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Atomic decomposition of vector Hardy spaces

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ABSTRACT

We study Banach-valued Hardy spaces $\mathfrak{h}^p_{\mathbb{X}}(\mathbb{R}^{n+1}_+)$ of harmonic functions in the upper half space of $\mathbb{R}^{n+1}$ defined in terms of maximal functions and the corresponding space of distributional boundary limits $\mathbb{H}^p_{\mathbb{X}}(\mathbb{R}^n)$, where $\mathbb{X}$ is an arbitrary real or complex Banach space. For $p > 1$ the elements of $\mathfrak{h}^p_{\mathbb{X}}(\mathbb{R}^{n+1}_+)$ are the Poisson transform of Borel measures with p-bounded variation and values in $\mathbb{X}$. For $p \leq 1$ we prove the existence of atomic decomposition of elements in $\mathbb{H}^p_{\mathbb{X}}(\mathbb{R}^n)$ where the atoms are vector measures with certain size and cancellation properties that generalize the atoms in the real valued Hardy spaces.

1. Introduction and preliminaries

Much work has been done in the study of Hardy spaces of holomorphic and harmonic functions in the disk and with values in a Banach space. Questions like the existence of boundary values, equivalences of the various definitions of Hardy spaces valid in the scalar case and atomic decompositions among others are linked to the properties of the Banach space like the Radon–Nikodým properties or UMD (see for example [1–3,5,10]).

The purpose of this work is to study Hardy spaces that we will denote by $\mathfrak{h}^p_{\mathbb{X}}(\mathbb{R}^{n+1}_+)$, consisting of harmonic functions in the upper half space $\mathbb{R}^{n+1}_+ = \{(x,t) : x \in \mathbb{R}^n, t > 0\}$ and values in a Banach space $\mathbb{X}$ with non-tangential maximal function in $L^p(\mathbb{R}^n)$. We will not assume any additional property of $\mathbb{X}$. Using the Poisson transform we will see in Section 2 that for $p > 1$, $\mathfrak{h}^p_{\mathbb{X}}(\mathbb{R}^{n+1}_+)$ is isomorphic to the space of measures of bounded p-variation and values in $\mathbb{X}$. For $0 < p \leq 1$, we study in Section 3 the boundary limits as $\mathbb{X}$-valued temperate distributions of the elements of $\mathfrak{h}^p_{\mathbb{X}}(\mathbb{R}^{n+1}_+)$. For these distributions denoted by $\mathbb{H}^p_{\mathbb{X}}(\mathbb{R}^n)$, we prove an extension of the theorems by Latter [11] and Coifman [6] giving atomic decompositions for the elements of the real Hardy spaces $H^p(\mathbb{R}^n)$. The atoms in this setting are vector measures having ad-hoc size and cancellation properties. The proof of this follows the strategy of the scalar case. First we notice that the complex modulus can be replaced directly by the norm in $\mathbb{X}$ so that the classical result by Fefferman–Stein can be reproduced in this context to show that various maximal functions can be used to define $\mathbb{H}^p_{\mathbb{X}}(\mathbb{R}^n)$, including the so called grand maximal function. Then the decomposition follows from a version of the Calderón–Zygmund decomposition. The construction will follow the lines of the proof of the scalar result presented in [15].

Prior to this work Blasco and García-Cuerva showed an atomic decomposition for elements in boundary Banach valued Hardy spaces on the disk in [4], and Pérez-Esteva and Rivera-Noriega made a general atomic decomposition in [13] for elements in boundary Banach valued Hardy spaces on Lipschitz domains in the case $p = 1$. In [3] Blasco obtained for $p \geq 1$ a representation for the Hardy spaces $\mathfrak{h}^p_{\mathbb{X}}(\mathbb{R}^{n+1}_+)$ in terms of cone p-summing operators.

Throughout this paper, $\mathbb{X}$ will always denote a complex Banach space. We will say that a function $u : \Omega \subset \mathbb{R}^n \to \mathbb{X}$ is harmonic if $\Delta u = 0$. A function u is harmonic if and only if it is weakly harmonic, namely if $e^* \circ u$ is a complex harmonic function for any $e^* \in \mathbb{X}^*$ (see [14]).

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Now we summarize the basic facts about $\mathbb{X}$-valued measures that will be used in the paper. We refer the reader to the monographs [8,7] for details on this topic. Consider $\mathcal{B}(\Omega)$ as the Borel algebra on $\Omega \subset \mathbb{R}^n$ and λ as the Lebesgue measure on $\mathbb{R}^n$. We will write $\lambda(A) = |A|$.

Let μ be a Borel measure in a Borel set $\Omega \subset \mathbb{R}^n$ with values in $\mathbb{X}$, which we will always assume countable additive. We will say that μ has bounded variation if

$$|\mu|(\Omega) = \sup\left\{\left\{\sum_{E\in\pi} \|\mu(E)\|_{\mathbb{X}}\right\} : \pi \text{ finite Borel partition of } \Omega\right\} < +\infty.$$

In this case, $|\mu|(E)$, defined as before, defines a positive countable additive measure in $\mathcal{B}(\Omega)$ (see [7]). We will say that such μ is regular if $|\mu|$ is a positive regular additive measure. Notice that this holds if $\Omega = \mathbb{R}^n$ since $\mathbb{R}^n$ is $\sigma - compact$ and $|\mu|$ is finite.

$\mathfrak{M}_{\mathbb{X}}(\Omega)$ will denote the space of all Borel $\mathbb{X}$-valued measures in $\mathcal{B}(\Omega)$ such that are regular, countable additive and with bounded variation.

Let $\mathbb{B}$ be a Banach space in duality with $\mathbb{X}$ and $\mu \in \mathfrak{M}_{\mathbb{X}}(\Omega)$, then for a simple measurable function f in Ω with values in either $\mathbb{B}$ or $\mathbb{C}$, we denote the vector integration

$$\int_\Omega f(x)\,\mathrm{d}\mu(x) \tag{1}$$

defined in the obvious way. This integration can be extended to a class of measurable functions called integrable with respect to μ (see [8, II.12]). In particular every $\phi \in C_{0,\mathbb{B}}(\mathbb{R}^n)$, the space of continuous functions in $\mathbb{R}^n$ with values in $\mathbb{B}$ and vanishing at infinity is integrable with respect to μ, and the same holds if $\phi \in C_0(\mathbb{R}^n) = C_{0,\mathbb{C}}(\mathbb{R}^n)$. In terms of (1), we have Singer's representation theorem (see [8,12])

$$C_{0,\mathbb{X}}(\mathbb{R}^n)^* = \mathfrak{M}_{\mathbb{X}^*}(\mathbb{R}^n) \tag{2}$$

with $\|\mu\|_{\mathfrak{M}_{\mathbb{X}}(\mathbb{R}^n)} = |\mu|(\mathbb{R}^n)$.

Next, let Ω a Borel set in $\mathbb{R}^n$. Denote by $L^p_{\mathbb{X}}(\Omega)$ the standard spaces of Bochner measurable functions such that $\|f(\cdot)\|_{\mathbb{X}} \in L^p(\Omega)$ with norm

$$\|f\|_p = \left\{\int_\Omega \|f(x)\|^p_{\mathbb{X}}\mathrm{d}x\right\}^{1/p}.$$

For $1 < p \le \infty$ and Q a closed cube, we denote by $V^p_{\mathbb{X}}(Q)$, the space of *measures of bounded p-variation,* that consists of the measures μ such that

$$|\mu|_p(Q) = \sup\left\{\left\{\sum_{E\in\pi} \frac{\|\mu(E)\|^p_{\mathbb{X}}}{|E|^{p-1}}\right\}^{\frac{1}{p}} : \pi \text{ finite partition of } Q\right\} < +\infty,$$

when $1 < p < \infty$, and

$$|\mu|_\infty(Q) = \inf\{C > 0 : \|\mu(E)\|_{\mathbb{X}} \le C|E|, E \subseteq Q, E \in \mathcal{B}\} < +\infty.$$

The spaces $V^p_{\mathbb{X}}(Q)$ for $p \in (1, \infty]$ are Banach spaces with norm (see [8])

$$\|\mu\|_{V^p_{\mathbb{X}}(Q)} = |\mu|_p(Q).$$

We have continuous inclusions $V^p_{\mathbb{X}}(Q) \subset V^q_{\mathbb{X}}(Q)$ if $p < q$; also we have the following.

Remark 1.1. Every measure $\mu \in V^p_{\mathbb{X}}(Q)$ is countably additive, λ-continuous, and has bounded variation.

If $f \in L^p_{\mathbb{X}}(Q)$, then $f\,\mathrm{d}x$ defines an element in $V^p_{\mathbb{X}}(Q)$ such that $\|f\|_{L^p_{\mathbb{X}}(Q)} = \|f\,\mathrm{d}x\|_{V^p_{\mathbb{X}}(Q)}$ and conversely if $\mu \in V^p_{\mathbb{X}}(Q)$ has a density f, then $f \in L^p_{\mathbb{X}}(Q)$. Hence there are isometric inclusions $L^p_{\mathbb{X}}(Q) \subset V^p_{\mathbb{X}}(Q)$ for $p > 1$.

Let $p > 1$ and q be the conjugate exponent of p, then for $\varphi \in L^q(Q)$ or $\varphi \in L^q_{\mathbb{B}}(Q)$, with $\mathbb{B}$ a Banach space in duality with $\mathbb{X}$, one can define (starting with simple functions φ), $\int_Q \varphi(x)\,\mathrm{d}\mu(x)$. Moreover we have the representation (see [8])

$$L^q_{\mathbb{X}}(Q)^* = V^p_{\mathbb{X}^*}(Q). \tag{3}$$

Now we extend this notions to $\mathbb{R}^n$.

Definition 1.2. We define $V^p_{\mathbb{X}}(\mathbb{R}^n)$, $p > 1$, as the space of regular measures in the Borel algebra of bounded sets, such that $\|\mu\|_{V^p_{\mathbb{X}}(Q)} \le A$ for any cube Q. Then define

$$\|\mu\|_{V^p_{\mathbb{X}}(\mathbb{R}^n)} = \sup_{Q\subseteq\mathbb{R}^n} |\mu|_p(Q)$$

and $V^1_{\mathbb{X}}(\mathbb{R}^n)$ as the measures $\mu \in \mathfrak{M}_{\mathbb{X}}(\mathbb{R}^n)$ that are λ-continuous.

For $\mu \in V^p_{\mathbb{X}}(\mathbb{R}^n)$ we can define $\int_{\mathbb{R}^n} f\,d\mu$ for scalar functions $f \in L^q(\mathbb{R}^n)$ or for $f \in L^q_{\mathbb{B}}(\mathbb{R}^n)$, with $\mathbb{B}$ a Banach space in duality with $\mathbb{X}$, and the representation (3) also holds in this case.

Remark 1.1 implies that if $\mu \in V^p_X(Q)$ with $p > 1$, then $|\mu|$ defines a finite Borel measure in Q absolutely continuous with respect to the Lebesgue measure and such that $\frac{d|\mu|}{d\lambda} \in L^p(Q)$. Using the inequality $\left\| \int_Q \varphi \, d\mu \right\|_{\mathbb{X}} \le \int_Q |\varphi| \, d|\mu|$, for any cube Q, we have the following characterization for the measures with bounded p-variation in $\mathbb{R}^n$ (compare with [1]).

Lemma 1.3. *Let μ a vector measure, then μ has bounded p-variation in $\mathbb{R}^n$ with $p > 1$ if and only if there exists a nonnegative function $g \in L^p(\mathbb{R}^n)$ such that for all functions φ in $L^q(\mathbb{R}^n)$,*

$$\left\| \int_{\mathbb{R}^n} \varphi \, d\mu \right\|_{\mathbb{X}} \le \int_{\mathbb{R}^n} |\varphi| g \, dx.$$

In this case it follows that $g = d|\mu|/d\lambda$.

We denote by $\mathcal{S}$ the Schwartz space of rapidly decreasing C^∞ functions in $\mathbb{R}^n$ with the topology of a Fréchet space provided by the seminorms $\|\phi\|_{\alpha\beta} = \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta \phi(x)|$, where α, β are n-tuples of non-negative integers. We call an $\mathbb{X}$-*valued distribution* to every continuous linear operator of $\mathcal{S}$ into $\mathbb{X}$. We denote by $\mathcal{S}'_{\mathbb{X}}$ the space of all $\mathbb{X}$-valued distribution and we will write $\langle f, \phi \rangle = f(\phi)$ for or $\phi \in \mathcal{S}$. Every $\mu \in \mathfrak{M}_{\mathbb{X}}(\mathbb{R}^n)$ and $\mu \in V^p_{\mathbb{X}}(\mathbb{R}^n)$, in particular $f \in L^p_{\mathbb{X}}(\mathbb{R}^n)$, defines an element in $\mathcal{S}'_{\mathbb{X}}$ by $\langle \mu, \phi \rangle = \int \phi d\mu$. We will say that $f_k \to f$ in $\mathcal{S}'_{\mathbb{X}}$ if $\langle f_k, \phi \rangle \to \langle f, \phi \rangle$ in $\mathbb{X}$ for all ϕ in $\mathcal{S}$.

For a function ϕ, we will use the standard operators of translation and reflexion: $\tau_x\phi(y) = \phi(y - x)$ and $\check{}\,\phi(y) = \phi(-y)$ *and* the convolution $(f * \phi)(\varphi) = \langle f, \check{}\,\phi * \varphi \rangle$, for $\varphi \in \mathcal{S}$ and $f \in \mathcal{S}'_{\mathbb{X}}$. As in the scalar case $f * \phi \in \mathcal{C}^\infty_{\mathbb{X}}(\mathbb{R}^n)$ with all its derivatives having polynomial growth. The Fourier transform in $\mathcal{S}'_{\mathbb{X}}$ is an isomorphism defined as usual by $\langle \hat{f}, \varphi \rangle = \langle f, \hat{\varphi} \rangle$ for $f \in \mathcal{S}'_{\mathbb{X}}$ and $\varphi \in \mathcal{S}$. We have

$$(f * \phi)\hat{}\,(\xi) = \hat{f}(\xi)\hat{\phi}(\xi), \quad \phi \in \mathcal{S}, \; f \in \mathcal{S}'_{\mathbb{X}}. \tag{4}$$

2. Vector Hardy spaces

We start this section by defining vector-valued Hardy spaces of harmonic functions.

Definition 2.1. For any $p > 0$, we define the $\mathbb{X}$-valued Hardy space $\mathfrak{h}^p_{\mathbb{X}}(\mathbb{R}^{n+1}_+)$ as the space of all $\mathbb{X}$-valued harmonic functions u defined in $\mathbb{R}^{n+1}_+$ such that its non-tangential maximal function

$$\mathcal{M}_{nt}(u)(x) = \sup_{|x-y|<t} \|u(y, t)\|_{\mathbb{X}}$$

belongs to the space $L^p(\mathbb{R}^n)$.

We provide $\mathfrak{h}^p_{\mathbb{X}}(\mathbb{R}^{n+1}_+)$ with the norm

$$\|u\|_{\mathfrak{h}^p_{\mathbb{X}}} = \|\mathcal{M}_{nt}(u)\|_p$$

when $p \ge 1$, and for $p < 1$ the p-norm

$$\|u\|_{\mathfrak{h}^p_{\mathbb{X}}} = \|\mathcal{M}_{nt}(u)\|^p_p\,.$$

As in the scalar case we notice that $u \in \mathfrak{h}^p_{\mathbb{X}}$ implies that

$$\|u(x, t)\|_{\mathbb{X}} \le (\|\mathcal{M}_{nt} u\|_p) t^{-\frac{n}{p}}, \quad \forall x \in \mathbb{R}^n, t > 0, \tag{5}$$

and

$$\|u(\cdot, t)\|_1 \le \|\mathcal{M}_{nt} u\|_p \, t^{n-\frac{n}{p}}, \quad t > 0. \tag{6}$$

When $p > 1$, $\mathfrak{h}^p_{\mathbb{X}}(\mathbb{R}^{n+1}_+)$ is related to $L^p_{\mathbb{X}}(\mathbb{R}^n)$ throughout the Poisson transform; however, that space is isomorphic to the larger space $V^p_{\mathbb{X}}(\mathbb{R}^n)$.

Let $P_t(x) = c_n t/(t^2 + |x|^2)^{\frac{n+1}{2}}$ be the Poisson kernel in $\mathbb{R}^n$, with $c_n = \Gamma((n + 1)/2)/\pi^{\frac{n+1}{2}}$. It is not difficult to see that the integral

$$u(x, t) = \int_{\mathbb{R}^n} P_t(x - y) \, d\mu(y), \tag{7}$$

$t > 0, x \in \mathbb{R}^n$, is well defined for $\mu \in V^p_{\mathbb{X}}(\mathbb{R}^n)$, $p > 1$, and $\mu \in \mathfrak{M}_{\mathbb{X}}(\mathbb{R}^n)$. We will write $u(x, t) = P_t * \mu(x)$.

Notice that if $e^* \in \mathbb{X}^*$, then $(e^* \circ u)(x,t) = \int_{\mathbb{R}^n} P_t(x-y)\, \mathrm{d}(e^* \circ \mu)(y)$ is harmonic since it is the Poisson integral of a complex measure; thus u is harmonic. Since

$$\|u(x,t)\|_{\mathbb{X}} \le \int_{\mathbb{R}^n} P_t(x-y)\, \mathrm{d}\,|\mu|\,(y),$$

we have that $\|u(x,t)\|_{\mathbb{X}}$ is bounded by the Poisson integral of a function in $L^p(\mathbb{R}^n)$ or a finite positive Borel measure. Then we conclude by the scalar theory that if $p > 1$, $\mathcal{M}_{nt}u \in L^p(\mathbb{R}^n)$ and the following isometric inclusion holds:

$$V^p_{\mathbb{X}}(\mathbb{R}^n) \hookrightarrow \mathfrak{h}^p_{\mathbb{X}}(\mathbb{R}^{n+1}_+). \tag{8}$$

If $\mu \in V^p_{\mathbb{X}}(\mathbb{R}^n)$ has a density f and $p \ge 1$, we have that $u(\cdot,t) \to f$ in $L^p(\mathbb{R}^n)$ and almost everywhere, since the Lebesgue differentiation theorem holds for the $\mathbb{X}$-valued functions and the proof of the scalar statement can be adapted to this setting.

Now we represent the Hardy spaces $\mathfrak{h}^p_{\mathbb{X}}(\mathbb{R}^{n+1}_+)$ for $p \ge 1$ as the Poisson integral of vector measures (compare with the result [3] where the representation is in terms of cone p-summing operators).

Proposition 2.2. *Let $p \ge 1$, then every $u \in \mathfrak{h}^p_{\mathbb{X}}(\mathbb{R}^{n+1}_+)$ is the Poisson integral of a measure $\mu \in V^p_{\mathbb{X}}(\mathbb{R}^n)$. Moreover for $u(x,t) = \int_{\mathbb{R}^n} P_t(x-y)\, \mathrm{d}\mu(y)$, we have that $u(\cdot,t) \to \mu$ in $\mathcal{S}'_{\mathbb{X}}$ as t tends to zero.*

Hence for $p > 1$, the Poisson transform is an isomorphism of $V^p_{\mathbb{X}}(\mathbb{R}^n)$ onto $\mathfrak{h}^p_{\mathbb{X}}(\mathbb{R}^{n+1}_+)$ and for $p = 1$ the Poisson transform is an isomorphism of a subspace of $V^1_{\mathbb{X}}(\mathbb{R}^n)$ onto $\mathfrak{h}^1_{\mathbb{X}}(\mathbb{R}^{n+1}_+)$.

Proof. Suppose that $p > 1$ and let $u \in \mathfrak{h}^p_{\mathbb{X}}(\mathbb{R}^{n+1}_+)$. By (3), the following inclusions hold:

$$L^p_{\mathbb{X}}(\mathbb{R}^n) \hookrightarrow L^p_{\mathbb{X}^{**}}(\mathbb{R}^n) \hookrightarrow V^p_{\mathbb{X}^{**}}(\mathbb{R}^n) \cong \left[L^q_{\mathbb{X}^*}(\mathbb{R}^n)\right]^*.$$

Consider the $\mathbb{X}$-valued functions $u_t(x) = u(x,t)$ defined in $\mathbb{R}^n$. Since $\mathcal{M}_{nt}(u) \in L^p(\mathbb{R}^n)$, the set $\{u_t\}_{t>0}$ is bounded in $L^p_{\mathbb{X}}(\mathbb{R}^n)$, implying that $\{u_t \mathrm{d}x\}_{t>0}$ is bounded in $V^p_{\mathbb{X}^{**}}(\mathbb{R}^n)$. The Alouglou Theorem implies the existence of a measure μ in $V^p_{\mathbb{X}^{**}}$ and a sequence $t_k \searrow 0$ such that $u_{t_k} \to \mu$ in the ω^* topology in $\left[L^q_{\mathbb{X}^*}(\mathbb{R}^n)\right]^*$. Now, if $e^* \in \mathbb{X}^*$ then $P_t(x-\cdot)\cdot e^* \in L^q_{\mathbb{X}^*}(\mathbb{R}^n)$, with $1/p + 1/q = 1$. Then

$$\langle u_{t_k}, P_t(x-\cdot)\cdot e^*\rangle \longrightarrow \langle \mu, P_t(x-\cdot)\cdot e^*\rangle$$

implying that

$$\begin{aligned}\left\langle u(x,t+t_k), e^*\right\rangle &= \left\langle \int P_t(x-y)u(y,t_k)\,\mathrm{d}y, e^*\right\rangle \\ &\to \left\langle \int P_t(x-y)\,\mathrm{d}\mu(y), e^*\right\rangle.\end{aligned}$$

Hence $u(x,t) = P_t * \mu(x)$.

When $p = 1$ we have instead

$$L^1_{\mathbb{X}}(\mathbb{R}^n) \hookrightarrow L^1_{\mathbb{X}^{**}}(\mathbb{R}^n) \hookrightarrow \mathfrak{M}_{\mathbb{X}^{**}}(\mathbb{R}^n) \cong \mathcal{C}_{0,\mathbb{X}^*}(\mathbb{R}^n)^*.$$

As before, we get a measure $\mu \in \mathfrak{M}_{\mathbb{X}^{**}}$ such that $u_{t_k} \to \mu$ in the weak star topology and $u(x,t) = P_t * \mu(x)$. Let us see that this measure μ is λ-continuous. For every $\ell \in \mathbb{X}^{***}$, the complex measure $\ell \circ \mu$ has a density in $L^1(\mathbb{R}^n)$ since it is the boundary limit of the scalar function $\ell \circ u \in \mathfrak{h}^1(\mathbb{R}^{n+1}_+) = \mathfrak{h}^1_{\mathbb{C}}(\mathbb{R}^{n+1}_+)$ and it is known that $\mathbb{H}^1_{\mathbb{C}}(\mathbb{R}^n)$ is a subspace of $L^1(\mathbb{R}^n)$. Hence, if a Borel set A has Lebesgue measure zero, then $\ell \circ \mu(A) = 0$ for every $\ell \in \mathbb{X}^{***}$; thus $\mu(A) = 0$ and μ is λ-continuous.

Now it remains to prove that the measure μ obtained above takes all its values in $\mathbb{X}$. Suppose that there exists a Borel set A such that $\mu(A) \in \mathbb{X}^{**} \setminus \mathbb{X}$ and $|A| < +\infty$. The Hahn–Banach theorem assures the existence of a functional $\ell \in (\mathbb{X}^{**})^*$ such that $\ell(\mathbb{X}) = 0$ and $\ell(\mu(A)) = 1$. Then the scalar function $v = \ell \circ u$ is harmonic and $\mathcal{M}_{nt}(v) \in L^p(\mathbb{R}^n)$, so that $v = P_t * g$ for some $g \in Ⱡ^p(\mathbb{R}^n)$ (see III. 4.1 of [15]). Since $v = \ell \circ u = P_t * (\ell \circ \mu)$ we obtain $g = \ell \circ \mu$. Hence $v(\cdot,t)$ converges to g in $L^p(\mathbb{R}^n)$ and

$$\lim_{t\to 0}\int_A v_t\,\mathrm{d}x = g(A). \tag{9}$$

Now we calculate

$$\int_A v(x,t)\,\mathrm{d}x = \int_A \ell \circ u(x,t)\,\mathrm{d}x = \ell\left(\int_A u(x,t)\,\mathrm{d}x\right) = 0, \tag{10}$$

because $u(x,t) \in \mathbb{X}$ for every point (x,t) in $\mathbb{R}^{n+1}_+$, while

$$g(A) = \int_A \mathrm{d}(\ell \circ \mu)(x) = \ell\left(\int_A \mathrm{d}\mu(x)\right) = 1. \tag{11}$$

From Eqs. (9)–(11) we obtain

$$0 = \lim_{t\to 0} \int_A v(x,t)\ \mathrm{d}x = \int_A g\ \mathrm{d}x = 1.$$

This contradiction yields to our claim that μ is $\mathbb{X}$-valued. The representation is proved for all $p \geq 1$ and by (8) the Poisson kernel maps $V^p_{\mathbb{X}}(\mathbb{R}^n)$ onto $\mathfrak{h}^p_{\mathbb{X}}(\mathbb{R}^{n+1}_+)$.

The representation is proved for all $p \geq 1$ and by (8) the Poisson kernel maps $V^p_{\mathbb{X}}(\mathbb{R}^n)$ onto $\mathfrak{h}^p_{\mathbb{X}}(\mathbb{R}^{n+1}_+)$ for $p > 1$. □

Remark 2.3. The precise description of the measures in the previous proposition for $p = 1$ is part of Theorem 3.2 below. Notice that from the proof of Proposition 2.2 we see that, similar to the scalar case, if we define the vector Hardy space in $\mathbb{R}^{n+1}_+$ as the space of harmonic functions such that

$$\sup_{t>0} \int_{\mathbb{R}^n} \|u(x,t)\|_{\mathbb{X}}\ \mathrm{d}x < +\infty,$$

then the boundary limits of functions in this vector Hardy space is the space $\mathfrak{M}_{\mathbb{X}}(\mathbb{R}^n)$.

Now we shall prove that for any $u \in \mathfrak{h}^p_{\mathbb{X}}$ with $p < 1$, $u(\cdot,t)$ converges in $S'_{\mathbb{X}}$ as $t \to 0$, and this distribution is *bounded*, namely, the convolution $f * \phi$ is a bounded function for all $\phi \in \mathcal{S}$.

When f is a bounded $\mathbb{X}$-valued distribution we can define $f * g \in \mathcal{S}'_{\mathbb{X}}$ for all $g \in L^1(\mathbb{R}^n)$, as

$$\langle f * g, \varphi\rangle = \langle f * \check{}\varphi, \check{}g\rangle.$$

Notice that functions $f \in L^p_{\mathbb{X}}$ and measures $\mu \in V^p_{\mathbb{X}}$ are bounded distributions. In fact $\|f * \varphi(x)\|_{\mathbb{X}} \leq \|f\|_{L^p_{\mathbb{X}}} \|\varphi\|_{L^q_{\mathbb{X}}}$ and $\|\mu * \varphi(x)\|_{\mathbb{X}} \leq \|\mu\|_{V^p_{\mathbb{X}}} \|\varphi\|_{L^q_{\mathbb{X}}}$ for all $x \in \mathbb{R}^n$.

Remark 2.4. If the scalar distribution $e^* \circ f$ is bounded for every $e^* \in \mathbb{X}^*$, then f is a bounded $\mathbb{X}$-valued distribution.

In fact, for any $e^* \in \mathbb{X}^*$ the scalar function $(e^* \circ f) * \varphi$ is uniformly bounded on $\mathbb{R}^n$ because $(e^* \circ f) * \varphi$ is continuously differentiable, but $(e^* \circ f) * \varphi(x) = e^*(f * \varphi(x))$. This implies that the set $\{f * \varphi(x) : x \in \mathbb{R}^n\} \subset \mathbb{X}$ is weakly bounded, and hence bounded, so that there exists a constant C_φ that satisfies $\|f * \varphi(x)\|_{\mathbb{X}} \leq C_\varphi$ for all $x \in \mathbb{R}^n$. Therefore f is a bounded distribution.

Theorem 2.5. *Let $u \in \mathfrak{h}^p_{\mathbb{X}}(\mathbb{R}^{n+1}_+)$, $p < 1$. If we consider $f_t = u(\cdot,t)$, then there exists a bounded $\mathbb{X}$-valued distribution f such that $\lim_{t\to 0} f_t = f$ in $S'_{\mathbb{X}}$ and f uniquely determines u.*

Proof. We consider the family of functions u_τ in $\mathbb{R}^{n+1}_+$, $\tau > 0$, defined by $u_\tau(x,t) = u(x,t+\tau)$. Then $\mathcal{M}_{nt}u_\tau \leq \mathcal{M}_{nt}u$; hence $u_\tau \in \mathfrak{h}^p_{\mathbb{X}}(\mathbb{R}^{n+1}_+)$, and by (6) we get $u_\tau \in \mathfrak{h}^1_{\mathbb{X}}(\mathbb{R}^{n+1}_+)$ for every $\tau > 0$.

Write $f_t(x) = u(x,t)$, then $f_t \in L^p_{\mathbb{X}}(\mathbb{R}^n)$ with $\|f_t\|_{L^p_{\mathbb{X}}} \leq \|\mathcal{M}_{nt}u\|_p$ for all $t > 0$; thus convolution $f_\tau * P_t$ is well defined, and because differentiability of u_τ in all $(x,t) \in \overline{\mathbb{R}^{n+1}_+}$ it follows that $u_\tau(x,t) = f_\tau * P_t(x)$ for all $(x,t) \in \mathbb{R}^{n+1}_+$. Hence $u(x,t) = u_\tau(x,t-\tau) = f_\tau * P_{t-\tau}(x)$; therefore,

$$f_t(x) = f_\tau * P_{t-\tau}(x).$$

This equality shows that f_t is the convolution of the Poisson kernel and the $\mathbb{X}$-valued function f_τ. From (4) we get

$$\hat{f}_t(\xi) = \hat{f}_\tau(\xi)(P_{t-\tau})\hat{}(\xi). \tag{12}$$

But $(P_{t-\tau})\hat{}(\xi) = \mathrm{e}^{-2\pi|\xi|(t-\tau)}$ and from (12) we have

$$\hat{f}_t(\xi)\ \mathrm{e}^{2\pi|\xi|t} = \hat{f}_\tau(\xi)\ \mathrm{e}^{2\pi|\xi|\tau}. \tag{13}$$

Now define the $\mathbb{X}$-valued function $\psi : \mathbb{R}^n \to \mathbb{X}$ as

$$\psi(\xi) = \hat{f}_t(\xi)\ \mathrm{e}^{2\pi|\xi|t}. \tag{14}$$

Hence

$$\left\|\psi(\xi)\ \mathrm{e}^{-2\pi|\xi|t}\right\|_{\mathbb{X}} = \left\|\hat{f}_t(\xi)\right\|_{\mathbb{X}} \leq \int_{\mathbb{R}^n} \|u(x,t)\|_{\mathbb{X}}\ \mathrm{d}x \leq Ct^{-N}$$

where $N = n/p - n > 0$, which implies $\|\psi(\xi)\|_{\mathbb{X}} \leq Ct^{-N}\ \mathrm{e}^{2\pi|\xi|t}$ for all $t > 0$; thus

$$\|\psi(\xi)\|_{\mathbb{X}} \leq C \inf_{t>0} t^{-N}\ \mathrm{e}^{2\pi|\xi|t} = C|\xi|^N.$$

Hence ψ defines an $\mathbb{X}$-valued distribution. Let $f \in \mathcal{S}'_{\mathbb{X}}$ be such that $\hat{f} = \psi$. Now we will prove that $f = \lim_{t\to 0} f_t$ in $\mathcal{S}'_{\mathbb{X}}$. Let $\phi \in \mathcal{S}$, then

$$\begin{aligned}\lim_{t\to 0}\langle f_t, \phi\rangle &= \lim_{t\to 0}\int_{\mathbb{R}^n} f_t(x)\phi(x)\,\mathrm{d}x \\ &= \lim_{t\to 0}\int_{\mathbb{R}^n} \psi(\xi)\,\mathrm{e}^{-2\pi|\xi|t}\mathfrak{F}^{-1}(\phi)(\xi)\,\mathrm{d}\xi \\ &= \int_{\mathbb{R}^n}\psi(\xi)\mathfrak{F}^{-1}(\phi)(\xi)\,\mathrm{d}\xi \\ &= \langle f, \phi\rangle.\end{aligned}$$

Moreover,

$$\|f * \varphi(x)\|_{\mathbb{X}} = \left\|\langle f, \check{\tau}_x\varphi\rangle\right\|_{\mathbb{X}} = \lim_{t\to 0}\left\|\langle f_t, \check{\tau}_x\varphi\rangle\right\|_{\mathbb{X}} \le \|\mathcal{M}_{nt}u\|_p\,\|\varphi\|_q$$

and f is a bounded distribution. Finally, from (14) and $\hat{f} = \psi$ we see that $u(x,t) = f * P_t(x)$; hence f determines uniquely u. □

Like in the scalar case, we can characterize the boundary values of vector Hardy spaces $\mathfrak{h}^p_{\mathbb{X}}(\mathbb{R}^{n+1}_+)$ by maximal functions. For the function $\Phi \in \mathcal{S}$ we define the maximal $M_\Phi f(x) = \sup_{t>0}\|f * \Phi_t\|_{\mathbb{X}}$. Let $\mathcal{F}$ be a finite family of seminorms $\|\cdot\|_{\alpha,\beta}$ in $\mathcal{S}$ and let $\mathcal{S}_{\mathcal{F}} = \{\varphi \in \mathcal{S} : \|\varphi\|_{\alpha,\beta} \le 1, \forall\,\|\cdot\|_{\alpha,\beta} \in \mathcal{F}\}$, we define the *grand maximal* $\mathcal{M}_{\mathcal{F}} f(x) = \sup_{\Phi\in\mathcal{S}_{\mathcal{F}}} M_\Phi f(x)$, defined in $\mathbb{R}^n$.

Theorem 2.6. *Let $f \in \mathcal{S}'_{\mathbb{X}}$ and $0 < p \le \infty$. Then the following statements are equivalent:*

(i) *there exists $\Phi \in \mathcal{S}$ such that $\int \Phi \ne 0$, then $M_\Phi f \in L^p(\mathbb{R}^n)$;*
(ii) *there exists $\mathcal{F}$, a finite family of seminorms of $\mathcal{S}$, such that $\mathcal{M}_{\mathcal{F}} f \in L^p(\mathbb{R}^n)$;*
(iii) *f is a bounded $\mathbb{X}$-valued distribution and $\mathcal{M}_{nt}u \in L^p(\mathbb{R}^n)$, where $u(x,t) = f * P_t(x)$.*

Replacing the complex modulus by the norm in $\mathbb{X}$, we see that the proof of Theorem 2.6 is the same as the scalar one, see for example [9,15], and it consists on proving that the gauges $\|M_\Phi f\|_p$, $\|\mathcal{M}_{\mathcal{F}} f\|$ and $\|\mathcal{M}_{nt}(f * P_t)\|_p$, are all comparable with constants that depend on p and n, only.

Definition 2.7. For $p > 0$ we define $\mathbb{H}^p_{\mathbb{X}}(\mathbb{R}^n)$ as the space of all $f \in \mathcal{S}'_{\mathbb{X}}$ satisfying any of the equivalent conditions (i)–(iii) in Theorem 2.6.

Proposition 2.2, Theorem 2.5 and (iii) of Theorem 2.6 imply the next result.

Theorem 2.8. *Let $p > 0$ and u an $\mathbb{X}$-valued harmonic function in $\mathbb{R}^{n+1}_+$. Then $u \in \mathfrak{h}^p_{\mathbb{X}}(\mathbb{R}^{n+1}_+)$ if and only if $u = f * P_t$ for some f in $\mathbb{H}^p_{\mathbb{X}}(\mathbb{R}^n)$.*

3. Atomic decomposition

We start by giving an estimate of a distribution applied to a bump function in terms of a maximal function.

Lemma 3.1. *Let $N_0 > 0$ fixed and consider $\mathcal{S}_{\mathcal{F}_{N_0}} = \{\phi \in \mathcal{S} : \|\phi\|_{\alpha,\beta} \le 1, \forall|\alpha|, |\beta| < N_0\}$. Let f be an $\mathbb{X}$-valued distribution and ζ a function in $\mathcal{S}$ with support in a ball $B = B(x_0, r)$, such that $|\partial^\alpha\zeta| \le r^{-n-|\alpha|}$ for all $|\alpha| \le N_0$, then*

$$\|\langle f, \zeta\rangle\|_{\mathbb{X}} \le C\mathcal{M}_{\mathcal{F}_{N_0}} f(x), \quad \forall x \in B.$$

Proof. For each $x \in B \subseteq \mathbb{R}^n$ we put $\zeta_x(y) = r^n\zeta(x - ry)$. Then

$$\partial^\alpha\zeta_x(y) = (-1)^{|\alpha|}r^{n+|\alpha|}\partial^\alpha\zeta(x - ry). \tag{15}$$

The last quantity is not zero only if $x - ry \in B$, which implies that $|y| \le 2$.

Now $|y| \le 2$ and (15) imply that

$$\begin{aligned}|y^\beta\partial^\alpha\zeta_x(y)| &\le 2^{N_0}r^{n+|\alpha|}\,|\partial^\alpha\zeta(x - ry)| \le 2^{N_0}r^{n+|\alpha|}r^{-n-|\alpha|} \\ &= 2^{N_0},\end{aligned}$$

for all multi indices such that $|\alpha|, |\beta| \le N_0$. It follows that $\|\zeta_x\|_{\alpha,\beta} \le 2^{N_0}$ and $(1/2^{N_0})\zeta_x \in \mathcal{F}_{N_0}$. By definition of ζ_x we have $(\zeta_x)_r(x - y) = \zeta(y)$. Then

$$\langle f, \zeta\rangle = \langle f, (\zeta_x)_r(x - \cdot)\rangle = \langle f, \check{\tau_x}((\zeta_x)_r)\rangle = f * (\zeta_x)_r(x).$$

Therefore

$$\|\langle f, \zeta\rangle\|_{\mathbb{X}} \le 2^{N_0}\mathcal{M}_{\mathcal{F}_{N_0}} f(x). \quad \square$$

Next we will make a Calderón–Zygmund type decomposition for vector measures. Denote the grand maximal by $\mathcal{M}f$ instead of $\mathcal{M}_{\mathcal{F}}f$, as the finite set $\mathcal{F}$ will be fixed. Φ will denote a function in $\mathcal{C}^\infty(\mathbb{R}^n)$ with support in the unit ball in $\mathbb{R}^n$ and satisfying $\int_{\mathbb{R}^n}\Phi(x)\,\mathrm{d}x \neq 0$.

We will say that the family of sets $\{A_k\}_{k\in\mathbb{N}}$ has the bounded intersection property when there exists an integer number $M>0$ such that any point in $\mathbb{R}^n$ belongs at the most to M sets of the family.

Theorem 3.2 (*Calderón–Zygmund Decomposition*). *Let $\mu \in \mathfrak{M}_{\mathbb{X}}(\mathbb{R}^n)$ be absolutely continuous with respect to the Lebesgue measure for every cube $Q \subseteq \mathbb{R}^n$. Suppose that $\mathcal{M}\mu \in L^p(\mathbb{R}^n)$ with $p>0$. Then for any fixed constant $A>0$ there exists a decomposition $\mu = \omega + \nu$, $\nu = \sum_k \nu_k$, and a collection of cubes $\{Q_k^*\}_{k\in\mathbb{N}}$, such that*

1. *$\{Q_k^*\}$ has the bounded intersection property, and*
$$\bigcup_k Q_k^* = \{x : \mathcal{M}\mu(x) > A\}.$$
2. *Each measure ν_k is supported in Q_k^*, $\nu_k(\mathbb{R}^n)=0$ and satisfies*
$$\int_{\mathbb{R}^n}(M_\Phi \nu_k)^p \leq C\int_{Q_k^*}(\mathcal{M}\mu)^p. \tag{16}$$
3. $$\|\omega\|_{V_{\mathbb{X}}^\infty(\mathbb{R}^n)} \leq CA. \tag{17}$$

Remark 3.3. If $0<p\leq 1$, then the measures ν and ω of Theorem 3.2 belong to $\mathbb{H}_{\mathbb{X}}^p$. In fact, (16) and Theorem 2.6 imply that $\nu_k \in \mathbb{H}_{\mathbb{X}}^p$ for all $k\in\mathbb{N}$, and

$$\|\nu\|_{\mathbb{H}_{\mathbb{X}}^p}^p \leq \sum_{k\in\mathbb{N}}\|\nu_k\|_{\mathbb{H}_{\mathbb{X}}^p}^p \leq c\sum_{k\in\mathbb{N}}\int_{Q_k^*}(\mathcal{M}\mu)^p \leq cM\int_{\Omega}(\mathcal{M}\mu)^p.$$

Thus, $\nu\in\mathbb{H}_{\mathbb{X}}^p$, and so does the measure $\omega = \mu - \nu$. The measure ω satisfies the inequality

$$\|\mu-\omega\|_{\mathbb{H}_{\mathbb{X}}^p}^p = \|\nu\|_{\mathbb{H}_{\mathbb{X}}^p}^p \leq cM\int_{\Omega}(\mathcal{M}\mu)^p. \tag{18}$$

Proof of Theorem 3.2. We define the open set $\Omega = \{x\in\mathbb{R}^n : \mathcal{M}\mu(x) > A\}$. Consider a *Whitney decomposition* for Ω, namely a family of closed cubes $\{Q_k\}$ such that (i) $\Omega = \bigcup_{i\in\mathbb{N}} Q_k$, (ii) $Q_k^\circ \bigcap Q_j^\circ = \emptyset$, for all $i\neq j$, where Q_k° is the interior of the cube Q_k and (iii) $d_k \leq d(Q_k, \mathbb{R}^n\setminus\Omega) \leq 4d_k$, d_k is the diagonal of the cube Q_k and $d(Q_k,\mathbb{R}^n\setminus\Omega)$ is the distance from cube Q_k to set F.

If we let $1<\tilde{a}<a^*$ and $\widetilde{Q}_k$ and Q_k^* are the result of dilating Q_k by $\widetilde{a}$ and a^*, we can choose a^* close enough to 1 so that $\bigcup_{i\in\mathbb{N}} Q_k^* = \Omega$. The families $\{Q_k^*\}$ and $\{\widetilde{Q}_k\}$ have the bounded intersection property. Let M be the biggest number of sets of these families containing a single point in $\mathbb{R}^n$. We have that $Q_k^*\bigcap Q_i^* \neq \emptyset$ implies $d_k \simeq d_i$.

Now we consider the standard partition of the unity for the set Ω subordinated to the covering $\{\widetilde{Q}_k\}$: Let φ be smooth function taking the value 1 in the cube of side length and centered at the origin and vanishing outside the cube with the same center and side $\tilde{a}$. We set $\varphi_k(x) = \varphi\left((x-x_k)/(d_k/\sqrt{n})\right)$, where x_k is the center of Q_k.

Now we define $\phi_k = \varphi_k/\sum_j \varphi_j$. The function ϕ_k satisfies

$$\int_{\mathbb{R}^n}\phi_k(x)\,\mathrm{d}x \simeq |Q_k^*| \simeq d_k^n. \tag{19}$$

We put $\widetilde{\phi}_k = \phi_k/\int\phi_k$, getting $\int\widetilde{\phi}_k = 1$ and

$$|\partial^\beta \phi_k(x)| \leq C_\beta d_k^{-|\beta|}. \tag{20}$$

Let $\mathcal{P}$ be the space of polynomials in $\mathbb{R}^n$ and $\mathcal{P}_d$ be the finite-dimensional vector space of polynomials of degree $\leq d$ in $\mathbb{R}^n$. We choose an integer d such that $p > n/(n+d+1)$. Let $\mathcal{H}$ be the Hilbert space $L^2(Q_k^*, \widetilde{\phi}_k\,\mathrm{d}x)$ and denote by $\mathcal{H}_d$ to $\mathcal{P}_d$ as a subspace of $\mathcal{H}$. The inner product $(f,g)_{L^2(Q_k^*)}$ is the integral $\int_{Q_k^*} f(x)g(x)\,\mathrm{d}x$, while $(f,g)_{\mathcal{H}}$ is $\int_{Q_k^*} f(x)g(x)\widetilde{\phi}_k(x)\,\mathrm{d}x$.

Let $\{q_1,q_2,\ldots,q_N\}$ be an orthonormal basis in $\mathcal{H}_d$, then $(q_i, q_j\widetilde{\phi}_k)_{L^2(Q_k^*)} = \delta_{ij}$. For $q\in\mathcal{P}$ and $x\in\mathbb{X}$ the product $q\cdot x \in \mathcal{P}\otimes\mathbb{X}$ defines the application in $\mathcal{S}_{\mathbb{X}}'$ given by

$$\phi \mapsto \langle q\cdot x, \phi\rangle = (q,\phi)_{L^2}\cdot x.$$

Define $P_k : \mathcal{S}'_{\mathbb{X}} \longrightarrow \mathcal{H}_d \otimes \mathbb{X}$ by

$$P_k(f) = \sum_{i=1}^{N} q_i \langle f, q_i \widetilde{\phi}_k \rangle. \tag{21}$$

P_k is a projection on $\mathcal{H}_d \otimes \mathbb{X}$ and it satisfies $\langle (\mu - P_k\mu), q\phi_k \rangle = 0$ for all $q \in \mathcal{H}_d$.

Now we define $h_k = P_k\mu$ and the block measure

$$\mathrm{d}\nu_k = \phi_k \,\mathrm{d}\mu - h_k\phi_k \,\mathrm{d}x.$$

The measures ν_k satisfy the moment condition $\langle \nu_k, q \rangle = \int_{Q_k} q(x)\,\mathrm{d}\nu_k(x) = 0$ for every polynomial q in $\mathcal{H}_d$.

To prove (16) we need the bounds

$$M_\Phi \nu_k(x) \le c\mathcal{M}\mu(x), \quad x \in Q_k^*, \tag{22}$$

$$M_\Phi \nu_k(x) \le cA \frac{d_k^{n+d+1}}{|x - x_k|^{n+d+1}}, \quad x \notin Q_k^*. \tag{23}$$

In fact, inequalities (22) and (23) imply that

$$\begin{aligned} \int (M_\Phi \nu_k)^p &= \left(\int_{Q_k^*} + \int_{(Q_k^*)^C} \right) (M_\Phi \nu_k)^p \\ &\le \int_{Q_k^*} (\mathcal{M}\mu)^p + \int_{(Q_k^*)^C} \left\{ cA \frac{d_k^{n+d+1}}{|x - x_k|^{n+d+1}} \right\}^p \,\mathrm{d}x. \end{aligned}$$

Calculating the last integral

$$\begin{aligned} (cA)^p d_k^{p(n+d+1)} \int_{(Q_k^*)^C} \frac{1}{|x - x_k|^{p(n+d+1)}} \,\mathrm{d}x &= c_n A^p d_k^{p(n+d+1)} \int_{d_k}^{\infty} \frac{1}{r^{p(n+d+1)}} r^{n-1} \,\mathrm{d}r \\ &= c_{n,p} A^p d_k^{p(n+d+1)} d_k^{n-p(n+d+1)} \\ &= c_{n,p,a^*} A^p |Q_k^*| \\ &\le c_{n,p,a^*} \int_{Q_k^*} (\mathcal{M}\mu)^p, \end{aligned}$$

since $p > n/(n + d + 1)$ and $\mathcal{M}\mu(x) > A$ when $x \in Q_k^* \subseteq \Omega$. Hence (16) follows.

Now we show (22) and (23). First we claim that the polynomial h_k satisfies

$$\|h_k(x)\phi_k(x)\|_{\mathbb{X}} \le c_d A \tag{24}$$

$$\|h_k(x)\phi_k(x)\|_{\mathbb{X}} \le c_d \mathcal{M}\mu(x) \quad \forall x \in Q_k^*. \tag{25}$$

From the fact that

$$\sup_{x \in Q_k^*} |\partial^\beta q(x)| \le A_\beta d_k^{-|\beta|} \|q\|_{\mathcal{H}}, \quad q \in \mathcal{P}_d, \tag{26}$$

(see [15, p. 104]), we have in particular that $|\partial^\beta q_i| \le A_{\beta,N} d_k^{-|\beta|}$ for $|\beta| \ge 0$ and $i = 1, 2, \ldots, N$. Inequality (20) and the Leibniz formula give us $|\partial^\beta q_i \widetilde{\phi}_k(x)| \le A_\beta d_k^{-n-|\beta|}$. We apply Lemma 3.1 for the function $q_i\widetilde{\phi}_k$ and $r = 5d_k$ to obtain that $\left\| \langle \mu, q_i\widetilde{\phi}_k \rangle \right\|_{\mathbb{X}} \le cA$ and $\left\| \langle \mu, q_i\widetilde{\phi}_k \rangle \right\|_{\mathbb{X}} \le c\mathcal{M}\mu(z)$ for $z \in Q_k^*$.

Then we have

$$\begin{aligned} \|h_k(x)\phi_k(x)\|_{\mathbb{X}} &= \left\| \left(\sum_{i=1}^{N} q_i(x) \langle \mu, q_i\widetilde{\phi}_k \rangle \right) \phi_k(x) \right\|_{\mathbb{X}} \\ &\le \sum_{i=1}^{N} |q_i(x)||\phi_k(x)| \left\| \langle \mu, q_i\widetilde{\phi}_k \rangle \right\|_{\mathbb{X}}. \end{aligned}$$

When $x \notin Q_k^*$ every term vanishes in the last sum. If $x \in Q_k^*$, the function ϕ_k is bounded by 1 and by (26) the polynomials q_i are uniformly bounded by a constant C, then

$$\|h_k(x)\phi_k(x)\|_{\mathbb{X}} \le c_d A,$$

proving the inequality (24).

In similar way we prove (25):

$$\|h_k(x)\phi_k(x)\|_{\mathbb{X}} \le \sum_{i=1}^{N} C\mathcal{M}\mu(x) = c_d\mathcal{M}\mu(x), \quad x \in Q_k^*.$$

Now we are ready for proving inequalities (22) and (23): We have

$$M_\Phi \nu_k(x) \le M_\Phi(\phi_k\, \mathrm{d}\mu)(x) + M_\Phi(h_k\phi_k)(x). \tag{27}$$

Let us analyze the first term. When $x \in Q_k^*$ the convolution $\phi_k\, \mathrm{d}\mu * \Phi_t(x)$ is given by $\langle \mu, \phi_k\Phi_t(x-\cdot)\rangle$. To apply Lemma 3.1, we set

$$\zeta(y) = \phi_k(y)\Phi_t(x-y). \tag{28}$$

Then by the Leibniz rule,

$$|\partial^\beta\zeta(y)| \le \begin{cases} c_{\beta,\phi_k,\Phi}t^{-n-|\beta|}, & \text{when } t \le d_k \\ c_{\beta,\phi_k,\Phi}d_k^{-n-|\beta|}, & \text{when } t > d_k. \end{cases}$$

Also

$$\mathrm{supp}\zeta = \mathrm{supp}\phi_k\Phi_t(x-\cdot) \subset Q_k^* \bigcap B(x,r),$$

where $B(x,r)$ is the ball centered at x and radius r with $r = t$ if $t \le d_k$ while $t = 2d_k$ if $t > d_k$. Therefore for any $t > 0$ we have

$$\|\phi_k\, \mathrm{d}\mu * \Phi_t(x)\|_{\mathbb{X}} = \|\langle \mu, \zeta\rangle\|_{\mathbb{X}} \le c\mathcal{M}\mu(x).$$

Hence

$$M_\Phi(\phi_k\, \mathrm{d}\mu)(x) \le c\mathcal{M}\mu(x), \quad x \in Q_k^*. \tag{29}$$

To estimate $M_\Phi(h_k\phi_k)(x)$, we have by (25)

$$\|h_k\phi_k * \Phi_t(x)\|_{\mathbb{X}} \le \int \|h_k(y)\phi_k(y)\|_{\mathbb{X}}\, \Phi_t(x-y)\, \mathrm{d}y \le c_d\mathcal{M}\mu(x), \quad x \in Q_k^*.$$

We take the supremum for $t > 0$ and we get

$$M_\Phi(h_k\phi_k)(x) \le c\mathcal{M}\mu(x), \quad x \in Q_k^*. \tag{30}$$

From (27), (29) and (30) inequality (22) follows. By the moment condition on the measure ν_k we have

$$\nu_k * \Phi_t(x) = \langle \nu_k, \Phi_t(x-\cdot)\rangle = \langle \nu_k, \Phi_t(x-\cdot) - q\rangle,$$

where q is the Taylor's polynomial of degree d for the function $y \mapsto \Phi_t(x-y)$ at $y = x_k$. Then

$$\nu_k * \Phi_t(x) = \langle \phi_k\, \mathrm{d}\mu, \Phi_t(x-\cdot) - q\rangle - \langle h_k\phi_k, \Phi_t(x-\cdot) - q\rangle,$$

thus

$$M_\Phi\nu_k(x) \le \sup_{t>0}\langle \phi_k\, \mathrm{d}\mu, \Phi_t(x-\cdot) - q\rangle + \sup_{t>0}\langle h_k\phi_k, \Phi_t(x-\cdot) - q\rangle = I_1 + I_2.$$

Then we have for any $\beta \in \mathbb{N}^n$ that

$$|\partial^\beta[\Phi_t(x-y) - q(y)]| \le A_\beta \frac{d_k^{d+1}}{|x-x_k|^{n+d+1}} d_k^{-|\beta|}, \tag{31}$$

thus, if we define the function

$$\zeta(y) = \frac{|x-x_k|^{n+d+1}}{d_k^{n+d+1}}\phi_k(y)[\Phi_t(x-y) - q(y)],$$

we get $|\partial^\beta\zeta| \le A_\beta d_k^{-n-|\beta|}$. We apply Lemma 3.1 to μ and ζ obtaining $\|\langle \mu, \zeta\rangle\|_{\mathbb{X}} \le cA$. Taking the supremum on $t > 0$ we get

$$I_1 \le cA\frac{d_k^{n+d+1}}{|x-x_k|^{n+d+1}}. \tag{32}$$

To estimate I_2 we use (31) with $\beta = 0$, then

$$\begin{aligned}\|\langle h_k\phi_k, \Phi_t(x-\cdot) - q\rangle\|_{\mathbb{X}} &\le \|h_k\|_{\mathbb{X}} \frac{d_k^{d+1}}{|x-x_k|^{n+d+1}} \int_{Q_k^*} \mathrm{d}y \\ &= cA\frac{d_k^{n+d+1}}{|x-x_k|^{n+d+1}}.\end{aligned}$$

Taking the supremum on $t > 0$ we get

$$I_2 \le cA\frac{d_k^{n+d+1}}{|x-x_k|^{n+d+1}}, \tag{33}$$

and the proof of (23) is complete.

Now we can define the *bad measure*

$$\nu = \sum_{k\in\mathbb{N}} \nu_k.$$

We define the *good measure* by

$$\mathrm{d}\omega(x) = \begin{cases}\mathrm{d}\mu(x), & x \in \mathbb{R}^n \setminus \Omega \\ \sum_k h_k(x)\phi_k(x)\,\mathrm{d}x, & x\in\Omega.\end{cases}$$

It is clear that $\omega + \nu = \mu$. Finally we prove that $\mu \in V_{\mathbb{X}}^{\infty}(\mathbb{R}^n)$. For $E \subseteq \Omega$, we have

$$\|\omega(E)\|_{\mathbb{X}} \le \int_E \left(\sum_k \|h_k\phi_k\|_{\mathbb{X}}\right) \mathrm{d}x \le c_d AM|E|.$$

If $E \subseteq \mathbb{R}^n \setminus \Omega$, we have $\omega(E) = \mu(E)$, so we will see that $\|\mu(E)\|_{\mathbb{X}}$ is bounded by $cA|E|$. In fact, consider a functional $e^* \in \mathbb{X}^*$, and let μ_{e^*} be the composition $e^* \circ \mu$. By the assumption on μ, there exists $f \in L^1_{loc}(\mathbb{R}^n)$ such that $\mathrm{d}\mu_{e^*}(x) = f(x)\,\mathrm{d}x$ on every cube Q. On the other hand, we have $|\langle e^*, \mu(E)\rangle| = \left|\int_E \lim_{t\to 0} f_{e^*} * \Phi_t(x)\,\mathrm{d}x\right|$. Fatou's lemma implies that $|\langle e^*, \mu(E)\rangle| \le \underline{\lim}_{t\to 0} \int_E |f_{e^*} * \Phi_t(x)|\,\mathrm{d}x$. Also, $f_{e^*} * \Phi_t(x) = \langle e^*, \mu * \Phi_t(x)\rangle$. Hence

$$\begin{aligned}\|\mu(E)\|_{\mathbb{X}} &\le \sup_{\|e^*\|_{\mathbb{X}^*}\le 1} \underline{\lim}_{t\to 0} \int_E |f_{e^*} * \Phi_t(x)|\,\mathrm{d}x \\ &= \sup_{\|e^*\|_{\mathbb{X}^*}\le 1} \underline{\lim}_{t\to 0} \int_E |\langle e^*, \mu * \Phi_t(x)\rangle|\,\mathrm{d}x \\ &\le \int_E \mathcal{M}\mu(x)\,\mathrm{d}x \\ &\le cA|E|.\end{aligned}$$

The proof of Theorem 3.2 is complete now. □

Observe that in Theorem 3.2 we can replace the measure μ by a distribution $f \in \mathbb{H}^p_{\mathbb{X}}$ and let $\nu_k = (f - h_k)\phi_k$, where the polynomials $h_k = P_k f$. We also have in this case that (16) holds. We define as before $\nu = \sum_k \nu_k$ and $\omega = f - \nu$ with next result.

Theorem 3.4. *Let f be a distribution such that $\mathcal{M}f \in L^p(\mathbb{R}^n)$. Then there exists a family of cubes $\{Q_k^*\}$, and a decomposition $f = \omega + \nu$ with $\nu = \sum \nu_k$, where ν_k are the distributions with compact support such that the points* (1) *and* (2) *of Theorem* 3.2 *hold with $\langle \nu_k, 1\rangle = 0$ and*

$$M_\Phi\omega(x) \le c\mathcal{M}f(x)\chi_{\Omega^C}(x) + cA\sum_k \frac{d_k^{n+1}}{(d_k + |x-x_k|)^{n+1}}. \tag{34}$$

Proof. As before we define $\omega = f - \nu$. It remains to prove (34). Assume first that $x \notin \Omega$, then we have that

$$\frac{d_k^{n+1}}{|x-x_k|^{n+1}} \le c\frac{d_k^{n+1}}{(d_k + |x-x_k|)^{n+1}}.$$

We have that $M_\Phi\omega \le M_\Phi f + \sum_{k\in\mathbb{N}} M_\Phi \nu_k$. But $M_\Phi f \le c\mathcal{M}f$, and this together with (23) with $d = 0$ implies that

$$M_\Phi \nu_k(x) \le cA\frac{d_k^{n+1}}{d_k + |x-x_k|^{n+1}}, \tag{35}$$

proving (34) when $x \notin \Omega$. Now assume that $x \in \Omega$ and let m such that $x \in Q_m^*$. Then

$$\omega = f - \sum_{k\in\mathbb{N}} \nu_k = f - \sum_N \nu_k - \sum_F \nu_k,$$

where $k \in N$ if and only if $Q_k^* \cap Q_m^* \neq \emptyset$ and $F = \mathbb{N} - N$. Recall that the set N that depends on m has finite cardinality bounded by a fixed number M. When $k \in F$ we have that $cd_k \leq |x - x_k|$; then as before

$$\sum_F M_\Phi \nu_k(x) \leq cA \sum_F \frac{d_k^{n+1}}{(d_k + |x - x_k|)^{n+1}}.$$

If $k \in N$ we can split

$$f - \sum_N \nu_k = f - \sum_N \phi_k f - \sum_N q_k\phi_k.$$

For the last sum we know that $\|q_k\phi_k\|_{\mathbb{X}} \leq cA$ and $1 \leq c_n d_m^{n+1}/(d_m + |x - x_m|)^{n+1}$. So we estimate

$$\sum_N M_\Phi(q_k\phi_k)(x) \leq \sum_N cA \leq cAM \leq c_{n,M} A \frac{d_m^{n+1}}{(d_m + |x - x_m|)^{n+1}}.$$

Let Φ the test function supported in the ball. Let us analyze the convolution of Φ_t with $f - \sum_N \phi_k f$. Consider a constant c_0 such that $c_0 d_m = \sup\{r > 0 : B(x, r) \subset Q_m^*\}$. Then, for $t \leq c_0 d_m$ we have $\left(f - \sum_N \phi_k f\right) * \Phi_t(x) = 0$ due to the fact that $1 - \sum_c \phi_k = 0$ in Q_m^* and the support of $\Phi_t(x - \cdot)$ is contained in $B(x, t) \subset Q_m^*$. For $t > c_0 d_m$ observe that

$$\left(f - \sum_N \phi_k f\right) * \Phi_t(x) = \left\langle f, \left(1 - \sum_N \phi_k\right) \Phi_t(x - \cdot)\right\rangle.$$

We consider the function $\psi = \left(1 - \sum_N \phi_k\right) \Phi_t(x - \cdot)$. Notice that $\partial^\beta \sum_N \phi_k = \sum_N d_k^{-|\beta|} \partial^\beta \phi$, and since the quantities d_k and d_m are comparable when $Q_k^* \bigcap Q_m^* \neq \emptyset$, then $|\partial^\beta \sum_N \phi_k| \leq A_\beta M d_m^{-|\beta|}$. We also have that $|\partial^\beta \Phi_t| \leq A_\beta t^{-n-|\beta|} |\partial^\beta \Phi|$, but $t > c_0 d_m$; hence $|\partial^\beta \phi| \leq A_{\beta,M} d_m^{-n-|\beta|}$. Applying Lemma 3.1 for the ball $B = B(x, \rho)$ and $\rho = cd_m$ with a constant c large enough, so that $B \bigcap \Omega^C \neq \emptyset$, we obtain

$$|\langle f, \psi\rangle| \leq \mathcal{M}f(y) \leq cA,$$

for $y \in B \bigcap \Omega^C$. Hence

$$\sup_{t>0} \left\| \left(f - \sum_N \phi_k f\right) * \Phi_t(x) \right\|_{\mathbb{X}} \leq cA \leq cA \frac{d_m^{n+1}}{(d_m + |x - x_m|)^{n+1}}.$$

Finally

$$\begin{aligned} M_\Phi \omega &\leq M_\Phi \left(f - \sum_N \phi_k f\right) + \sum_N M_\Phi(q_k\phi_k) + \sum_F M_\Phi \nu_k \\ &\leq (c + c_M) A \frac{d_m^{n+1}}{(d_m + |x - x_m|)^{n+1}} + cA \sum_{k\in F} \frac{d_k^{n+1}}{(d_k + |x - x_k|)^{n+1}}, \end{aligned}$$

proving (34) when $x \in \Omega$. $\square$

Corollary 3.5. *The distribution ω of Theorem 3.4 belongs to the space $\mathbb{H}^1_{\mathbb{X}}(\mathbb{R}^n)$.*

Proof. We integrate (34),

$$\begin{aligned} \int_{\mathbb{R}^n} M_\Phi \omega(x)\, dx &\leq c \int_{\mathbb{R}^n} \mathcal{M}f(x) \chi_{\Omega^C}(x)\, dx + cA \int_{\mathbb{R}^n} \sum_{k\in\mathbb{N}} \frac{d_k^{n+1}}{(d_k + |x - x_k|)^{n+1}}\, dx \\ &\leq A^{1-p} \int_{\Omega^C} \mathcal{M}f^p(x)\, dx + cA|\Omega| < \infty \end{aligned}$$

and Theorem 2.6 implies $\omega \in \mathbb{H}^1_{\mathbb{X}}$. $\square$

Theorem 3.6. *Let $p < 1$, then the space $V^1_{\mathbb{X}}(\mathbb{R}^n) \bigcap \mathbb{H}^p_{\mathbb{X}}(\mathbb{R}^n)$ is dense in $\mathbb{H}^p_{\mathbb{X}}(\mathbb{R}^n)$.*

Proof. Let $f \in \mathbb{H}^p_{\mathbb{X}}$. If we set $A_k = k$ for every $k \in \mathbb{N}$, then by Theorem 3.4 there exist ν_k and ω_k distributions in $\mathbb{H}^p_{\mathbb{X}}$ such that $f = \omega_k + \nu_k$ and $\omega_k \to f$ as $k \to \infty$ in $\mathbb{H}^p_{\mathbb{X}}$ by (18). Then Corollary 3.5 implies that $\{\omega_k\}_{k\in\mathbb{N}} \subseteq V^1_{\mathbb{X}} \bigcap \mathbb{H}^p_{\mathbb{X}}$. □

Definition 3.7. Let $p \leq 1$. We say that a Borel $\mathbb{X}$-valued measure μ is a p-atom in $\mathbb{X}$ if it satisfies

1. supp $\mu \subset B$, where B is a ball.
2. $\|\mu\|_{V^\infty_{\mathbb{X}}} \leq 1/|B|^{\frac{1}{p}}$.
3. μ satisfies the moment condition $\int_{\mathbb{R}^n} x^\alpha \, \mathrm{d}\mu(x) = 0$, for all $|\alpha| \leq n\,[1/p - 1]$.

With the same proof as in the scalar case we have the following.

Proposition 3.8. *Let $p \leq 1$, then*

(a) *if μ is a p-atom in $\mathbb{X}$, then μ belongs to $\mathbb{H}^p_{\mathbb{X}}(\mathbb{R}^n)$. Moreover, evaluated at μ, any of the gauges defining $\mathbb{H}^p_{\mathbb{X}}(\mathbb{R}^n)$ according to Theorem 2.6 is bounded by a constant independent of μ.*
(b) *If $\{\mu_j\}$ is countable family of p-atoms in $\mathbb{X}$ and $\{\lambda_j\}$ a sequence of complex numbers in $\ell^p(\mathbb{N})$, then the series*

$$\mu = \sum_{j\in\mathbb{N}} \lambda_j \mu_j \tag{36}$$

belongs to $\mathbb{H}^p_{\mathbb{X}}(\mathbb{R}^n)$.

Now we will prove the converse of this proposition.

Proposition 3.9. *For $\mu \in V^1_{\mathbb{X}} \bigcap \mathbb{H}^p_{\mathbb{X}}$, $p < 1$, there exist a sequence $\{\lambda_j\}_{j\in\mathbb{N}}$ in ℓ^p and a family $\{\mu_j\}_{j\in\mathbb{N}}$ of p-atoms such that*

$$\mu = \sum_{j\in\mathbb{N}} \lambda_j \mu_j,$$

converging in $\mathcal{S}'_X$ and

$$\sum |\lambda_i|^p \leq C\|\mu\|^p_{\mathbb{H}^p_{\mathbb{X}}}.$$

Proof. For each $j \in \mathbb{Z}$ Theorem 3.2 we decompose $\mu = \omega^{(j)} + \nu^{(j)}$, where $\nu^{(j)} = \sum_{k\in\mathbb{N}} \nu^{(j)}_k$, and $\Omega^{(j)} = \{x \in \mathbb{R}^n : \mathcal{M}\mu(x) > 2^j\} = \bigcup_{k\in\mathbb{N}} (Q^{(j)}_k)^*$. Notice that $\Omega^{(j+1)} \subset \Omega^{(j)}$. We know that $\omega^{(j)} \to \mu$ in $\mathbb{H}^p_{\mathbb{X}}$ when $j \to \infty$, also $\left\|\omega^{(j)}\right\|_{V^\infty_{\mathbb{X}}} \leq c2^j$, so $\omega^{(j)} \to 0$ when $j \to -\infty$; therefore

$$\mu = \lim_{N\to\infty} \sum_{j=-\infty}^{N} \omega^{(j+1)} - \omega^{(j)}. \tag{37}$$

The terms $\omega^{(j+1)} - \omega^{(j)}$ satisfy $\left\|\omega^{(j+1)} - \omega^{(j)}\right\|_{V^\infty_{\mathbb{X}}} \leq c2^j$ and $\mathrm{supp}(\omega^{(j+1)} - \omega^{(j)}) \subseteq \Omega^{(j)}$.

Recall that the block measures are given by $\mathrm{d}\nu^{(j)}_k = \phi^{(j)}_k(\mathrm{d}\mu - h^{(j)}_k \mathrm{d}x)$ and $h^{(j)}_k = P^{(j)}_k \mu$. We now define the polynomials $H_{k,l}$ with degree d at the most as

$$H_{k,l} = P^{(j+1)}_l[\phi^{(j)}_k(\mathrm{d}\mu - h^{(j+1)}_l \,\mathrm{d}x)]. \tag{38}$$

The polynomials $H_{k,l}$ satisfy

(i) $H_{k,l} \neq 0$ when $(Q^{(j)}_k)^* \bigcap (Q^{(j+1)}_k)^* \neq \emptyset$.
To see this, notice that $H_{k,l} = \sum_{i=1}^N \langle \phi^{(j)}_k(\mathrm{d}\mu - h^{(j+1)}_l \,\mathrm{d}x), q_i \widetilde{\phi}^{(j+1)}_l \rangle q_i$.
(ii) $\mathrm{diam}[Q^{(j)}_k] \geq c \cdot \mathrm{diam}[Q^{(j+1)}_l]$ when $(Q^{(j)}_k)^* \bigcap (Q^{(j+1)}_l)^* \neq \emptyset$.
(iii) $\left\| H_{k,l}(x)\phi^{(j+1)}_l(x) \right\|_{\mathbb{X}} \leq c2^j$.
To see this we write $H_{k,l} = \sum_{i=1}^N \langle \phi^{(j)}_k(\mathrm{d}\mu - h^{(j+1)}_l \,\mathrm{d}x), q_i \widetilde{\phi}^{(j+1)}_l \rangle q_i$, where the polynomials $q_1, q_2, \ldots, q_N$ are a basis of the Hilbert space $L^2((Q^{(j+1)}_l)^*, \widetilde{\phi}^{(j+1)}_l \,\mathrm{d}x)$.
Now we estimate $\left\| \langle \phi^{(j)}_k(\mathrm{d}\mu - h^{(j+1)}_l \,\mathrm{d}x), q_i \widetilde{\phi}^{(j+1)}_l \rangle \right\|_{\mathbb{X}}$:
Let $\zeta = \phi^{(j)}_k q_i \widetilde{\phi}^{(j+1)}_l$. Since for some constants A_β we have $|\partial^\beta q_i(x)| \leq A_\beta (d^{(j+1)}_l)^{-|\beta|}$, $|\partial^\beta \widetilde{\phi}^{(j+1)}_l(x)| \leq A_\beta (d^{(j+1)}_l)^{-n-|\beta|}$, $|\partial^\beta \phi^{(j)}_k(x)| \leq A_\beta (d^{(j)}_k)^{-|\beta|} \leq c(d^{(j+1)}_l)^{-|\beta|}$, (see (26)) and by (ii), we get

$$\left\| \langle \phi^{(j)}_k \,\mathrm{d}\mu, q_i \widetilde{\phi}^{(j+1)}_l \rangle \right\|_{\mathbb{X}} \leq c2^{j+1}. \tag{39}$$

To estimate $\langle h_l^{(j+1)}\phi_k^{(j)}\,\mathrm{d}x, q_i\widetilde{\phi}_l^{(j+1)}\rangle$, we have

$$\left\|\langle h_l^{(j+1)}\phi_k^{(j)}\,\mathrm{d}x, q_i\widetilde{\phi}_l^{(j+1)}\rangle\right\|_{\mathbb{X}} \le \sum_{m=1}^{N}|\langle q_m\phi_k^{(j)}\,\mathrm{d}x, q_i\widetilde{\phi}_l^{(j+1)}\rangle|\left\|\langle \mu, q_m\widetilde{\phi}_l^{(j+1)}\rangle\right\|_{\mathbb{X}},$$

but $\left\|\langle \mu, q_m\widetilde{\phi}_l^{(j+1)}\rangle\right\|_{\mathbb{X}} \le c2^{j+1}$ for $m = 1, 2, \ldots, N$, and

$$\sum_{m=1}^{N}|\langle q_m\phi_k^{(j)}\,\mathrm{d}x, q_i\widetilde{\phi}_l^{(j+1)}\rangle| = |\phi_k^{(j)}|\sum_{m=1}^{N}|\langle q_m\,\mathrm{d}x, q_i\widetilde{\phi}_l^{(j+1)}\rangle| = |\phi_k^{(j)}|\sum_{m=1}^{N}\delta_{m,i} \le 1.$$

We conclude that

$$\left\|\langle h_l^{(j+1)}\phi_k^{(j)}\,\mathrm{d}x, q_i\widetilde{\phi}_l^{(j+1)}\rangle\right\|_{\mathbb{X}} \le c2^{j+1}. \tag{40}$$

Inequalities (39) and (40) imply (iii).

(iv) $\sum_{k\in\mathbb{N}} H_{k,l} = 0$. In fact,

$$\sum_{k\in\mathbb{N}} H_{k,l} = P_l^{(j+1)}[(\mu - h_l^{(j+1)})\chi_{\Omega^{(j)}}] = P_l^{(j+1)}[(\mu - P_l^{(j+1)}\mu)] = 0.$$

Next we will construct the atoms. Notice that

$$\begin{aligned}\omega^{(j+1)} - \omega^{(j)} &= \nu^{(j)} - \nu^{(j+1)}\\ &= \sum_{k\in\mathbb{N}}\phi_{j,k}(\mathrm{d}\mu - h_k^{(j)}\,\mathrm{d}x) - \sum_{l\in\mathbb{N}}\phi_l^{(j+1)}(\mathrm{d}\mu - h_l^{(j+1)}\,\mathrm{d}x).\end{aligned}$$

Then we let

$$\omega^{(j+1)} - \omega^{(j)} = \sum_{k\in\mathbb{N}} A_{j,k}, \tag{41}$$

where the measures $A_{j,k}$ are defined as

$$\mathrm{d}A_{j,k} = \phi_k^{(j)}(\mathrm{d}\mu - h_k^{(j)}\,\mathrm{d}x) - \sum_{l\in\mathbb{N}}\phi_l^{(j+1)}\phi_k^{(j)}(\mathrm{d}\mu - h_l^{(j+1)}\,\mathrm{d}x) + \sum_{l\in\mathbb{N}} H_{k,l}\phi_l^{(j+1)}\,\mathrm{d}x. \tag{42}$$

Eq. (41) holds since $\sum_{k\in\mathbb{N}}\phi_k^{(j)} = 1$ in $\mathrm{supp}\{\phi_l^{(j+1)}\}$. We have the following:

1. The measures $A_{j,k}$ have support in a ball $B_k^{(j)}$ that contains the cube $(Q_k^{(j)})^*$ and all cubes $(Q_l^{(j+1)})^*$ that intersect $(Q_k^{(j)})^*$. Moreover, by (ii), we may assume that $|B_k^{(j)}| = c|Q_k^{(j)}|$
2. The measures $A_{j,k}$ are bounded by $c2^j$. In fact,

$$\begin{aligned}\left\|\left[\phi_k^{(j)}\mu - \sum_{l\in\mathbb{N}}\phi_l^{(j+1)}\phi_k^{(j)}\mu\right](E)\right\|_{\mathbb{X}} &= \left\|\int_E \chi_{\Omega_{j+1}^C}(x)\phi_k^{(j)}(x)\,\mathrm{d}\mu\right\|_{\mathbb{X}}\\ &\le c2^j|E|,\end{aligned}$$

due to (17). Moreover

$$\left\|(\phi_k^{(j)}h_k^{(j)}\,\mathrm{d}x)(E)\right\|_{\mathbb{X}} \le \int_E \left\|h_k^{(j)}(x)\phi_k^{(j)}(x)\right\|_{\mathbb{X}}\,\mathrm{d}x \le c2^j|E|,$$

and

$$\begin{aligned}\left\|\left(\sum_{l\in\mathbb{N}}\phi_l^{(j+1)}\phi_k^{(j)}h_l^{(j+1)}\,\mathrm{d}x\right)(E)\right\|_{\mathbb{X}} &\le \int_E\left(\phi_k^{(j)}(x)\sum_{l\in\mathbb{N}}\left\|h_l^{(j+1)}(x)\phi_l^{(j+1)}(x)\right\|_{\mathbb{X}}\right)\,\mathrm{d}x\\ &\le \int_E \phi_k^{(j)}(x)c2^{j+1}M\,\mathrm{d}x\\ &\le c2^j|E|,\end{aligned}$$

since every point x belongs to M cubes $(Q_l^{(j+1)})^*$ at the most. Finally

$$\begin{aligned}\left\|\left(\sum_{l\in\mathbb{N}} H_{k,l}\phi_l^{(j+1)}\,\mathrm{d}x\right)(E)\right\|_{\mathbb{X}} &\le \int_E \sum_{l\in\mathbb{N}} \left\|H_{k,l}(x)\phi_l^{(j+1)}(x)\right\|_{\mathbb{X}}\,\mathrm{d}x\\ &\le \int_E c2^{j+1}M\,\mathrm{d}x\\ &\le c2^j|E|.\end{aligned}$$

3. The $A_{j,k}$ satisfy the moment condition $\int_{\mathbb{R}^n} x^\alpha\,\mathrm{d}A_{j,k}(x)$, for all $|\alpha|\le n\,[1/p-1]$. In fact, $\mathrm{d}\nu_k^{(j)}=\phi_k^{(j)}(\mathrm{d}\mu-h_k^{(j)}\,\mathrm{d}x)$ satisfies this property. Also
$$-\sum_{l\in\mathbb{N}}\phi_l^{(j+1)}\phi_k^{(j)}(\mathrm{d}\mu-h_k^{(j)}\,\mathrm{d}x)+\sum_{l\in\mathbb{N}}H_{k,l}\phi_l^{(j+1)}\,\mathrm{d}x=-\sum_{l\in\mathbb{N}}\phi_l^{(j+1)}[\phi_k^{(j)}(\mathrm{d}\mu-h_l^{(j+1)}\,\mathrm{d}x)-H_{k,l}\,\mathrm{d}x],$$
and the terms in the right hand side of this equation are of the form $\phi_l^{(j+1)}(\mu-P_l^{(j+1)}\mu)$ which satisfy the moment condition.

Now define

$$\lambda_k^{(j)}=2^j|B_k^{(j)}|^{\frac{1}{p}}$$
$$a_{j,k}=\frac{1}{\lambda_k^{(j)}}A_{j,k}.$$

Notice that the measures $a_{j,k}$ have support contained in $B_{j,k}$,

$$\left\|a_{j,k}\right\|_{V_{\mathbb{X}}^\infty}=\frac{\left\|A_{j,k}\right\|_{V_{\mathbb{X}}^\infty}}{\lambda_{j,k}}\le\frac{2^j}{2^j|B_{j,k}|^{\frac{1}{p}}}=|B_{j,k}|^{-\frac{1}{p}},$$

and they satisfy the required moment conditions. By (37) and (41), we have

$$\mu=\sum_{\substack{j\in\mathbb{Z}\\k\in\mathbb{N}}}\lambda_{j,k}a_{j,k}.$$

Finally

$$\begin{aligned}\sum_{\substack{j\in\mathbb{Z}\\k\in\mathbb{N}}}|\lambda_{j,k}|^p &= c\sum_{\substack{j\in\mathbb{Z}\\k\in\mathbb{N}}}2^{jp}|Q_{j,k}^*|\le cM\sum_{j\in\mathbb{Z}}2^{jp}|\Omega^{(j)}|\\ &\le c_{M,p}\sum_{j\in\mathbb{Z}}\int_{2^{j-1}}^{2^j}\lambda^{p-1}|\{x\in\mathbb{R}^n:\mathcal{M}\mu(x)>\lambda\}|\,\mathrm{d}\lambda\\ &= c_{M,p}\int\mathcal{M}\mu^p=c_{M,p}\,\|\mu\|_{\mathbb{H}_{\mathbb{X}}^p}^p.\quad\square\end{aligned}$$

Now we are ready to prove the existence of atomic decompositions for any element in $\mathbb{H}_{\mathbb{X}}^p$.

Theorem 3.10. *Let $p\le 1$, then for any $f\in\mathbb{H}_{\mathbb{X}}^p$ there exist a sequence $\{\lambda_j\}$ in ℓ^p and a sequence of p-atoms $\{a_j\}$ such that*

$$\mu=\sum_{j\in\mathbb{N}}\lambda_j a_j,$$

converging in $\mathcal{S}_X'$ and

$$\sum|\lambda_i|^p\le C\|f\|_{\mathbb{H}_{\mathbb{X}}^p}^p.$$

Proof. Let $f\in\mathbb{H}_{\mathbb{X}}^p$; then by Theorem 3.6 there exists a sequence $\{\mu_j\}_{j\in\mathbb{N}}$ of measures in $V_{\mathbb{X}}^1\bigcap\mathbb{H}_{\mathbb{X}}^p$ such that $\mu_j\to\mu$ in $\mathbb{H}_{\mathbb{X}}^p$ as $j\to\infty$ and $\left\|\mu_{j+1}-\mu_j\right\|_{\mathbb{H}_{\mathbb{X}}^p}^p\le 2^{-j}\,\|\mu\|_{\mathbb{H}_{\mathbb{X}}^p}^p$. Letting $\mu_0=0$ we have

$$\mu=\sum_{j=0}^{\infty}\mu_{j+1}-\mu_j.$$

By Proposition 3.9, for each $j \in \mathbb{N}$ we have an atomic decomposition of measure $\mu_{j+1} - \mu_j \in V_{\mathbb{X}} \bigcap \mathbb{H}^p_{\mathbb{X}}$,

$$\mu_{j+1} - \mu_j = \sum_{k \in \mathbb{N}} \lambda_{j,k} a_{j,k},$$

with

$$\sum_{k \in \mathbb{N}} |\lambda_{j,k}|^p \le C \left\| \mu_{j+1} - \mu_j \right\|^p_{\mathbb{H}^p_{\mathbb{X}}}.$$

Then

$$\mu = \sum_{j,k \in \mathbb{N}} \lambda_{j,k} a_{j,k}, \tag{43}$$

and

$$\sum_{j,k \in \mathbb{N}} |\lambda_{j,k}|^p \le C \left\| \mu \right\|^p_{\mathbb{H}^p_{\mathbb{X}}}. \quad \square$$

Contrasting with the scalar case, we should not expect in vector-valued Hardy spaces to have atomic decompositions consisting of integrable functions. Recall that a Banach space $\mathbb{X}$ has the Radon–Nikodým property if every measure $\mu \in V^1_{\mathbb{X}}(Q)$ has a density in $L^1_{\mathbb{X}}(Q)$ for some cube Q (and then for every cube).

Proposition 3.11. *All the elements of $\mathbb{H}^1_{\mathbb{X}}$ are functions (have a density in $L^1_X(\mathbb{R}^n)$) if and only if $\mathbb{X}$ has the Radon–Nikodým property.*

Proof. Suppose that every element of $\mathbb{H}^1_{\mathbb{X}}(\mathbb{R}^n)$ has a density. Let Q be a cube in $\mathbb{R}^n$ and ν a measure in $V^1_{\mathbb{X}}(Q)$. Define the measure $\mu = \nu - \nu_Q \,\mathrm{d}x$, where $\nu_Q = \nu(Q)/|Q|$. Then μ is a multiple of a 1-*atom*, and thus it belongs to $\mathbb{H}^1_{\mathbb{X}}(\mathbb{R}^n)$. Our assumption implies that μ has a density so that μ has a density too. From this it follows that $\mathbb{X}$ has the Radon–Nikodým property. The other direction is clear if we remember that every 1-*atom* belongs to $V^1_{\mathbb{X}}(\mathbb{R}^n)$. $\square$

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