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Multilinear Fourier multipliers related to time–frequency localization

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ABSTRACT

We consider multilinear multipliers associated in a natural way with localization operators. Boundedness and compactness results are obtained. In particular, we get a geometric condition on a subset $A \subset \mathbb{R}^{2d}$ which guarantees that, for a fixed *synthesis* window $\psi \in L^2(\mathbb{R}^d)$, the family of localization operators $L^A_{\varphi,\psi}$ obtained when the *analysis* window φ varies on the unit ball of $L^2(\mathbb{R}^d)$ is a relatively compact set of compact operators.

1. Introduction

Localization operators or, more generally, pseudodifferential operators on $L^2(\mathbb{R}^d)$ have recently received attention (see for instance [1–11] and the references therein). Boundedness, Schatten class conditions and compactness have been thoroughly investigated. Each localization operator is determined by a *symbol* F and two *windows* $\varphi,\ \psi$ called the analysis and the synthesis windows. In this paper we are interested in the behavior of these operators as multilinear functions, that is in their dependence on the signals but also on the windows. For a given symbol F we consider the operator

$$(\psi, \varphi, f) \to L^F_{\varphi,\psi}(f).$$

We first represent the previous operator as a multilinear Fourier multiplier, where the multiplier is given by $m(x, y, z) = a(x+y, y+z)$ and a is the partial Fourier transform of F with respect to the first d variables. This representation suggests the definition of a special class of Fourier multipliers $\mathcal{T}_a$. The study of multilinear multipliers was started by Coifman and Meyer for smooth symbols [12]. After giving some continuity results, we concentrate on the bilinear operator $\mathcal{T}_a(\psi, \cdot, \cdot)$ obtained when the synthesis window is fixed. This is a bilinear pseudodifferential operator as in [13] but the results obtained in that paper cannot be applied in the present context. We show that, under a mild hypothesis on a and ψ,

$$\mathcal{T}_a(\psi, \cdot, \cdot) : L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$$

is a compact bilinear pseudodifferential operator. Then we turn our attention to the behavior of the family of localization operators obtained when the synthesis window ψ remains fixed and the analysis window varies on the unit ball of $L^2(\mathbb{R}^d)$. It turns out that $\{L^F_{\varphi,\psi} : \varphi \in B_{L^2}\}$ is a relatively compact set of compact operators on $L^2(\mathbb{R}^d)$ whenever $F \in L^p(\mathbb{R}^{2d})$ and $1 \le p < \infty$. However, for $F \in L^\infty(\mathbb{R}^{2d})$ the relative compactness of $\{L^F_{\varphi,\psi} : \varphi \in B_{L^2}\}$ in the Banach space of compact operators on $L^2(\mathbb{R}^d)$ is equivalent to the bilinear pseudodifferential operator $\mathcal{T}_a(\psi, \cdot, \cdot)$ being compact. An example shows that this is not always the case even if $L^F_{\varphi,\psi}$ is a compact operator for every pair of windows. At this point, it is worth noting

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that the multilinear map $(\psi,\varphi,f)\to L^F_{\varphi,\psi}(f)$ is never compact for $F\in L^p(\mathbb{R}^{2d}),\ 1\le p\le\infty$. In the special case where F is the characteristic function of a subset $A\subset\mathbb{R}^{2d}$, we get a geometric condition on A which suffices to guarantee that $\{L^A_{\varphi,\psi}:\varphi\in B_{L^2}\}$ is a relatively compact set of compact operators for every $\psi\in L^2(\mathbb{R}^d)$. That is, in a certain sense, the compactness of $L^A_{\varphi,\psi}$ is *uniform* with respect to the analysis window.

2. Notation and preliminaries

We start with the basic definitions and we recall the properties of the time–frequency representations that will be used throughout the paper. We refer the reader to [14] for the necessary background.

Given $\varphi\in L^2(\mathbb{R}^d)\setminus\{0\}$, the short time Fourier transform (also called the continuous Gabor transform) of $f\in L^2(\mathbb{R}^d)$ with respect to the window φ is

$$V_\varphi f(x,\omega)=\int_{\mathbb{R}^d} f(t)\overline{\varphi(t-x)}e^{-2\pi i\omega t}\,dt.$$

The short time Fourier transform has the property that

$$V_\varphi f\in L^2(\mathbb{R}^{2d})\cap L^\infty(\mathbb{R}^{2d})$$

when $f,\varphi\in L^2(\mathbb{R}^d)$ and $\|V_\varphi f\|_2=\|f\|_2\|\varphi\|_2$, whereas $\|V_\varphi f\|_\infty\le\|f\|_2\|\varphi\|_2$. Clearly, we may also write $V_\varphi f(x,\omega)=\langle f,M_\omega T_x\varphi\rangle$, where

$$M_\omega f(t)=e^{2\pi i\omega t}f(t)\quad\text{and}\quad T_xf(t)=f(t-x)$$

are the modulation and translation operators. Hence $V_\varphi f$ can also be defined for $f\in\mathcal{S}'(\mathbb{R}^d)$ and $\varphi\in\mathcal{S}(\mathbb{R}^d)$. Here we denote by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of $\mathcal{C}^\infty$ and rapidly decreasing functions and by $\mathcal{S}'(\mathbb{R}^d)$ its dual space.
Moreover, f can be recovered from $V_\varphi f$ by means of the following formula:

$$f=\frac{1}{\langle\varphi,\psi\rangle}\int_{\mathbb{R}^{2d}}V_\varphi f(x,\omega)M_\omega T_x\psi\,dxd\omega$$

whenever $\langle\varphi,\psi\rangle\neq 0$, interpreting the integral in a weak sense. If we multiply $V_\varphi f$ by a suitable function F before reconstructing f, we recover a filtered version of the original function. The operators obtained in this way,

$$L^F_{\varphi,\psi}f=\int_{\mathbb{R}^{2d}}F(x,\omega)V_\varphi f(x,\omega)M_\omega T_x\psi\,dxd\omega,$$

are called *localization operators* and were defined in 1988 by Daubechies [5] in order to localize a signal both in time and in frequency. The function F is called the symbol of the operator, whereas φ is the analysis window and ψ is the synthesis window. When F is the characteristic function of a measurable set A in $\mathbb{R}^{2d}$ we write $L^A_{\varphi,\psi}$.

The expression above can also be written as $\langle L^F_{\varphi,\psi}f,h\rangle=\langle F,V_\psi h\overline{V_\varphi f}\rangle$. Hence, it is clear that $L^F_{\varphi,\psi}$ is a bounded linear operator from $L^2(\mathbb{R}^d)$ into itself whenever $F\in L^p(\mathbb{R}^{2d})$ and $\varphi,\psi\in L^2(\mathbb{R}^d)$, and

$$\|L^F_{\varphi,\psi}f\|_2\le\|F\|_p\|f\|_2\|\varphi\|_2\|\psi\|_2. \tag{1}$$

The class of symbols can be enlarged if we restrict the class of windows: taking $\varphi,\psi\in\mathcal{S}(\mathbb{R}^d)$, the boundedness of the localization operator is guaranteed if the symbol F belongs to the modulation space $M^\infty(\mathbb{R}^{2d})$ (see [3] where it is also shown that this class is, in a certain sense, optimal). To define this modulation space, fix $\Phi\in\mathcal{S}(\mathbb{R}^{2d})\setminus\{0\}$ and put

$$M^\infty(\mathbb{R}^{2d}):=\{T\in\mathcal{S}'(\mathbb{R}^{2d}):V_\Phi T\in L^\infty(\mathbb{R}^{4d})\}.$$

Equipped with the norm $\|T\|_{M^\infty}=\|V_\Phi T\|_\infty$, $M^\infty(\mathbb{R}^{2d})$ is a Banach space whose definition is independent on the window Φ, and different windows give rise to equivalent norms. $M^\infty(\mathbb{R}^{2d})$ contains $L^p(\mathbb{R}^{2d})$ with continuous inclusion for all $1\le p\le\infty$ and some tempered distributions like the point evaluations δ_z. The closure of $\mathcal{S}(\mathbb{R}^{2d})$ in $M^\infty(\mathbb{R}^{2d})$ is denoted as $M^0(\mathbb{R}^{2d})$ and it can be characterized as the set of those $T\in\mathcal{S}'(\mathbb{R}^{2d})$ such that $V_\Phi T$ vanishes at infinity. Again [14] is the standard reference for modulation spaces.

Localization operators are special pseudodifferential operators. In fact, $L^F_{\varphi,\psi}$ can also be interpreted as a Weyl operator L_σ with Weyl symbol $\sigma=F*W(\psi,\varphi)$. Here $L_\sigma:\mathcal{S}(\mathbb{R}^d)\to\mathcal{S}'(\mathbb{R}^d)$ is given by (see for instance [1])

$$\langle L_\sigma f,h\rangle=\langle\sigma,W(h,f)\rangle.$$

The Wigner cross-distribution of $f,\ \varphi\in L^2(\mathbb{R}^d)$ is

$$W(f,\varphi)(x,\omega)=\int_{\mathbb{R}^d}f\left(x+\frac{t}{2}\right)\overline{\varphi\left(x-\frac{t}{2}\right)}e^{-2i\pi\omega t}\,dt.$$

The mixed Lebesgue spaces $L^{p,q}(\mathbb{R}^{2d}),\ 1 \leq p, q \leq \infty$, consist of all measurable functions F in $\mathbb{R}^{2d}$ such that

$$\|F\|_{p,q} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |F(x,y)|^p dx \right)^{q/p} dy \right)^{1/q} < \infty$$

with obvious modifications if p or q are ∞. The spaces $(L^{p,q}(\mathbb{R}^{2d}), \|\cdot\|_{p,q})$ are Banach spaces and $L^{p,p}(\mathbb{R}^{2d}) = L^p(\mathbb{R}^{2d})$. The Schwartz class is dense in $L^{p,q}(\mathbb{R}^{2d})$ for $p, q < \infty$.
The Fourier transform for $f \in \mathcal{S}(\mathbb{R}^d)$ is

$$\widehat{f}(\omega) = \int_{\mathbb{R}^d} f(t) e^{-2\pi i \omega t}\, dt, \quad \omega \in \mathbb{R}^d.$$

It defines an isomorphism $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d), f \mapsto \widehat{f}$, which extends to an isomorphism $\mathcal{F} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$. We still denote by $\widehat{\sigma} = \mathcal{F}(\sigma)$ the Fourier transform of the tempered distribution σ. We will also need to consider the Banach space

$$\mathcal{F}L^1(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d) : \widehat{f} \in L^1(\mathbb{R}^d)\}$$

endowed with the norm $\|f\|_{\mathcal{F}L^1} = \|\widehat{f}\|_1$. We note that the map $(\psi, \varphi, f) \mapsto L^F_{\varphi,\psi}(f)$ is bilinear in (ψ, f) but conjugate linear in φ. Hence

$$T_F(\psi, \varphi, f) := L^F_{\overline{\varphi},\psi}(f)$$

is a multilinear map.
From now on B_{L^2} will denote the closed unit ball of $L^2(\mathbb{R}^d)$.

3. Multilinear Fourier multipliers

We first express the multilinear map T_F in terms of a multilinear Fourier multiplier where the multiplier is given by $m(x, y, z) = a(x + y, y + z)$ and a is related to the symbol F. Then some boundedness and compactness results for such a particular class of Fourier multipliers are given. As a consequence we get a geometric property on a subset $A \subset \mathbb{R}^{2d}$ which guarantees that, for a fixed *synthesis* window $\psi \in L^2(\mathbb{R}^d)$, the family of localization operators $L^A_{\varphi,\psi}$ obtained when the *analysis* window φ varies on the unit ball of $L^2(\mathbb{R}^d)$ is a relatively compact set of compact operators. That is, in a certain sense, the compactness of the localization operator is uniform with respect to the analysis window used when filtering the signal.
In order to avoid technicalities let us first assume that both the symbol F and the windows φ, ψ are functions in the Schwartz class. We put

$$F(x, \omega) = \int_{\mathbb{R}^d} a(v, \omega) e^{2\pi i v x}\, dv.$$

That is, $a \in \mathcal{S}(\mathbb{R}^{2d})$ is the partial Fourier transform of F with respect to the first d variables. Then, for $f \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\begin{aligned} L^F_{\varphi,\psi}(f)(t) &= \iint_{\mathbb{R}^{2d}} F(x, \omega) V_\varphi f(x, \omega) \psi(t - x) e^{2\pi i \omega t}\, d(x, \omega) \\ &= \iiint_{\mathbb{R}^{3d}} a(v, \omega) V_\varphi f(x, \omega) \psi(t - x) e^{2\pi i(\omega t + vx)}\, d(x, \omega, v). \end{aligned}$$

On the other hand, by the Parseval identity,

$$V_\varphi f(x, \omega) = \int_{\mathbb{R}^d} f(t) \overline{\varphi(t - x)} e^{-2\pi i \omega t}\, dt = e^{-2\pi i \omega x} \int_{\mathbb{R}^d} \widehat{f}(u) \overline{\widehat{\varphi}(u - \omega)} e^{2\pi i u x}\, du$$

and we finally obtain

$$L^F_{\varphi,\psi}(f)(t) = \iiiint_{\mathbb{R}^{4d}} a(v, \omega) \widehat{f}(u) \overline{\widehat{\varphi}(u - \omega)} \psi(t - x) e^{2\pi i(\omega t + vx + ux - \omega x)}\, d(x, \omega, u, v).$$

Since

$$\int_{\mathbb{R}^d} \psi(t - x) e^{2\pi i(-\omega x + ux + vx)}\, dx = e^{2\pi i t(u + v - \omega)} \widehat{\psi}(u + v - \omega)$$

we have

$$L^F_{\varphi,\psi}(f)(t) = \iiint_{\mathbb{R}^{3d}} a(v, \omega) \widehat{f}(u) \overline{\widehat{\varphi}(u - \omega)} \widehat{\psi}(u + v - \omega) e^{2\pi i t \omega} e^{2\pi i t(u + v - \omega)}\, d(u, v, \omega).$$

Then the coordinate change

$$u + v - \omega = x, \qquad \omega - u = y, \quad u = z$$

gives

$$L^F_{\varphi,\psi}(f)(t) = \iiint_{\mathbb{R}^{3d}} a(x+y, y+z)\widehat{\psi}(x)\,\overline{\widehat{\varphi}(-y)}\,\widehat{f}(z)\,e^{2\pi it(x+y+z)}\,d(x,y,z).$$

This identity suggests the following definition.

Definition 3.1. Let $a \in L^{p,q}(\mathbb{R}^{2d}),\ 1 \le p, q \le \infty$, be given. Then

$$\mathcal{T}_a : \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$$

is defined by

$$\mathcal{T}_a(\psi, \varphi, f)(t) := \iiint_{\mathbb{R}^{3d}} a(x+y, y+z)\widehat{\psi}(x)\widehat{\varphi}(y)\widehat{f}(z)\,e^{2\pi it(x+y+z)}\,d(x,y,z).$$

Denoting by p' and q' the conjugate exponents of p and q,

$$\iiint_{\mathbb{R}^{3d}} \left|a(x+y, y+z)\widehat{\psi}(x)\widehat{\varphi}(y)\widehat{f}(z)\right|\,d(x,y,z)$$

is less than or equal to

$$\iint_{\mathbb{R}^{2d}} \|a(\cdot, y+z)\|_p \left\|\widehat{\psi}\right\|_{p'} \left|\widehat{\varphi}(y)\widehat{f}(z)\right|\,d(y,z) \le \|a\|_{p,q} \left\|\widehat{\psi}\right\|_{p'} \|\widehat{\varphi}\|_{q'} \left\|\widehat{f}\right\|_1.$$

When $a \in L^\infty(\mathbb{R}^{2d})$ we can make the following extension:

$$\mathcal{T}_a : \mathcal{F}L^1(\mathbb{R}^d) \times \mathcal{F}L^1(\mathbb{R}^d) \times \mathcal{F}L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d).$$

$\mathcal{T}_a$ coincides with the multilinear Fourier multiplier as defined in [15] for

$$m(x,y,z) = a(x+y, y+z).$$

If we fix ψ then $\mathcal{T}_a(\psi, \cdot, \cdot)$ is a particular case of the bilinear pseudodifferential operators considered in [13,16]. According to the previous discussion, in the case where $F \in \mathcal{S}(\mathbb{R}^{2d})$, a is the partial Fourier transform of F with respect to the first d variables and $\varphi, \psi, f \in \mathcal{S}(\mathbb{R}^d)$, we have

$$L^F_{\varphi,\psi}(f) = \mathcal{T}_a(\psi, \overline{\varphi}, f).$$

For $a \in L^{p,q}(\mathbb{R}^{2d})$ and $\psi, f \in \mathcal{S}(\mathbb{R}^d)$ we have

$$a * \left(M_{-t}\widehat{\check{\psi}} \otimes M_{-t}\widehat{\check{f}}\,\right)(y,y) = \iint_{\mathbb{R}^{2d}} a(u,v)\widehat{\psi}(u-y)\widehat{f}(v-y)e^{2\pi it(u+v-2y)}\,d(u,v)$$

where $\check{f}(t) = f(-t)$, and hence

$$\mathcal{T}_a(\psi, \varphi, f)(t) = \int_{\mathbb{R}^d} a * \left(M_{-t}\widehat{\check{\psi}} \otimes M_{-t}\widehat{\check{f}}\right)(y,y)\left(M_t\widehat{\varphi}\right)(y)\,dy. \tag{2}$$

Since, for $a \in \mathcal{S}'(\mathbb{R}^{2d})$,

$$b(t,y) := a * \left(M_{-t}\widehat{\check{\psi}} \otimes M_{-t}\widehat{\check{f}}\right)(y,y)$$

is a $\mathcal{C}^\infty$ function dominated by a polynomial, the expression (2) can be used to define

$$\mathcal{T}_a : \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$$

even if $a \in \mathcal{S}'(\mathbb{R}^{2d})$ is an arbitrary tempered distribution.

Proposition 3.2. *Let us assume that $a \in L^{\infty,1}(\mathbb{R}^{2d})$; then $\mathcal{T}_a$ can be extended as a continuous multilinear operator $\mathcal{T}_a : L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.*

Proof. We put $\rho(\omega) = \|a(\cdot, \omega)\|_\infty,\ \rho \in L^1(\mathbb{R}^d)$. Then, for $\psi, \varphi, f, \chi \in \mathcal{S}(\mathbb{R}^d)$,

$$\langle \mathcal{T}_a\widehat{(\psi, \varphi}, f), \chi\rangle = \langle \mathcal{T}_a(\psi, \varphi, f), \widehat{\chi}\rangle$$

is given by

$$\iiint_{\mathbb{R}^{3d}} a(x+y, y+z)\widehat{\psi}(x)\widehat{\varphi}(y)\widehat{f}(z)\,\chi(x+y+z)\,d(x,y,z).$$

The coordinate change $t = x + y + z$ (with fixed y and z) gives

$$\langle \mathcal{T}_a(\widehat{\psi, \varphi, f}), \chi \rangle = \iiint_{\mathbb{R}^{3d}} a(t-z, y+z)\widehat{\psi}(t-y-z)\widehat{\varphi}(y)\widehat{f}(z)\ \chi(t)\ d(t, y, z),$$

that is,

$$\mathcal{T}_a(\widehat{\psi, \varphi, f})(t) = \iint_{\mathbb{R}^{2d}} a(t-z, y+z)\widehat{\psi}(t-y-z)\widehat{\varphi}(y)\widehat{f}(z)d(y, z).$$

Therefore

$$\begin{aligned}|\mathcal{T}_a(\widehat{\psi, \varphi, f})(t)| &\leq \iint_{\mathbb{R}^{2d}} \rho(u)|\widehat{\psi}(t-u)\widehat{\varphi}(y)\widehat{f}(u-y)|d(y, u)\\ &\leq \int_{\mathbb{R}^d} \rho(u)|\widehat{\psi}(t-u)|(|\widehat{\varphi}| * |\widehat{f}|)(u)du\\ &\leq \|\varphi\|_2\ \|f\|_2(\rho * |\widehat{\psi}|)(t)\end{aligned}$$

and hence, as the Fourier transform is an isometry on $L^2(\mathbb{R}^d)$, $\mathcal{T}_a(\psi, \varphi, f) \in L^2(\mathbb{R}^d)$ and its norm is controlled by $\|a\|_{L^{\infty,1}}\|\psi\|_2\|\varphi\|_2\|f\|_2$. □

We now obtain continuity results for the case where a or $\widehat{a} \circ \ell$ is in $L^{\infty,2}(\mathbb{R}^{2d})$, where $\ell : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ denotes the linear transformation $\ell(v, u) = (u, v)$. In particular, we show that for ψ in appropriate Banach spaces, $\mathcal{T}_a(\psi, \cdot, \cdot)$ defines a bilinear pseudodifferential operator on $L^2(\mathbb{R}^d)$ and we obtain an estimate for its norm.

Theorem 3.3. *Let $a \in L^{\infty,2}(\mathbb{R}^{2d})$ be given. Then, $\mathcal{T}_a$ can be extended to a multilinear continuous operator*

$$\mathcal{T}_a : \mathcal{F}L^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d).$$

Moreover,

$$\|\mathcal{T}_a\| \leq \|a\|_{\infty,2}.$$

Proof. Let us fix $\psi, \varphi, f \in \mathcal{S}(\mathbb{R}^d)$, and define $\rho(v) = \|a(\cdot, v)\|_\infty$, so $\|a\|_{\infty,2} = \|\rho\|_2$. Then, as in the proof of Proposition 3.2,

$$\mathcal{T}_a(\widehat{\psi, \varphi, f})(t) = \iint_{\mathbb{R}^{2d}} a(t-z, y+z)\widehat{\psi}(t-y-z)\widehat{\varphi}(y)\widehat{f}(z)\ d(y, z).$$

We note that Fubini's theorem gives the existence of this last Lebesgue integral for almost all values of $t \in \mathbb{R}^d$. Then

$$\begin{aligned}|\mathcal{T}_a(\widehat{\psi, \varphi, f})(t)| &\leq \int_{\mathbb{R}^d} |\widehat{\psi}(s)|\left(\int_{\mathbb{R}^d} |a(y+s, t-s)|\ |\widehat{\varphi}(y)\widehat{f}(t-y-s)|\ dy\right) ds\\ &\leq \|\varphi\|_2\ \|f\|_2 \int_{\mathbb{R}^d} |\widehat{\psi}(s)|\ \rho(t-s)\ ds.\end{aligned}$$

Now, Plancherel's and Minkowski's theorems give

$$\|\mathcal{T}_a(\psi, \varphi, f)\|_2 \leq \|\varphi\|_2\ \|f\|_2\ \|\widehat{\psi}\|_1\ \ \|a\|_{\infty,2}$$

and we obtain the conclusion. □

We recall that $\ell : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ denotes the linear transformation $\ell(v, u) = (u, v)$.

Theorem 3.4. *Let $a \in \mathcal{S}'(\mathbb{R}^{2d})$ be given such that $\widehat{a} \circ \ell \in L^{\infty,2}(\mathbb{R}^{2d})$. Then, $\mathcal{T}_a$ can be extended to a multilinear continuous operator*

$$\mathcal{T}_a : L^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d).$$

Moreover,

$$\|\mathcal{T}_a\| \leq \|\widehat{a} \circ \ell\|_{\infty,2}\ .$$

Proof. Here $\rho(u) = \|\widehat{a}(u, \cdot)\|_\infty$ and, as before, we consider $\psi, \varphi, f \in \mathcal{S}(\mathbb{R}^d)$. The Fourier transform of

$$\Phi_{x,t}(y, z) = a(x+y, y+z)e^{2\pi it(y+z)}$$

is given by

$$\widehat{\Phi_{x,t}}(v, \omega) = e^{2\pi ix(v-\omega)}\widehat{a}(v-\omega, \omega-t).$$

Hence,

$$\begin{aligned}\mathcal{T}_a(\psi,\varphi,f)(t) &= \iiint_{\mathbb{R}^{3d}} \widehat{\Phi_{x,t}}(v,\omega)\widehat{\psi}(x)(\varphi\otimes f)(v,\omega)e^{2\pi i t x}\, d(x,v,\omega)\\ &= \iiint_{\mathbb{R}^{3d}} \widehat{a}(v-\omega,\omega-t)e^{2\pi i x(v-\omega+t)}\widehat{\psi}(x)\varphi(v)f(\omega)\, d(x,v,\omega)\\ &= \iint_{\mathbb{R}^{2d}} \widehat{a}(v-\omega,\omega-t)\psi(v-\omega+t)\varphi(v)f(\omega)\, d(v,\omega).\end{aligned}$$

As in Theorem 3.3,

$$\begin{aligned}|\mathcal{T}_a(\psi,\varphi,f)(t)| &\le \int_{\mathbb{R}^d} |\psi(s)|\left(\int_{\mathbb{R}^d} |\widehat{a}(s-t,v-s)|\,|\varphi(v)f(v+t-s)|\,dv\right)ds\\ &\le \|f\|_2\,\|\varphi\|_2 \int_{\mathbb{R}^d} |\psi(s)|\,\rho(s-t)\,ds = \|f\|_2\,\|\varphi\|_2\,(\psi*\check{\rho})(t),\end{aligned}$$

and thus $\mathcal{T}_a(\psi,\varphi,f)\in L^2(\mathbb{R}^d)$ and

$$\|\mathcal{T}_a(\psi,\varphi,f)\|_2 \le \|f\|_2\,\|\varphi\|_2\,\|\psi\|_1\,\|\rho\|_2\,. \quad \square$$

The two previous results give sufficient conditions on a for the continuity of the bilinear operator $\mathcal{T}_a(\psi,\cdot,\cdot)$ on $L^2(\mathbb{R}^d)$. When these two conditions hold simultaneously, it is compact. We remark that the next result does not require that $\psi\in L^1(\mathbb{R}^d)\cap\mathcal{F}L^1(\mathbb{R}^d)$.

We recall that a subset $C\subset L^2(\mathbb{R}^d)$ is relatively compact if, and only if, for any $\epsilon>0$ there is an $A>0$ such that

$$\int_{|u|\ge A} |f(u)|^2\,du \le \epsilon \quad\text{and}\quad \int_{|u|\ge A} |\widehat{f}(u)|^2\,du \le \epsilon$$

for every $f\in C$. See for instance [17] and also [18] for a more general result.

Theorem 3.5. *Let a be a measurable function such that $a,\widehat{a}\circ\mathcal{I}\in L^{\infty,2}(\mathbb{R}^{2d})$ and $\psi\in L^1(\mathbb{R}^d)\cup\mathcal{F}L^1(\mathbb{R}^d)$. Then,*

$$\mathcal{T}_a(\psi,\cdot,\cdot): L^2(\mathbb{R}^d)\times L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$$

is a bilinear compact operator.

Proof. We first assume that $\psi\in\mathcal{S}(\mathbb{R}^d)$ and we consider two subcases. (i) In the case where a also belongs to $L^\infty(\mathbb{R}^{2d})$ we fix $\epsilon>0$ and take $R>0$ such that

$$\|\widehat{\psi}\|_1\left(\int_{|v|\ge R}\|a(\cdot,v)\|_\infty^2\,dv\right)^{\frac12} \le \frac{\epsilon}{2}.$$

Now we define $b(u,v):=a(u,v)$ if $|v|\le R$ and $b(u,v):=0$ elsewhere. According to Theorem 3.3 we have

$$\|\mathcal{T}_a(\psi,\cdot,\cdot)-\mathcal{T}_b(\psi,\cdot,\cdot)\| \le \frac{\epsilon}{2}. \tag{3}$$

On the other hand, whenever $|t|\ge 2R$ and for every φ,f in the unit ball of $L^2(\mathbb{R}^d)$ we have

$$\begin{aligned}\left|\mathcal{T}_b(\widehat{\psi,\varphi,f})(t)\right| &\le \iint_{|y+z|\le\frac{|t|}{2}} \left|a(t-z,y+z)\widehat{\psi}(t-y-z)\widehat{\varphi}(y)\widehat{f}(z)\right|\,d(y,z)\\ &\le \|a\|_\infty \int_{|s|\ge\frac{|t|}{2}} |\widehat{\psi}(s)|\,ds.\end{aligned}$$

Since $\psi\in\mathcal{S}(\mathbb{R}^d)$ we can find $M>0$ large enough such that

$$\left(\int_{|t|>M} |\mathcal{T}_b(\widehat{\psi,\varphi,f})(t)|^2\,dt\right)^{\frac12}$$

is less than or equal to

$$\|a\|_\infty\left(\int_{|t|\ge M}\left(\int_{|s|\ge\frac{|t|}{2}}|\widehat{\psi}(s)|\,ds\right)^2 dt\right)^{\frac12} \le \frac{\epsilon}{2} \tag{4}$$

for every φ, f in the unit ball of $L^2(\mathbb{R}^d)$. From the estimates (3) and (4) we obtain that

$$\sup\left\{\left(\int_{|t|>M} |\mathcal{T}_a(\widehat{\psi,\varphi,f})(t)|^2 dt\right)^{\frac{1}{2}} : \|\varphi\|_2 \le 1,\ \ \|f\|_2 \le 1\right\} \le \varepsilon.$$

(ii) In the case where $a \notin L^\infty(\mathbb{R}^{2d})$, define $a_n(u,v) = a(u,v)$ if $\|a(\cdot,v)\|_\infty \le n$ and $a_n(u,v) = 0$ elsewhere. Then, according to Theorem 3.3, given $\epsilon > 0$ we find n_0 such that for $n \ge n_0$ we have

$$\left\| \mathcal{T}_a(\psi,\cdot,\cdot) - \mathcal{T}_{a_n}(\psi,\cdot,\cdot) \right\| \le \frac{\epsilon}{2}. \tag{5}$$

Combining (5) and the subcase (i) we finally obtain that

$$\sup\left\{\left(\int_{|t|>M} |\mathcal{T}_a(\widehat{\psi,\varphi,f})(t)|^2 dt\right)^{\frac{1}{2}} : \|\varphi\|_2 \le 1,\ \ \|f\|_2 \le 1\right\}$$

goes to zero as $M \to \infty$.

A similar argument but using Theorem 3.4 instead of Theorem 3.3 gives that also

$$\sup\left\{\left(\int_{|A|>M} |\mathcal{T}_a(\psi,\varphi,f)(A)|^2 dA\right)^{\frac{1}{2}} : \|\varphi\|_2 \le 1,\ \ \|f\|_2 \le 1\right\}$$

goes to zero as $M \to \infty$. The theorem is proved when $\psi \in \mathcal{S}(\mathbb{R}^d)$.

Now, if $\psi \in \mathcal{F}L^1(\mathbb{R}^d)$ (resp. $\psi \in L^1(\mathbb{R}^d)$), by 3.3 (resp. by 3.4) $\mathcal{T}_a(\psi,\cdot,\cdot)$ is the limit of the sequence of compact bilinear operators $(\mathcal{T}_a(\psi_n,\cdot,\cdot))_n$ where $(\psi_n)_n$ converges to ψ in $\mathcal{F}L^1(\mathbb{R}^d)$ (resp. in $L^1(\mathbb{R}^d)$) and consequently it is compact. □

It is well known that for symbols $F \in L^p(\mathbb{R}^{2d}),\ 1 \le p < \infty$, or vanishing at infinity, the localization operator $L^F_{\varphi,\psi}$ is compact on $L^2(\mathbb{R}^{2d})$ for all $\varphi, \psi \in L^2(\mathbb{R}^{2d})$ (see for instance [11]). The relation between localization operators and the multilinear multipliers $\mathcal{T}_a$ mentioned at the beginning of this section can be used to show that the family of localization operators $L^F_{\varphi,\psi}$ obtained when the *synthesis* window is fixed and the *analysis* window φ varies on the unit ball of $L^2(\mathbb{R}^d)$ is a relatively compact set of compact operators.

Proposition 3.6. *Fix $F \in L^p(\mathbb{R}^{2d}),\ 1 \le p < \infty$, and $\psi \in L^2(\mathbb{R}^d)$. Then $\{L^F_{\varphi,\psi} : \varphi \in B_{L^2}\}$ is a relatively compact set of compact operators.*

Proof. By an approximation argument it suffices to prove the proposition in the case where $F \in \mathcal{S}(\mathbb{R}^{2d})$ and $\psi \in \mathcal{S}(\mathbb{R}^d)$. By Theorem 3.5, the bilinear operator $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d),\ (\varphi, f) \to L^F_{\overline{\varphi},\psi}(f)$, is compact. According to [19, Theorem 1.11], in order to conclude it suffices to check that for all $h \in L^2(\mathbb{R}^d)$ the set

$$\{(L^F_{\varphi,\psi})^*(h) : \varphi \in B_{L^2}\}$$

is relatively compact in $L^2(\mathbb{R}^d)$. But this follows from the identities

$$\left(L^F_{\varphi,\psi}\right)^*(h) = L^{\overline{F}}_{\psi,\varphi}(h) = L^G_{\psi,h}(\varphi)$$

where $G(u,v) = \widehat{\widehat{\overline{F}}}(v,-u)$, and the fact that $L^G_{\psi,h}$ is a compact operator on $L^2(\mathbb{R}^d)$. □

We put $\mathcal{U} : \mathbb{R}^{2d} \to \mathbb{R}^{2d},\ \mathcal{U}(a,b) := (b,-a)$. Then $\mathcal{U}^{-1} = -\mathcal{U}$.

Lemma 3.7. *Let $F \in L^\infty(\mathbb{R}^{2d})$ be given and $G = \widehat{\widehat{\overline{F}}} \circ \mathcal{U} \in \mathcal{S}'(\mathbb{R}^{2d})$. Then,*

$$L^G_{\psi,h} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$$

is a compact operator for every $\psi, h \in \mathcal{S}(\mathbb{R}^d)$.

Proof. We fix $\Phi \in \mathcal{S}(\mathbb{R}^{2d})$ and define $\Psi \in \mathcal{S}(\mathbb{R}^{2d})$ by $\widehat{\Psi}(a,b) = (\Phi \circ \mathcal{U})(-a,-b)$. A routine calculation gives

$$(V_\Phi G)(x_1,x_2;\omega_1,\omega_2) = (V_{\widehat{\Psi}}\widehat{\widehat{\overline{F}}})(x_2,-x_1;\omega_2,-\omega_1).$$

Then, we apply [14, p. 40] to get

$$|(V_\Phi G)(x_1,x_2;\omega_1,\omega_2)| = \left|(V_\Psi \overline{F})(\omega_2,-\omega_1;-x_2,x_1)\right| = \left|\mathcal{F}\left(\overline{F}T_{(\omega_2,-\omega_1)}\Psi\right)(-x_2,x_1)\right|.$$

Since the map

$$\mathbb{R}^{2d} \to L^1(\mathbb{R}^{2d}),\ (\omega_1,\omega_2) \mapsto T_{(\omega_2,-\omega_1)}\Psi,$$

is continuous, the set

$$\left\{T_{(\omega_2,-\omega_1)}\Psi : |\omega| \le R\right\}$$

is compact in $L^1(\mathbb{R}^{2d})$. Consequently $\left\{\mathcal{F}\left(\overline{F}T_{(\omega_2,-\omega_1)}\Psi\right) : |\omega| \le R\right\}$ is a compact set in the Banach space $C_0(\mathbb{R}^{2d})$ of continuous functions vanishing at infinity. In particular,

$$\lim_{|x|\to\infty} \sup_{|\omega|\le R} |(V_\Phi G)(x,\omega)| = 0. \tag{6}$$

From [6, Theorem 3.15] we conclude that $L^G_{\psi,h}$ is a compact operator for every $\psi, h \in \mathcal{S}(\mathbb{R}^d)$. □

Proposition 3.8. *Let $F \in L^\infty(\mathbb{R}^{2d})$ and $\psi \in L^2(\mathbb{R}^d)$ be given. The following conditions are equivalent:*

(1) *The bilinear map $B : L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, $B(\varphi, f) = L^F_{\varphi,\psi}(f)$, is compact.*

(2) *$\{L^F_{\varphi,\psi} : \varphi \in B_{L^2}\}$ is a relatively compact set of compact operators on $L^2(\mathbb{R}^d)$.*

Proof. Clearly (2) ⇒ (1). Let us now assume that condition (1) is satisfied. In order to prove (2), according to [19, Theorem 1.11], it suffices to check that for all $h \in L^2(\mathbb{R}^d)$ the set

$$\{(L^F_{\varphi,\psi})^*(h) : \varphi \in B_{L^2}\}$$

is relatively compact in $L^2(\mathbb{R}^d)$. When $\psi, h \in \mathcal{S}(\mathbb{R}^d)$, this follows from the previous lemma and the identities

$$\left(L^F_{\varphi,\psi}\right)^*(h) = L^{\overline{F}}_{\psi,\varphi}(h) = L^G_{\psi,h}(\varphi)$$

where $G(u,v) = \widehat{\widehat{\overline{F}}}(v,-u)$. For $\psi, h \in L^2(\mathbb{R}^d)$ we use an approximation argument and (1). □

Remark 3.9. (a) Condition (2) in the previous result is in general stronger than (1). In fact, if we take $F = \delta$ then $B(\varphi, f) = \langle f, \overline{\varphi}\rangle\psi$; therefore it is compact. As $\left(L^F_{\varphi,\psi}\right)^*(h) = \langle h, \psi\rangle\varphi$ it is clear that $\{(L^F_{\varphi,\psi})^*(h) : \varphi \in B_{L^2}\}$ is not relatively compact in $L^2(\mathbb{R}^d)$, except for $\langle h, \psi\rangle = 0$. Then $\{L^F_{\varphi,\psi} : \varphi \in B_{L^2}\}$ is not a relatively compact set of compact operators on $L^2(\mathbb{R}^d)$.

(b) Results similar to those established in Propositions 3.6 and 3.8 are also true when we fix the analysis window and the synthesis window varies in the unit ball. Just keep in mind that the adjoint operator of $L^F_{\varphi,\psi}$ is $L^{\overline{F}}_{\psi,\varphi}$.

The following example shows that Proposition 3.6 cannot be extended to the case $p = \infty$.

Example 3.10. There is an $F \in L^\infty(\mathbb{R}^2)$ that meets the condition that $L^F_{\varphi,\psi}$ is a compact operator on $L^2(\mathbb{R})$ for every pair of windows $\varphi, \psi \in L^2(\mathbb{R})$ and yet for some $\psi \in L^2(\mathbb{R})$ the set $\{L^F_{\varphi,\psi} : \varphi \in B_{L^2}\}$ is not relatively compact.

Proof. We consider $F(s,t) = e^{i\pi(s^2+t^2)}$. From [20, Theorem 3.1], $L^F_{\varphi,\psi}$ is a compact operator on $L^2(\mathbb{R})$ for every pair of windows $\varphi, \psi \in L^2(\mathbb{R})$. Let us now assume that $\{L^F_{\varphi,\psi} : \varphi \in B_{L^2}\}$ is a relatively compact set of compact operators for every $\psi \in L^2(\mathbb{R})$. From [20, Theorem 4.4] we obtain that

$$\{\sigma = F * W(\psi,\varphi) : \varphi \in \mathcal{S}(\mathbb{R}^d) \cap B_{L^2}\}$$

is a relatively compact set in the modulation space $M^0(\mathbb{R}^{2d})$ for every $\psi \in L^2(\mathbb{R})$. That is, for a fixed $\psi \in L^2(\mathbb{R})$ and $\Phi \in \mathcal{S}(\mathbb{R}^2)$ and every $\epsilon > 0$ there is an $R > 0$ such that

$$|V_\Phi(F * W(\psi,\varphi))(x,\omega)| < \epsilon$$

whenever $(x,\omega) \in \mathbb{R}^4$, $\|(x,\omega)\| > R$ and $\varphi \in \mathcal{S}(\mathbb{R}) \cap B_{L^2}$. By [14, 4.3.2], $W(\psi, T_u\varphi)(x,\omega) = e^{2\pi i\omega u}W(\psi,\varphi)(x - \frac{u}{2}, \omega)$. In view of the expression for F, we deduce that

$$F * W(\psi, T_u\varphi)(x,\omega) = e^{i\pi u^2}T_{(\frac{u}{2},u)}M_{(0,u)}(F * W(\psi,\varphi))(x,\omega).$$

Consequently, from [14, 3.1.3], we get

$$|V_\Phi(F * W(\psi, T_u\varphi))(x_1,x_2,\omega_1,\omega_2)| = \left|V_\Phi\left(F * W(\psi,\varphi)\right)\left(x_1 - \frac{u}{2}, x_2 - u, \omega_1, \omega_2 - u\right)\right|.$$

We now fix $(x_1^0, x_2^0, \omega_1^0, \omega_2^0) \in \mathbb{R}^{4d}$ and take $u \in \mathbb{R}^d$ satisfying $\left\|x_2^0 + u\right\| > R$. Then

$$\left|V_\Phi\left(F * W(\psi,\varphi)\right)(x_1^0, x_2^0, \omega_1^0, \omega_2^0)\right|$$

equals

$$\left|V_\Phi\left(F * W(\psi, T_u\varphi)\right)\left(x_1^0 + \frac{u}{2}, x_2^0 + u, \omega_1^0, \omega_2^0 + u\right)\right| < \epsilon$$

for every $\varphi \in \mathcal{S}(\mathbb{R}^d) \cap B_{L^2}$. Since $\epsilon > 0$ and $\psi \in \mathcal{S}(\mathbb{R}^d)$ are arbitrary we finally conclude that $V_\Phi\big(F * W(\psi, \varphi)\big) = 0$, and hence $F * W(\psi, \varphi) = 0$ for all $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d) \cap B_{L^2}$ and $F = 0$. This is a contradiction. □

We now restrict our attention to localization operators $L^A_{\varphi,\psi}$ whose symbol is the characteristic function of a set $A \subset \mathbb{R}^{2d}$. We denote by

$$A_x = \{y \in \mathbb{R}^d : (x, y) \in A\} \quad \text{and} \quad A^y = \{x \in \mathbb{R}^d : (x, y) \in A\}$$

the sections of the set A and by m_d the Lebesgue measure on $\mathbb{R}^d$.

Theorem 3.11. *Let $A \subset \mathbb{R}^{2d}$ satisfy*

$$\int_{\mathbb{R}^d} m_d(A_x)^2\, dx < +\infty \quad \text{or} \quad \int_{\mathbb{R}^d} m_d(A^y)^2\, dy < +\infty.$$

Then, for each $\psi \in L^2(\mathbb{R}^d)$, the bilinear operator $(\varphi, f) \to L^A_{\varphi,\psi}(f)$ is compact.

Proof. Let us first assume that $\int_{\mathbb{R}^d} m_d(A^y)^2\, dy < +\infty$. This means that $\chi_A \in L^{1,2}(\mathbb{R}^{2d})$ and we can select a sequence $(F_n) \subset \mathcal{S}(\mathbb{R}^{2d})$ which converges to χ_A in $L^{1,2}(\mathbb{R}^{2d})$. We now denote by a and b_n the partial Fourier transforms with respect to the first d variables of χ_A and F_n respectively. According to Theorem 3.3,

$$\mathcal{T}_a, \mathcal{T}_{b_n} : \mathcal{F}L^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$$

and

$$\left\| \mathcal{T}_a - \mathcal{T}_{b_n} \right\| \le \|a - b_n\|_{\infty,2} \le \|\chi_A - F_n\|_{1,2}\, .$$

Now, as an application of Theorem 3.5 we get that $\mathcal{T}_{b_n}(\psi, \cdot, \cdot)$, and hence also $\mathcal{T}_a(\psi, \cdot, \cdot)$, is compact for every $\psi \in \mathcal{S}(\mathbb{R}^d)$. This means that $(\varphi, f) \to L^A_{\overline{\varphi},\psi}(f)$ is compact whenever $\psi \in \mathcal{S}(\mathbb{R}^d)$. For arbitrary $\psi \in L^2(\mathbb{R}^d)$ we use an approximation argument and (1).
Finally, if A satisfies $\int_{\mathbb{R}^d} m_d(A_x)^2\, dx < +\infty$, as

$$V_{h_1}h_2(x, \omega) = e^{-2\pi i x\omega} V_{\hat{h}_1}\hat{h}_2(\omega, -x),$$

one has

$$\langle L^A_{\varphi,\psi} f, g\rangle = \langle L^B_{\hat{\varphi},\hat{\psi}} \hat{f}, \hat{g}\rangle$$

where $B := \{(x, \omega) : (-\omega, x) \in A\}$. Hence $m_d(B^\omega) = m_d(A_{-\omega})$ and as the Fourier transform is an isometry on $L^2(\mathbb{R}^d)$, we conclude. □

The sets satisfying the hypothesis of the previous theorem are thin at infinity in the sense of [21, Definition 3.2]. We observe that for a set $A \subset \mathbb{R}^{2d}$ with finite measure, the conclusion of Theorem 3.11 follows from Proposition 3.6. However, Theorem 3.11 applies to sets not having necessarily finite measure.

Theorem 3.12. *For a fixed $F \in L^p(\mathbb{R}^{2d})$, $F \neq 0$ and $1 \le p \le \infty$, the multilinear operator*

$$L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d), \qquad (\varphi, \psi, f) \to L^F_{\varphi,\psi}(f)$$

is not compact.

Proof. We proceed by contradiction. Assume that, on the contrary, the multilinear operator is compact. Then

$$\{L^{\overline{F}}_{\varphi,\psi} : \varphi,\ \psi \in B_{L^2}\}$$

is a relatively compact set of compact operators. In fact, according to Ruess [19, Theorem 1.11], we only have to check that for each $h \in L^2(\mathbb{R}^d)$ the set $\{h \circ L^{\overline{F}}_{\varphi,\psi} : \varphi,\ \psi \in B_{L^2}\}$ is compact in $L^2(\mathbb{R}^d)$. But this follows from

$$h \circ L^{\overline{F}}_{\varphi,\psi} = L^F_{\psi,\varphi}(h).$$

As in Example 3.10, we obtain from [20, Theorem 4.4] that

$$\{\sigma = \overline{F} * W(\psi, \varphi) : \varphi, \psi \in \mathcal{S}(\mathbb{R}^d) \cap B_{L^2}\}$$

is a relatively compact set in the modulation space $M^0(\mathbb{R}^{2d})$, that is, for a fixed $\Phi \in \mathcal{S}(\mathbb{R}^{2d})$ and every $\epsilon > 0$ there is an $R > 0$ such that

$$|V_\Phi(\overline{F} * W(\psi, \varphi))(x, \omega)| < \epsilon$$

whenever $(x, \omega) \in \mathbb{R}^{4d}$ and $\|(x, \omega)\| > R$ and $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d) \cap B_{L^2}$. We now fix $(x_0, \omega_0) \in \mathbb{R}^{4d}$ and take $y = (u, \eta) \in \mathbb{R}^{2d}$ satisfying $\|x_0 - y\| > R$. Then

$$\begin{aligned} |V_\Phi(\overline{F} * W(\psi, \varphi))(x_0, \omega_0)| &= |V_\Phi(\overline{F} * T_y W(\psi, \varphi))(x_0 - y, \omega_0)| \\ &= |V_\Phi(\overline{F} * W(T_u M_\eta \psi, T_u M_\eta \varphi))(x_0 - y, \omega_0)| < \epsilon \end{aligned}$$

for every $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d) \cap B_{L^2}$. Since $\epsilon > 0$ is arbitrary we finally conclude that $V_\Phi\left(\overline{F} * W(\psi, \varphi)\right) = 0$, and hence $\overline{F} * W(\psi, \varphi) = 0$ for all $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d) \cap B_{L^2}$ and $F = 0$. This is a contradiction. $\square$

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