Wang, Well-posedness for the fourth-order Schrödinger equations, J. Math. Anal. Appl. 320 (2006) 246–265.
[17] C. Hao, L. Hsiao, B.X. Wang, Well-posedness of the Cauchy problem for the fourth-order Schrödinger equations in high dimensions, J. Math. Anal. Appl. 328 (2007) 58–83.
[18] J. Segata, Modified wave operators for the fourth-order nonlinear Schrödinger type equation with cubic nonlinearity, Math. Methods Appl. Sci. 29 (2006) 1785–1800.
[19] C.X. Miao, B. Zhang, Global well-posedness of the Cauchy problem for nonlinear Schrödinger-type equations, Discrete Contin. Dyn. Syst. 17 (2007) 181–200.
[20] C.X. Miao, H.G. Wu, J.Y. Zhang, Scattering theory below energy for the cubic fourth-order Schrödinger equation, arXiv: 0907.5047v2.
[21] J. Bergh, J. Löfström, Interpolation Spaces, Springer-Verlag, 1976.