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Rational time–frequency super Gabor frames and their duals[☆]

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ABSTRACT

This paper addresses general super Gabor systems with rational time–frequency product lattices. A Zak transform matrix method is developed for such super Gabor systems. We characterize complete super Gabor systems and super Gabor frames. Given a super Gabor frame, we obtain an explicit expression of its canonical dual and a parametrization of all its super Gabor duals and prove that the canonical dual is the norm-minimal one among all super Gabor duals. We also prove that a super Gabor frame has a unique super Gabor dual if and only if it is a Riesz basis.

1. Introduction

The notion of frame was introduced by Duffin and Schaeffer in [24]. It is a generalization of basis, and can be viewed as some kind of "overcomplete basis". A basis in a Hilbert space allows one to represent each element in a unique way. An overcomplete frame also allows one to represent each element via it, but the representation is not unique. This property plays a significant role in mathematics (such as nonlinear approximation), signal transmission and modern time–frequency analysis. In the past more than twenty years, the theory of frames has been growing rapidly. Gabor frames are a class of important frames among all kinds of frames. Given $\eta, \mu \in \mathbb{R}^d$, define the *modulation* operator M_η and *translation* operator T_μ on $L^2(\mathbb{R}^d)$ respectively by

$$M_\eta f(\cdot) = e^{2\pi i\langle \eta, \cdot\rangle} f(\cdot) \quad \text{and} \quad T_\mu f(\cdot) = f(\cdot - \mu) \tag{1.1}$$

for $f \in L^2(\mathbb{R}^d)$, where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product in $\mathbb{R}^d$. For $\alpha, \beta > 0$ and $g \in L^2(\mathbb{R}^d)$, we denote by $\mathcal{G}(g, \alpha, \beta)$ the Gabor system:

$$\mathcal{G}(g, \alpha, \beta) = \{M_{\beta n} T_{\alpha k} g : n, k \in \mathbb{Z}^d\}, \tag{1.2}$$

where $\mathbb{Z}^d$ is the set of d-dimensional integer points in $\mathbb{R}^d$. $\mathcal{G}(g, \alpha, \beta)$ is called a *(Gabor) frame* for $L^2(\mathbb{R}^d)$ if there exist constants $C_1, C_2 > 0$ such that

$$C_1\|f\|^2 \le \sum_{n,k\in\mathbb{Z}^d} |\langle f, M_{\beta n} T_{\alpha k} g\rangle|^2 \le C_2\|f\|^2 \tag{1.3}$$

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for $f \in L^2(\mathbb{R}^d)$, where C_1 and C_2 are called frame bounds. In particular, the frame is called a *tight (Gabor) frame* for $L^2(\mathbb{R}^d)$ if $C_1 = C_2$; called a *Parseval (Gabor) frame* for $L^2(\mathbb{R}^d)$ if $C_1 = C_2 = 1$; called a *(Gabor) Riesz basis* for $L^2(\mathbb{R}^d)$ if it ceases to be a frame for $L^2(\mathbb{R}^d)$ whenever any one of its elements is removed. $\mathcal{G}(g, \alpha, \beta)$ is called a *(Gabor) Bessel sequence* in $L^2(\mathbb{R}^d)$ if the right-hand side inequality in (1.3) holds, where C_2 is called Bessel bound. The fundamentals of frames can be found in [6,7,11,12,16,24].

In this paper, we focus on super Gabor frames. Given a positive integer L, the direct sum Hilbert space $\bigoplus_{l=1}^{L} L^2(\mathbb{R}^d)$ is exactly the vector-valued Hilbert space $L^2(\mathbb{R}^d, \mathbb{C}^L)$ endowed with the inner product defined by

$$\langle \mathbf{f}, \mathbf{g}\rangle = \sum_{l=1}^{L} \int_{\mathbb{R}^d} f_l(x)\overline{g_l(x)}dx \quad \text{for } \mathbf{f} = (f_1, f_2, \ldots, f_L), \mathbf{g} = (g_1, g_2, \ldots, g_L) \in L^2(\mathbb{R}^d, \mathbb{C}^L).$$

In what follows, for $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^L)$ and $1 \le l \le L$, we always denote by f_l the l-th component of $\mathbf{f}$. Similarly to (1.1), for η, $\mu \in \mathbb{R}^d$, we define the *modulation* operator M_η and *translation* operator T_μ on $L^2(\mathbb{R}^d, \mathbb{C}^L)$ respectively by

$$M_\eta \mathbf{f}(\cdot) := \left(M_\eta f_1(\cdot), M_\eta f_2(\cdot), \ldots, M_\eta f_L(\cdot)\right),$$
$$T_\mu \mathbf{f}(\cdot) := \left(T_\mu f_1(\cdot), T_\mu f_2(\cdot), \ldots, T_\mu f_L(\cdot)\right)$$

for $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^L)$. Obviously, they are both unitary operators on $L^2(\mathbb{R}^d, \mathbb{C}^L)$, and it is easy to check that

$$M_\eta T_\mu \mathbf{f}(\cdot) = e^{2\pi i\langle \eta, \mu\rangle} T_\mu M_\eta \mathbf{f}(\cdot) \quad \text{for } \mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^L). \tag{1.4}$$

For $\alpha, \beta > 0$ and $\mathbf{g} \in L^2(\mathbb{R}^d, \mathbb{C}^L)$, define the Gabor system $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ generated by $\mathbf{g}$ as

$$\mathcal{G}(\mathbf{g}, \alpha, \beta) = \{M_{\beta n} T_{\alpha k}\mathbf{g} : n, k \in \mathbb{Z}^d\}. \tag{1.5}$$

$\mathcal{G}(\mathbf{g}, \alpha, \beta)$ is called a *super (Gabor) frame* for $L^2(\mathbb{R}^d, \mathbb{C}^L)$ if it is a frame for $L^2(\mathbb{R}^d, \mathbb{C}^L)$, i.e., there exist constants $C_1, C_2 > 0$ such that

$$C_1\|\mathbf{f}\|^2 \le \sum_{n,k\in\mathbb{Z}^d} |\langle \mathbf{f}, M_{\beta n} T_{\alpha k}\mathbf{g}\rangle|^2 \le C_2\|\mathbf{f}\|^2 \tag{1.6}$$

for $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^L)$, where C_1 and C_2 are called frame bounds, L is called the *length* of super (Gabor) frame. *Tight (Parseval) super Gabor frame*, *super Gabor Riesz basis* and *super Gabor Bessel sequence* are defined similarly. We always denote by I the identity operator, regardless of its acting space. Let $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ and $\mathcal{G}(\mathbf{h}, \alpha, \beta)$ be both Bessel sequences in $L^2(\mathbb{R}^d, \mathbb{C}^L)$. We define $\mathcal{S}_{\mathbf{h},\mathbf{g}} : L^2(\mathbb{R}^d, \mathbb{C}^L) \to L^2(\mathbb{R}^d, \mathbb{C}^L)$ by

$$\mathcal{S}_{\mathbf{h},\mathbf{g}}\mathbf{f} = \sum_{n,k\in\mathbb{Z}^d} \langle \mathbf{f}, M_{\beta n} T_{\alpha k}\mathbf{h}\rangle M_{\beta n} T_{\alpha k}\mathbf{g} \quad \text{for } \mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^L). \tag{1.7}$$

It is well-known that, if $\mathcal{S}_{\mathbf{h},\mathbf{g}} = I$, then $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ and $\mathcal{G}(\mathbf{h}, \alpha, \beta)$ are both frames for $L^2(\mathbb{R}^d, \mathbb{C}^L)$ by the theory of frames. In this case, we say $\mathbf{h}$ is a *super Gabor dual* of $\mathbf{g}$. In particular, $\gamma = \mathcal{S}_{\mathbf{g},\mathbf{g}}^{-1}\mathbf{g}$ is a super Gabor dual of $\mathbf{g}$ when $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ is a frame for $L^2(\mathbb{R}^d, \mathbb{C}^L)$, which is so-called the *canonical dual* of $\mathbf{g}$. It is noteworthy that there exist dual frames not having the Gabor structure for an overcomplete frame (not a Riesz basis) $\mathcal{G}(\mathbf{g}, \alpha, \beta)$, which can be proved by Lemma 5.6.1 in [6].

The notion of superframe in general Hilbert spaces was introduced by Balan [1] in the context of “multiplexing”, which has been widely used in mobile communication network, satellite communication network and computer area network. The idea of “multiplexing” is to encode L independent signals f_l, $l = 1, 2, \ldots, L$, as a single sequence that captures the time–frequency information of each f_l. Given a Gabor system $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ of the form (1.5), for $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^L)$, one then considers the sequence of numbers $\langle \mathbf{f}, M_{\beta n} T_{\alpha k}\mathbf{g}\rangle = \sum_{l=1}^{L}\langle f_l, M_{\beta n} T_{\alpha k} g_l\rangle$ for $n, k \in \mathbb{Z}^d$. One requires that $\mathbf{f}$ is completely determined by this sequence, and that there exists a stable reconstruction. This requirement leads to the definition of super Gabor frame. In recent years, super wavelet and Gabor frames in $L^2(\mathbb{R}^d, \mathbb{C}^L)$ have interested some mathematicians and engineering specialists. The details can be found in [2–4,9,10,13,17–19,22] and references therein, where super wavelet and Gabor frames were sometimes called vector-valued wavelet and Gabor frames. In 2008, Führ in [13] derived frame bound estimates for super Gabor system in $L^2(\mathbb{R}, \mathbb{C}^L)$ with window functions belonging to Schwartz space, and obtained estimates for the window $\mathbf{h} = \begin{pmatrix} h_0, & h_1, & \ldots, & h_L\end{pmatrix} \in L^2(\mathbb{R}, \mathbb{C}^{L+1})$composed of the first $L + 1$ Hermite functions, where super Gabor systems were called vector-valued Gabor systems. The proof is based on a sampling estimate for the Paley–Wiener space established by Führ and Gröchenig in [14]. In 2009, using growth estimates for the Weierstrass σ-function and a new type of interpolation problem for entire functions on the Bargmann–Fock space, Gröchenig and Lyubarskii in [17] characterized all lattices $\Lambda \subset \mathbb{R}^2$ such that the Gabor system $\{M_{\lambda_2} T_{\lambda_1}\mathbf{h} : \lambda = (\lambda_1, \lambda_2) \in \Lambda\}$ is a frame for $L^2(\mathbb{R}, \mathbb{C}^L)$. However, less is known about a general super Gabor system. Necessary density conditions were studied in [2] by Balan. A sufficient and necessary density condition was obtained in [22] by Li and Han for the rational time–frequency lattice case. Motivated by the above works, we in this paper investigate general Gabor systems $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ of the form (1.5) under the following assumptions:

Assumption 1. L is a positive integer.

Assumption 2. $\alpha, \beta > 0$, and $\alpha\beta = \frac{p}{q}$ with p and q being relatively prime positive integers.

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers, by A^* its conjugate transpose for a complex matrix A, by Q_η the set $[0, \eta)^d$ for $\eta > 0$, by E_t the set $\{0, 1, \ldots, t-1\}^d$ for $t \in \mathbb{N}$, by $\mathbb{C}^{E_t}$ the complex Euclidean space consisting of complex vectors indexed by E_t, by $\mathbb{C}^{LE_t}$ the complex Euclidean space consisting of the vector $X = \begin{pmatrix} x^{(1)} \\ x^{(2)} \\ \vdots \\ x^{(L)} \end{pmatrix}$ with $x^{(m)} \in \mathbb{C}^{E_t}$, $1 \le m \le L$, and by $\|X\|_{\mathbb{C}^{LE_t}}$ the Euclidean norm of X. The relation such as inclusion or equality between two (Lebesgue) measurable sets in $\mathbb{R}^d$ means that it holds up to a set of measure zero.

From (1.7), we have

$$\mathbf{f} = \sum_{n,k\in\mathbb{Z}^d} \langle \mathbf{f}, M_{\beta n} T_{\alpha k} \mathbf{h} \rangle M_{\beta n} T_{\alpha k} \mathbf{g} \quad \text{for } \mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^L)$$

if $\mathbf{h}$ is a super Gabor dual of $\mathbf{g}$. In particular, we have much freedom to represent $\mathbf{f}$ when $\mathbf{g}$ admits many Gabor duals. Such freedom is exactly why frames attract many researchers, and has important applications in signal transmission. A detailed argument can be found in [8]. Our main goal in this paper is to establish a general theory of super Gabor dual frames. We first obtain a characterization of super Gabor frames which allows us to easily design super Gabor frames, and then derive an explicit expression of super Gabor duals (see Theorems 4.2–4.5 below). Our main novelty is to introduce a new Zak transform matrix, and to establish all theorems in terms of such matrix. This matrix method allows us to easily obtain super Gabor dual frames. Our Zak transform matrix here is related to but different from the Zibulski–Zeevi matrix in [25]. It is more convenient for our purposes.

The rest of this paper is organized as follows. Section 2 is an auxiliary one. In this section, we associate a Gabor system $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ with a $q^d \times Lp^d$ matrix-valued function $\mathbf{G}(x, w)$ via a Zak transform which was first introduced in [15]. Some properties of $\mathbf{G}$ are presented. In Section 3, we obtain necessary and sufficient conditions for $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ to be complete, a Bessel sequence, a frame for $L^2(\mathbb{R}^d, \mathbb{C}^L)$, and prove that a super Gabor frame is a Riesz basis for $L^2(\mathbb{R}^d, \mathbb{C}^L)$ if and only if $(\alpha\beta)^d = \frac{1}{L}$. This allows us to easily construct super Gabor frames and Riesz bases by designing corresponding Zak transform matrices. In Section 4, for an arbitrary Gabor frame $\mathcal{G}(\mathbf{g}, \alpha, \beta)$, we derive an explicit expression of the canonical dual and a parametrization of all super Gabor duals of $\mathbf{g}$, and prove that the canonical dual is the norm-minimal one among all super Gabor duals. We also prove that a super Gabor frame has a unique super Gabor dual if and only if it is a Riesz basis for $L^2(\mathbb{R}^d, \mathbb{C}^L)$.

2. Zak transform matrix

This section is an auxiliary one to following sections. To begin with we introduce some notions. Given a measurable set S in $\mathbb{R}^d$, a collection $\{S_k : k \in \mathbb{Z}^d\}$ of measurable sets in $\mathbb{R}^d$ is called a *partition* of S if

$$\bigcup_{k\in\mathbb{Z}^d} S_k = S \quad \text{and} \quad S_k \cap S_{k'} = \emptyset \text{ for } k \neq k' \text{ in } \mathbb{Z}^d.$$

For $\lambda > 0$ and measurable sets $S, S' \subset \mathbb{R}^d$, we say S is $\lambda\mathbb{Z}^d$*-congruent* to S' if there exists a partition $\{S_k : k \in \mathbb{Z}^d\}$ of S such that $\{S_k + \lambda k : k \in \mathbb{Z}^d\}$ is a partition of S'. In particular, only finitely many S_k among S_k, $k \in \mathbb{Z}^d$, are nonempty if both S and S' are bounded in addition. Obviously, S' is also $\lambda\mathbb{Z}^d$-congruent to S if S is $\lambda\mathbb{Z}^d$-congruent to S'. So we usually say S and S' are $\lambda\mathbb{Z}^d$-congruent in this case. Similarly, for a set S in $\mathbb{Z}^d$, a collection $\{S_k : k \in \mathbb{Z}^d\}$ of subsets of $\mathbb{Z}^d$ is called a partition of S if

$$\bigcup_{k\in\mathbb{Z}^d} S_k = S \quad \text{and} \quad S_k \cap S_{k'} = \emptyset \text{ for } k \neq k' \text{ in } \mathbb{Z}^d.$$

For $\lambda \in \mathbb{N}$ and $S, S' \subset \mathbb{Z}^d$, we say S is $\lambda\mathbb{Z}^d$-congruent to S' if there exists a partition $\{S_k : k \in \mathbb{Z}^d\}$ of S such that $\{S_k + \lambda k : k \in \mathbb{Z}^d\}$ is a partition of S'. It has properties similar to the above sets in $\mathbb{R}^d$. The *Zak transform* $Z_{q\alpha} f$ of $f \in L^2(\mathbb{R}^d)$ is defined by

$$Z_{q\alpha} f(x, w) = \sum_{k\in\mathbb{Z}^d} f(x - q\alpha k) e^{2\pi i \langle k, w \rangle}$$

for a.e. $(x, w) \in \mathbb{R}^{2d}$. It is easy to check that $Z_{q\alpha}$ has *quasi-periodicity*:

$$Z_{q\alpha} f(x + q\alpha m, w + n) = e^{2\pi i \langle m, w \rangle} Z_{q\alpha} f(x, w) \tag{2.1}$$

for $f \in L^2(\mathbb{R}^d)$, $m, n \in \mathbb{Z}^d$ and a.e. $(x, w) \in \mathbb{R}^{2d}$. Let $\mathcal{H}_p$ be the Hilbert space $\mathcal{H}_p = L^2(Q_{\frac{1}{\beta}} \times Q_1, \mathbb{C}^{E_p})$. It consists of all column vector-valued functions $s(x, w) = \big(s_j(x, w)\big)_{j\in E_p}$ indexed by E_p with the components $s_j \in L^2(Q_{\frac{1}{\beta}} \times Q_1)$ and with the norm $\|s\|_{\mathcal{H}_p} = \left(\sum_{j\in E_p}\int_{Q_{\frac{1}{\beta}}\times Q_1}|s_j(x,w)|^2dxdw\right)^{\frac{1}{2}}$. We define the *vector-valued Zak transform* $\mathcal{Z}_{q\alpha}f$ of $f\in L^2(\mathbb{R}^d)$ by $\mathcal{Z}_{q\alpha}f(x,w)=\left(Z_{q\alpha}f(x+\frac{j}{\beta},w)\right)_{j\in E_p}$ for a.e. $(x,w)\in\mathbb{R}^{2d}$.

By simple arguments, we have the following two lemmas:

Lemma 2.1. (i) *For* $(\mu,\eta)\in\mathbb{R}^{2d}$, *we have*

$$Z_{q\alpha}(M_\eta T_\mu f)(x,w)=e^{2\pi i\langle\eta,x\rangle}Z_{q\alpha}f(x-\mu,w-q\alpha\eta)$$

for $f\in L^2(\mathbb{R}^d)$ *and a.e.* $(x,w)\in\mathbb{R}^{2d}$*;*
(ii) $Z_{q\alpha}$ *is a unitary operator from* $L^2(\mathbb{R}^d)$ *onto* $L^2(Q_{q\alpha}\times Q_1)$*;*
(iii) $\mathcal{Z}_{q\alpha}$ *is a unitary operator from* $L^2(\mathbb{R}^d)$ *onto* $\mathcal{H}_p$*.*

Lemma 2.2. *The set* $K=\bigcup_{j\in E_p,l\in E_q}\left(Q_{\frac{1}{q\beta}}+\frac{j}{\beta}-\alpha l\right)$ *is* $q\alpha\mathbb{Z}^d$*-congruent to* $Q_{q\alpha}$*.*

Remark 2.1. By (ii) in Lemma 2.1 and (2.1), for $f\in L^2(\mathbb{R}^d)$, $Z_{q\alpha}f\in L^2((Q_{q\alpha}+q\alpha k)\times Q_1)$ for $k\in\mathbb{Z}^d$. So $Z_{q\alpha}f\in L^2(E\times Q_1)$ for an arbitrary bounded and measurable set E in $\mathbb{R}^d$ since E can be covered by finitely many such $Q_{q\alpha}+q\alpha k$. This fact will be used frequently in the context, and we will not specify it.

Given $g\in L^2(\mathbb{R}^d)$, we denote by G the $q^d\times p^d$ matrix-valued function:

$$G(x,w)=\left(\overline{Z_{q\alpha}g\left(x+\frac{j}{\beta}-\alpha l,w\right)}\right)_{l\in E_q,j\in E_p}\quad\text{for a.e. }(x,w)\in\mathbb{R}^{2d}.\tag{2.2}$$

For $\mathbf{g}\in L^2(\mathbb{R}^d,\mathbb{C}^L)$, define $q^d\times Lp^d$ matrix-valued function $\mathbf{G}$ by

$$\mathbf{G}(x,w)=\left(G_1(x,w),G_2(x,w),\ldots,G_L(x,w)\right)\quad\text{for a.e. }(x,w)\in\mathbb{R}^{2d},\tag{2.3}$$

where G_m, $1\le m\le L$, are defined as in (2.2), i.e., $G_m(x,w)=\left(\overline{Z_{q\alpha}g_m(x+\frac{j}{\beta}-\alpha l,w)}\right)_{l\in E_q,j\in E_p}$. Similarly, we associate f, h and $\gamma\in L^2(\mathbb{R}^d)$ with $F(x,w)$, $H(x,w)$ and $\Gamma(x,w)$, and associate $\mathbf{f}$, $\mathbf{h}$ and $\gamma\in L^2(\mathbb{R}^d,\mathbb{C}^L)$ with $\mathbf{F}(x,w)$, $\mathbf{H}(x,w)$ and $\boldsymbol{\Gamma}(x,w)$ respectively. By (ii) in Lemmas 2.1 and 2.2 and (2.1), $\mathbf{g}$ is uniquely determined by the values of $\mathbf{G}(x,w)$ on $Q_{\frac{1}{q\beta}}\times Q_1$. So an arbitrary $q^d\times Lp^d$ matrix-valued function $\mathbf{G}(x,w)$ defined on $Q_{\frac{1}{q\beta}}\times Q_1$ with entries belonging to $L^2(Q_{\frac{1}{q\beta}}\times Q_1)$ determines a unique $\mathbf{g}\in L^2(\mathbb{R}^d,\mathbb{C}^L)$ such that $\mathbf{g}$ is related to $\mathbf{G}$ by (2.3). For $\mathbf{f}\in L^2(\mathbb{R}^d,\mathbb{C}^L)$, we always denote by $\mathcal{F}(x,w)$ the vector-valued function

$$\mathcal{F}(x,w)=\begin{pmatrix}\mathcal{Z}_{q\alpha}f_1(x,w)\\ \mathcal{Z}_{q\alpha}f_2(x,w)\\ \vdots\\ \mathcal{Z}_{q\alpha}f_L(x,w)\end{pmatrix}\quad\text{for a.e. }(x,w)\in\mathbb{R}^{2d}.\tag{2.4}$$

Then we have the following lemma and theorem by Lemma 2.1 and (2.1):

Lemma 2.3. *Let* $\mathbf{f}\in L^2(\mathbb{R}^d,\mathbb{C}^L)$*. Then* $\|\mathbf{f}\|^2=\int_{Q_{\frac{1}{\beta}}\times Q_1}\|\mathcal{F}(x,w)\|^2_{\mathbb{C}^{LEp}}dxdw$*.*

Theorem 2.1. *For* $\mathbf{f},\mathbf{g}\in L^2(\mathbb{R}^d,\mathbb{C}^L)$*, we have*

$$\langle\mathbf{f},M_{\beta n}T_{\alpha(qk+l)}\mathbf{g}\rangle=\int_{Q_{\frac{1}{\beta}}\times Q_1}(\mathbf{G}(x,w)\mathcal{F}(x,w))_l\,e^{-2\pi i\beta\langle n,x\rangle}e^{2\pi i\langle k,w\rangle}dxdw$$

for $(n,k,l)\in\mathbb{Z}^d\times\mathbb{Z}^d\times E_q$*, where* $(\mathbf{G}(x,w)\mathcal{F}(x,w))_l$ *is the* l*-th component of* $\mathbf{G}(x,w)\mathcal{F}(x,w)$*:*

$$(\mathbf{G}(x,w)\mathcal{F}(x,w))_l=\sum_{m=1}^{L}\sum_{j\in E_p}\overline{Z_{q\alpha}g_m\left(x+\frac{j}{\beta}-\alpha l,w\right)}Z_{q\alpha}f_m\left(x+\frac{j}{\beta},w\right).$$

Let $\mathbf{g},\mathbf{h}\in L^2(\mathbb{R}^d,\mathbb{C}^L)$. Next we show how the values of $(\mathbf{G}^*\mathbf{H})(x,w)$ on $Q_{\frac{1}{q\beta}}\times Q_1$ determine its behavior on $\mathbb{R}^{2d}$. Similarly to Lemma 2.3 in [23], we have the following lemma:

Lemma 2.4. *For every $j \in \mathbb{Z}^d$, there exists a unique $(k_j, l_j, m_j) \in E_p \times E_q \times \mathbb{Z}^d$ such that*

$$j = qk_j + pl_j + pqm_j.$$

Lemma 2.5. *For every $j \in \mathbb{Z}^d$, there exists a $\mathbb{Z}^d$-periodic $p^d \times p^d$ unitary matrix-valued function $U_j(w)$ such that*

$$(G^*H)\left(x + \frac{j}{q\beta}, w\right) = U_j^*(w)(G^*H)(x, w)U_j(w) \tag{2.5}$$

for $g, h \in L^2(\mathbb{R}^d)$ and a.e. $(x, w) \in \mathbb{R}^{2d}$.

Proof. For an arbitrary $j \in \mathbb{Z}^d$, there exists a unique $(k_j, l_j, m_j) \in E_p \times E_q \times \mathbb{Z}^d$ such that $\frac{j}{q\beta} = \frac{k_j}{\beta} + \alpha l_j + q\alpha m_j$ by Lemma 2.4 and the fact that $\alpha\beta = \frac{p}{q}$. It follows that

$$\left((G^*H)\left(x + \frac{j}{q\beta}, w\right)\right)_{j_1, j_2} = \sum_{l \in E_q} Z_{q\alpha} g\left(x + \frac{j_1 + k_j}{\beta} - \alpha(l - l_j), w\right) \overline{Z_{q\alpha} h\left(x + \frac{j_2 + k_j}{\beta} - \alpha(l - l_j), w\right)} \tag{2.6}$$

for $(j_1, j_2) \in E_p \times E_p$ and a.e. $(x, w) \in \mathbb{R}^{2d}$ by (2.1). Again by $q\mathbb{Z}^d$-congruence between E_q and $E_q - l_j$ and (2.1), (2.6) can be rewritten as

$$\left((G^*H)\left(x + \frac{j}{q\beta}, w\right)\right)_{j_1, j_2} = \sum_{l \in E_q} Z_{q\alpha} g\left(x + \frac{j_1 + k_j}{\beta} - \alpha l, w\right) \overline{Z_{q\alpha} h\left(x + \frac{j_2 + k_j}{\beta} - \alpha l, w\right)}.$$

Observe that to every $m \in \{\xi : \xi = x + y, x, y \in E_p\}$ there corresponds a unique $\lambda_m \in \{0, 1\}^d$ such that $m - p\lambda_m \in E_p$. There exist unique $\lambda_{j_1}^{(j)}, \lambda_{j_2}^{(j)} \in \{0, 1\}^d$ such that $j_1 + k_j - p\lambda_{j_1}^{(j)}, j_2 + k_j - p\lambda_{j_2}^{(j)} \in E_p$. So

$$\left((G^*H)\left(x + \frac{j}{q\beta}, w\right)\right)_{j_1, j_2} = e^{2\pi i \langle \lambda_{j_1}^{(j)} - \lambda_{j_2}^{(j)}, w\rangle} \left((G^*H)(x, w)\right)_{j_1 + k_j - p\lambda_{j_1}^{(j)}, j_2 + k_j - p\lambda_{j_2}^{(j)}} \tag{2.7}$$

for a.e. $(x, w) \in \mathbb{R}^{2d}$ by (2.1). For $w \in \mathbb{R}^d$, define $U_j(w) : \mathbb{C}^{E_p} \to \mathbb{C}^{E_p}$ by

$$\left(U_j(w)\xi\right)_n = e^{-2\pi i \langle \tilde{\lambda}_n^{(j)}, w\rangle} \xi_{n - k_j + p\tilde{\lambda}_n^{(j)}}$$

for $\xi \in \mathbb{C}^{E_p}$ and $n \in E_p$, where $\tilde{\lambda}_n^{(j)} \in \{0, 1\}^d$ is such that $n - k_j + p\tilde{\lambda}_n^{(j)} \in E_p$. Then it is easy to check that $U_j(w)$ is $\mathbb{Z}^d$-periodic and unitary for $w \in \mathbb{R}^d$. Also observe that

$$\{j_1 + k_j - p\lambda_{j_1}^{(j)} : j_1 \in E_p\} = \{j_2 + k_j - p\lambda_{j_2}^{(j)} : j_2 \in E_p\} = E_p.$$

We have $\langle (G^*H)(x + \frac{j}{q\beta}, w)\xi, \zeta\rangle = \langle U_j^*(w)(G^*H)(x, w)U_j(w)\xi, \zeta\rangle$ for $\xi, \zeta \in \mathbb{C}^{E_p}$ and a.e. $(x, w) \in \mathbb{R}^{2d}$ by (2.7). This leads to (2.5). The proof is completed. □

Theorem 2.2. *For every $j \in \mathbb{Z}^d$, there exists a $\mathbb{Z}^d$-periodic $Lp^d \times Lp^d$ unitary matrix-valued function $\mathbf{U}_j(w)$ such that*

$$(\mathbf{G}^*\mathbf{H})\left(x + \frac{j}{q\beta}, w\right) = \mathbf{U}_j^*(w)(\mathbf{G}^*\mathbf{H})(x, w)\mathbf{U}_j(w) \quad \text{for } \mathbf{g}, \mathbf{h} \in L^2(\mathbb{R}^d, \mathbb{C}^L) \text{ and a.e. } (x, w) \in \mathbb{R}^{2d}.$$

Proof. By Lemma 2.5, to each $j \in \mathbb{Z}^d$ there corresponds a $\mathbb{Z}^d$-periodic $p^d \times p^d$ unitary matrix-valued function $U_j(w)$ such that

$$(G_m^*H_{m'})\left(x + \frac{j}{q\beta}, w\right) = U_j^*(w)(G_m^*H_{m'})(x, w)U_j(w) \quad \text{for } 1 \le m, m' \le L \text{ and a.e. } (x, w) \in \mathbb{R}^{2d}.$$

Define

$$\mathbf{U}_j(w) = \begin{pmatrix} U_j(w) & & & \\ & U_j(w) & & \\ & & \ddots & \\ & & & U_j(w) \end{pmatrix}.$$

Then it is as desired. The proof is completed. □

3. Super Gabor frame characterization

This section focuses on the characterization of $\mathbf{g} \in L^2(\mathbb{R}^d, \mathbb{C}^L)$ with $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ being complete, a Bessel sequence, a frame for $L^2(\mathbb{R}^d, \mathbb{C}^L)$. For the situation $L = 1$, a similar characterization was obtained in [5,20,25] using Zibulski–Zeevi matrices. We also prove that a frame $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ is a Riesz basis for $L^2(\mathbb{R}^d, \mathbb{C}^L)$ if and only if $(\alpha\beta)^d = \frac{1}{L}$, which is well-known when $L = 1$.

Lemma 3.1. *For $s \in L^1(Q_{\frac{1}{\beta}} \times Q_1)$, we have*

$$\sum_{n,k\in\mathbb{Z}^d} \left| \int_{Q_{\frac{1}{\beta}} \times Q_1} s(x, w) e^{-2\pi i\beta\langle n,x\rangle} e^{2\pi i\langle k,w\rangle} dxdw \right|^2 = \beta^{-d} \int_{Q_{\frac{1}{\beta}} \times Q_1} |s(x, w)|^2 dxdw. \tag{3.1}$$

Proof. Write $c_{n,k} = \int_{Q_{\frac{1}{\beta}} \times Q_1} s(x, w) e^{-2\pi i\beta\langle n,x\rangle} e^{2\pi i\langle k,w\rangle} dxdw$ for $(n, k) \in \mathbb{Z}^d \times \mathbb{Z}^d$. If $s \in L^2(Q_{\frac{1}{\beta}} \times Q_1)$, (3.1) holds since $\{\beta^{\frac{d}{2}} e^{2\pi i\beta\langle n,x\rangle} e^{-2\pi i\langle k,w\rangle} : n, k \in \mathbb{Z}^d\}$ is an orthonormal basis for $L^2(Q_{\frac{1}{\beta}} \times Q_1)$. If $s \notin L^2(Q_{\frac{1}{\beta}} \times Q_1)$, then $\beta^{-d} \int_{Q_{\frac{1}{\beta}} \times Q_1} |s(x, w)|^2 dxdw = \infty$, we only need to prove $\sum_{n,k\in\mathbb{Z}^d} |c_{n,k}|^2 = \infty$ to finish the proof. We argue by contradiction. Suppose $\sum_{n,k\in\mathbb{Z}^d} |c_{n,k}|^2 < \infty$. Then

$$t(x, w) = \sum_{n,k\in\mathbb{Z}^d} c_{n,k} \beta^{\frac{d}{2}} e^{2\pi i\beta\langle n,x\rangle} e^{-2\pi i\langle k,w\rangle} \in L^2\left(Q_{\frac{1}{\beta}} \times Q_1\right)$$

due to $\{\beta^{\frac{d}{2}} e^{2\pi i\beta\langle n,x\rangle} e^{-2\pi i\langle k,w\rangle} : n, k \in \mathbb{Z}^d\}$ being an orthonormal basis for $L^2(Q_{\frac{1}{\beta}} \times Q_1)$. Observe that two functions $\beta^{-\frac{d}{2}} s(x, w)$ and $t(x, w)$ in $L^1(Q_{\frac{1}{\beta}} \times Q_1)$ have the same Fourier coefficients $c_{n,k}$, $(n, k) \in \mathbb{Z}^d \times \mathbb{Z}^d$. It follows that $\beta^{-\frac{d}{2}} s(x, w) = t(x, w)$ for a.e. $(x, w) \in Q_{\frac{1}{\beta}} \times Q_1$, and thus $s \in L^2(Q_{\frac{1}{\beta}} \times Q_1)$ due to $t \in L^2(Q_{\frac{1}{\beta}} \times Q_1)$. This is a contradiction. The proof is completed. □

By Theorem 2.1 and Lemma 3.1, we have the following lemma:

Lemma 3.2. *For $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^L)$, we have*

$$\sum_{n,k\in\mathbb{Z}^d} |\langle \mathbf{f}, M_{\beta n} T_{\alpha k} \mathbf{g}\rangle|^2 = \beta^{-d} \int_{Q_{\frac{1}{\beta}} \times Q_1} \langle (\mathbf{G}^*\mathbf{G})(x, w)\mathcal{F}(x, w), \mathcal{F}(x, w)\rangle_{\mathbb{C}^{LEp}} dxdw.$$

By Theorem 2.2 and $\mathbb{Z}^d$-periodicity of $\mathbf{G}(x, w)$ with respect to w, we have the following lemma:

Lemma 3.3. *For $\mathbf{g} \in L^2(\mathbb{R}^d, \mathbb{C}^L)$ and constants C_1, C_2, C_3, we have*
(i) *$rank(\mathbf{G}(x, w)) = C_1$ for a.e. $(x, w) \in Q_{\frac{1}{q\beta}} \times Q_1$ if and only if $rank(\mathbf{G}(x, w)) = C_1$ for a.e. $(x, w) \in \mathbb{R}^{2d}$;*
(ii) *$(\mathbf{G}^*\mathbf{G})(x, w) \geq C_2 I \ (\leq C_3 I)$ for a.e. $(x, w) \in Q_{\frac{1}{q\beta}} \times Q_1$ if and only if $(\mathbf{G}^*\mathbf{G})(x, w) \geq C_2 I \ (\leq C_3 I)$ for a.e. $(x, w) \in \mathbb{R}^{2d}$.*

Theorem 3.1. *$\mathcal{G}(\mathbf{g}, \alpha, \beta)$ is complete in $L^2(\mathbb{R}^d, \mathbb{C}^L)$ if and only if $rank(\mathbf{G}(x, w)) = Lp^d$ for a.e. $(x, w) \in Q_{\frac{1}{q\beta}} \times Q_1$.*

Proof. By Lemma 3.3, we only need to prove that $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ is complete in $L^2(\mathbb{R}^d, \mathbb{C}^L)$ if and only if

$$rank(\mathbf{G}(x, w)) = Lp^d \quad \text{for a.e. } (x, w) \in Q_{\frac{1}{\beta}} \times Q_1. \tag{3.2}$$

And by Lemma 3.2, the completeness of $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ in $L^2(\mathbb{R}^d, \mathbb{C}^L)$ is equivalent to $\mathbf{f} = 0$ being a unique solution to the equation

$$\int_{Q_{\frac{1}{\beta}} \times Q_1} \langle (\mathbf{G}^*\mathbf{G})(x, w)\mathcal{F}(x, w), \mathcal{F}(x, w)\rangle_{\mathbb{C}^{LEp}} dxdw = 0 \tag{3.3}$$

in $L^2(\mathbb{R}^d, \mathbb{C}^L)$. Since the integrand $\langle (\mathbf{G}^*\mathbf{G})(x, w)\mathcal{F}(x, w), \mathcal{F}(x, w)\rangle_{\mathbb{C}^{LEp}}$ in (3.3) is nonnegative, (3.3) holds if and only if

$$\langle (\mathbf{G}^*\mathbf{G})(x, w)\mathcal{F}(x, w), \mathcal{F}(x, w)\rangle_{\mathbb{C}^{LEp}} = 0 \quad \text{for a.e. } (x, w) \in Q_{\frac{1}{\beta}} \times Q_1. \tag{3.4}$$

Therefore, to finish the proof, we only need to prove that $\mathbf{f} = 0$ is a unique solution to the Eq. (3.4) if and only if (3.2) holds.

Suppose (3.2) holds, and $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^L)$ satisfies (3.4). By (3.4), we have $\mathbf{G}(x, w)\mathcal{F}(x, w) = \mathbf{0}$ for a.e. $(x, w) \in Q_{\frac{1}{\beta}} \times Q_1$. This together with (3.2) implies that $\mathcal{F}(x, w) = \mathbf{0}$ for a.e. $(x, w) \in Q_{\frac{1}{\beta}} \times Q_1$. It follows that $\mathbf{f} = 0$ by (iii) in Lemma 2.1.

Now suppose $\mathbf{f} = 0$ is a unique solution to the Eq. (3.4) in $L^2(\mathbb{R}^d, \mathbb{C}^L)$. Next we prove (3.2) by contradiction. Suppose $rank(\mathbf{G}(\cdot, \cdot)) < Lp^d$ on some measurable subset of $Q_{\frac{1}{\beta}} \times Q_1$ with positive measure. Let $\mathbf{P}(x, w)$ be the orthogonal projection operator of $\mathbb{C}^{LE_p}$ onto the kernel $ker(\mathbf{G}(x, w))$ of $\mathbf{G}(x, w)$ for a.e. $(x, w) \in Q_{\frac{1}{\beta}} \times Q_1$. Then $\mathbf{P}(x, w) = \lim_{n\to\infty} \exp(-n(\mathbf{G}^*\mathbf{G})(x, w))$ for a.e. $(x, w) \in Q_{\frac{1}{\beta}} \times Q_1$ by an easy application of the spectral theorem of self-adjoint matrices (see also [8, p. 978]). So $\mathbf{P}(\cdot, \cdot)$ is measurable by the measurability of $\mathbf{G}(\cdot, \cdot)$. Now we claim that there exists $\mathbf{x_0} \in \mathbb{C}^{LE_p}$ such that $\mathbf{P}(x, w)\mathbf{x}_0 \neq \mathbf{0}$ on some $E \subset Q_{\frac{1}{\beta}} \times Q_1$ with $|E| > 0$. Indeed, if for an arbitrary $\mathbf{x} \in \mathbb{C}^{LE_p}$, $\mathbf{P}(x, w)\mathbf{x} = \mathbf{0}$ for a.e. $(x, w) \in Q_{\frac{1}{\beta}} \times Q_1$, then $ker(\mathbf{G}(\cdot, \cdot)) = \{\mathbf{0}\}$ a.e. on $Q_{\frac{1}{\beta}} \times Q_1$. This implies that $rank(\mathbf{G}(\cdot, \cdot)) = Lp^d$ a.e. on $Q_{\frac{1}{\beta}} \times Q_1$, which is a contradiction. Define $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^L)$ by

$$\mathcal{F}(x, w) = \begin{cases} \mathbf{P}(x, w)\mathbf{x}_0, & \text{if } (x, w) \in E; \\ \mathbf{0}, & \text{if } (x, w) \in Q_{\frac{1}{\beta}} \times Q_1 \setminus E \end{cases}$$

for a.e. $(x, w) \in Q_{\frac{1}{\beta}} \times Q_1$. Then $\mathcal{F}(x, w)$ is measurable since $\mathbf{P}(x, w)$ does, and $\|\mathcal{F}(x, w)\|_{\mathbb{C}^{LE_p}} \leq \|\mathbf{x}_0\|_{\mathbb{C}^{LE_p}}$ for a.e. $(x, w) \in Q_{\frac{1}{\beta}} \times Q_1$, which implies that $\mathbf{f}$ is well-defined and $\mathbf{f} \neq 0$ by (iii) in Lemma 2.1. It is obvious that $\mathbf{G}(x, w)\mathcal{F}(x, w) = \mathbf{0}$ for a.e. $(x, w) \in Q_{\frac{1}{\beta}} \times Q_1$. So $\mathbf{f}$ is a nonzero solution to the Eq. (3.4) in $L^2(\mathbb{R}^d, \mathbb{C}^L)$, which contradicts the assumption that $\mathbf{f} = 0$ is a unique solution to (3.4) in $L^2(\mathbb{R}^d, \mathbb{C}^L)$. The proof is completed. □

Remark 3.1. Observe that $\mathbf{G}$ is a $q^d \times Lp^d$ matrix-valued function. By Theorem 3.1, $Lp^d \leq q^d$ if $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ is complete in $L^2(\mathbb{R}^d, \mathbb{C}^L)$, which implies that $(\alpha\beta)^d \leq \frac{1}{L}$. So $(\alpha\beta)^d \leq \frac{1}{L}$ is necessary for the existence of complete super Gabor systems in $L^2(\mathbb{R}^d, \mathbb{C}^L)$. In fact, it is sufficient for the existence of complete super Gabor systems (super Gabor frames) in $L^2(\mathbb{R}^d, \mathbb{C}^L)$, which was proved in [22, Theorem 1.1].

Theorem 3.2. *$\mathcal{G}(\mathbf{g}, \alpha, \beta)$ is a Bessel sequence in $L^2(\mathbb{R}^d, \mathbb{C}^L)$ with Bessel bound B if and only if*

$$(\mathbf{G}^*\mathbf{G})(x, w) \leq \beta^d BI \quad \textit{for a.e. } (x, w) \in Q_{\frac{1}{q\beta}} \times Q_1.$$

Proof. By Lemma 3.3, we only need to prove that $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ is a Bessel sequence in $L^2(\mathbb{R}^d, \mathbb{C}^L)$ with Bessel bound B if and only if

$$(\mathbf{G}^*\mathbf{G})(x, w) \leq \beta^d BI \quad \text{for a.e. } (x, w) \in Q_{\frac{1}{\beta}} \times Q_1. \tag{3.5}$$

By Lemmas 2.3 and 3.2, $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ is a Bessel sequence in $L^2(\mathbb{R}^d, \mathbb{C}^L)$ with Bessel bound B if and only if

$$\int_{Q_{\frac{1}{\beta}} \times Q_1} \langle (\mathbf{G}^*\mathbf{G})(x, w)\mathcal{F}(x, w), \mathcal{F}(x, w)\rangle_{\mathbb{C}^{LE_p}} dxdw \leq \beta^d B \int_{Q_{\frac{1}{\beta}} \times Q_1} \|\mathcal{F}(x, w)\|^2_{\mathbb{C}^{LE_p}} dxdw \tag{3.6}$$

for $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^L)$. It is obvious (3.5) implies (3.6). To finish the proof, we only need to prove (3.6) implies (3.5). We argue by contradiction. Suppose (3.6) holds, while (3.5) fails to hold. Then there exists $E \subset Q_{\frac{1}{\beta}} \times Q_1$ with $|E| > 0$ such that $(\mathbf{G}^*\mathbf{G})(x, w) > \beta^d BI$ for $(x, w) \in E$. Define $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^L)$ by $f_2 = f_3 = \cdots = f_L = 0$ and $\mathcal{Z}_{q\alpha} f_1(x, w) = \begin{pmatrix} \chi_E(x, w) \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ for a.e. $(x, w) \in Q_{\frac{1}{\beta}} \times Q_1$. Then $\mathbf{f}$ is well-defined and $\mathbf{f} \neq \mathbf{0}$ by (iii) in Lemma 2.1, and

$$\int_{Q_{\frac{1}{\beta}} \times Q_1} \langle (\mathbf{G}^*\mathbf{G})(x, w)\mathcal{F}(x, w), \mathcal{F}(x, w)\rangle_{\mathbb{C}^{LE_p}} dxdw > \beta^d B \int_{Q_{\frac{1}{\beta}} \times Q_1} \langle \mathcal{F}(x, w), \mathcal{F}(x, w)\rangle_{\mathbb{C}^{LE_p}} dxdw,$$

which is a contradiction to (3.6). The proof is completed. □

Remark 3.2. We denote by $\|\mathbf{G}(x, w)\|_{\mathbb{C}^{LE_p} \to \mathbb{C}^{E_q}}$ the norm of $\mathbf{G}(x, w)$ as an operator from $\mathbb{C}^{LE_p}$ to $\mathbb{C}^{E_q}$ for a.e. $(x, w) \in Q_{\frac{1}{q\beta}} \times Q_1$. Theorem 3.2 implies the equivalence between $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ being a Bessel sequence in $L^2(\mathbb{R}^d, \mathbb{C}^L)$ and $\|\mathbf{G}(x, w)\|_{\mathbb{C}^{LE_p} \to \mathbb{C}^{E_q}} \in L^\infty(Q_{\frac{1}{q\beta}} \times Q_1)$. Observe that, for the linear space consisting of all complex $q^d \times Lp^d$ matrices, the maximum of moduli of all entries of a matrix also define a norm, which is of course equivalent to the norm when a matrix is viewed as an operator from $\mathbb{C}^{LE_p}$ to $\mathbb{C}^{E_q}$. Therefore, $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ is a Bessel sequence in $L^2(\mathbb{R}^d, \mathbb{C}^L)$ if and only if all entries of $\mathbf{G}(x, w)$ belong to $L^\infty(Q_{\frac{1}{q\beta}} \times Q_1)$.

Theorem 3.3. *$\mathcal{G}(\mathbf{g}, \alpha, \beta)$ is a frame for $L^2(\mathbb{R}^d, \mathbb{C}^L)$ with frame bounds A and B if and only if*

$$\beta^d AI \le (\mathbf{G}^*\mathbf{G})(x, w) \le \beta^d BI \quad \textit{for a.e. } (x, w) \in Q_{\frac{1}{q\beta}} \times Q_1.$$

Proof. By Theorem 3.2, we may as well assume that $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ is a Bessel sequence in $L^2(\mathbb{R}^d, \mathbb{C}^L)$ with Bessel bound B. Similarly to the beginning proof of Theorem 3.2, to finish the proof, we only need to prove the equivalence between

$$\beta^d AI \le (\mathbf{G}^*\mathbf{G})(x, w) \quad \text{for a.e. } (x, w) \in Q_{\frac{1}{\beta}} \times Q_1 \tag{3.7}$$

and

$$\beta^d A \int_{Q_{\frac{1}{\beta}} \times Q_1} \|\mathcal{F}(x, w)\|^2_{\mathbb{C}^{LE_p}} dxdw \le \int_{Q_{\frac{1}{\beta}} \times Q_1} \langle (\mathbf{G}^*\mathbf{G})(x, w)\mathcal{F}(x, w), \mathcal{F}(x, w)\rangle_{\mathbb{C}^{LE_p}} dxdw \tag{3.8}$$

for $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^L)$. It is obvious that (3.7) implies (3.8). Next we prove the converse implication by contradiction. Suppose (3.8) holds, and there exists $E \subset Q_{\frac{1}{\beta}} \times Q_1$ with $|E| > 0$ such that $(\mathbf{G}^*\mathbf{G})(x, w) < \beta^d AI$ for $(x, w) \in E$. Define $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^L)$ by $f_2 = f_3 = \cdots = f_L = 0$ and $\mathcal{Z}_{q\alpha} f_1(x, w) = \begin{pmatrix} \chi_E(x, w) \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ for a.e. $(x, w) \in Q_{\frac{1}{\beta}} \times Q_1$. Similarly to Theorem 3.2, we can prove that (3.8) fails to hold for such $\mathbf{f}$. The proof is completed. □

Remark 3.3. Note that $\beta^d AI \le (\mathbf{G}^*\mathbf{G})(x, w) \le \beta^d BI$ for a.e. $(x, w) \in Q_{\frac{1}{q\beta}} \times Q_1$ if and only if

$$(\mathbf{G}^*\mathbf{G}(x, w))^{-1} \le \beta^{-d} A^{-1} I \quad \text{and} \quad (\mathbf{G}^*\mathbf{G})(x, w) \le \beta^d BI \text{ for a.e. } (x, w) \in Q_{\frac{1}{q\beta}} \times Q_1.$$

By an argument similar to Remark 3.2, $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ is a frame for $L^2(\mathbb{R}^d, \mathbb{C}^L)$ if and only if all entries of $(\mathbf{G}^*\mathbf{G}(x, w))^{-1}$ and $(\mathbf{G}^*\mathbf{G})(x, w)$ (or $\mathbf{G}(x, w)$) belong to $L^\infty(Q_{\frac{1}{q\beta}} \times Q_1)$.

Next we turn to the characterization of super Gabor frames being Riesz bases for $L^2(\mathbb{R}^d, \mathbb{C}^L)$. For this purpose, we introduce two lemmas:

Lemma 3.4. *Let $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ be complete in $L^2(\mathbb{R}^d, \mathbb{C}^L)$. Then rank$(\mathbf{G}(x, w)) = q^d$ for a.e. $(x, w) \in Q_{\frac{1}{q\beta}} \times Q_1$ if and only if $(\alpha\beta)^d = \frac{1}{L}$.*

Proof. Since $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ is complete in $L^2(\mathbb{R}^d, \mathbb{C}^L)$, we have $rank(\mathbf{G}(x, w)) = Lp^d$ for a.e. $(x, w) \in Q_{\frac{1}{q\beta}} \times Q_1$ by Theorem 3.1. So, if $rank(\mathbf{G}(x, w)) = q^d$ for a.e. $(x, w) \in Q_{\frac{1}{q\beta}} \times Q_1$, we have $Lp^d = q^d$, which implies that $(\alpha\beta)^d = \frac{1}{L}$ due to the fact that $\alpha\beta = \frac{p}{q}$; and if $(\alpha\beta)^d = \frac{1}{L}$, then $rank(\mathbf{G}(x, w)) = \frac{p^d}{(\alpha\beta)^d} = q^d$ for a.e. $(x, w) \in Q_{\frac{1}{q\beta}} \times Q_1$. The proof is completed. □

Lemma 3.5. *Let $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ be a Bessel sequence in $L^2(\mathbb{R}^d, \mathbb{C}^L)$. Define $\mathcal{T} : l^2(\mathbb{Z}^d \times \mathbb{Z}^d) \to L^2(\mathbb{R}^d, \mathbb{C}^L)$ by $\mathcal{T}(\{c_{k,n}\}_{k,n\in\mathbb{Z}^d}) = \sum_{n,k\in\mathbb{Z}^d} c_{k,n} M_{\beta n} T_{\alpha k}\mathbf{g}$ for $\{c_{k,n}\}_{k,n\in\mathbb{Z}^d} \in l^2(\mathbb{Z}^d \times \mathbb{Z}^d)$, and denote by $ker(\mathcal{T})$ the kernel of $\mathcal{T}$. Then*

$$ker(\mathcal{T}) = \{\{c_{k,n}\}_{k,n\in\mathbb{Z}^d} \in l^2(\mathbb{Z}^d \times \mathbb{Z}^d) : \mathbf{G}^*(x, w)\mathcal{V}_c(x, w) = \mathbf{0} \textit{ for a.e. } (x, w) \in Q_{\frac{1}{\beta}} \times Q_1\},$$

where $\mathcal{V}_c(x, w) = \left(\sum_{n,k\in\mathbb{Z}^d} c_{qk+l,n} e^{2\pi i\beta\langle n,x\rangle} e^{-2\pi i\langle k,w\rangle}\right)_{l\in E_q}$.

Proof. Since $\{c_{k,n}\}_{k,n\in\mathbb{Z}^d} \in ker(\mathcal{T})$ if and only if $\sum_{l\in E_q}\sum_{k,n\in\mathbb{Z}^d} \bar{c}_{qk+l,n}\langle \mathbf{f}, M_{\beta n} T_{\alpha(qk+l)}\mathbf{g}\rangle = 0$ for $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^L)$, which is equivalent to

$$\sum_{l\in E_q}\sum_{k,n\in\mathbb{Z}^d} \bar{c}_{qk+l,n} \int_{Q_{\frac{1}{\beta}} \times Q_1} (\mathbf{G}(x, w)\mathcal{F}(x, w))_l\, e^{-2\pi i\beta\langle n,x\rangle} e^{2\pi i\langle k,w\rangle} dxdw = 0 \tag{3.9}$$

for $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^L)$ by Theorem 2.1. Suppose $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ is a Bessel sequence in $L^2(\mathbb{R}^d, \mathbb{C}^L)$ with Bessel bound B. Then

$$\|\mathbf{G}(x, w)\mathcal{F}(x, w)\|^2_{\mathbb{C}^{E_q}} = \langle (\mathbf{G}^*\mathbf{G})(x, w)\mathcal{F}(x, w), \mathcal{F}(x, w)\rangle_{\mathbb{C}^{LE_p}} \le \beta^d B\|\mathcal{F}(x, w)\|^2_{\mathbb{C}^{LE_p}}$$

for $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^L)$ and a.e. $(x, w) \in Q_{\frac{1}{\beta}} \times Q_1$ by Theorem 3.2 and Lemma 3.3, which implies that $(\mathbf{G}(x, w)\mathcal{F}(x, w))_l \in L^2(Q_{\frac{1}{\beta}} \times Q_1)$ for $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^L)$ and each $l \in E_q$. It follows that

$$\sum_{k,n\in\mathbb{Z}^d} \bar{c}_{qk+l,n} \int_{Q_{\frac{1}{\beta}} \times Q_1} (\mathbf{G}(x, w)\mathcal{F}(x, w))_l\, e^{-2\pi i\beta\langle n,x\rangle} e^{2\pi i\langle k,w\rangle} dxdw$$
$$= \int_{Q_{\frac{1}{\beta}} \times Q_1} (\mathbf{G}(x, w)\mathcal{F}(x, w))_l \sum_{k,n\in\mathbb{Z}^d} \bar{c}_{qk+l,n} e^{-2\pi i\beta\langle n,x\rangle} e^{2\pi i\langle k,w\rangle} dxdw$$

for each $l \in E_q$. So (3.9) can be rewritten as $\int_{Q_{\frac{1}{\beta}} \times Q_1} \langle \mathbf{G}(x, w)\mathcal{F}(x, w), \mathcal{V}_c(x, w)\rangle_{\mathbb{C}^{E_q}} dxdw = 0$, equivalently,

$$\int_{Q_{\frac{1}{\beta}} \times Q_1} \langle \mathcal{F}(x, w), \mathbf{G}^*(x, w)\mathcal{V}_c(x, w)\rangle_{\mathbb{C}^{LE_p}} dxdw = 0 \tag{3.10}$$

for $\mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^L)$. By (iii) in Lemma 2.1, when $\mathbf{f}$ runs over $L^2(\mathbb{R}^d, \mathbb{C}^L)$, $\mathcal{F}(x, w)$ runs over the orthogonal direct sum $\mathcal{H}_p \bigoplus \mathcal{H}_p \bigoplus \cdots \bigoplus \mathcal{H}_p$ with multiplicity L. So (3.10) is equivalent to

$$\mathbf{G}^*(x, w)\mathcal{V}_c(x, w) = \mathbf{0} \quad \text{for a.e. } (x, w) \in Q_{\frac{1}{\beta}} \times Q_1.$$

Therefore, $\{c_{k,n}\}_{k,n\in\mathbb{Z}^d} \in ker(\mathcal{T})$ if and only if $\mathbf{G}^*(x, w)\mathcal{V}_c(x, w) = \mathbf{0}$ for a.e. $(x, w) \in Q_{\frac{1}{\beta}} \times Q_1$. The proof is completed. □

Theorem 3.4. *Let $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ be a frame for $L^2(\mathbb{R}^d, \mathbb{C}^L)$. Then $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ is a Riesz basis for $L^2(\mathbb{R}^d, \mathbb{C}^L)$ if and only if $(\alpha\beta)^d = \frac{1}{L}$.*

Proof. Since $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ is a frame for $L^2(\mathbb{R}^d, \mathbb{C}^L)$, it is a complete Bessel sequence in $L^2(\mathbb{R}^d, \mathbb{C}^L)$. So Lemmas 3.4 and 3.5 both work for $\mathcal{G}(\mathbf{g}, \alpha, \beta)$. Also observe that the frame $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ is a Riesz basis if and only if $ker(\mathcal{T}) = \{0\}$. So, by Lemmas 3.4 and 3.5, to prove the theorem, we only need to prove that $\{c_{k,n}\}_{k,n\in\mathbb{Z}^d} = 0$ is a unique solution to the equation

$$\mathbf{G}^*(x, w)\mathcal{V}_c(x, w) = \mathbf{0} \quad \text{for a.e. } (x, w) \in Q_{\frac{1}{\beta}} \times Q_1 \tag{3.11}$$

in $l^2(\mathbb{Z}^d \times \mathbb{Z}^d)$ if and only if $rank(\mathbf{G}(x, w)) = q^d$ for a.e. $(x, w) \in Q_{\frac{1}{q\beta}} \times Q_1$, equivalently,

$$rank(\mathbf{G}^*(x, w)) = q^d \quad \text{for a.e. } (x, w) \in Q_{\frac{1}{\beta}} \times Q_1 \tag{3.12}$$

by Lemma 3.3 and the fact that $rank(\mathbf{G}^*(x, w)) = rank(\mathbf{G}(x, w))$. It is obvious that (3.12) implies that $\{c_{k,n}\}_{k,n\in\mathbb{Z}^d} = 0$ is a unique solution to (3.11) in $l^2(\mathbb{Z}^d \times \mathbb{Z}^d)$. Next we prove the converse implication by contradiction. Suppose $\{c_{k,n}\}_{k,n\in\mathbb{Z}^d} = 0$ is a unique solution to (3.11) in $l^2(\mathbb{Z}^d \times \mathbb{Z}^d)$, and $rank(\mathbf{G}^*(x, w)) < q^d$ (equivalent to $rank(\mathbf{G}(x, w)) < q^d$) on some measurable subset of $Q_{\frac{1}{\beta}} \times Q_1$ with positive measure. Let $\mathbf{Q}(x, w)$ be the orthogonal projection operator of $\mathbb{C}^{E_q}$ onto the kernel $ker(\mathbf{G}^*(x, w))$ of $\mathbf{G}^*(x, w)$ for a.e. $(x, w) \in Q_{\frac{1}{\beta}} \times Q_1$. Then, by the same procedure as in Theorem 3.1, there exists $\mathbf{y}_0 \in \mathbb{C}^{E_q}$ such that $\mathbf{Q}(x, w)\mathbf{y}_0 \neq \mathbf{0}$ on some $E_0 \subset Q_{\frac{1}{\beta}} \times Q_1$ with $|E_0| > 0$, and that $\{c_{k,n}\}_{k,n\in\mathbb{Z}^d} \in l^2(\mathbb{Z}^d \times \mathbb{Z}^d)$ defined by

$$\mathcal{V}_c(x, w) = \begin{cases} \mathbf{Q}(x, w)\mathbf{y}_0, & \text{if } (x, w) \in E_0; \\ \mathbf{0}, & \text{if } (x, w) \in Q_{\frac{1}{\beta}} \times Q_1 \setminus E_0 \end{cases} \quad \text{for } (x, w) \in Q_{\frac{1}{\beta}} \times Q_1$$

is a nonzero solution to (3.11) in $l^2(\mathbb{Z}^d \times \mathbb{Z}^d)$. This is a contradiction. The proof is completed. □

4. Super Gabor dual

Let $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ be a frame for $L^2(\mathbb{R}^d, \mathbb{C}^L)$. In this section, we investigate the super Gabor duals of $\mathbf{g}$. For the situation $L = 1$, a Zibulski–Zeevi matrix characterization of them can be found in [5,21,20,26,25]. By Theorems 2.1 and 3.2 and Lemmas 2.1, 2.3 and 3.3, we can easily obtain the following lemma:

Lemma 4.1. *Let $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ and $\mathcal{G}(\mathbf{h}, \alpha, \beta)$ be both Bessel sequences in $L^2(\mathbb{R}^d, \mathbb{C}^L)$. Then*

$$\begin{pmatrix} \mathcal{Z}_{q\alpha}(\mathcal{S}_{\mathbf{h},\mathbf{g}}\mathbf{f})_1(x, w) \\ \mathcal{Z}_{q\alpha}(\mathcal{S}_{\mathbf{h},\mathbf{g}}\mathbf{f})_2(x, w) \\ \vdots \\ \mathcal{Z}_{q\alpha}(\mathcal{S}_{\mathbf{h},\mathbf{g}}\mathbf{f})_L(x, w) \end{pmatrix} = \beta^{-d}(\mathbf{G}^*\mathbf{H})(x, w)\mathcal{F}(x, w) \quad \textit{for } \mathbf{f} \in L^2(\mathbb{R}^d, \mathbb{C}^L) \textit{ and a.e. } (x, w) \in \mathbb{R}^{2d}.$$

Theorem 4.1. *Let $\mathcal{G}(\mathbf{g},\alpha,\beta)$ and $\mathcal{G}(\mathbf{h},\alpha,\beta)$ be both Bessel sequences in $L^2(\mathbb{R}^d,\mathbb{C}^L)$. Then $\mathcal{S}_{\mathbf{h},\mathbf{g}}=I$ if and only if $(\mathbf{G}^*\mathbf{H})(x,w)=\beta^dI$ a.e. on $Q_{\frac{1}{q\beta}}\times Q_1$.*

Proof. By Theorem 2.2, $(\mathbf{G}^*\mathbf{H})(x,w)=\beta^dI$ for a.e. $(x,w)\in Q_{\frac{1}{q\beta}}\times Q_1$ if and only if

$$(\mathbf{G}^*\mathbf{H})(x,w)=\beta^dI \quad \text{for a.e. } (x,w)\in Q_{\frac{1}{\beta}}\times Q_1. \tag{4.1}$$

By Lemma 4.1 and (iii) in Lemma 2.1, $\mathcal{S}_{\mathbf{h},\mathbf{g}}\mathbf{f}=\mathbf{f}$ for $\mathbf{f}\in L^2(\mathbb{R}^d,\mathbb{C}^L)$ if and only if

$$(\mathbf{G}^*\mathbf{H})(x,w)\mathcal{F}(x,w)=\beta^d\mathcal{F}(x,w) \quad \text{for } \mathbf{f}\in L^2(\mathbb{R}^d,\mathbb{C}^L) \text{ and a.e. } (x,w)\in Q_{\frac{1}{\beta}}\times Q_1. \tag{4.2}$$

So, to finish the proof, we only need to prove the equivalence between (4.1) and (4.2). It is obvious that (4.1) implies (4.2). Next we prove the converse implication. Suppose (4.2) holds, and $\mathbf{e}$ is an arbitrary vector in $\mathbb{C}^{LE_p}$ with only one component being 1 and the others being 0. Define $\mathbf{f}\in L^2(\mathbb{R}^d,\mathbb{C}^L)$ by $\mathcal{F}(x,w)=\mathbf{e}$ for $(x,w)\in Q_{\frac{1}{\beta}}\times Q_1$. Then $\mathbf{f}$ is well-defined by (iii) in Lemma 2.1. Applying (4.2) to all such $\mathbf{f}$, we obtain (4.1). The proof is completed. □

Lemma 4.2. *Let $\mathcal{G}(\mathbf{g},\alpha,\beta)$ and $\mathcal{G}(\mathbf{h},\alpha,\beta)$ be both Bessel sequences in $L^2(\mathbb{R}^d,\mathbb{C}^L)$, and $\gamma\in L^2(\mathbb{R}^d,\mathbb{C}^L)$. Then there exists $\mathbf{f}\in L^2(\mathbb{R}^d,\mathbb{C}^L)$ such that $\gamma=S_{\mathbf{h},\mathbf{g}}\mathbf{f}$ if and only if*

$$\boldsymbol{\Gamma}(x,w)=\beta^{-d}\mathbf{F}(x,w)\mathbf{H}^*(x,w)\mathbf{G}(x,w) \quad \textit{for a.e. } (x,w)\in Q_{\frac{1}{q\beta}}\times Q_1.$$

Proof. By Lemmas 2.2 and 4.1 and (iii) in Lemma 2.1, $\gamma=S_{\mathbf{h},\mathbf{g}}\mathbf{f}$ if and only if

$$\begin{pmatrix} \mathcal{Z}_{q\alpha}\gamma_1(x-\alpha l,w)\\ \mathcal{Z}_{q\alpha}\gamma_2(x-\alpha l,w)\\ \vdots\\ \mathcal{Z}_{q\alpha}\gamma_L(x-\alpha l,w)\end{pmatrix}=\beta^{-d}(\mathbf{G}^*\mathbf{H})(x-\alpha l,w)\mathcal{F}(x-\alpha l,w) \tag{4.3}$$

for $l\in E_q$ and a.e. $(x,w)\in Q_{\frac{1}{q\beta}}\times Q_1$. Observe that

$$-pl=q\begin{pmatrix}0\\0\\ \vdots\\0\end{pmatrix}+p\left[q\begin{pmatrix}1\\1\\ \vdots\\1\end{pmatrix}-l\right]+pq\begin{pmatrix}-1\\-1\\ \vdots\\-1\end{pmatrix}$$

for each $l\in E_q$, which is the unique decomposition as in Lemma 2.4. By an argument similar to the proof of (2.7), we have $(G_m^*H_{m'})(x-\alpha l,w)=(G_m^*H_{m'})(x-\frac{pl}{q\beta},w)=(G_m^*H_{m'})(x,w)$ for $1\le m,m'\le L$, $l\in E_q$ and a.e. $(x,w)\in\mathbb{R}^{2d}$. So (4.3) can be rewritten as

$$\begin{pmatrix} \mathcal{Z}_{q\alpha}\gamma_1(x-\alpha l,w)\\ \mathcal{Z}_{q\alpha}\gamma_2(x-\alpha l,w)\\ \vdots\\ \mathcal{Z}_{q\alpha}\gamma_L(x-\alpha l,w)\end{pmatrix}=\beta^{-d}(\mathbf{G}^*\mathbf{H})(x,w)\mathcal{F}(x-\alpha l,w) \quad \text{for } l\in E_q \text{ and a.e. } (x,w)\in Q_{\frac{1}{q\beta}}\times Q_1.$$

This is equivalent to $\boldsymbol{\Gamma}^*(x,w)=\beta^{-d}(\mathbf{G}^*\mathbf{H})(x,w)\mathbf{F}^*(x,w)$ for a.e. $(x,w)\in Q_{\frac{1}{q\beta}}\times Q_1$ by the definition of $\boldsymbol{\Gamma}$ and $\mathbf{F}$, and thus $\gamma=S_{\mathbf{h},\mathbf{g}}\mathbf{f}$ if and only if $\boldsymbol{\Gamma}(x,w)=\beta^{-d}\mathbf{F}(x,w)\mathbf{H}^*(x,w)\mathbf{G}(x,w)$ for a.e. $(x,w)\in Q_{\frac{1}{q\beta}}\times Q_1$. The proof is completed. □

Lemma 4.3. *Let $\mathcal{G}(\mathbf{g},\alpha,\beta)$ be a frame for $L^2(\mathbb{R}^d,\mathbb{C}^L)$. Then, for an arbitrary $\mathbf{h}\in L^2(\mathbb{R}^d,\mathbb{C}^L)$, $\mathcal{G}(\mathbf{h},\alpha,\beta)$ is a Bessel sequence in $L^2(\mathbb{R}^d,\mathbb{C}^L)$ if and only if there exists a $q^d\times q^d$ matrix-valued function $\mathcal{A}(x,w)$ defined on $Q_{\frac{1}{q\beta}}\times Q_1$ with each entry being in $L^\infty(Q_{\frac{1}{q\beta}}\times Q_1)$ such that*

$$\mathbf{H}(x,w)=\mathcal{A}(x,w)\mathbf{G}(x,w) \quad \textit{for a.e. } (x,w)\in Q_{\frac{1}{q\beta}}\times Q_1.$$

Proof. The sufficiency holds by Remark 3.2. Next we turn to the necessity. Define $\mathbf{f}=\mathcal{S}_{\mathbf{g},\mathbf{g}}^{-1}\mathbf{h}$. Then $\mathbf{h}=\mathcal{S}_{\mathbf{g},\mathbf{g}}\mathbf{f}$, and thus $\mathbf{H}(x,w)=\mathcal{A}(x,w)\mathbf{G}(x,w)$ for a.e. $(x,w)\in Q_{\frac{1}{q\beta}}\times Q_1$ by Lemma 4.2, where $\mathcal{A}(x,w)=\beta^{-d}\mathbf{F}(x,w)\mathbf{G}^*(x,w)$. Since $\mathcal{G}(\mathbf{g},\alpha,\beta)$ is a frame for $L^2(\mathbb{R}^d,\mathbb{C}^L)$, $S_{\mathbf{g},\mathbf{g}}$ is a bounded and invertible operator. This implies that $\mathcal{G}(\mathbf{f},\alpha,\beta)=S_{\mathbf{g},\mathbf{g}}^{-1}\mathcal{G}(\mathbf{h},\alpha,\beta)$ is a Bessel sequence in $L^2(\mathbb{R}^d,\mathbb{C}^L)$ since $\mathcal{G}(\mathbf{h},\alpha,\beta)$ is. By Remark 3.2, all entries of $\mathbf{F}(x,w)$ and $\mathbf{G}(x,w)$ belong to $L^\infty(Q_{\frac{1}{q\beta}}\times Q_1)$. So all entries of $\mathcal{A}(x,w)$ do. The proof is completed. □

Theorem 4.2. *Let $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ be a frame for $L^2(\mathbb{R}^d, \mathbb{C}^L)$. Define $\gamma \in L^2(\mathbb{R}^d, \mathbb{C}^L)$ by*

$$\boldsymbol{\Gamma}(x, w) = \beta^d \mathbf{G}(x, w)((\mathbf{G}^*\mathbf{G})(x, w))^{-1}$$

for a.e. $(x, w) \in Q_{\frac{1}{q\beta}} \times Q_1$. Then γ is the canonical dual of $\mathbf{g}$.

Proof. For an arbitrary $\mathbf{h} \in L^2(\mathbb{R}^d, \mathbb{C}^L)$, $\mathbf{h}$ is the canonical dual of $\mathbf{g}$ if and only if $\mathbf{g} = S_{\mathbf{g},\mathbf{g}}\mathbf{h}$, which is equivalent to

$$\mathbf{G}(x, w) = \beta^{-d}\mathbf{H}(x, w)(\mathbf{G}^*\mathbf{G})(x, w) \quad \text{for a.e. } (x, w) \in Q_{\frac{1}{q\beta}} \times Q_1 \tag{4.4}$$

by Lemma 4.2. Since $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ is a frame for $L^2(\mathbb{R}^d, \mathbb{C}^L)$, $(\mathbf{G}^*\mathbf{G})(x, w)$ is bounded and invertible by Theorem 3.3. So (4.4) can be rewritten as $\mathbf{H}(x, w) = \beta^d\mathbf{G}(x, w)((\mathbf{G}^*\mathbf{G})(x, w))^{-1} = \boldsymbol{\Gamma}(x, w)$ for a.e. $(x, w) \in Q_{\frac{1}{q\beta}} \times Q_1$, which is equivalent to $\mathbf{h} = \gamma$. So γ is the canonical dual of $\mathbf{g}$. The proof is completed. □

Theorem 4.3. *Let $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ be a frame for $L^2(\mathbb{R}^d, \mathbb{C}^L)$, and let γ^0 be the canonical dual of $\mathbf{g}$. Then $\|\gamma^0\| \le \|\gamma\|$ for an arbitrary Gabor dual γ of $\mathbf{g}$, with equality if and only if $\gamma = \gamma^0$.*

Proof. Suppose γ is a Gabor dual of $\mathbf{g}$. Then $S_{\gamma-\gamma^0,\mathbf{g}} = 0$. It follows that

$$\begin{aligned} 0 &= \sum_{k,n\in\mathbb{Z}^d} \langle \gamma^0, M_{\beta n}T_{\alpha k}(\gamma - \gamma^0)\rangle\langle M_{\beta n}T_{\alpha k}\mathbf{g}, \gamma^0\rangle \\ &= \sum_{k,n\in\mathbb{Z}^d} \langle M_{\beta n}T_{\alpha k}\gamma^0, \gamma - \gamma^0\rangle\langle \mathbf{g}, M_{\beta n}T_{\alpha k}\gamma^0\rangle \end{aligned}$$

by (1.4), i.e.,

$$0 = \left\langle \sum_{k,n\in\mathbb{Z}^d} \langle \mathbf{g}, M_{\beta n}T_{\alpha k}S_{\mathbf{g},\mathbf{g}}^{-1}\mathbf{g}\rangle M_{\beta n}T_{\alpha k}\gamma^0, \gamma - \gamma^0 \right\rangle. \tag{4.5}$$

Since $M_{\beta n}T_{\alpha k}S_{\mathbf{g},\mathbf{g}}^{-1} = S_{\mathbf{g},\mathbf{g}}^{-1}M_{\beta n}T_{\alpha k}$ for $n, k \in \mathbb{Z}^d$, and $S_{\mathbf{g},\mathbf{g}}^{-1}$ is self-adjoint, (4.5) can be rewritten as

$$0 = \left\langle \sum_{k,n\in\mathbb{Z}^d} \langle S_{\mathbf{g},\mathbf{g}}^{-1}\mathbf{g}, M_{\beta n}T_{\alpha k}\mathbf{g}\rangle M_{\beta n}T_{\alpha k}\gamma^0, \gamma - \gamma^0 \right\rangle = \langle S_{\mathbf{g},\mathbf{g}}^{-1}\mathbf{g}, \gamma - \gamma^0\rangle = \langle \gamma^0, \gamma - \gamma^0\rangle.$$

So $\|\gamma\|^2 = \|(\gamma - \gamma^0) + \gamma^0\|^2 = \|\gamma - \gamma^0\|^2 + \|\gamma^0\|^2$. The theorem therefore follows. □

Theorem 4.4. *Let $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ be a frame for $L^2(\mathbb{R}^d, \mathbb{C}^L)$. Then $\mathbf{g}$ has a unique super Gabor dual if and only if $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ is a Riesz basis for $L^2(\mathbb{R}^d, \mathbb{C}^L)$.*

Proof. The sufficiency is trivial. Next we prove the necessity. Suppose $\mathbf{g}$ has a unique super Gabor dual. Then we claim that $\mathbf{h} = 0$ is a unique solution to the equation

$$(\mathbf{G}^*\mathbf{H})(x, w) = \mathbf{0} \quad \text{for a.e. } (x, w) \in Q_{\frac{1}{q\beta}} \times Q_1 \tag{4.6}$$

with $\mathcal{G}(\mathbf{h}, \alpha, \beta)$ being a Bessel sequence in $L^2(\mathbb{R}^d, \mathbb{C}^L)$. Indeed, if $\mathbf{h}$ is a nonzero solution to (4.6). Then $\mathbf{G}^*(x, w)(\boldsymbol{\Gamma}(x, w) + \mathbf{H}(x, w)) = \beta^d I$ for a.e. $(x, w) \in Q_{\frac{1}{q\beta}} \times Q_1$ by Theorem 4.1, where γ is the canonical dual of $\mathbf{g}$. This implies that $\gamma + \mathbf{h}$ is another Gabor dual of $\mathbf{g}$ by Theorem 4.1, contradicting the fact that $\mathbf{g}$ has a unique super Gabor dual. By Lemma 3.4 and Theorem 3.4, $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ is a Riesz basis for $L^2(\mathbb{R}^d, \mathbb{C}^L)$ if and only if $rank(\mathbf{G}(x, w)) = q^d$ for a.e. $(x, w) \in Q_{\frac{1}{q\beta}} \times Q_1$, equivalently,

$$rank(\mathbf{G}^*(x, w)) = q^d \quad \text{for a.e. } (x, w) \in Q_{\frac{1}{q\beta}} \times Q_1. \tag{4.7}$$

Therefore, to finish the proof, we only need to prove that $\mathbf{h} = 0$ being a unique solution to (4.6) implies (4.7). We argue by contradiction. Suppose $rank(\mathbf{G}^*(x, w)) < q^d$ on some measurable subset in $Q_{\frac{1}{q\beta}} \times Q_1$ with positive measure. Let $\mathbf{Q}(x, w)$ be the orthogonal projection of $\mathbb{C}^{E_q}$ onto the kernel $ker(\mathbf{G}^*(x, w))$ of $\mathbf{G}^*(x, w)$ for a.e. $(x, w) \in Q_{\frac{1}{q\beta}} \times Q_1$. Then by the same procedure as in Theorem 3.1, $\mathbf{Q}(x, w)$ is measurable, and there exists $\mathbf{y}_0 \in \mathbb{C}^{E_q}$ such that $\mathbf{Q}(x, w)\mathbf{y}_0 \ne \mathbf{0}$ on some $E \subset Q_{\frac{1}{q\beta}} \times Q_1$ with $|E| > 0$. Take $\mathbf{H}(x, w)$ as a $q^d \times Lp^d$ matrix-valued function defined on $Q_{\frac{1}{q\beta}} \times Q_1$ with one column being $\chi_E(x, w)\mathbf{Q}(x, w)\mathbf{y}_0$ and the others being zero. Then the entries of $\mathbf{H}(x, w)$ are in $L^\infty(Q_{\frac{1}{q\beta}} \times Q_1)$, $\mathbf{H}(x, w)$ corresponds to a nonzero $\mathbf{h} \in L^2(\mathbb{R}^d, \mathbb{C}^L)$, and thus $\mathcal{G}(\mathbf{h}, \alpha, \beta)$ is a Bessel sequence in $L^2(\mathbb{R}^d, \mathbb{C}^L)$ by Remark 3.2. It is obvious that $(\mathbf{G}^*\mathbf{H})(x, w) = \mathbf{0}$ for a.e. $(x, w) \in Q_{\frac{1}{q\beta}} \times Q_1$. So $\mathbf{h}$ is a nonzero solution to (4.6), which is a contradiction. The proof is completed. □

Theorem 4.5. *Let $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ be a frame for $L^2(\mathbb{R}^d, \mathbb{C}^L)$. Then, for an arbitrary $\mathbf{h} \in L^2(\mathbb{R}^d, \mathbb{C}^L)$, $\mathbf{h}$ is a super Gabor dual of $\mathbf{g}$ if and only if there exists a $q^d \times q^d$ matrix-valued function $\mathcal{A}(x, w)$ defined on $Q_{\frac{1}{q\beta}} \times Q_1$ with entries being in $L^\infty(Q_{\frac{1}{q\beta}} \times Q_1)$ such that*

$$\mathbf{H}(x, w) = \beta^d \mathbf{G}(x, w)((\mathbf{G}^*\mathbf{G})(x, w))^{-1}(I - \beta^{-d}\mathbf{G}^*(x, w)\mathcal{A}(x, w)\mathbf{G}(x, w)) + \mathcal{A}(x, w)\mathbf{G}(x, w) \tag{4.8}$$

for a.e. $(x, w) \in Q_{\frac{1}{q\beta}} \times Q_1$.

Proof. The sufficiency holds by Remark 3.2 and Theorem 4.1. Next we turn to the necessity. Suppose $\mathbf{h}$ is a super Gabor dual of $\mathbf{g}$. By Lemma 4.3, there exists a $q^d \times q^d$ matrix-valued function $\mathcal{C}(x, w)$ defined on $Q_{\frac{1}{q\beta}} \times Q_1$ with entries being in $L^\infty(Q_{\frac{1}{q\beta}} \times Q_1)$ such that $\mathbf{H}(x, w) = \mathcal{C}(x, w)\mathbf{G}(x, w)$ for a.e. $(x, w) \in Q_{\frac{1}{q\beta}} \times Q_1$. Define a $q^d \times q^d$ matrix-valued function

$$\mathcal{A}(x, w) := \mathcal{C}(x, w) - \beta^d \mathbf{G}(x, w)((\mathbf{G}^*\mathbf{G})(x, w))^{-2}\mathbf{G}^*(x, w) \quad \text{for a.e. } (x, w) \in Q_{\frac{1}{q\beta}} \times Q_1.$$

Then all its entries are in $L^\infty(Q_{\frac{1}{q\beta}} \times Q_1)$ by Remark 3.2 and Theorem 3.3, and

$$\mathcal{A}(x, w)\mathbf{G}(x, w) = \mathbf{H}(x, w) - \beta^d \mathbf{G}(x, w)((\mathbf{G}^*\mathbf{G})(x, w))^{-1}, \tag{4.9}$$

equivalently,

$$\mathbf{H}(x, w) = \mathcal{A}(x, w)\mathbf{G}(x, w) + \beta^d \mathbf{G}(x, w)((\mathbf{G}^*\mathbf{G})(x, w))^{-1} \quad \text{for a.e. } (x, w) \in Q_{\frac{1}{q\beta}} \times Q_1.$$

However, $\mathbf{G}^*(x, w)\mathcal{A}(x, w)\mathbf{G}(x, w) = \mathbf{0}$ by (4.9) and Theorem 4.1. So (4.8) holds. The proof is completed. □

Remark 4.1. Theorem 4.5 provides us with a parametrization of computing all super Gabor duals, which together with Theorem 4.2 shows that the computation of super Gabor duals can be reduced to the computation of canonical duals.

Next we conclude this paper with some examples. By Theorems 3.1 and 3.4 and Remark 3.3, we have

Example 4.1. Let $L = 2$, $\alpha, \beta > 0$ with $\alpha\beta = \frac{1}{2}$, and let $\mathbf{A}(x, w)$ be a 2×2 matrix-valued function defined on $Q_{\frac{1}{q\beta}} \times Q_1$ with its entries belonging to $L^2(Q_{\frac{1}{q\beta}} \times Q_1)$. Define $\mathbf{g} \in L^2(\mathbb{R}, \mathbb{C}^2)$ by

$$\mathbf{G}(x, w) = \mathbf{A}(x, w) \quad \text{for a.e. } (x, w) \in Q_{\frac{1}{q\beta}} \times Q_1.$$

Then

(i) $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ is complete if and only if $\det(\mathbf{A}(x, w)) \neq 0$ for a.e. $(x, w) \in Q_{\frac{1}{q\beta}} \times Q_1$;

(ii) $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ is a Riesz basis for $L^2(\mathbb{R}, \mathbb{C}^2)$ if and only if all entries of $(\mathbf{A}^*\mathbf{A})(x, w)$ and $((\mathbf{A}^*\mathbf{A})(x, w))^{-1}$ belong to $L^\infty(Q_{\frac{1}{q\beta}} \times Q_1)$.

Example 4.2. In Example 4.1, take $\alpha = \frac{1}{2}$, $\beta = 1$, and $\mathbf{A}(x, w) = \begin{pmatrix} 1 & 1 - (2+\sqrt{3})e^{-2\pi i w} \\ e^{2\pi i w} & 1 + e^{2\pi i w} \end{pmatrix}$. Then $\mathbf{g}(x) = \Big(\chi_{[0,1)}(x), \chi_{[-\frac{1}{2},1)}(x) - (2+\sqrt{3})\chi_{[-1,-\frac{1}{2})}(x)\Big)$, and $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ is a Riesz basis for $L^2(\mathbb{R}, \mathbb{C}^2)$. Moreover, the canonical dual γ of $\mathbf{g}$ is given by

$$\begin{aligned}
\gamma_1(x) &= \frac{1}{3+\sqrt{3}}\chi_{\left[-1,-\frac{1}{2}\right)}(x) - \frac{1}{3+\sqrt{3}}\chi_{\left[-\frac{1}{2},0\right)}(x) + \frac{1}{3+\sqrt{3}}\chi_{\left[0,\frac{1}{2}\right)}(x) + \frac{9+5\sqrt{3}}{(3+\sqrt{3})^2}\chi_{\left[\frac{1}{2},1\right)}(x), \\
\gamma_2(x) &= -\frac{1}{3+\sqrt{3}}\chi_{\left[-1,-\frac{1}{2}\right)}(x) + \frac{1}{3+\sqrt{3}}\chi_{\left[-\frac{1}{2},0\right)}(x).
\end{aligned}$$

Example 4.3. Let $L = 1$, $\alpha = \frac{1}{2}$, $\beta = 1$. Define $g \in L^2(\mathbb{R})$ by $g(x) = \chi_{[-\frac{1}{2},1)}(x) - (2+\sqrt{3})\chi_{[-1,-\frac{1}{2})}(x)$. Then $\mathcal{G}(g, \frac{1}{2}, 1)$ is a frame for $L^2(\mathbb{R})$ but not a Riesz basis for $L^2(\mathbb{R})$, and the canonical dual γ of g is given by

$$\begin{aligned}
\boldsymbol{\Gamma}(x, w) &= \mathbf{G}(x, w)((\mathbf{G}^*\mathbf{G})(x, w))^{-1} \\
&= \frac{1}{10 + 4\sqrt{3} - (1+\sqrt{3})(e^{-2\pi i w} + e^{2\pi i w})} \begin{pmatrix} 1 - (2+\sqrt{3})e^{-2\pi i w} \\ 1 + e^{2\pi i w} \end{pmatrix}
\end{aligned}$$

for $(x, w) \in Q_{\frac{1}{2}} \times Q_1$.

Proof. It is obvious that $p = 1, q = 2$. A simple computation shows that $Z_1 g(x, w) = 1-(2+\sqrt{3})e^{2\pi i w}$ and $Z_1 g(x-\frac{1}{2}, w) = 1+e^{-2\pi i w}$ for $(x, w) \in Q_{\frac{1}{2}} \times Q_1$. Then $\mathbf{G}(x, w) = \begin{pmatrix} 1-(2+\sqrt{3})e^{-2\pi i w} \\ 1+e^{2\pi i w} \end{pmatrix}$, and $(\mathbf{G}^*\mathbf{G})(x, w) = 10+4\sqrt{3}-(1+\sqrt{3})(e^{-2\pi i w}+e^{2\pi i w})$ for $(x, w) \in Q_{\frac{1}{2}} \times Q_1$. Observe that $(\mathbf{G}^*\mathbf{G})(x, w)$ and $((\mathbf{G}^*\mathbf{G})(x, w))^{-1}$ are both continuous and have no zero on $[0, \frac{1}{2}] \times [0, 1]$. It follows that $\mathcal{G}(g, \frac{1}{2}, 1)$ is a frame for $L^2(\mathbb{R})$ by Remark 3.3. By Theorem 3.4, $\mathcal{G}(g, \frac{1}{2}, 1)$ is not a Riesz basis for $L^2(\mathbb{R})$ since $\alpha\beta \neq \frac{1}{L}$. By Theorem 4.2, we can obtain the canonical dual γ. $\square$

Remark 4.2. Observe that $\frac{1}{10+4\sqrt{3}-(1+\sqrt{3})(e^{-2\pi i w}+e^{2\pi i w})}$ in Example 4.3 has infinitely many nonzero Fourier coefficients. It follows that the canonical dual γ of g is not compactly supported, although g is compactly supported. Interestingly, such g is a component of $\mathbf{g}$ in Example 4.2, while $\mathbf{g}$ and its canonical dual are both compactly supported. This shows that, to some extent, super Gabor frames enjoy more advantages in the computation of duals than usual Gabor frames.

Example 4.4. In Example 4.1, take $\alpha = \frac{1}{2}$, $\beta = 1$, and $\mathbf{A}(x, w) = \begin{pmatrix} 1 & \lambda x+\mu \\ \lambda x-\mu & 1+\lambda^2 x^2-\mu^2 \end{pmatrix}$, where λ, μ are two complex constants. Then

$$\mathbf{g}(x) = \left(\chi_{\left[0,\frac{1}{2}\right)}(x) + (\overline{\lambda}x-\overline{\mu})\chi_{\left[-\frac{1}{2},0\right)}(x),\ (\overline{\lambda}x+\overline{\mu})\chi_{\left[0,\frac{1}{2}\right)}(x) + (1+\overline{\lambda^2}x^2-\overline{\mu^2})\chi_{\left[-\frac{1}{2},0\right)}(x)\right),$$

and $\mathcal{G}(\mathbf{g}, \alpha, \beta)$ is a Riesz basis for $L^2(\mathbb{R}, \mathbb{C}^2)$. Moreover, the canonical dual γ of $\mathbf{g}$ is given by

$$\gamma(x) = \left((\lambda^2x^2+1-\mu^2)\chi_{\left[0,\frac{1}{2}\right)}(x) - (\lambda x+\mu)\chi_{\left[-\frac{1}{2},0\right)}(x),\quad (-\lambda x+\mu)\chi_{\left[0,\frac{1}{2}\right)}(x) + \chi_{\left[-\frac{1}{2},0\right)}(x)\right).$$

The following example is an immediate consequence of Remark 3.3, Theorems 3.4 and 4.2.

Example 4.5. Given $L = 2$, $\alpha = \frac{1}{3}$, $\beta = 1$, let

$$\mathbf{g} = \left(\chi_{[0,1)}(x),\ \lambda\chi_{\left[0,\frac{1}{3}\right)}(x) + \mu\chi_{\left[\frac{1}{3},\frac{2}{3}\right)}(x) + \eta\chi_{\left[\frac{2}{3},1\right)}(x)\right),$$

where λ, μ and η are three not all equal complex constants. Then $\mathcal{G}(\mathbf{g}, \frac{1}{3}, 1)$ is a frame for $L^2(\mathbb{R}, \mathbb{C}^2)$ but not a Riesz basis for $L^2(\mathbb{R}, \mathbb{C}^2)$, and the canonical dual γ of $\mathbf{g}$ is given by

$$\begin{aligned}
\gamma_1(x) &= \frac{|\eta|^2+|\mu|^2-\lambda\overline{\eta}-\lambda\overline{\mu}}{|\lambda-\eta|^2+|\lambda-\mu|^2+|\eta-\mu|^2}\chi_{\left[0,\frac{1}{3}\right)}(x) + \frac{|\lambda|^2+|\mu|^2-\eta\overline{\lambda}-\eta\overline{\mu}}{|\lambda-\eta|^2+|\lambda-\mu|^2+|\eta-\mu|^2}\chi_{\left[\frac{2}{3},1\right)}(x) \\
&\quad + \frac{|\lambda|^2+|\eta|^2-\mu\overline{\lambda}-\mu\overline{\eta}}{|\lambda-\eta|^2+|\lambda-\mu|^2+|\eta-\mu|^2}\chi_{\left[\frac{1}{3},\frac{2}{3}\right)}(x), \\
\gamma_2(x) &= \frac{2\lambda-\eta-\mu}{|\lambda-\eta|^2+|\lambda-\mu|^2+|\eta-\mu|^2}\chi_{\left[0,\frac{1}{3}\right)}(x) + \frac{2\eta-\lambda-\mu}{|\lambda-\eta|^2+|\lambda-\mu|^2+|\eta-\mu|^2}\chi_{\left[\frac{2}{3},1\right)}(x) \\
&\quad + \frac{2\mu-\lambda-\eta}{|\lambda-\eta|^2+|\lambda-\mu|^2+|\eta-\mu|^2}\chi_{\left[\frac{1}{3},\frac{2}{3}\right)}(x).
\end{aligned}$$

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