J. Math. Anal. Appl. 401 (2013) 672–681

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

On nonlinear Schrödinger–Poisson equations with general potentials

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ARTICLE INFO

Article history:
Received 10 August 2012
Available online 29 December 2012
Submitted by Jie Xiao

Keywords:
Nonlinear Schrödinger–Poisson equations
Variational methods

ABSTRACT

We study the existence of infinitely many finite energy radial solutions to the nonlinear Schrödinger–Poisson equations

$$\begin{cases} \Delta u - u - \phi(x)u + f(u) = 0 & \text{in } \mathbb{R}^3 \\ \Delta\phi + u^2 = 0, \quad \lim_{|x|\to\infty} \phi(x) = 0 & \text{in } \mathbb{R}^3 \end{cases}$$

(NSPE for short) under some structure conditions on the nonlinearity function f. As consequences of the main result, we can provide examples of f which guarantee the existence of infinitely many finite energy solutions but

(i) $f(t)$ grows faster than t^2 and slower than t^p for all $p > 2$ or
(ii) $f(t)$ is the same as $|t|t$ when $|t| \le t_0$ for arbitrarily given $t_0 > 0$.

If $f(t) = |t|^{p-1}t$, it is known that (NSPE) admits no finite energy nontrivial solutions when $p \in (1, 2]$ and admits infinitely many finite energy solutions when $p \in (2, 5)$ so examples (i) and (ii) show some interesting features of (NSPE).

1. Introduction and statement of the main result

Consider the following system of PDEs

$$\begin{cases} \Delta u - u - \lambda\phi(x)u + |u|^{p-1}u = 0 & \text{in } \mathbb{R}^3 \\ \Delta\phi + u^2 = 0, \quad \lim_{|x|\to\infty} \phi(x) = 0 & \text{in } \mathbb{R}^3 \end{cases} \tag{1.1}$$

where $\lambda \in (0, \infty)$, $p \in (1, 5)$ and u, ϕ are real valued unknown functions defined on $\mathbb{R}^3$. This is called the system of Schrödinger–Poisson equations because it consists of a Schrödinger equation coupled with a Poisson term. It describes systems of identical charged particles interacting each other in the case that effects of magnetic field could be ignored and its solution represents, in particular, a standing wave for such a system. For the details, we refer to [4,15].

In this paper, we restrict our interest to the solutions of finite energy, i.e., solutions (u, ϕ) belonging to the energy space $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$, where

$$H^1(\mathbb{R}^3) := \left\{u \in L^2(\mathbb{R}^3) \mid \nabla u \in L^2(\mathbb{R}^3)\right\}$$

and

$$D^{1,2}(\mathbb{R}^3) := \left\{u \in L^6(\mathbb{R}^3) \mid \nabla u \in L^2(\mathbb{R}^3)\right\}.$$

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doi:10.1016/j.jmaa.2012.12.054

Then, one can easily reduce the Eq. (1.1) to a single equation by solving ϕ in terms of u. It is just the convolution of the fundamental solution of Laplace equation $\frac{1}{4\pi|x|}$ with u^2, which is well defined since $u \in H^1(\mathbb{R}^3)$ so one can rewrite the Eq. (1.1) as

$$\Delta u - u - \lambda\left(\frac{1}{4\pi|x|} * u^2\right)u + |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^3, u \in H^1(\mathbb{R}^3). \tag{1.2}$$

This is a semilinear elliptic PDE with a nonlocal term.

It turns out that the solution structure of (1.2) is more complicated than that of the case where the nonlocal term is discarded. If we just consider the equation

$$\Delta u - u + |u|^{p-1}u = 0 \quad \text{in } \mathbb{R}^3,$$

there is a simple threshold $p = 5$ between existence and nonexistence for nontrivial finite energy solutions; there are infinitely many finite energy solutions if $p \in (1, 5)$ and there is no nontrivial finite energy solution if $p \geq 5$. It is also known that the positive radial solution is unique.

On the other hand, such a simple characterization no longer holds when we consider the Eq. (1.2). While it is still true that there is no nontrivial finite energy solution for $p \geq 5$, the existence range of p depends on the parameter $\lambda > 0$. In fact, if $\lambda > 1/4$, there is no nontrivial solution when $p \in (1, 2]$ and there are infinitely many solutions when $p \in (2, 5)$. On the other hand, if $\lambda > 0$ is sufficiently small, one can find at least two positive radial solutions when $p \in (1, 2)$ thus there is no uniqueness of positive radial solution at all in the range of $p \in (1, 2)$. See [1,15] for all above results. We also refer to [3,7,9,8,12] for further results about (1.2).

In this paper, we put $\lambda = 1$ and consider the Schrödinger–Poisson equations with general potential;

$$\Delta u - u - \left(\frac{1}{4\pi|x|} * u^2\right)u + f(u) = 0 \quad \text{in } \mathbb{R}^3, \tag{1.3}$$

where $f : \mathbb{R} \to \mathbb{R}$ is an odd continuous function. Then one can ask the following questions:

Question 1. Is there an example of $f(t)$ which grows faster than t^2 but slower than t^p for all $p > 2$ as $t \to \infty$ and guarantees the existence of infinitely many solutions to (1.3)?

Question 2. For an arbitrarily given $t_0 > 0$, is there an example of $f(t)$ which satisfies $f(t) = |t|t$ if $|t| \leq t_0$ and guarantees the existence of infinitely many solutions to (1.3)?

We remind that when $f(t)$ is $|t|t$, (1.3) admits only the trivial solution and when $f(t)$ is $|t|^{p-1}t$, $p > 2$, (1.3) admits infinitely many solutions so it seems interesting to ask such questions.

We prove in this paper that the answers are affirmative. Furthermore, we propose the following general conditions on f:

(F1) $f \in C(\mathbb{R}, \mathbb{R})$ is odd and $\lim_{t\to 0^+} f(t)/t = 0$;
(F2) $\limsup_{t\to\infty} |f(t)/t^p| < \infty$ for some $p \in (1, 5)$;
(F3) $2f(t)/t^2 - 3F(t)/t^3$ is monotone increasing to ∞ on $(0, \infty)$ where $F(t) := \int_0^t f(s)\,ds$,

and prove the following:

Theorem 1.1. *Under the conditions* (F1)–(F3), *there are infinitely many finite energy radial solutions to* (1.3). *In addition, at least one of them is positive everywhere.*

Remark 1.1. In fact, the oddness of f is not needed when proving the existence of at least one positive solution to (1.3). See the last section of the paper.

We can give two examples of f satisfying (F1)–(F3). By elementary computations, one can see that odd continuous functions f_1 and f_2 defined as

$$\begin{cases} f_1(t) = t^2 \log t, & t > 0, \\ \quad\;\; = 0, & t = 0, \\ \quad\;\; = -f(-t), & t < 0, \end{cases} \tag{1.4}$$

and for given $t_0 > 0$,

$$\begin{cases} f_2(t) = t^2, & 0 \leq t \leq t_0, \\ \quad\;\; = t^4/t_0^2, & t \geq t_0, \\ \quad\;\; = -f(-t), & t < 0, \end{cases} \tag{1.5}$$

fulfill the conditions (F1)–(F3). It is also clear that f_1 and f_2 satisfy the hypothesis of Questions 1 and 2 respectively. Thus we obtain the following corollary.

Corollary 1.1. *There are examples of f which positively answer the Questions 1 and 2.*

Also, we believe that Theorem 1.1 is a first step toward answering the following question, which is more fundamental than the Questions 1 and 2 are.

Question 3. Are there optimal assumptions for f to guarantee the existence of a (or infinitely many) nontrivial finite energy solution(s) to (1.3).

This question is motivated by the celebrated work of Berestycki and Lions [5,6], in which they found the almost optimal conditions of f for the equation

$$\Delta u - u + f(u) = 0 \quad \text{in } \mathbb{R}^N \tag{1.6}$$

to have a (or infinitely many) nontrivial finite energy solution(s). Under the conditions on f:

(F1$'$) (superlinearity near zero) $f \in C(\mathbb{R}, \mathbb{R})$ and $\lim_{t\to 0^+} f(t)/t = 0$;
(F2$'$) (subcriticality near infinity) $\limsup_{t\to\infty} |f(t)/t^p| < \infty$ for some $p \in (1, (N+2)/(N-2))$;
(F3$'$) (the Berestycki–Lions condition) there exists $T > 0$ such that $\frac{1}{2}T^2 < F(T)$, where $F(t) = \int_0^t f(s)ds$,

they proved that the Eq. (1.6) admits a nontrivial least energy solution. They also showed that if f is odd, the Eq. (1.6) admits infinitely many finite energy radial solutions. Note that the condition (F3$'$) is necessary for existence since every finite energy solutions to (1.6) must satisfy the following Pohozaev's identity:

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2\, dx = N \int_{\mathbb{R}^N} F(u) - \frac{1}{2}u^2\, dx.$$

For the Eq. (1.3), it is proved by Azzollini, dAvenia and Pomponio [2] that there is a finite energy solution of (1.3) under the assumptions (F1$'$)–(F3$'$) when the parameter $\lambda > 0$ is inserted again and sufficiently small. We can check the condition (F3$'$) is also necessary for (1.3) because we also have a Pohozaev type identity for (1.3) (see Proposition 4.1):

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2\, dx + \frac{5}{4}\int_{\mathbb{R}^3} \phi_u u^2\, dx = 3\int_{\mathbb{R}^3} F(u) - \frac{1}{2}u^2\, dx.$$

However, the nonexistence result in the range of $p \in (1, 2]$ tells us that (F1$'$)–(F3$'$) are not sufficient for (1.3) if λ is larger than 1/4. Instead of (F1$'$)–(F3$'$), Theorem 1.1 provides sufficient conditions (F1)–(F3) although it is hard to expect that they are optimal.

The rest of the paper is organized as follows. We will use the variational approach to prove Theorem 1.1. We first prove the existence of infinitely many solutions. Then the existence of a positive solution follows by mildly modifying the argument. In Section 2, we introduce various definitions, variational settings and preliminaries. In Section 3, we construct a sequence of $\mathbb{Z}_2$-homotopy stable families (see Section 2 for definition) satisfying some linking structure, which plays a crucial role in obtaining our main result. In Section 4, we prove Theorem 1.1.

2. Variational settings and preliminaries

Let $H_r^1(\mathbb{R}^3)$ and $D_r^{1,2}(\mathbb{R}^3)$ be the set of radially symmetric functions in $H^1(\mathbb{R}^3)$ and $D^{1,2}(\mathbb{R}^3)$ respectively. Denote the Sobolev norm of $H_r^1(\mathbb{R}^3)$ as

$$\|u\| := \left(\int_{\mathbb{R}^3} |\nabla u|^2 + u^2\, dx\right)^{1/2}.$$

The following lemma is well known in literature.

Lemma 2.1. (i) *For any $u \in H_r^1(\mathbb{R}^3)$, there is a unique solution $\phi_u(x) \in D_r^{1,2}(\mathbb{R}^3)$ of*

$$\Delta\phi + u^2 = 0, \qquad \lim_{x\to\infty} \phi(x) = 0,$$

such that it is represented as

$$\phi_u(x) = \int_{\mathbb{R}^3} \frac{1}{4\pi}\frac{u^2(y)}{|x-y|}\, dy.$$

(ii) *Let $\{u_n\}$ be a sequence such that $u_n \rightharpoonup u$ weakly in $H_r^1(\mathbb{R}^3)$. Then, $\phi_{u_n} \to \phi_u$ in $D_r^{1,2}(\mathbb{R}^3)$ and*

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n^2\, dx \to \int_{\mathbb{R}^3} \phi_u u^2\, dx,$$

as $n \to \infty$ up to a subsequences.

We refer to [15] for the proof.

Now, define a functional $I : H_r^1(\mathbb{R}^3) \to \mathbb{R}$

$$I(u) := \int_{\mathbb{R}^3} \frac{1}{2}|\nabla u|^2 + \frac{1}{2}u^2 + \frac{1}{4}\phi_u u^2 - F(u)\,dx, \quad u \in H_r^1(\mathbb{R}^3). \tag{2.1}$$

By elementary computations and Palais's principle of symmetric criticality [14], it is easy to see that (2.1) is a well-defined C^1 functional on $H_r^1(\mathbb{R}^3)$, whose critical points are radial solutions of the (1.3).

In the paper, we will find infinitely many critical points of I under the conditions (F1)–(F3). To do this, we introduce several terminologies adopted from the survey paper [10].

Definition 2.1. Let X be a Banach space and G be a compact Lie group acting on X.

(i) A functional I on X is called G-invariant if $I(gx) = I(x)$ for all $g \in G$;
(ii) A subset A of X is called G-invariant if $gx \in A$ for all $g \in G$ and $x \in A$;
(iii) A map $\eta \in C(X, X)$ is called G-equivariant if $\eta(gx) = g\eta(x)$ for all $g \in G$ and $x \in A$;
(iv) A map $\eta \in C(X, X)$ is called a deformation of identity if it is homotopic to the identity, i.e., there exists a $h \in C([0, 1] \times X, X)$ such that $h(0, \cdot) = \mathrm{Id}$ and $h(1, \cdot) = \eta$;
(v) A homotopy $h \in C([0, 1] \times X, X)$ is G-equivariant if $h(t, \cdot)$ is G-equivariant for all $t \in [0, 1]$;
(vi) Let B be a closed subset of X. Then, a class $\mathcal{F}$ of compact subsets of X is called a G-homotopy stable family with boundary B if
   (a) every set in $\mathcal{F}$ is G-invariant and contains B;
   (b) $\eta(A) \in \mathcal{F}$ for all $A \in \mathcal{F}$ and all G-equivariant deformation of identity η that leaves B invariant (i.e. $\eta(x) = x$ if $x \in B$).

The following result is essentially known (see [11,16]) although this specific version does not appear in literature.

Theorem 2.1. *Let $J \subset \mathbb{R}$ be a compact interval and I_λ be a family of G-invariant C^1 functionals on X parametrized by $\lambda \in J$. Suppose that there exists a G-homotopy stable family $\mathcal{F}$ with boundary B such that*

$$c(\lambda) := \inf_{A\in\mathcal{F}} \max_{x\in A} I_\lambda(x) > \sup_{x\in B} I_\lambda(x) \tag{2.2}$$

for all $\lambda \in J$. Suppose also that I_λ satisfy the following property (H)*:*

(H) *Let $\lambda_0 \in J$ and $\{(\lambda_n, u_n)\}$ be a sequence in $J \times X$ such that $\{\lambda_n\}$ is strictly increasing to λ_0. If*

$$-I(\lambda_0, u_n), \qquad I(\lambda_n, u_n), \qquad \frac{I(\lambda_n, u_n) - I(\lambda_0, u_n)}{\lambda_0 - \lambda_n}$$

are all bounded above, then $\{\|u_n\|\}$ is bounded and for every $\varepsilon > 0$, there exists $N > 0$ such that

$$I(\lambda_0, u_n) \le I(\lambda_n, u_n) + \varepsilon \quad \text{for all } n \ge N.$$

Then, for all $\lambda_0 \in J$ satisfying

$$\frac{c(\lambda_n) - c(\lambda_0)}{\lambda_0 - \lambda_n} \le M(\lambda_0) \quad \text{for some } M(\lambda_0) > 0 \quad \text{and} \quad \lambda_n \uparrow \lambda_0,$$

I_{λ_0} has a bounded (PS) sequence at level $c(\lambda_0)$. In particular, I_λ has a bounded (PS) sequence at level $c(\lambda)$ for almost every $\lambda \in J$ by Denjoy's theorem [11].

A very close version appears in [18] in which the author shows the almost identical result with Theorem 2.1 in a case where the functional I_λ is of the form

$$I_\lambda(u) := A(u) - \lambda B(u),$$

where $B(u)$ is sign definite. Instead of this, Theorem 2.1 imposes the property (H) in [11], which does not require the sign definiteness. One can prove this theorem by combining ideas of proofs of Theorem 2.1 in [11] and Proposition 2.3 in [1] with minor modifications. We omit the proof for the sake of simplicity.

To make use of Theorem 2.1, we consider the following functional with parameter $\lambda \in [1/2, 1]$,

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + u^2\,dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2\,dx - \lambda \int_{\mathbb{R}^3} F(u)\,dx. \tag{2.3}$$

By the condition (F1), I_λ is an even functional. In the next section, we construct a sequence of $\mathbb{Z}_2$-homotopy stable families for I_λ where $\mathbb{Z}_2$ acts on $H_r^1(\mathbb{R}^3)$ as $\{\mathrm{id}, -\mathrm{id}\}$ and apply Theorem 2.1 to find bounded (PS) sequences of I_λ for almost every $\lambda \in [1/2, 1]$.

3. Constructing $\mathbb{Z}_2$-homotopy stable families for I_λ

For each $u \in H_r^1(\mathbb{R}^3)$, we define a curve u_t in $H_r^1(\mathbb{R}^3)$ such that

$$u_t(x) := t^2 u(tx).$$

Then for fixed λ, we obtain a function $c_u(t) : [0, \infty) \to \mathbb{R}$

$$c_u(t) := I_\lambda(u_t) = \frac{1}{2}t^3 \int_{\mathbb{R}^3} |\nabla u|^2\, dx + \frac{1}{2}t \int_{\mathbb{R}^3} u^2\, dx + \frac{1}{4}t^3 \int_{\mathbb{R}^3} \phi_u u^2\, dx - \frac{\lambda}{t^3} \int_{\mathbb{R}^3} F(t^2 u)\, dx.$$

Lemma 3.1. *For each fixed nonzero $u \in H_r^1$, there exist $t_0 > 0$ such that*

(i) *$c_u(t)$ is strictly increasing on $(0, t_0)$ and $\lim_{t\to 0} c_u(t) = 0$;*
(ii) *$c_u(t)$ attains the global maximum at $t = t_0$;*
(iii) *$c_u(t)$ is strictly decreasing on (t_0, ∞) and $\lim_{t\to\infty} c_u(t) = -\infty$.*

Proof. By differentiating $c_u(t)$, we get

$$c_u'(t) = \frac{3}{2}t^2 \int_{\mathbb{R}^3} |\nabla u|^2\, dx + \frac{1}{2} \int_{\mathbb{R}^3} u^2\, dx + \frac{3}{4}t^2 \int_{\mathbb{R}^3} \phi_u u^2\, dx - \lambda \int_{\mathbb{R}^3} \frac{2}{t^2} f(t^2 u)u - \frac{3}{t^4} F(t^2 u)\, dx.$$

We define

$$\begin{aligned} P(t) &:= \frac{3}{2}t^2 \int_{\mathbb{R}^3} |\nabla u|^2\, dx + \frac{1}{2} \int_{\mathbb{R}^3} u^2\, dx + \frac{3}{4}t^2 \int_{\mathbb{R}^3} \phi_u u^2\, dx \\ &= \left(\frac{3}{2} \int_{\mathbb{R}^3} |\nabla u|^2\, dx + \frac{3}{4} \int_{\mathbb{R}^3} \phi_u u^2\, dx\right) t^2 + \frac{1}{2} \int_{\mathbb{R}^3} u^2\, dx \\ &:= at^2 + b. \end{aligned}$$

Note that $P(t)$ is a polynomial of degree two with $P(0) = b = \frac{1}{2}\int_{\mathbb{R}^3} u^2 > 0$. The remaining part of $c_u'(t)$ is

$$\lambda \int_{\mathbb{R}^3} \frac{2}{t^2} f(t^2 u)u - \frac{3}{t^4} F(t^2 u)\, dx = t^2 \int_{\{u\neq 0\}} \lambda \left(\frac{2f(t^2 u)}{(t^2 u)^2} - \frac{3F(t^2 u)}{(t^2 u)^3}\right) u^3\, dx := t^2 g(t).$$

Let us denote

$$c_u'(t) = P(t) - t^2 g(t) = t^2 \left(\frac{b}{t^2} + a - g(t)\right).$$

While the function $b/t^2 + a$ is strictly decreasing to a and has the value ∞ at $t = 0$, the condition (F3) tells us that $g(t)$ is monotone increasing to infinity so that $g(0) < \infty$. Therefore we can conclude that there exists a $t_0 > 0$ such that $c_u'(t) > 0$ for $0 < t < t_0$, $c_u'(t_0) = 0$ and $c_u'(t) < 0$ for $t > t_0$. Thus, $c_u(t)$ is strictly increasing on $(0, t_0)$, attains the global maximum at t_0 and is strictly decreasing on (t_0, ∞).

It remains to show that $\lim_{t\to 0} c_u(t) = 0$ and $\lim_{t\to\infty} c_u(t) = -\infty$. To prove the former, it is sufficient to see that

$$\lim_{t\to 0} \frac{1}{t^3} \int_{\mathbb{R}^3} F(t^2 u)\, dx = 0.$$

By the condition (F1) and L'Hospital theorem, we have

$$\frac{1}{t^3} \int_{\mathbb{R}^3} F(t^2 u)\, dx = \frac{1}{t^3} \int_{\{u\neq 0\}} \frac{F(t^2 u)}{(t^2 u)^2} t^4 u^2\, dx = t \int_{\{u\neq 0\}} \frac{F(t^2 u)}{(t^2 u)^2} u^2\, dx \to 0 \quad \text{as } t \to 0.$$

Also, observe that $c_u'(t) = P(t) - t^2 g(t) \to -\infty$ as $t \to \infty$, which implies the latter. This completes the proof. □

Now, we construct a sequence of $\mathbb{Z}_2$-homotopy stable families for I_λ satisfying (2.2). As a first step, we prove I_λ satisfies certain kind of linking structure (see [17]). Let $\{u_i\}_{i=1}^\infty$ be an orthonormal basis of $H_r^1(\mathbb{R}^3)$, Y_k be the subspace of $H_r^1(\mathbb{R}^3)$ spanned by $\{u_1, u_2, \ldots, u_k\}$ and Z_k be spanned by $\{u_k, u_{k+1}, \ldots\}$. For any nonzero $u \in H_r^1(\mathbb{R}^3)$, define $T(u) > 0$ as a unique positive real number t satisfying

$$\|t^{-2} u(t^{-1}x)\| = 1. \tag{3.1}$$

Such a $t > 0$ should exist and be unique because the Eq. (3.1) is equivalent to

$$t^3 - \left(\int_{\mathbb{R}^3} u^2\, dx\right) t - \int_{\mathbb{R}^3} |\nabla u|^2\, dx = 0$$

and it is easy to see that this polynomial has the unique positive root. We define $T(0) = 0$ to make $T(u)$ continuous.

Lemma 3.2. *There exist real sequences ρ_j, r_j and k_j such that $k_j \to \infty$ as $j \to \infty$, $\rho_j > r_j > 0$ and*

(i) $a_j(\lambda) := \inf_{\{u \in Z_{k_j} \mid T(u)=r_j\}} I_\lambda(u) \geq j$ *for all* $\lambda \in [1/2, 1]$ *and all* j;
(ii) $b_j(\lambda) := \max_{\{u \in Y_{k_j} \mid T(u)=\rho_j\}} I_\lambda(u) \leq 0$ *for all* $\lambda \in [1/2, 1]$ *and all* j.

Proof. (i) We see from the conditions (F1) and (F2) that there exists $C > 0$ such that

$$|F(u)| \leq \frac{1}{4}|u|^2 + C|u|^{p+1}.$$

Thus, we have

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^3} u^2\,dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2\,dx - \lambda\int_{\mathbb{R}^3} F(u)\,dx \tag{3.2}$$

$$\geq \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \left(\frac{1}{2} - \frac{\lambda}{4}\right)\int_{\mathbb{R}^3} u^2\,dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2\,dx - \lambda C\int_{\mathbb{R}^3} |u|^{p+1}\,dx \tag{3.3}$$

$$\geq \frac{1}{4}\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \frac{1}{4}\int_{\mathbb{R}^3} u^2\,dx - C\int_{\mathbb{R}^3} |u|^{p+1}\,dx. \tag{3.4}$$

For given any $t > 0$, we define

$$W_{k,t} := \{v \in H_r^1(\mathbb{R}^3) \mid \|v\| = 1,\ v_t = t^2 v(t\cdot) \in Z_k\}.$$

Note that for each $t > 0$, the map $L_t : H_r^1 \to H_r^1$ defined by $v \mapsto v_t$ is a linear isomorphism so that $L_t^{-1}(Z_k)$ is a vector space with codimension $k-1$ and $W_{k,t}$ is the unit sphere in $L_t^{-1}(Z_k)$. Now, we represent a function $u \in Z_k$ as

$$u(x) = T^2(u)v(T(u)x), \quad \text{where } v \in W_{k,T(u)}.$$

Then, we see for $u \in Z_k$ such that $T(u) > 1$

$$I_\lambda(u) \geq \frac{1}{4}T(u)^3\int_{\mathbb{R}^3} |\nabla v|^2\,dx + \frac{1}{4}T(u)\int_{\mathbb{R}^3} v^2\,dx - CT(u)^{2p-1}\int_{\mathbb{R}^3} |v|^{p+1}\,dx$$
$$\geq \frac{1}{4}T(u) - CT(u)^{2p-1}\beta_{k,T(u)},$$

where

$$\beta_{k,t} := \sup_{v \in W_{k,t}} \int_{\mathbb{R}^3} |v|^{p+1}\,dx.$$

We claim that for fixed $t > 0$, $\beta_{k,t} \to 0$ as $k \to \infty$. To prove the claim, suppose the contrary. By the definition of $\beta_{k,t}$, it holds that $\beta_{k,t} \geq \beta_{k+1,t}$ for all k. Thus, there exists a $\beta_t > 0$ such that $\beta_{k,t} \to \beta_t$ as $k \to \infty$. Choose $v_{k,t} \in W_{k,t}$ satisfying

$$\int_{\mathbb{R}^3} |v_{k,t}|^{p+1}\,dx > \frac{\beta_{k,t}}{2}.$$

Since the vector space $L_t^{-1}(Z_k)$ has codimension $k-1$ and $\|v_{k,t}\| = 1$, the sequence $v_{k,t}$ converges weakly in $H_r^1(\mathbb{R}^3)$ and strongly in $L^{p+1}(\mathbb{R}^3)$ to 0, up to a subsequences, as $k \to \infty$. However, this makes a contradiction because $\lim_{k\to\infty}\int_{\mathbb{R}^3} |v_{k,t}|^{p+1}\,dx \geq \beta_t/2 > 0$. Now, we take $r_j = 8j$ to see

$$I_\lambda(u) \geq 2j - C(8j)^{2p-1}\beta_{k,8j}$$

for every $u \in Z_k$ such that $T(u) = r_j$. Then, we are able to choose k_j satisfying $C(8j)^{2p-1}\beta_{k_j,8j} < j$ by the claim. This shows (i).

(ii) For given any $t > 0$, we define

$$V_{k,t} := \{v \in H_r^1(\mathbb{R}^3) \mid \|v\| = 1,\ v_t = t^2 v(t\cdot) \in Y_k\},$$

which is homeomorphic to

$$U_{k,t} := \{u \in Y_k \mid T(u) = t\}.$$

It is clear that $U_{k,t}$ is compact because it is closed and bounded in the finite dimensional space Y_k so that $V_{k,t}$ is compact. Also, Lemma 3.1 says that $I_\lambda(v_t) \to -\infty$ as $t \to \infty$. By combining these two facts and compactness of interval $[1/2, 1]$, we deduce that there exists a $\rho_j > 0$ such that $\rho_j > r_j > 0$ and $I_\lambda(v_{\rho_j}) \leq 0$ for all $v \in V_{k_j,\rho_j}$ and all $\lambda \in [1/2, 1]$. This proves (ii). □

Let us define

$$B_j := \{u \in Y_{k_j} \mid T(u) \le \rho_j\}, \qquad N_j := \{u \in Z_{k_j} \mid T(u) = r_j\}$$

for ρ_j, r_j and k_j of Lemma 3.2.

Lemma 3.3 (*Intersection Lemma*)**.** *If $\gamma \in \Gamma_j$, where Γ_j denotes the set of continuous maps $\gamma \in C(B_j, H_r^1(\mathbb{R}^3))$ satisfying $\gamma|_{\partial B_j} =$ id, the intersection $\gamma(B_j) \cap N_j$ is nonempty.*

Proof. Let us define

$$U := \{u \in \mathring{B}_j \mid T(\gamma(u)) < r_j\},$$

where $\mathring{B}_j$ denotes the interior of B_j in Y_{k_j}. We claim that U is a symmetric bounded neighborhood of 0 in Y_{k_j} satisfying $T(\gamma(\partial U)) = r_j$. It is clear that U is a symmetric open set containing 0 because γ is odd continuous and T is even continuous so that $\gamma(0) = 0$ and $T(\gamma(u)) = T(\gamma(-u))$. To see U is bounded, it is suffices to show B_j is bounded in Y_{k_j}. On the contrary suppose that B_j is not bounded. Then there exists a sequence $\{u_n\}$ such that $\|u_n\| \to \infty$ but $T(u_n) \le \rho_j$ for all n. However, this contradicts the relation,

$$\int_{\mathbb{R}^3} |\nabla u_n|^2\,dx = T(u_n)^3 - \left(\int_{\mathbb{R}^3} u_n^2\,dx\right) T(u_n).$$

Finally, to prove $T(\gamma(\partial U)) = r_j$, we will suppose two cases:

(i) there is a $u_0 \in \partial U$ such that $T(\gamma(u_0)) > r_j$;
(ii) there is a $u_0 \in \partial U$ such that $T(\gamma(u_0)) < r_j$;

and get a contradiction. Note that the case (i) immediately contradicts the continuity of T and γ. Suppose the case (ii). Due to the continuity of T and γ, there is a neighborhood V of u_0 in Y_{k_j} such that $T(\gamma(V)) < r_j$. Then for all $v \in V$ such that $v \notin U$, $v \notin B_j$, which shows that $u_0 \in \partial B_k$. However, this contradicts the fact that γ leaves ∂B_j invariant and the definition of B_j. This proves the claim.

Now, let P_j be the projection map from $H_r^1(\mathbb{R}^3)$ onto Y_{k_j-1} and consider a continuous odd map $P_j \circ \gamma : \partial U \to Y_{k_j-1}$. The famous Borsuk–Ulam theorem implies that there is a $u_0 \in \partial U$ such that $P_j \circ \gamma(u_0) = 0$, which means $u_0 \in \gamma(B_j) \cap N_j$. This completes the proof. □

At this point, we can give a sequence of $\mathbb{Z}_2$-homotopy stable families $\mathcal{F}_j$ for I_λ as the following:

$$\mathcal{F}_j := \{\gamma(B_j) \mid \gamma \in \Gamma_j\}.$$

Observe that $\mathcal{F}_j$ has the boundary ∂B_j. Then, by applying Theorem 2.1, we can obtain the following result.

Theorem 3.1. *Let*

$$C_j(\lambda) := \inf_{A \in \mathcal{F}_j} \max_{u \in A} I_\lambda(u).$$

Then for each j, $C_j(\lambda) \ge j$ for all $\lambda \in [1/2, 1]$ and there is a critical point $u_{j,\lambda}$ of I_λ such that $I_\lambda(u_{j,\lambda}) = C_j(\lambda)$ for almost every $\lambda \in [1/2, 1]$.

Proof. Lemmas 3.2 and 3.3 imply that $C_j(\lambda) \ge j$ and $\sup_{u \in \partial B_j} I_\lambda(u) \le 0$ for all j. Thus the condition (2.2) of Theorem 2.1,

$$C_j(\lambda) = \inf_{A \in \mathcal{F}_j} \max_{u \in A} I_\lambda(u) > \sup_{u \in \partial B_j} I_\lambda(u),$$

is satisfied. To prove the functional I_λ has the property (H) of Theorem 2.1, suppose that for some $\lambda_0 \in [1/2, 1]$ and $\{(\lambda_n, u_n)\} \subset [1/2, 1] \times X$ such that $\{\lambda_n\}$ is strictly increasing to λ_0, the sequences

$$-I_{\lambda_0}(u_n), \qquad I_{\lambda_n}(u_n), \qquad \frac{I_{\lambda_n}(u_n) - I_{\lambda_0}(u_n)}{\lambda_0 - \lambda_n}$$

are all bounded above. This means that there is a $C > 0$ such that

$$-\left(\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 + u_n^2\,dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_n} u_n^2\,dx - \lambda_0 \int_{\mathbb{R}^3} F(u_n)\,dx\right) < C,$$
$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 + u_n^2\,dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_{u_n} u_n^2\,dx - \lambda_n \int_{\mathbb{R}^3} F(u_n)\,dx < C,$$
$$\int_{\mathbb{R}^3} F(u_n)\,dx < C,$$

from which we deduce $\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u_n|^2+u_n^2\,dx+\frac{1}{4}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\,dx<2C$ and $-2C<\int_{\mathbb{R}^3}F(u_n)\,dx$. Thus we see $\|u_n\|$ is bounded and for fixed $\varepsilon>0$,

$$\begin{aligned}I_{\lambda_0}(u_n)&=I_{\lambda_n}(u_n)+(\lambda_n-\lambda_0)\int_{\mathbb{R}^3}F(u_n)\,dx\\&\le I_{\lambda_n}(u_n)+(\lambda_0-\lambda_n)2C\le I_{\lambda_n}(u_n)+\varepsilon\end{aligned}$$

if n is sufficiently large. Therefore the property (H) is satisfied by I_λ so we can apply Theorem 2.1 to conclude that for each j, there exists a bounded (PS) sequence $\{u_n\}$ of I_λ at the level $C_j(\lambda)$ for almost every $\lambda\in[1/2,1]$. Since $\{u_n\}$ is bounded, there exists a $u\in H_r^1(\mathbb{R}^3)$ such that a subsequence u_n, still denoted by u_n, converges to u weakly in $H_r^1(\mathbb{R}^3)$. To complete the proof, it is sufficient to show that u_n converges to u strongly in $H_r^1(\mathbb{R}^3)$ up to a subsequence. This can be done by the standard argument. Decompose $I'_\lambda(u):H_r^1(\mathbb{R}^3)\to H_r^1(\mathbb{R}^3)^*$ as

$$I'_\lambda(u)=Lu+Ku,$$

where L is a boundedly invertible linear operator defined as

$$L(u)[v]:=\int_{\mathbb{R}^3}\nabla u\cdot\nabla v+uv\,dx$$

and K is an operator defined as

$$K(u)[v]:=\int_{\mathbb{R}^3}\phi_u uv\,dx-\int_{\mathbb{R}^3}f(u)v\,dx,$$

which is compact by Lemma 2.1. Thus we have

$$u_n=L^{-1}(-Ku_n+o(1)),$$

which shows the compactness of u_n in $H_r^1(\mathbb{R}^3)$. This completes the proof. □

4. Proof of Theorem 1.1

We need an integral identity called the Pohozaev's identity to prove Theorem 1.1.

Proposition 4.1 (*Pohozaev's Identity*). *Suppose that $u\in H^1(\mathbb{R}^3)$ is a critical point of I_λ. Then u satisfies the integral identity*

$$\int_{\mathbb{R}^3}\frac{1}{2}|\nabla u|^2+\frac{3}{2}u^2+\frac{5}{4}\phi_u u^2-3\lambda F(u)\,dx=0.$$

This identity was firstly proved in [8] when $f(t)=|t|^{p-1}t$. We refer to [2] for general f.

Now, we are ready to prove Theorem 1.1. Fix $j\in\mathbb{N}$. Theorem 3.1 implies that there are sequences $\{u_n\}\in H_r^1(\mathbb{R}^3)$ and $\{\lambda_n\}\to 1$ such that each u_n is a critical point of I_{λ_n} satisfying $I_{\lambda_n}(u_n)=C_j(\lambda_n)$.

First of all, we prove $C_j(\lambda_n)$ is bounded for n. Take a $A\in\mathcal{F}$ such that

$$\max_{u\in A}I_1(u)\le C_j(1)+\frac{1}{2}.$$

Since A is compact, we have

$$C_j(\lambda_n)\le\max_{u\in A}I_{\lambda_n}(u)\le\max_{u\in A}I_1(u)+(1-\lambda_n)\max_{u\in A}\int_{\mathbb{R}^3}F(u)\,dx\le C_j(1)+\frac{1}{2}+\frac{1}{2}=C_j(1)+1$$

if n is large.

Secondly, we prove $\{u_n\}$ is bounded. Let $v_n(x):=T(u_n)^{-2}u_n(T(u_n)^{-1}x)$ so that $\|v_n\|=1$. Then, $\{v_n\}$ converges to a function v weakly in H_r^1 and strongly in L^q for $q\in(2,6)$ up to a subsequence. We claim that v is not identically zero. To the contrary, suppose that $v\equiv 0$. Then, for every $R>0$ it holds that

$$\int_{\mathbb{R}^3}F(R^2v_n(Rx))\,dx\to\int_{\mathbb{R}^3}F(R^2v(Rx))\,dx=0$$

as $n\to\infty$. By Proposition 4.1 we see that

$$\begin{aligned}\frac{d}{dt}I_{\lambda_n}(t^2u_n(t\cdot))\bigg|_{t=1}&=\int_{\mathbb{R}^3}\frac{3}{2}|\nabla u_n|^2+\frac{1}{2}u_n^2+\frac{3}{4}\phi_{u_n}u_n^2-\lambda_n(2f(u_n)u_n-3F(u_n))\,dx\\&=2I'_{\lambda_n}(u_n)u_n-\int_{\mathbb{R}^3}\frac{1}{2}|\nabla u_n|^2+\frac{3}{2}u^2+\frac{5}{4}\phi_{u_n}u_n^2-3\lambda F(u_n)\,dx=0.\end{aligned}\tag{4.1}$$

Then Lemma 3.1 implies that the function $c_{u_n}(t) = I_{\lambda_n}(t^2 u_n(t\cdot))$ attains its maximum at $t = 1$. Thus for any $R > 1$ we have

$$\begin{aligned} I_{\lambda_n}(u_n) &\geq I_{\lambda_n}\left(\left(\frac{R}{T(u_n)}\right)^2 u_n\left(\frac{R}{T(u_n)}\cdot\right)\right) = I_{\lambda_n}(R^2 v_n(R\cdot)) \\ &= \frac{R^3}{2}\int_{\mathbb{R}^3}|\nabla v_n|^2\,dx + \frac{R}{2}\int_{\mathbb{R}^3} v_n^2\,dx + \frac{R^3}{4}\int_{\mathbb{R}^3}\phi_{v_n}v_n^2\,dx - \lambda_n\int_{\mathbb{R}^3}F(R^2 v_n(Rx))\,dx \\ &> \frac{R}{2}\int_{\mathbb{R}^3}|\nabla v_n|^2\,dx + \frac{R}{2}\int_{\mathbb{R}^3} v_n^2\,dx - \lambda_n\int_{\mathbb{R}^3}F(R^2 v_n(Rx))\,dx \\ &= \frac{R}{2} - \lambda_n\int_{\mathbb{R}^3}F(R^2 v_n(Rx))\,dx. \end{aligned}$$

Since $I_{\lambda_n}(u_n) = C_j(\lambda_n)$ is bounded for n, by choosing R and n large, we get a contradiction so that v is not identically zero. Now, suppose that $\|u_n\| \to \infty$. From the Eq. (4.1), we see

$$\begin{aligned} &\frac{3}{2}\int_{\mathbb{R}^3}|\nabla v_n|^2\,dx + \frac{1}{2T(u_n)^2}\int_{\mathbb{R}^3} v_n^2\,dx + \frac{3}{4}\int_{\mathbb{R}^3}\phi_{v_n}v_n^2\,dx \\ &\quad = \lambda_n\int_{\mathbb{R}^3}\left(\frac{2f(T(u_n)^2 v_n)}{(T(u_n)^2 v_n)^2} - \frac{3F(T(u_n)^2 v_n)}{(T(u_n)^2 v_n)^3}\right)v_n^3\,dx. \end{aligned} \tag{4.2}$$

Since $\|u_n\| \to \infty$, we can easily check $T(u_n) \to \infty$ as $n \to \infty$. Thus the left hand side of (4.2) remains bounded as $n \to \infty$ but the right hand side goes to infinity from the condition (F3) and the fact $v \neq 0$. This is a contradiction so that $\{u_n\}$ is bounded. We notice that this type of reasoning appears in [13].

Thirdly, from the boundedness of $\{u_n\}$, it is not difficult to deduce that $\{u_n\}$ is a (PS) sequence of $I_1(u)$. Then, by using the same argument in the proof of Theorem 3.1, we obtain a critical point u^j of $I_1(u)$. Since $j \leq C_j(\lambda_n) = I_{\lambda_n}(u_n) \leq C_j(1) + 1$, it holds that $j \leq I_1(u^j) \leq C_j(1) + 1$. This proves the existence of infinitely many radial solutions to (1.3).

Finally, it remains to show that there is also a positive solution to (1.3). To do this, we replace the nonlinearity f with f_+ defined as

$$\begin{cases} f_+(t) := f(t) & \text{if } t \geq 0, \\ f_+(t) := 0 & \text{if } t \leq 0. \end{cases}$$

Let $u \in H_r^1(\mathbb{R}^3)$ be an arbitrary solution to (1.3) with the replaced nonlinearity f_+. Then, by multiplying the both side of (1.3) by u_-, the negative part of u, and integrating by parts, we easily see that u_- is zero. This means that every solution to (1.3) with f_+ is a nonnegative solution to (1.3) with f. Note that every nonnegative solution to (1.3) is positive everywhere by the strong maximum principle.

Now, we will find a nontrivial critical point of I_λ with the term $F_+(u) = \int_0^u f_+(s)\,ds$ replacing $F(u)$. As a group action G, we take the trivial group $\{\mathrm{id}\}$ and as a boundary B, we take $\{0, v_0\}$, where $v_0 \in H_r^1(\mathbb{R}^3)$ is a function satisfying $I_\lambda(v_0) \leq 0$ for all $\lambda \in [1/2, 1]$. Such a function v_0 exists by Lemma 3.1. Then, we can define a G-homotopy stable family $\mathcal{F}$ with boundary B as

$$\mathcal{F} := \{\gamma([0, 1]) \mid \gamma \in \Gamma\},$$

where

$$\Gamma := \{\gamma \in C([0, 1], H_r^1(\mathbb{R}^3)) \mid \gamma(0) = 0, \gamma(1) = v_0\}.$$

By (3.2) and Sobolev embedding, we have that

$$c(\lambda) := \inf_{A\in\mathcal{F}}\max_{x\in A} I_\lambda(x) > \sup_{x\in B} I_\lambda(x).$$

We have already shown that I_λ has the property (H) so we can apply Theorem 2.1. Then, the proof of remaining part is exactly the same with the argument of Sections 3 and 4. This completes the entire proof of Theorem 1.1.

Acknowledgments

The author was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) (No. 2012-0000941).

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