J. Math. Anal. Appl. 396 (2012) 578–589

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Pointwise multipliers of Besov spaces on Carnot–Carathéodory spaces

Yanchang Han

School of Mathematical Sciences, South China Normal University, Guangzhou, 510631, China

ARTICLE INFO

Article history:
Received 4 April 2012
Available online 5 July 2012
Submitted by Jie Xiao

Keywords:
Pointwise multipliers
Besov spaces
Calderón reproducing formula
Carnot–Carathéodory space
Space of homogeneous type

ABSTRACT

Let $(\mathcal{X}, d, \mu)$ be a Carnot–Carathéodory space, namely, $\mathcal{X}$ is a smooth manifold, d is a control, or Carnot–Carathéodory, metric induced by a collection of vector fields of finite type. μ is a nonnegative Borel regular measure on $\mathcal{X}$ satisfying that there exists constant $C_0 \in [1, \infty)$ such that for all $x \in \mathcal{X}$ and $0 < r <$diam$\mathcal{X}$,

$$\begin{aligned}\mu(B(x, 2r)) &:= \mu(\{y \in \mathcal{X} : d(x, y) < 2r\})\\ &\le C_0\mu(B(x, r)) < \infty \quad \text{(doubling property)}.\end{aligned}$$

Using the discrete Calderón reproducing formula and the Plancherel–Pôlya characterization of the inhomogeneous Besov spaces developed by Han et al. [12], and Han et al. (2008) [10], pointwise multipliers of inhomogeneous Besov spaces are obtained.

© 2012 Elsevier Inc. All rights reserved.

1. Introduction

The multiplier theory of function spaces has been studied for a long time and a lot of results have been obtained. We know that the multiplier theory is one of the important parts in the studies of the Gleason problem, function space properties and the general operator theory. The pointwise multipliers on $\mathbb{R}^d$ are studied as a part of the research of function spaces in several monographs, cf. [1–8]. Pointwise multipliers have been found many important applications in partial differential equations.

However, it was not clear how to generalize the pointwise multipliers on $\mathbb{R}^n$ to spaces of homogeneous type introduced by Coifman and Weiss (see [9]) because the Fourier transform is no longer available. The main purpose of this paper is to establish pointwise multipliers on inhomogeneous Besov spaces in the setting of Carnot–Carathéodory spaces. To be more precisely, we first recall some necessary definitions. In this paper, we always assume that $(\mathcal{X}, d)$ is a metric space with a regular Borel measure μ such that all balls defined by d have finite and positive measures. In what follows, set diam $(\mathcal{X}) \equiv \sup\{d(x, y) : x, y \in \mathcal{X}\}$ and for any $x \in \mathcal{X}$ and $r > 0$, set $B(x, r) \equiv \{y \in \mathcal{X} : d(x, y) < r\}$.

Definition 1.1 (*[10]*)**.** Let $(\mathcal{X}, d)$ be a metric space with a Borel regular measure μ such that all balls defined by d have finite and positive measures. The triple $(\mathcal{X}, d, \mu)$ is called a space of homogeneous type if there exists a constant $C_0 \in [1, \infty)$ such that for all $x \in \mathcal{X}$ and $r > 0$,

$$\mu(B(x, 2r)) \le C_0\mu(B(x, r)) \quad \text{(doubling property)}. \tag{1.1}$$

Remark 1.2. We point out that the doubling condition (1.1) implies that there exist positive constants C and n such that for all $x \in \mathcal{X}$ and $\lambda \ge 1$,

$$\mu((B(x, \lambda r))) \le C\lambda^n\mu(B(x, r)), \tag{1.2}$$

where C is independent of x and r. Denote by n the homogeneous "dimension" of $\mathcal{X}$ as in [10].

E-mail addresses: hanyc291scnu@hotmail.com, hanych@scnu.edu.cn.

0022-247X/$ – see front matter © 2012 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2012.06.052

A space of homogeneous type is called a RD-space, if there exist constants $a_0, \widetilde{C}_0 \in (1, \infty)$ such that for all $x \in \mathcal{X}$ and $0 < r < \operatorname{diam}(\mathcal{X})/a_0$,

$$\widetilde{C}_0 \mu(B(x, r)) \le \mu(B(x, a_0 r))$$

i.e., some "reverse" doubling condition holds.

Clearly, any Ahlfors n-regular metric measure space $(\mathcal{X}, d, \mu)$ (which means that there exists some $n > 0$ such that $\mu(B(x, r)) \sim r^n$ for $x \in \mathcal{X}$ and $0 < r < \operatorname{diam}(\mathcal{X})/2$) is a (n, n)-space, also is a RD-space and a space of homogeneous type in the sense of Coifman in [10]. In other words, μ satisfies the doubling condition which is weaker than Ahlfors n-regular metric measure spaces and RD-spaces.

Another such typical space is the Carnot–Carathéodory space. One example with unbounded total measure studied in [11] is that $\mathcal{X}$ arises as the boundary of an unbounded model polynomial domain in $\mathbb{C}^2$. Let $\Omega = \{(z, w) \in \mathbb{C}^2 : \operatorname{Im}(w) > P(z)\}$, where P is a real, subharmonic, non-harmonic polynomial of degree m. Then $\mathcal{X} = \partial\Omega$ can be identified with $\mathbb{C} \times \mathbb{R} = \{(z, t) : z \in \mathbb{C}, t \in \mathbb{R}\}$. The basic (0, 1) Levi vector field is then $\bar{Z} = \frac{\partial}{\partial \bar{z}} - i\frac{\partial P}{\partial \bar{z}}\frac{\partial}{\partial t}$, and we write $\bar{Z} = \mathbb{X}_1 + i\mathbb{X}_2$. The real vector fields $\{\mathbb{X}_1, \mathbb{X}_2\}$ and their commutators of order $\le m$ span the tangent space to $\mathcal{X}$ at each point. See [12,10] for more details and references therein.

We also shall suppose that $\mu(\mathcal{X}) = \infty$, $\mu(\{x\}) = 0$ for all $x \in \mathcal{X}$. For any $x, y \in \mathcal{X}$ and $\delta > 0$, set $V_\delta(x) \equiv \mu(B(x, \delta))$ and $V(x, y) \equiv \mu(B(x, d(x, y)))$. It follows from (1.1) that $V(x, y) \sim V(y, x)$. The following notion of approximations of the identity on RD-spaces was first introduced in [10].

We begin with recalling the definition of an approximation to the identity, which plays the same role as the heat kernel $H(s, x, y)$ does in Nagel–Stein's theory [11]. Let $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

Definition 1.3 (*[12,10]*). A sequence $\{S_k\}_{k\in\mathbb{Z}_+}$ of operators is said to be an approximation to the identity (in short, ATI) if there exists constant $C_1 > 0$ such that for all $k \in \mathbb{Z}_+$ and all x, x', y and $y' \in \mathcal{X}$, $S_k(x, y)$, the kernel of S_k satisfies the following conditions:

$$S_k(x, y) = 0 \quad \text{if } \rho(x, y) \ge C_1 2^{-k} \quad \text{and} \quad |S_k(x, y)| \lesssim \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y)}; \tag{1.3}$$

$$|S_k(x, y) - S_k(x', y)| \lesssim 2^k \rho(x, x') \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y)} \tag{1.4}$$

for $\rho(x, x') \le \max\{C_1, 1\}2^{-k}$;

$$|S_k(x, y) - S_k(x, y')| \lesssim 2^k \rho(y, y') \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y)} \tag{1.5}$$

for $\rho(y, y') \le \max\{C_1, 1\}2^{-k}$;

$$|[S_k(x, y) - S_k(x, y')] - [S_k(x', y) - S_k(x', y')]| \lesssim 2^{2k} \rho(x, x')\rho(y, y') \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y)} \tag{1.6}$$

for $\rho(x, x') \le \max\{C_1, 1\}2^{-k}$ and $\rho(y, y') \le \max\{C_1, 1\}2^{-k}$;

$$\int_{\mathcal{X}} S_k(x, y) d\mu(y) = \int_{\mathcal{X}} S_k(x, y) d\mu(x) = 1. \tag{1.7}$$

The space of test functions plays a key role in this paper; see [10].

Definition 1.4. Fix two exponents $0 < \beta \le 1$ and $\gamma > 0$. A function f defined on $\mathcal{X}$ is said to be a *test function* of type (β, γ) centered at $x_0 \in \mathcal{X}$ with width $r > 0$ if there exists a nonnegative constant C such that f satisfies the following conditions:

$$|f(x)| \le C \frac{1}{(V_r(x_0) + V(x, x_0))} \frac{r^\gamma}{(r + d(x, x_0))^\gamma}; \tag{1.8}$$

$$|f(x) - f(x')| \le C \left(\frac{d(x, x')}{r + d(x, x_0)}\right)^\beta \frac{1}{(V_r(x_0) + V(x, x_0))} \frac{r^\gamma}{(r + d(x, x_0))^\gamma} \tag{1.9}$$

for $d(x, x') \le \frac{1}{2}(r + d(x, x_0))$.

If f is a test function of type (β, γ) centered at x_0 with width $r > 0$, we write $f \in \mathcal{M}(x_0, r, \beta, \gamma)$, and the norm of f in $\mathcal{M}(x_0, r, \beta, \gamma)$ is defined by

$$\|f\|_{\mathcal{M}(x_0, r, \beta, \gamma)} = \inf\{C \ge 0 : \text{(1.8) and (1.9) hold}\}.$$

We denote by $\mathcal{M}(\beta,\gamma)$ the class of all $f \in \mathcal{M}(x_0, 1, \beta, \gamma)$. It is easy to see that $\mathcal{M}(x_1, r, \beta, \gamma) = \mathcal{M}(\beta,\gamma)$ with the equivalent norms for all $x_1 \in \mathcal{X}$ and $r > 0$. Furthermore, it is also easy to check that $\mathcal{M}(\beta,\gamma)$ is a Banach space with respect to the norm in $\mathcal{M}(\beta,\gamma)$.

In what follows, for given $\epsilon \in (0, 1]$, we let $\widetilde{\mathcal{M}}(\beta,\gamma)$ be the completion of the space $\mathcal{M}(\epsilon,\epsilon)$ in $\mathcal{M}(\beta,\gamma)$ when $0 < \beta, \gamma \le \epsilon$. Obviously $\widetilde{\mathcal{M}}(\epsilon,\epsilon) = \mathcal{M}(\epsilon,\epsilon)$. Moreover, $f \in \widetilde{\mathcal{M}}(\beta,\gamma)$ if and only if $f \in \mathcal{M}(\beta,\gamma)$ when $0 < \beta, \gamma \le \epsilon$ and there exists $\{f_j\}_{j\in\mathbb{N}} \subset \mathcal{M}(\epsilon,\epsilon)$ such that $\|f - f_j\|_{\mathcal{M}(\beta,\gamma)} \to 0$ as $j \to \infty$. If $f \in \widetilde{\mathcal{M}}(\beta,\gamma)$, we then define $\|f\|_{\widetilde{\mathcal{M}}(\beta,\gamma)} = \|f\|_{\mathcal{M}(\beta,\gamma)}$. Obviously $\widetilde{\mathcal{M}}(\beta,\gamma)$ is a Banach space and we also have $\|f\|_{\widetilde{\mathcal{M}}(\beta,\gamma)} = \lim_{j\to\infty} \|f_j\|_{\mathcal{M}(\beta,\gamma)}$ for the above chosen $\{f_j\}_{j\in\mathbb{N}}$.

We denote by $(\widetilde{\mathcal{M}}(\beta,\gamma))'$ the dual space of $\widetilde{\mathcal{M}}(\beta,\gamma)$ consisting of all linear functionals $\mathcal{L}$ from $\widetilde{\mathcal{M}}(\beta,\gamma)$ to $\mathbb{C}$ with the property that there exists a constant C such that for all $f \in \widetilde{\mathcal{M}}(\beta,\gamma)$,

$$|\mathcal{L}(f)| \le C\|f\|_{\widetilde{\mathcal{M}}(\beta,\gamma)}.$$

We denote by $\langle h, f\rangle$ the natural pairing of elements $h \in (\widetilde{\mathcal{M}}(\beta,\gamma))'$ and $f \in \widetilde{\mathcal{M}}(\beta,\gamma)$. Since $\widetilde{\mathcal{M}}(x_1, r, \beta, \gamma) = \widetilde{\mathcal{M}}(\beta,\gamma)$ with the equivalent norms for all $x_1 \in \mathcal{X}$ and $r > 0$, thus, for all $h \in (\widetilde{\mathcal{M}}(\beta,\gamma))'$, $\langle h, f\rangle$ is well defined for all $f \in \widetilde{\mathcal{M}}(x_0, r, \beta, \gamma)$ with $x_0 \in \mathcal{X}$ and $r > 0$.

The following constructions, which provide an analogue of the grid of Euclidean dyadic cubes on spaces of homogeneous type, were given by Christ in [13].

Lemma 1.5. *Let $\mathcal{X}$ be a space of homogeneous type. Then there exist a collection $\{Q_\alpha^k \subset \mathcal{X} : k \in \mathbb{Z}_+, \alpha \in I_k\}$ of open subsets, where I_k is some (possible finite) index set, and constants $\delta \in (0, 1)$ and $C_5, C_6 > 0$ such that*

(i) $\mu(\mathcal{X} \setminus \cup_\alpha Q_\alpha^k) = 0$ *for each fixed k and $Q_\alpha^k \cap Q_\beta^k = \emptyset$ if $\alpha \neq \beta$;*
(ii) *for any α, β, k, l with $l \ge k$, either $Q_\beta^l \subset Q_\alpha^k$ or $Q_\beta^l \cap Q_\alpha^k = \emptyset$;*
(iii) *for each (k, α) and each $l < k$ there is a unique β such that $Q_\alpha^k \subset Q_\beta^l$;*
(iv) diam $(Q_\alpha^k) \le C_2\delta^k$;
(v) *each Q_α^k contains some ball $B(z_\alpha^k, C_3\delta^k)$, where $z_\alpha^k \in \mathcal{X}$.*

In fact, we can think of Q_α^k as being a dyadic cube with a diameter roughly δ^k and centered at z_α^k. In what follows, we always suppose $\delta = 1/2$. See [14] for how to remove this restriction. Also, in the following, for $k \in \mathbb{Z}_+, \tau \in I_k$, we will denote by $Q_\tau^{k,\nu}$, $\nu = 1, \ldots, N(k, \tau, M)$, the set of all cubes $Q_{\tau'}^{k+M} \subset Q_\tau^k$, where M is a fixed large positive integer.

Now, we can introduce the inhomogeneous Besov spaces $B_p^{\alpha,q}(\mathcal{X})$ via the approximation in Definition 1.3. Note that the Besov spaces have been already investigated for decades in the study of partial differential equations, interpolation theory and approximation theory.

Definition 1.6. Suppose that $\{S_k\}_{k\in\mathbb{Z}_+}$ is an ATI and let $D_0 = S_0$, and $D_k = S_k - S_{k-1}$ for $k \in \mathbb{N}$. Let M be a fixed large positive integer, $Q_\tau^{0,\nu}$ be as above. Suppose that $-1 < s < 1$.

The *inhomogeneous Besov space* $B_p^{s,q}(\mathcal{X})$ for $\max(\frac{n}{n+1}, \frac{n}{n+1+s}) < p \le \infty, 0 < q \le \infty$ is the collection of all $f \in (\widetilde{\mathcal{M}}(\beta,\gamma))'$, for some β and γ satisfying

$$\max\left(0, s, -s + n\left(\frac{1}{\min\{p,1\}} - 1\right)\right) < \beta < 1, \qquad \frac{n}{\min\{p,1\}} - n < \gamma < 1 \tag{1.10}$$

such that

$$\|f\|_{B_p^{s,q}(\mathcal{X})} = \left\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu})[m_{Q_\tau^{0,\nu}}(|D_0(f)|)]^p\right\}^{\frac{1}{p}} + \left\{\sum_{k=1}^{\infty}[2^{ks}\|D_k(f)\|_{L^p(\mathcal{X})}]^q\right\}^{\frac{1}{q}} < \infty$$

where $m_{Q_\tau^{0,\nu}}(D_0(f))$ are averages of $D_0(f)$ over $Q_\tau^{0,\nu}$.

The restrictions (1.10) guarantee that the definitions of the *inhomogeneous Besov spaces* $B_p^{s,q}(\mathcal{X})$ for $\max(\frac{n}{n+1}, \frac{n}{n+1+s}) < p \le \infty, 0 < q \le \infty$ are independent of the choices of β and γ satisfying these conditions and $B_p^{s,q}(\mathcal{X}) \subset (\widetilde{\mathcal{M}}(\beta,\gamma))'$, $\widetilde{\mathcal{M}}(\beta,\gamma) \subset \mathcal{M}(\beta,\gamma) \subset B_p^{s,q}(\mathcal{X})$ in [10].

The classical scale of inhomogeneous Besov spaces contains many well-known function spaces. For example, if $\alpha > 0, p = q = \infty$, one recovers the Hölder–Zygmund spaces $\mathcal{C}^\alpha(\mathcal{X})$, i.e. $B_\infty^{\alpha,\infty}(\mathcal{X}) = \mathcal{C}^\alpha(\mathcal{X}), \alpha > 0$. The space $\mathcal{C}^\alpha(\mathcal{X})$ is defined as the collection of f such that

$$\|f\|_{\mathcal{C}^\alpha(\mathcal{X})} = \|f\|_\infty + \sup_{x\neq y}\frac{|f(x) - f(y)|}{d(x,y)^\alpha} < \infty.$$

If $1 > s > 0$, $1 < p < \infty$ and $1 \le q \le \infty$, then $B_p^{s,q}(\mathcal{X})$ coincides with the classical Besov–Lipschitz spaces $A_p^{s,q}(\mathcal{X})$.

The inhomogeneous Besov spaces have the following Plancherel–Pôlya characterizations in [10], which will be one of the basic tools to prove the main result of this paper.

Lemma 1.7. *Let $\{D_k\}_{k\in\mathbb{Z}_+}$ be as in Definition 1.6, $-1 < s < 1$. Then, if $\max(\frac{n}{n+1}, \frac{n}{n+1+s}) < p \le \infty$, $0 < q \le \infty$, for all $f \in (\widetilde{\mathcal{M}}(\beta,\gamma))'$ with β, γ satisfying (1.10), we have*

$$\|f\|_{B^{s,q}_p(\mathcal{X})} \sim \left\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu})[m_{Q_\tau^{0,\nu}}(|D_0(f)|)]^p\right\}^{\frac{1}{p}} + \left\{\sum_{k=1}^{\infty}\left[\sum_{\tau\in I_k}\sum_{\nu=1}^{N(\tau,k,M)} \mu(Q_\tau^{k,\nu})\left(2^{ks}\inf_{z\in Q_\tau^{k,\nu}}|D_k(f)(z)|\right)^p\right]^{\frac{q}{p}}\right\}^{\frac{1}{q}}$$
$$\sim \left\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu})[m_{Q_\tau^{0,\nu}}(|D_0(f)|)]^p\right\}^{\frac{1}{p}} + \left\{\sum_{k=1}^{\infty}\left[\sum_{\tau\in I_k}\sum_{\nu=1}^{N(\tau,k,M)} \mu(Q_\tau^{k,\nu})\left(2^{ks}\sup_{z\in Q_\tau^{k,\nu}}|D_k(f)(z)|\right)^p\right]^{\frac{q}{p}}\right\}^{\frac{1}{q}}.$$

We now introduce the following definition of the pointwise multiplier.

Definition 1.8. Suppose that g is a given function on $\mathcal{X}$. Then g is called a pointwise multiplier for $B^{s,q}_p(\mathcal{X})$ if $f \to gf$ admits a bounded linear mapping from $B^{s,q}_p(\mathcal{X})$ into itself.

The main result in this paper is the following.

Theorem 1.9. *Let $-1 < s < 1$, $\max(\frac{n}{n+1}, \frac{n}{n+1+s}) < p \le \infty$, $0 < q \le \infty$, then $g \in \mathcal{C}^\alpha(\mathcal{X})$ with $1 > \alpha > \max(s, \frac{n}{\min\{p,1\}} - n - s)$, is a multiplier for $B^{s,q}_p(\mathcal{X})$. In other words, $f \to gf$ yields a bounded linear mapping from $B^{s,q}_p(\mathcal{X})$ into itself and there is a positive constant C such that*

$$\|gf\|_{B^{s,q}_p(\mathcal{X})} \le C\|g\|_{\mathcal{C}^\alpha(\mathcal{X})}\|f\|_{B^{s,q}_p(\mathcal{X})} \tag{1.11}$$

holds for all $g \in \mathcal{C}^\alpha(\mathcal{X})$ and $f \in B^{s,q}_p(\mathcal{X})$.

We would like to point out that the study of pointwise multipliers is one of the important problems in the theory of function spaces. It has attracted a lot of attention in the decades since starting with [7]. Pointwise multipliers in general spaces $B^{s,q}_p(\mathbb{R}^n)$ where $0 < p \le \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$ have been studied in great detail in [5,8] and in the more recent paper [4].

Theorem 1.9 was proved in [8] for pointwise multipliers of inhomogeneous Besov spaces on $\mathbb{R}^n$ based on the Fourier transform. In the present setting, however, we do not have the Fourier transform at our disposal. Since the Fourier transform on Carnot–Carathéodory spaces is not available and hence the idea used in [8] does not work for this more general setting, a new idea to prove Theorem 1.9 is to use the discrete Calderón reproducing formula, which was developed in [10]. Therefore this scheme easily extends to geometrical settings where the Fourier transform does not exist. The Fourier transform is missing but a version of the pointwise multiplier is still present.

We would also like to point out that the above restrictions for α are sharp in the following sense. Let $s \in (-1, 1)$, max $(\frac{n}{n+1}, \frac{n}{n+1+s}) < p \le \infty$, $0 < q \le \infty$ and $\alpha > \max(s, \frac{n}{\min\{p,1\}} - n - s)$, then there exists a function $g \in \mathcal{C}^\alpha(\mathbb{R}^n)$ which is not a pointwise multiplier for $B^{s,q}_p(\mathbb{R}^n)$.

Throughout, we also denote by C a positive constant independent of main parameters involved, which may vary at different occurrences. Constants with subscripts do not change through the whole paper. We use $f \lesssim g$ and $f \gtrsim g$ to denote $f \le Cg$ and $f \ge Cg$, respectively. If $f \lesssim g \lesssim f$, we then write $f \sim g$. For any $a, b \in \mathbb{R}$, set $a \wedge b \doteq \min\{a, b\}$, $a \vee b \doteq \max\{a, b\}$. If $p > 1$, set $\frac{1}{p} + \frac{1}{p'} = 1$.

2. Proof of Theorem 1.9

In this section, we will prove Theorem 1.9. Since there are no Fourier transforms on spaces of homogeneous type, the proof of Theorem 1.9 is quite different from the proof of Theorem 2.8.2 in [8]. The key new ingredient in the proof of Theorem 1.9 is to apply the following discrete Calderón reproducing formula established in [12,10]. This formula can be stated as follows.

Lemma 2.1. *Suppose that $\{S_k\}_{k\in\mathbb{Z}_+}$ is an approximation to the identity as in Definition 1.3. Set $D_k = S_k - S_{k-1}$ for $k \in \mathbb{N}$ and $D_0 = S_0$. Then for any fixed $M \in \mathbb{N}$ large enough, there exists a family of functions $\{\widetilde{D}_k(x,y)\}_{k\in\mathbb{Z}_+}$ and $\{\widetilde{\widetilde{D}}_k(x,y)\}_{k\in\mathbb{Z}_+}$ such* *that for any fixed* $y_\tau^{k,\nu} \in Q_\tau^{k,\nu}$, $k \in \mathbb{N}$, $\tau \in I_k$ *and* $\nu \in \{1, \ldots, N(k, \tau, M)\}$ *and all* $f \in (\widetilde{\mathcal{M}}(\beta, \gamma))'$ *with* $0 < \beta, \gamma < \epsilon$ *and* $x \in \mathcal{X}$

$$f(x) = \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0,\tau,M)} \int_{Q_\tau^{0,\nu}} \widetilde{D}_0(x, y)d\mu(y)m_{Q_\tau^{0,\nu}}(D_0(f)) + \sum_{k\in\mathbb{N}} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau,M)} \mu(Q_\tau^{k,\nu})\widetilde{D}_k(x, y_\tau^{k,\nu})D_k(f)(y_\tau^{k,\nu}) \tag{2.1}$$

$$= \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0,\tau,M)} \int_{Q_\tau^{0,\nu}} D_0(x, y)d\mu(y)m_{Q_\tau^{0,\nu}}(\widetilde{\widetilde{D}}_0(f)) + \sum_{k\in\mathbb{N}} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau,M)} \mu(Q_\tau^{k,\nu})D_k(x, y_\tau^{k,\nu})\widetilde{\widetilde{D}}_k(f)(y_\tau^{k,\nu}) \tag{2.2}$$

where the series converges in the norm of $B_p^{s,q}(\mathcal{X})$ *with* $\max(\frac{n}{n+1}, \frac{n}{n+1+s}) < p < \infty$, $0 < q \le \infty$, $-1 < s < 1$, *and* $\widetilde{\mathcal{M}}(\beta', \gamma')$ *for* $f \in \widetilde{\mathcal{M}}(\beta, \gamma)$ *with* $\beta' < \beta$ *and* $\gamma' < \gamma$, *and* $(\widetilde{\mathcal{M}}(\beta', \gamma'))'$ *for* $f \in (\widetilde{\mathcal{M}}(\beta, \gamma))'$ *with* $\epsilon > \beta' > \beta$ *and* $\epsilon > \gamma' > \gamma$. *Moreover,* $\widetilde{D}_k(x, y)$ *and* $\widetilde{\widetilde{D}}_k(x, y)$, *the kernels of* $\widetilde{D}_k$ *and* $\widetilde{\widetilde{D}}_k$, *satisfy the similar estimates but with x and y interchanged in (2.4): for* $0 < \epsilon < 1$,

$$|\widetilde{D}_k(x, y)| \le C \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y) + V(x, y)} \frac{2^{-k\epsilon}}{(2^{-k} + d(x, y))^\epsilon}; \tag{2.3}$$

$$|\widetilde{D}_k(x, y) - \widetilde{D}_k(x', y)| \le C \left(\frac{d(x, x')}{2^{-k} + d(x, y)}\right)^\epsilon \times \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y) + V(x, y)} \frac{2^{-k\epsilon}}{(2^{-k} + d(x, y))^\epsilon}, \tag{2.4}$$

for $d(x, x') \le (2^{-k} + d(x, y))/2$;

$$\int_{\mathcal{X}} \widetilde{D}_k(x, y)d\mu(y) = \int_{\mathcal{X}} \widetilde{D}_k(x, y)d\mu(x) = 0, \tag{2.5}$$

when $k \in \mathbb{N}$; $\int_{\mathcal{X}} \widetilde{D}_k(x, y)d\mu(y) = \int_{\mathcal{X}} \widetilde{D}_k(x, y)d\mu(x) = 1$ *when* $k = 0$.

To prove Theorem 1.9, we first show the following lemma.

Lemma 2.2. *Let* $\{S_k(x, y)\}_{k\in\mathbb{Z}_+}$ *and* $\{G_k(x, y)\}_{k\in\mathbb{Z}_+}$ *be two approximations to the identity as in Lemma 2.1 above and* $D_k = S_k - S_{k-1}$, $E_k = G_k - G_{k-1}$ *for* $k \in \mathbb{N}$ *and* $D_0 = S_0$, $E_0 = G_0$. *For any given* $\epsilon \in (0, 1)$ *and* $g \in \mathcal{C}^\alpha(\mathcal{X})$ *with* $0 < \alpha < \epsilon$, *then*

$$|E_k g\widetilde{D}_{k'}(x, y)| \lesssim \|g\|_{\mathcal{C}^\alpha(\mathcal{X})} 2^{-|k-k'|\alpha} \frac{1}{V_{2^{-(k\wedge k')}}(x) + V_{2^{-(k\wedge k')}}(y) + V(x, y)} \times \frac{2^{-(k\wedge k')\epsilon}}{(2^{-(k\wedge k')} + d(x, y))^\epsilon} \tag{2.6}$$

where $k, k' \in \mathbb{Z}_+$.

Proof. We only consider the case $k' \ge k > 0$, other cases are similar or easier. We write

$$\begin{aligned}
|E_k g\widetilde{D}_{k'}(x, y)| &= \left|\int_{\mathcal{X}} [E_k(x, z)g(z) - E_k(x, y)g(y)]\widetilde{D}_{k'}(z, y)d\mu(z)\right| \\
&\le \int_{\mathcal{X}} |E_k(x, z) - E_k(x, y)||g(z)||\widetilde{D}_{k'}(z, y)|d\mu(z) \\
&\quad + \int_{\mathcal{X}} |E_k(x, y)||g(z) - g(y)||\widetilde{D}_{k'}(z, y)|d\mu(z) \\
&\lesssim \|g\|_{\mathcal{C}^\alpha(\mathcal{X})} 2^{-(k'-k)\epsilon} \frac{1}{V_{2^{-(k\wedge k')}}(x) + V_{2^{-(k\wedge k')}}(y) + V(x, y)} \frac{2^{-k\epsilon}}{(2^{-k} + d(x, y))^\epsilon} \\
&\quad + \|g\|_{\mathcal{C}^\alpha(\mathcal{X})} \frac{1}{V_{2^{-k}}(x) + V_{2^{-k}}(y) + V(x, y)} \frac{2^{-k\epsilon}}{(2^{-k} + d(x, y))^\epsilon}
\end{aligned}$$

$$\times \int_{\mathcal{X}} d(z,y)^{\alpha} \frac{1}{V_{2^{-k'}}(z)+V_{2^{-k'}}(y)+V(z,y)} \frac{2^{-k'\epsilon}}{(2^{-k'}+d(z,y))^{\epsilon}} d\mu(z)$$
$$\lesssim \|g\|_{\mathcal{C}^{\alpha}(\mathcal{X})} \frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)+V(x,y)} \frac{2^{-k\epsilon}}{(2^{-k}+d(x,y))^{\epsilon}} [2^{-(k'-k)\epsilon}+2^{-k'\alpha}]$$
$$\lesssim \|g\|_{\mathcal{C}^{\alpha}(\mathcal{X})} \frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)+V(x,y)} \frac{2^{-k\epsilon}}{(2^{-k}+d(x,y))^{\epsilon}} 2^{-(k'-k)\alpha}.$$

This finishes the proof of Lemma 2.2. □

Now we show the following technical version of Theorem 1.9.

Proposition 2.3. *Suppose that* $-1 < s < 1$, $\max(\frac{n}{n+1}, \frac{n}{n+1+s}) < p \le \infty$, $0 < q \le \infty$, $\alpha > \max(s, \frac{n}{p\wedge 1} - n - s)$. *For any* $g \in \mathcal{C}^{\alpha}(\mathcal{X})$ *with* $0 < \alpha < 1$, $f \in \widetilde{\mathcal{M}}(\beta,\gamma)$ *for* β *and* γ *satisfying* (1.10), *then*

$$\|fg\|_{B_p^{s,q}(\mathcal{X})} \le C\|g\|_{\mathcal{C}^{\alpha}(\mathcal{X})} \|f\|_{B_p^{s,q}(\mathcal{X})}. \tag{2.7}$$

Proof. For any given $\epsilon \in (0,1)$ and $g \in \mathcal{C}^{\alpha}(\mathcal{X})$ with $0 < \alpha < \epsilon$, $f \in \widetilde{\mathcal{M}}(\beta,\gamma)$ we have

$$\|fg\|_{B_p^{s,q}(\mathcal{X})} = \left\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu})[m_{Q_\tau^{0,\nu}}(|E_0(fg)|)]^p\right\}^{\frac{1}{p}} + \left\{\sum_{k=1}^{\infty}[2^{ks}\|E_k(fg)\|_{L^p(\mathcal{X})}]^q\right\}^{\frac{1}{q}}$$
$$:= G + H.$$

Using the discrete Calderón reproducing formula and Lemma 2.2 implies

$$G \lesssim \|g\|_{\mathcal{C}^{\alpha}(\mathcal{X})} \left\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu}) \left[\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau',M)} \mu(Q_{\tau'}^{0,\nu'})|m_{Q_{\tau'}^{0,\nu'}}(D_0(f))|\right.\right.$$
$$\left.\left.\times \inf_{x\in Q_\tau^{0,\nu}} \inf_{y\in Q_{\tau'}^{0,\nu'}} \frac{1}{V_1(x)+V_1(y)+V(x,y)} \frac{1}{(1+d(x,y))^{\epsilon}}\right]^p\right\}^{\frac{1}{p}}$$
$$+ \|g\|_{\mathcal{C}^{\alpha}(\mathcal{X})} \left\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu}) \left[\sum_{k'=1}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)} 2^{-k'\alpha}\mu(Q_{\tau'}^{k',\nu'})|D_{k'}(f)(y_{\tau'}^{k',\nu'})|\right.\right.$$
$$\left.\left.\times \inf_{x\in Q_\tau^{0,\nu}} \frac{1}{V_1(x)+V_1(y_{\tau'}^{k',\nu'})+V(x,y_{\tau'}^{k',\nu'})} \frac{1}{(1+d(x,y_{\tau'}^{k',\nu'}))^{\epsilon}}\right]^p\right\}^{\frac{1}{p}}.$$

Applying the Hölder inequality for $p, \frac{q}{p} > 1$ and

$$\left(\sum_k |a_k|\right)^p \le \sum_k |a_k|^p \tag{2.8}$$

for all $a_k \in \mathbb{C}$ and $p, \frac{q}{p} \le 1$, it follows that

$$G \lesssim \|g\|_{\mathcal{C}^{\alpha}(\mathcal{X})} \left\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu}) \sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau',M)} \mu(Q_{\tau'}^{0,\nu'})|m_{Q_{\tau'}^{0,\nu'}}(D_0(f))|^p\right.$$
$$\left.\times\ \mu(Q_{\tau'}^{0,\nu'})^{p\wedge 1-1} \inf_{x\in Q_\tau^{0,\nu}} \inf_{y\in Q_{\tau'}^{0,\nu'}} \left[\frac{1}{V_1(x)+V_1(y)+V(x,y)} \frac{1}{(1+d(x,y))^{\epsilon}}\right]^{p\wedge 1}\right\}^{\frac{1}{p}}$$
$$+ \|g\|_{\mathcal{C}^{\alpha}(\mathcal{X})} \left\{\sum_{\tau\in I_0}\sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu}) \sum_{k'=1}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)} 2^{-k'[\alpha+s](p\wedge 1)}|2^{k's}D_{k'}(f)(y_{\tau'}^{k',\nu'})|^p\right.$$

$$\times\ \mu(Q_{\tau'}^{k',\nu'})^{p\wedge 1}\inf_{x\in Q_\tau^{0,\nu}}\left[\frac{1}{V_1(x)+V_1(y_{\tau'}^{k',\nu'})+V(x,y_{\tau'}^{k',\nu'})}\frac{1}{(1+d(x,y_{\tau'}^{k',\nu'}))^\epsilon}\right]^{p\wedge 1}\Bigg\}^{\frac{1}{p}}$$

$$\lesssim \|g\|_{\mathcal{C}^\alpha(\mathcal{X})}\left\{\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau',M)}\mu(Q_{\tau'}^{0,\nu'})|m_{Q_{\tau'}^{0,\nu'}}(D_0(f))|^p\mu(Q_{\tau'}^{0,\nu'})^{p\wedge 1-1}[V_1(y_{\tau'}^{0,\nu'})]^{1-p\wedge 1}\right\}^{\frac{1}{p}}$$

$$+\|g\|_{\mathcal{C}^\alpha(\mathcal{X})}\left\{\sum_{k'=1}^{\infty}2^{-k'[\alpha+s](p\wedge 1)}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)}\left[\frac{V_1(y_{\tau'}^{k',\nu'})}{\mu(Q_{\tau'}^{k',\nu'})}\right]^{1-p}\mu(Q_{\tau'}^{k',\nu'})|2^{k's}D_{k'}(f)(y_{\tau'}^{k',\nu'})|^p\right\}^{\frac{1}{p}}$$

$$\lesssim \|g\|_{\mathcal{C}^\alpha(\mathcal{X})}\left\{\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau',M)}\mu(Q_{\tau'}^{0,\nu'})|m_{Q_{\tau'}^{0,\nu'}}(D_0(f))|^p\right\}^{\frac{1}{p}}$$

$$+\|g\|_{\mathcal{C}^\alpha(\mathcal{X})}\left\{\sum_{k'=1}^{\infty}\left[\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)}\mu(Q_{\tau'}^{k',\nu'})|2^{k's}D_{k'}(f)(y_{\tau'}^{k',\nu'})|^p\right]^{\frac{q}{p}}\right\}^{\frac{1}{q}}$$

$$\lesssim \|g\|_{\mathcal{C}^\alpha(\mathcal{X})}\|f\|_{B_p^{s,q}(\mathcal{X})},$$

where we used the fact that for any $y_{\tau'}^{0,\nu'}\in Q_{\tau'}^{0,\nu'}$, $y_{\tau'}^{k',\nu'}\in Q_{\tau'}^{k',\nu'}$, $V_1(y_{\tau'}^{0,\nu'})\lesssim\mu(Q_{\tau'}^{0,\nu'})$, $V_1(y_{\tau'}^{k',\nu'})\lesssim 2^{k'n}\mu(Q_{\tau'}^{k',\nu'})$ and $s>-\alpha$ if $p\ge 1$ and $\frac{n}{p}-n-s<\alpha$ if $p<1$, and Lemma 1.7.

Similarly, by Lemmas 2.1 and 2.2, we have

$$H\le\left\{\sum_{k=1}^{\infty}\left[2^{ks}\left\|\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau',M)}m_{Q_{\tau'}^{0,\nu'}}(D_0(f))\int_{Q_{\tau'}^{0,\nu'}}E_kg\widetilde{D}_0(\cdot,y)d\mu(y)\right\|_{L^p(\mathcal{X})}\right]^q\right\}^{\frac{1}{q}}$$

$$+\left\{\sum_{k=1}^{\infty}\left[2^{ks}\left\|\sum_{k'=1}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)}\mu(Q_{\tau'}^{k',\nu'})E_kg\widetilde{D}_{k'}(\cdot,y_{\tau'}^{k',\nu'})D_{k'}(f)(y_{\tau'}^{k',\nu'})\right\|_{L^p(\mathcal{X})}\right]^q\right\}^{\frac{1}{q}}$$

$$\lesssim \|g\|_{\mathcal{C}^\alpha(\mathcal{X})}\left\{\sum_{k=1}^{\infty}2^{-k(\alpha-s)q}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau,M)}\mu(Q_\tau^{k,\nu})\left[\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau',M)}[m_{Q_{\tau'}^{0,\nu'}}(|D_0(f)|)]^p\right.\right.$$

$$\left.\left.\times\ \mu(Q_{\tau'}^{0,\nu'})^{p\wedge 1}\inf_{x\in Q_\tau^{k,\nu}}\inf_{y\in Q_{\tau'}^{0,\nu'}}\left[\frac{1}{V_1(x)+V_1(y)+V(x,y)}\frac{1}{(1+d(x,y))^\epsilon}\right]^{p\wedge 1}\right]^{\frac{q}{p}}\right\}^{\frac{1}{q}}$$

$$+\|g\|_{\mathcal{C}^\alpha(\mathcal{X})}\left\{\sum_{k=1}^{\infty}\left[\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau,M)}\mu(Q_\tau^{k,\nu})\sum_{k'=1}^{\infty}2^{-|k-k'|\alpha(p\wedge 1)}2^{ks(p\wedge 1)}2^{-k's(p\wedge 1)}\right.\right.$$

$$\times\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)}\mu(Q_{\tau'}^{k',\nu'})^{p\wedge 1}|2^{k's}D_{k'}(f)(y_{\tau'}^{k',\nu'})|^p$$

$$\left.\left.\times\inf_{x\in Q_\tau^{k,\nu}}\left[\frac{1}{V_{2^{-(k\wedge k')}}(x)+V_{2^{-(k\wedge k')}}(y_{\tau'}^{k',\nu'})+V(x,y_{\tau'}^{k',\nu'})}\frac{2^{-(k\wedge k')\epsilon}}{(2^{-(k\wedge k')}+d(x,y_{\tau'}^{k',\nu'}))^\epsilon}\right]^{p\wedge 1}\right]^{\frac{q}{p}}\right\}^{\frac{1}{q}}$$

$$\lesssim \|g\|_{\mathcal{C}^\alpha(\mathcal{X})}\left\{\sum_{k=1}^{\infty}2^{-k(\alpha-s)q}\left[\sum_{\tau'\in I_0}\sum_{\nu'=1}^{N(0,\tau',M)}\left[\frac{V_1(y_{\tau'}^{0,\nu'})}{\mu(Q_{\tau'}^{0,\nu'})}\right]^{1-p\wedge 1}\mu(Q_{\tau'}^{0,\nu'})[m_{Q_{\tau'}^{0,\nu'}}(|D_0(f)|)]^p\right]^{\frac{q}{p}}\right\}^{\frac{1}{q}}$$

$$+\|g\|_{\mathcal{C}^\alpha(\mathcal{X})}\left\{\sum_{k=1}^{\infty}\left[\sum_{k'=1}^{\infty}2^{-|k-k'|\alpha(p\wedge 1)}2^{ks(p\wedge 1)}2^{-k's(p\wedge 1)}\mu(Q_{\tau'}^{k',\nu'})^{p\wedge 1-1}V_{2^{-(k\wedge k')}}(y_{\tau'}^{k',\nu'})^{1-p\wedge 1}\right.\right.$$

$$\left.\left.\times\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau',M)}\mu(Q_{\tau'}^{k',\nu'})|2^{k's}D_{k'}(f)(y_{\tau'}^{k',\nu'})|^p\right]^{\frac{q}{p}}\right\}^{\frac{1}{q}}$$

$$\lesssim \|g\|_{\mathcal{C}^\alpha(\mathcal{X})}\|f\|_{B_p^{s,q}(\mathcal{X})},$$

where we used $V_{2^{-(k\wedge k')}}(y_{\tau'}^{k',\nu'})\lesssim 2^{[k'-(k\wedge k')]n}V_{2^{-(k')}}(y_{\tau'}^{k',\nu'})\lesssim 2^{[k'-(k\wedge k')]n}\mu(Q_{\tau'}^{k',\nu'})$, and $s<\alpha$ if $p\ge 1$, $\frac{n}{p}-n-s<\alpha$ if $p<1$, which verifies Proposition 2.3. $\square$

Since for $f\in B_p^{s,q}(\mathcal{X})$, in general, f could be a distribution and hence the multiplication gf, even for $g\in\mathcal{C}^\alpha(\mathcal{X})$, does not make sense. For this purpose, we need the following lemma. The proof of Theorem 1.9 then follows from Proposition 2.3 and this lemma.

Lemma 2.4. *For any $f\in B_p^{s,q}(\mathcal{X})$ with $\max(\frac{n}{n+1},\frac{n}{n+1+s})<p\le\infty$, $0<q\le\infty$, $-1<s<1$, and $g\in\mathcal{C}^\alpha(\mathcal{X})$ with $1>\epsilon>\alpha>\max(s,\frac{n}{p\wedge 1}-n-s)$. There exists a sequence $\{f_j\}_{j\in\mathbb{N}}$ such that $f_j\in\widetilde{\mathcal{M}}(\epsilon,\epsilon)$, $\|f_j\|_{B_p^{s,q}(\mathcal{X})}\lesssim\|f\|_{B_p^{s,q}(\mathcal{X})}$ and $\lim_{j\to\infty}\langle gf_j,h\rangle$ converges for any $h\in\widetilde{\mathcal{M}}(\beta,\gamma)$ with β and γ satisfying (1.10).*

Assuming Lemma 2.4 for the moment, for $g\in\mathcal{C}^\alpha(\mathcal{X})$ and $f\in B_p^{s,q}(\mathcal{X})$, $\lim_{j\to\infty}\langle gf_j,h\rangle$ exists, where f_j is given by Lemma 2.4. Therefore, for $g\in\mathcal{C}^\alpha(\mathcal{X})$, $f\in B_p^{s,q}(\mathcal{X})$, we can define

$$\langle gf,h\rangle=\lim_{j\to\infty}\langle gf_j,h\rangle$$

for $h\in\mathcal{M}(\beta,\gamma)$ with (β,γ) satisfying (1.10), f_j is a sequence given by Lemma 2.4 and the limit is independent of the choice of f_j.

Fatou's lemma and Proposition 2.3 imply

$$\|gf\|_{B_p^{s,q}(\mathcal{X})}\le\liminf_{j\to\infty}\|gf_j\|_{B_p^{s,q}(\mathcal{X})}\lesssim\|g\|_{\mathcal{C}^\alpha(\mathcal{X})}\|f\|_{B_p^{s,q}(\mathcal{X})},$$

which gives the proof of Theorem 1.9.

Therefore, it remains to show Lemma 2.4. To this end, we need the following lemma.

Lemma 2.5. *Let $\{S_k(x,y)\}_{k\in\mathbb{Z}_+}$ be a approximation to the identity as in Lemma 2.1 above and $D_k=S_k-S_{k-1}$ for $k\in\mathbb{N}$ and $D_0=S_0$. For any $g\in\mathcal{C}^\alpha(\mathcal{X})$ with $0<\alpha<1$, $h\in\widetilde{\mathcal{M}}(\beta,\gamma)$ with β and γ satisfying (1.10). Then*

$$|\langle D_k(\bullet,y)g,h\rangle|\le C\|g\|_{\mathcal{C}^\alpha(\mathcal{X})}\|h\|_{\mathcal{M}(\beta,\gamma)}2^{-k(\beta\wedge\alpha)}\frac{1}{V_1(x_0)+V(x_0,y)}\frac{1}{(1+d(y,x_0))^\gamma},\tag{2.9}$$

where $k\in\mathbb{Z}_+$.

Proof. We only prove the case $k\in\mathbb{N}$, the case $k=0$ is similar. We write

$$\begin{aligned}|\langle D_k(\bullet,y)g,h\rangle|\le&\int_{\mathcal{X}}|D_k(x,y)||g(x)-g(y)||h(x)|d\mu(x)+\int_{\mathcal{X}}|D_k(x,y)||g(y)||h(x)-h(y)|d\mu(x)\\ :=&\,I+II.\end{aligned}$$

For I, since $D_k(x,y)=0$ if $d(x,y)\ge 2C_12^{-k}$, we obtain

$$\begin{aligned}I\lesssim&\|g\|_{\mathcal{C}^\alpha(\mathcal{X})}\|h\|_{\mathcal{M}(\beta,\gamma)}2^{-k\alpha}\\ &\times\int_{d(x,y)<2C_12^{-k}}\frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)}\frac{1}{V_1(x_0)+V(x_0,x)}\frac{1}{(1+d(x,x_0))^\gamma}d\mu(x)\\ \lesssim&\|g\|_{\mathcal{C}^\alpha(\mathcal{X})}\|h\|_{\mathcal{M}(\beta,\gamma)}2^{-k\alpha}\left[\int_{d(x,y)<2C_12^{-k},\ d(x,y)\le\frac{1}{2}(1+d(y,x_0))}\cdots+\int_{2C_12^{-k}>d(x,y)>\frac{1}{2}(1+d(y,x_0))}\cdots\right]\\ :=&\|g\|_{\mathcal{C}^\alpha(X)}\|h\|_{\mathcal{M}(\beta,\gamma)}2^{-k\alpha}[I_1+I_2].\end{aligned}$$

For I_1, $d(x,y)\le\frac{1}{2}(1+d(y,x_0))$ implies that $d(y,x_0)\le 2(d(x,x_0)+1)$ and

$$\frac{1}{V_1(x_0)+V(x_0,x)}\lesssim\frac{1}{V_1(x_0)+\mu(B(x_0,d(x_0,y)))}=\frac{1}{V_1(x_0)+V(x_0,y)}.$$

Thus

$$I_1 \lesssim \frac{1}{V_1(x_0)+V(x_0,y)}\frac{1}{(1+d(y,x_0))^{\gamma}}\int_{d(x,y)<2C_1 2^{-k}}\frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)}d\mu(x)$$
$$\lesssim \frac{1}{V_1(x_0)+V(x_0,y)}\frac{1}{(1+d(y,x_0))^{\gamma}}.$$

For I_2, $2C_1 2^{-k} > d(x,y) > \frac{1}{2}(1+d(y,x_0))$ implies that

$$1+d(y,x_0) < C, \qquad \frac{1}{V_{2^{-k}}(x)+V_{2^{-k}}(y)} \lesssim \frac{1}{V(y,x)} \lesssim \frac{1}{V_1(x_0)+V(x_0,y)}.$$

Thus

$$I_2 \lesssim \frac{1}{V_1(x_0)+V(x_0,y)}\frac{1}{(1+d(x_0,y))^{\gamma}}\int_{\mathcal{X}}\frac{1}{V_1(x_0)+V(x_0,x)}\frac{1}{(1+d(x,x_0))^{\gamma}}d\mu(x)$$
$$\lesssim \frac{1}{V_1(x_0)+V(x_0,y)}\frac{1}{(1+d(x_0,y))^{\gamma}}.$$

Similarly we can deal with II

$$II \lesssim \frac{2^{-k\beta}}{V_1(x_0)+V(x_0,y)}\frac{1}{(1+d(y,x_0))^{\gamma}}.$$

Combining the estimates of I and II, we obtain that for $k \in \mathbb{N}$

$$|\langle D_k(\bullet,y)g,h\rangle| \le C\|g\|_{\mathcal{C}^{\alpha}(\mathcal{X})}\|h\|_{\mathcal{M}(\beta,\gamma)}2^{-k(\beta\wedge\alpha)}\frac{1}{V_1(x_0)+V(x_0,y)}\frac{1}{(1+d(y,x_0))^{\gamma}}.$$

Thus, (2.9) also holds. This finishes the proof of Lemma 2.5. □

Now we show Lemma 2.4.

Proof of Lemma 2.4. For any $f \in B_p^{s,q}(\mathcal{X})$, with $\max(\frac{n}{n+1},\frac{n}{n+1+s}) < p \le \infty$, $0 < q \le \infty$, $-1 < s < 1$, denote

$$f_j = \sum_{\tau=1}^{j}\sum_{\nu=1}^{N(0,\tau,M)}\mu(Q_\tau^{0,\nu})m_{Q_\tau^{0,\nu}}(\widetilde{\widetilde{D}}_0(f))D_{Q_\tau^{0,\nu}}(x)$$
$$+\sum_{k=1}^{j}\sum_{\tau=1}^{j}\sum_{\nu=1}^{N(k,\tau,M)}\mu(Q_\tau^{k,\nu})D_k(x,y_\tau^{k,\nu})\widetilde{\widetilde{D}}_k(f)(y_\tau^{k,\nu}),$$

where $D_{Q_\tau^{0,\nu}}(x) = \frac{1}{\mu(Q_\tau^{0,\nu})}\int_{Q_\tau^{0,\nu}} D_0(x,y)d\mu(y)$.

It is easy to see that $f_j \in \widetilde{\mathcal{M}}(\epsilon,\epsilon)$. Applying a similar proof as in Proposition 2.3 with $g = 1$ and $f = f_j$ gives $\|f_j\|_{B_p^{s,q}(\mathcal{X})} \le C\|f\|_{B_p^{s,q}(\mathcal{X})}$.

Next we prove that $\lim_{n\to\infty}\langle gf_n,h\rangle$ converges for any $h \in \widetilde{\mathcal{M}}(\beta,\gamma)$ with β and γ satisfying (1.10). For $j, m \in \mathbb{N}$, $m < j$, we can write

$$|\langle f_j - f_m, gh\rangle| \le \left|\sum_{\tau=m+1}^{j}\sum_{\nu=1}^{N(0,\tau,M)}\mu(Q_\tau^{0,\nu})m_{Q_\tau^{0,\nu}}(\widetilde{\widetilde{D}}_0(f))\langle D_{Q_\tau^{0,\nu}},gh\rangle\right|$$
$$+\left|\sum_{k=m+1}^{j}\sum_{\tau=1}^{j}\sum_{\nu=1}^{N(k,\tau,M)}\mu(Q_\tau^{k,\nu})\langle D_k(\bullet,y_\tau^{k,\nu}),gh\rangle\widetilde{\widetilde{D}}_k(f)(y_\tau^{k,\nu})\right|$$
$$+\left|\sum_{k=1}^{j}\sum_{\tau=m+1}^{j}\sum_{\nu=1}^{N(k,\tau,M)}\mu(Q_\tau^{k,\nu})\langle D_k(\bullet,y_\tau^{k,\nu}),gh\rangle\widetilde{\widetilde{D}}_k(f)(y_\tau^{k,\nu})\right|.$$

We consider the following four cases respectively:

(I) $1 < p < \infty$ and $1 < q < \infty$;
(II) $1 < p < \infty$ and $0 < q \le 1$;
(III) $1 < p < \infty$ and $q = \infty$ or $p = \infty$ and $0 < q \le \infty$;
(IV) $\max(\frac{n}{n+1},\frac{n}{n+1+s}) < p \le 1$.

We now consider case (I). By the duality and Proposition 2.3, we have

$$|\langle f_n - f_m, gh\rangle| \le \|f_n - f_m\|_{B^{s,q}_p}\|gh\|_{B^{-s,q'}_{p'}} \lesssim \|g\|_{\mathcal{C}^\alpha(\mathcal{X})}\|h\|_{B^{-s,q'}_{p'}(\mathcal{X})}\|f_n - f_m\|_{B^{s,q}_p}.$$

Note that $\|h\|_{B^{-s,q'}_{p'}(\mathcal{X})} \lesssim \|h\|_{\widetilde{\mathcal{M}}(\beta,\gamma)}$ and $\|f_n - f_m\|_{B^{s,q}_p}$ tends to zero as n, m tend to infinity. This implies that $|\langle f_n - f_m, gh\rangle| \to 0$ as $n, m \to \infty$ when $s \in (-1, 1)$, $1 < p < \infty$ and $1 < q < \infty$ and hence case (I) is concluded.

For case (II), the fact (2.9) implies

$$\begin{aligned}
&\left\{\sum_{\tau=m+1}^{j}\sum_{\nu=1}^{N(0,\tau,M)} \mu(Q^{0,\nu}_\tau)|D_{Q^{0,\nu}_\tau}(gh)|^{p'}\right\}^{\frac{1}{p'}}\\
&\quad\lesssim \|g\|_{\mathcal{C}^\alpha(\mathcal{X})}\|h\|_{\mathcal{M}(\beta,\gamma)}\left\{\sum_{\tau\ge \log_2^m}\int_{\{y:2^\tau<d(x_0,y)\le 2^{\tau+1}\}}\left[\frac{1}{V_1(x_0)+V(x_0,y)}\right]^{p'}\frac{1}{(1+d(y,x_0))^{\gamma p'}}d\mu(y)\right\}^{\frac{1}{p'}}\\
&\quad\lesssim \|g\|_{\mathcal{C}^\alpha(\mathcal{X})}\|h\|_{\mathcal{M}(\beta,\gamma)}\left\{\sum_{\tau\ge \log_2^m} 2^{-\tau\gamma p'}\right\}^{\frac{1}{p'}}\\
&\quad= m^{-\gamma}\|g\|_{\mathcal{C}^\alpha(\mathcal{X})}\|h\|_{\mathcal{M}(\beta,\gamma)},
\end{aligned}$$

and

$$\begin{aligned}
\sup_{k\in\mathbb{N}, m+1\le k\le j} 2^{-ks}\|D^*_k(gh)\|_{L^{p'}(\mathcal{X})} &\lesssim \|g\|_{\mathcal{C}^\alpha(\mathcal{X})}\|h\|_{\mathcal{M}(\beta,\gamma)}\sup_{k\in\mathbb{N}, m+1\le k\le j} 2^{-k(s+\beta\wedge\alpha)}\\
&\quad\times\left\{\int_{\mathcal{X}}\left[\frac{1}{V_1(x_0)+V(x_0,y)}\right]^{p'}\frac{1}{(1+d(y,x_0))^{\gamma p'}}d\mu(y)\right\}^{\frac{1}{p'}}\\
&\lesssim \|g\|_{\mathcal{C}^\alpha(\mathcal{X})}\|h\|_{\mathcal{M}(\beta,\gamma)}2^{-m(s+\beta\wedge\alpha)},
\end{aligned}$$

and

$$\begin{aligned}
\sup_{k\in\mathbb{N},\ 1\le k\le j}\left(\sum_{\tau=m+1}^{j}\sum_{\nu=1}^{N(k,\tau,M)}\mu(Q^{k,\nu}_\tau)|2^{-ks}D^*_k(gh)(y^{k,\nu}_\tau)|^{p'}\right)^{\frac{1}{p'}} &\lesssim \|g\|_{\mathcal{C}^\alpha(\mathcal{X})}\|h\|_{\mathcal{M}(\beta,\gamma)}\sup_{k\in\mathbb{N},\ 1\le k\le j}2^{-k(s+\beta\wedge\alpha)}m^{-\gamma}\\
&\lesssim \|g\|_{\mathcal{C}^\alpha(\mathcal{X})}\|h\|_{\mathcal{M}(\beta,\gamma)}m^{-\gamma}.
\end{aligned}$$

Applying the Hölder inequality for $p > 1$ and (2.8) for $q \le 1$, it follows that

$$\begin{aligned}
|\langle f_j - f_m, gh\rangle| &\le \left\{\sum_{\tau=m+1}^{j}\sum_{\nu=1}^{N(0,\tau,M)}\mu(Q^{0,\nu}_\tau)[m_{Q^{0,\nu}_\tau}(|\widetilde{\widetilde{D}}_0(f)|)]^p\right\}^{\frac{1}{p}}\\
&\quad\times\left\{\sum_{\tau=m+1}^{j}\sum_{\nu=1}^{N(0,\tau,M)}\mu(Q^{0,\nu}_\tau)|D_{Q^{0,\nu}_\tau}(gh)|^{p'}\right\}^{\frac{1}{p'}}\\
&\quad+\left\{\sum_{k=m+1}^{j}[2^{ks}\|\widetilde{\widetilde{D}}_k(f)\|_{L^p(\mathcal{X})}]^q\right\}^{\frac{1}{q}}\sup_{k\in\mathbb{N}, m+1\le k\le j}2^{-ks}\|D^*_k(gh)\|_{L^{p'}(\mathcal{X})}\\
&\quad+\left\{\sum_{k=1}^{j}\left(\sum_{\tau=m+1}^{j}\sum_{\nu=1}^{N(k,\tau,M)}\mu(Q^{k,\nu}_\tau)|2^{ks}\widetilde{\widetilde{D}}_k(f)(y^{k,\nu}_\tau)|^p\right)^{\frac{q}{p}}\right\}^{\frac{1}{q}}\\
&\quad\times\sup_{k\in\mathbb{N},\ 1\le k\le j}\left(\sum_{\tau=m+1}^{j}\sum_{\nu=1}^{N(k,\tau,M)}\mu(Q^{k,\nu}_\tau)|2^{-ks}D^*_k(gh)(y^{k,\nu}_\tau)|^{p'}\right)^{\frac{1}{p'}}\\
&\lesssim \|g\|_{\mathcal{C}^\alpha(\mathcal{X})}\|f\|_{B^{s,q}_p(\mathcal{X})}\|h\|_{\mathcal{M}(\beta,\gamma)}\{2^{-m(s+\beta\wedge\alpha)}+m^{-\gamma}\}.
\end{aligned}$$

Using the fact that $s+\beta\wedge\alpha > 0$, this proves $|\langle f_j - f_m, gh\rangle| \to 0$ as $j, m \to \infty$ when $s \in (-1, 1)$, $1 < p < \infty$ and $0 < q \le 1$.

For case (III) , if $p = \infty$, $q = \infty$, we obtain

$$|\langle f_j - f_m, gh\rangle| \le \|f\|_{B^{s,\infty}_\infty(\mathcal{X})} \left| \sum_{\tau=m+1}^{j} \sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu}) D_{Q_\tau^{0,\nu}}(gh) \right| + \|f\|_{B^{s,\infty}_\infty(\mathcal{X})} \sum_{k=m+1}^{j} \int_{\mathcal{X}} |2^{-ks} D_k^*(gh)(x)| d\mu(x) + \|f\|_{B^{s,\infty}_\infty(\mathcal{X})} \left| \sum_{k=1}^{j} \sum_{\tau=m+1}^{j} \sum_{\nu=1}^{N(k,\tau,M)} \mu(Q_\tau^{k,\nu}) 2^{-ks} D_k^*(gh)(y_\tau^{k,\nu}) \right|.$$

Applying Proposition 2.3, we have $gh \in B_1^{-s,1}(\mathcal{X})$. Note that the terms in brace are remainder of $(gh)_j - (gh)_m$ in the norm of $B_1^{-s,1}(\mathcal{X})$, which go to zero as n, m tend to infinity. Thus $|\langle f_j - f_m, gh\rangle| \to 0$ as $j, m \to \infty$ when $s \in (-1, 1)$, $p = \infty$ and $q = \infty$.

The estimate of case $\infty > p > 1$, $q = \infty$ or $p = \infty$, $q \ne \infty$ is similar to case (III) above.

For case (IV), applying (2.8) for $\max(\frac{n}{n+1}, \frac{n}{n+1+s}) < p \le 1$, we have

$$|\langle f_j - f_m, gh\rangle| \lesssim \left\{ \sum_{\tau=m+1}^{j} \sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu}) [m_{Q_\tau^{0,\nu}}(|\widetilde{\widetilde{D}}_0(f)|)]^p \right\}^{\frac{1}{p}} \times \sup_{m < d(x_0, y_\tau^{0,\nu}) \le j} \frac{\mu(Q_\tau^{0,\nu})^{1-\frac{1}{p}}}{V_1(x_0) + V(x_0, y_\tau^{0,\nu})} \frac{1}{(1 + d(y_\tau^{0,\nu}, x_0))^\gamma} + \sum_{k=m+1}^{j} 2^{ks} \|\widetilde{\widetilde{D}}_k(f)\|_{L^p(\mathcal{X})} \sup_{y_\tau^{k,\nu} \in \mathcal{X}} 2^{-ks} \mu(Q_\tau^{k,\nu})^{1-\frac{1}{p}} |D_k^*(gh)(y_\tau^{k,\nu})| + \sum_{k=1}^{j} 2^{ks} \|\widetilde{\widetilde{D}}_k(f)\|_{L^p(\mathcal{X})} \sup_{m < d(x_0, y_\tau^{k,\nu}) \le j} 2^{-ks} \mu(Q_\tau^{k,\nu})^{1-\frac{1}{p}} |D_k^*(gh)(y_\tau^{k,\nu})|.$$

By (2.9), for $i \in \mathbb{N}$ we have

$$\sup_{y_\tau^{k,\nu} \in \mathcal{X}} 2^{-ks} \mu(Q_\tau^{k,\nu})^{1-\frac{1}{p}} |D_k^*(gh)({}_\tau^{k,\nu})| \lesssim \|g\|_{\mathcal{C}^\alpha(\mathcal{X})} \|h\|_{\mathcal{M}(\beta,\gamma)} 2^{-k[s+\beta\wedge\alpha - n(\frac{1}{p}-1)]},$$

and

$$\sup_{m < d(x_0, y_\tau^{k,\nu}) \le j} 2^{-ks} \mu(Q_\tau^{k,\nu})^{1-\frac{1}{p}} |D_k^*(gh)({}_\tau^{k,\nu})| \lesssim \|g\|_{\mathcal{C}^\alpha(\mathcal{X})} \|h\|_{\mathcal{M}(\beta,\gamma)} \sup_{m < d(x_0, y_\tau^{k,\nu}) \le j} \frac{(V_1(y_\tau^{k,\nu}) + V(x_0, y_\tau^{k,\nu}))^{\frac{1}{p}-1}}{\mu(Q_\tau^{k,\nu})^{\frac{1}{p}-1}} \frac{1}{V_1(x_0)^{\frac{1}{p}}} \frac{2^{-ks} 2^{-k(\beta\wedge\alpha)}}{(1 + d(y_\tau^{k,\nu}, x_0))^\gamma}$$
$$\lesssim \|g\|_{\mathcal{C}^\alpha(\mathcal{X})} \|h\|_{\mathcal{M}(\beta,\gamma)} 2^{-k[s+\beta\wedge\alpha]} \sup_{\log_2^m < i} \sup_{2^i < d(x_0, y_\tau^{k,\nu}) \le 2^{i+1}} \frac{V(x_0, y_\tau^{k,\nu})^{\frac{1}{p}-1}}{\mu(Q_\tau^{k,\nu})^{\frac{1}{p}-1}} \frac{1}{(1 + d(y_\tau^{k,\nu}, x_0))^\gamma}$$
$$\lesssim \|g\|_{\mathcal{C}^\alpha(\mathcal{X})} \|h\|_{\mathcal{M}(\beta,\gamma)} 2^{-k[s+\beta\wedge\alpha - n(\frac{1}{p}-1)]} m^{-[\gamma - n(\frac{1}{p}-1)]},$$

where $i \in \mathbb{N}$.

Similar, by the Hölder inequality for $\infty \ge q > 1$, it follows that

$$|\langle f_j - f_m, gh\rangle| \lesssim \left\{ \sum_{\tau=m+1}^{j} \sum_{\nu=1}^{N(0,\tau,M)} \mu(Q_\tau^{0,\nu}) [m_{Q_\tau^{0,\nu}}(|\widetilde{\widetilde{D}}_0(f)|)]^p \right\}^{\frac{1}{p}} \times \sup_{y_\tau^{0,\nu}, m < d(x_0, y_\tau^{0,\nu}) \le j} \frac{\mu(Q_\tau^{0,\nu})^{1-\frac{1}{p}}}{V_1(x_0) + V(x_0, y_\tau^{0,\nu})} \frac{1}{(1 + d(y_\tau^{0,\nu}, x_0))^\gamma} + \left\{ \sum_{k=m+1}^{j} [2^{ks} \|\widetilde{\widetilde{D}}_k(f)\|_{L^p(\mathcal{X})}]^q \right\}^{\frac{1}{q}} \left\{ \sum_{k=m+1}^{j} \left[\sup_{y_\tau^{k,\nu} \in \mathcal{X}} 2^{-ks} \mu(Q_\tau^{k,\nu})^{1-\frac{1}{p}} |D_k^*(gh)(y_\tau^{k,\nu})| \right]^{q'} \right\}^{\frac{1}{q'}}$$

$$+\left\{\sum_{k=1}^{j}[2^{ks}\|\widetilde{\widetilde{D}}_k(f)\|_{L^p(X)}]^q\right\}^{\frac{1}{q}}\left\{\sum_{k=1}^{j}\left[\sup_{m<d(x_0,y_\tau^{k,\nu})\le j}2^{-ks}\mu(Q_\tau^{k,\nu})^{1-\frac{1}{p}}|D_k^*(gh)(y_\tau^{k,\nu})|\right]^{q'}\right\}^{\frac{1}{q'}}$$

$$\lesssim \|g\|_{\mathcal{C}^\alpha(\mathcal{X})}\|h\|_{\mathcal{M}(\beta,\gamma)}\|f\|_{B_p^{s,q}(X)}m^{-\gamma}$$

$$+\|g\|_{\mathcal{C}^\alpha(\mathcal{X})}\|h\|_{\mathcal{M}(\beta,\gamma)}\|f\|_{B_p^{s,q}(\mathcal{X})}\left\{\sum_{k=m+1}^{j}[2^{-k[s-n(\frac{1}{p}-1)]}2^{-k(\alpha\wedge\beta)}]^{q'}\right\}^{\frac{1}{q'}}$$

$$+\|g\|_{\mathcal{C}^\alpha(\mathcal{X})}\|h\|_{\mathcal{M}(\beta,\gamma)}\|f\|_{B_p^{s,q}(\mathcal{X})}\left\{\sum_{k=1}^{j}[2^{-k[s-n(\frac{1}{p}-1)]}2^{-k(\alpha\wedge\beta)}]^{q'}\right\}^{\frac{1}{q'}}m^{-[\gamma-n(\frac{1}{p}-1)]}$$

$$\lesssim \|g\|_{\mathcal{C}^\alpha(\mathcal{X})}\|h\|_{\mathcal{M}(\beta,\gamma)}\|f\|_{B_p^{s,q}(\mathcal{X})}[m^{-\gamma}+2^{-m[s-n(\frac{1}{p}-1)]}2^{-m(\alpha\wedge\beta)}+m^{-[\gamma-n(\frac{1}{p}-1)]}].$$

From (2.8) and (2.9) for $0<q\le 1$, we also have

$$|\langle f_j-f_m,gh\rangle|\le\left\{\sum_{\tau=m+1}^{j}\sum_{\nu=1}^{N(0,\tau,M)}\mu(Q_\tau^{0,\nu})[m_{Q_\tau^{0,\nu}}(|\widetilde{\widetilde{D}}_0(f)|)]^p\right\}^{\frac{1}{p}}\sup_{y_\tau^{0,\nu}\in\{m<d(x_0,y_\tau^{0,\nu})\le j\}}|D_{Q_\tau^{0,\nu}}(gh)|$$

$$+\left\{\sum_{k=m+1}^{j}[2^{ks}\|\widetilde{\widetilde{D}}_k(f)\|_{L^p(\mathcal{X})}]^q\right\}^{\frac{1}{q}}\sup_{k\in\mathbb{N},m<k\le j}\sup_{y_\tau^{k,\nu}\in\mathcal{X}}2^{-ks}\mu(Q_\tau^{k,\nu})^{\frac{1}{p}-1}|D_k^*(gh)(y_\tau^{k,\nu})|$$

$$+\left\{\sum_{k=1}^{j}[2^{ks}\|\widetilde{\widetilde{D}}_k(f)\|_{L^p(\mathcal{X})}]^q\right\}^{\frac{1}{q}}\sup_{k\in\mathbb{N},1\le k\le j}\sup_{m<d(x_0,y_\tau^{k,\nu})\le j}2^{-ks}\mu(Q_\tau^{k,\nu})^{\frac{1}{p}-1}|D_k^*(gh)(y_\tau^{k,\nu})|$$

$$\lesssim \|g\|_{\mathcal{C}^\alpha(\mathcal{X})}\|h\|_{\mathcal{M}(\beta,\gamma)}\|f\|_{B_p^{s,q}(\mathcal{X})}[2^{-m[s+\alpha\wedge\beta-n(\frac{1}{p}-1)]}+m^{-\gamma}+m^{-[\gamma-n(\frac{1}{p}-1)]}]$$

where we use the arbitrariness of $y_\tau^{k,\nu}$, and $\beta\wedge\alpha>\frac{n}{p}-n-s$ when $p\le 1$. This proves $|\langle f_j-f_m,gh\rangle|\to 0$ as $j,m\to\infty$ when $s\in(-1,1)$, $\max(\frac{n}{n+1},\frac{n}{n+1+s})<p\le 1$ and $0<q\le\infty$, and hence the proof of Lemma 2.4 is concluded. □

References

[1] N.S. Feldman, Pointwise multipliers from the Hardy space to the Bergman space, Illinois J. Math. 43 (1999) 211–221.
[2] K. Herbert, S. Winfried, Pointwise multipliers of Besov spaces of smoothness zero and spaces of continuous functions, Rev. Mat. Iberoamericana 18 (2002) 587–626.
[3] R.A. Macias, C. Segovia, Lipschitz function on spaces of homogeneous type, Adv. Math. 33 (1979) 257–270.
[4] V.G. Mazya, T.O. Shaposhnikova, Theory of Sobolev Multipliers, in: Grundlehren der Mathematischen Wissenschaften, vol. 337, Springer, Berlin, 2009.
[5] T. Runst, W. Sickel, Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations, W. de Gruyter, Berlin, 1996.
[6] W. Sickel, On pointwise multipliers for $F_p^{s,q}(\mathbb{R}^d)$ in case $\sigma_{p,q}<s<\frac{n}{p}$, Ann. Mat. Pura Appl., IV. Ser. 176 (1999) 209–250.
[7] R.S. Strichartz, Multipliers on fractional Sobolev spaces, J. Math. Mech. 16 (1967) 1031–1060.
[8] H. Triebel, Theory of Function Spaces, in: Monographs in Mathematics, vol. 78, Birkhäuser, Basel, 1983.
[9] R.R. Coifman, G. Weiss, Analyse harmonique noncommutative sur certains espaces homogenes, in: Lecture Notes in Math., vol. 242, Springer-Verlag, Berlin and New York, 1971.
[10] Y.S. Han, D. Müller, D.C. Yang, A theory of Besov and Triebel–Lizorkin spaces on metric measure spaces modeled on Carnot–Carathodory spaces, Abstr. Appl. Anal. (2008) 250. Art. ID 893409.
[11] A. Nagel, E.M. Stein, The $\bar{\partial}_b$-complex on decoupled boundarise in $\mathbb{C}^n$, Ann. of Math. (2) 164 (2006) 649–713.
[12] Y.S. Han, J. Li and, G.Z. Lu, Multiparameter Hardy spaces theory on product spaces of homogeneous type, Trans. Amer. Math. Soc., Preprint.
[13] M. Christ, A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral, Colloq. Math. 60/61 (1990) 601–628.
[14] Y.S. Han, E.T. Sawyer, Littlewood–Paley theory on spaces of homogeneous type and classical function spaces, Mem. Amer. Math. Soc. 110 (1994) 1–126.