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Stability of steady states for one dimensional parabolic equations with nonlinear boundary conditions

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ABSTRACT

We consider one dimensional parabolic equations with nonlinear boundary conditions: $u_t = u_{xx} - qu^{2q-1}$ in $\mathbb{R}_+ \times (0, T)$, $\partial_\nu u = u^q$ on $\{0\} \times (0, T)$, $u(x, 0) = u_0(x) \geq 0$ in $\mathbb{R}_+$. This equation admits a family of positive stationary solutions $\{\phi_\alpha(x)\}_{\alpha>0}$ ($\phi_\alpha(0) = \alpha$) such that $\phi_{\alpha_1}(x) < \phi_{\alpha_2}(x)$ if $\alpha_1 < \alpha_2$. The main purpose of this paper is to study the stability of these stationary solutions. Furthermore we discuss the large time behavior of global solutions. In particular, we prove that every global solution is uniformly bounded and converges to one of the stationary solutions.

1. Introduction

We consider one dimensional semilinear parabolic equations:

$$\begin{cases} u_t = u_{xx} - au^p, & (x,t) \in I \times (0,T), \\ \partial_\nu u = u^q, & (x,t) \in \partial I \times (0,T), \\ u(x,0) = u_0(x) \geq 0, & x \in I, \end{cases} \tag{1}$$

where $I = (-1, 1)$ or $I = \mathbb{R}_+ = \{x \in \mathbb{R}; x > 0\}$, $p, q > 1$, $a > 0$ and ∂_ν denotes the outward normal derivative on the boundary. A finite time blow-up and global solvability of (1) for the case $I = (-1, 1)$ are studied in [3,7]. It is known that the dynamics of (1) is classified into three cases:

$$\text{(i)}\, p > \hat{q} \quad \text{or} \quad p = \hat{q}, \qquad a > q, \qquad \text{(ii)}\, p < \hat{q} \quad \text{or} \quad p = \hat{q}, \qquad a < q, \qquad \text{(iii)}\, p = \hat{q}, \qquad a = q,$$

where $\hat{q} = 2q - 1$. For the case $I = (-1, 1)$ (possible for the case $I = \mathbb{R}_+$), every solution is globally defined and uniformly bounded for the case (i), while solutions blow up in a finite time if the initial data is large enough for the case (ii) [3,7]. As for the critical case (iii), the only case $I = (-1, 1)$ has been studied in [3]. They [3] proved that every positive solution is globally defined and converges to the unique positive singular solution of

$$\begin{cases} \Psi'' = q\Psi^{2q-1} & \text{in } (-1,1), \\ \Psi(\pm 1) = \infty. \end{cases} \tag{2}$$

Therefore every positive solution becomes unbounded at $t = \infty$. Furthermore for such a case, the following grow-up rate of positive solutions of (1) is derived in [5]:

$$\|u(t)\|_{L^\infty(-1,1)} \sim t^{1/2}.$$

In this paper, we study the large time behavior of positive solutions of (1) for the case (iii) with $I = \mathbb{R}_+$. In this case, there appear a family of stationary solutions $\{\phi_\alpha\}_{\alpha>0}$ ($\phi_\alpha(0) = \alpha$) and a positive singular solution ϕ_∞ ($\phi_\infty(0) = \infty$). In particular,

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these stationary solutions are completely ordered: $\phi_{\alpha_1}(x) < \phi_{\alpha_2}(x)$ if $\alpha_1 < \alpha_2$ (see Section 2). The first purpose of this paper is to study the stability of these stationary solutions.

Next we study a boundedness of global solutions of (1). As for the case (ii) with $I = (-1, 1)$ (possible for the case $I = \mathbb{R}_+$), it is shown in [4] that every global solution is uniformly bounded and satisfies

$$\|u(t)\|_{L^\infty(-1,1)} \le c(\|u_0\|_{L^\infty(-1,1)}), \quad t > 0.$$

On the other hand, as is stated above, for the case (iii) with $I = (-1, 1)$, every positive solution is globally defined but becomes unbounded at $t = \infty$. However, as for the case (iii) with $I = \mathbb{R}_+$, since the stationary solutions exist, the large time behavior of global solutions for the case $I = \mathbb{R}_+$ seems to be different from that for the case $I = (-1, 1)$. Here we discuss the possibility of global but unbounded solutions for the case (iii) with $I = \mathbb{R}_+$. Furthermore we study the ω-limit set of bounded global solutions. Let $u(x, t)$ be a bounded global solution of (1). Then by the compactness of the orbit $\{u(\cdot, t); t \ge 0\}$ in $C_{\text{loc}}(\overline{\mathbb{R}}_+)$, the solution $u(x, t)$ approaches to the ω-limit set:

$$\omega(u) := \{\xi \in BC(\overline{\mathbb{R}}_+); u(\cdot, t_k) \to \xi \text{ in } C_{\text{loc}}(\overline{\mathbb{R}}_+) \text{ for some sequence } t_k \to \infty\}.$$

In general, as for one dimensional (radially symmetric) semilinear parabolic equations on a bounded interval (a ball with a radius $R > 0$), from the view point of the intersection comparison argument, the ω-limit set consists of one of stationary solutions, i.e. $\omega(u) = \{\xi\}$, where ξ is a stationary solution. This implies that $u(\cdot, t) \to \xi$ as $t \to \infty$. However, for a unbounded domain case, the ω-limit set is not always given by one of stationary solutions in general. In fact, Poláčik–Yanagida [9] constructed a solution u of $u_t = \Delta u + u^p$ on the whole space $\mathbb{R}^n$ such that $\omega(u) = \{\varphi_\alpha; \alpha \in [\beta_1, \beta_2]\}$ for some $\beta_1 < \beta_2$, where φ_α is a positive radial symmetric stationary solution with $\varphi_\alpha(0) = \alpha$. Namely this solution u is oscillating between two stationary solutions φ_{β_1} and φ_{β_2}. As for (1), we will see that no oscillating solutions exist, in other words, the ω-limit set is given by one of stationary solutions.

Finally we study the large time behavior of sign changing solutions of (1) for the case (iii) with $I = (-1, 1)$. We note that (1.6) has two types of singular solutions. One is positive (negative) singular solutions $\pm\Psi$ satisfying (2), the others are sign changing singular solutions $\pm\Psi_s$ satisfying

$$\begin{cases} \Psi_s'' = q|\Psi_s|^{2q-2}\Psi_s & \text{in } (-1, 1), \\ \Psi_s(\pm 1) = \pm\infty. \end{cases}$$

As is stated above, the positive singular solution is stable in the sense that every positive solution converges to the positive singular solution $\Psi(x)$ as $t \to \infty$. Then here arises a natural question: "Are there solutions which converge to the sign changing singular solutions $\pm\Psi_s(x)$ as $t \to \infty$?" To provide a complete description of the large time behavior of sign changing solutions, this question is crucial. The last purpose of this paper is to answer this question.

The rest of this paper is organized as follows. In Section 2, we recall various type of comparison lemmas and collect some fundamental properties of zeros of solutions of one dimensional parabolic equations. Furthermore we introduce a family of positive stationary solutions and a positive singular solution. In Section 3, we study the stability of stationary solutions. A boundedness of global solutions is discussed in Section 4. Furthermore we study the large time behavior of bounded global solutions in Section 5. Finally in Section 6, we study the asymptotic behavior of sign changing solutions for the case (iii) with $I = (-1, 1)$.

Throughout this paper, we fix $p = 2q - 1$ and $a = q$. For simplicity, we denote a norm of $L^r(\mathbb{R}_+)$ by $\|\cdot\|_r$ and define $BC(I) = C(I) \cap L^\infty(I)$.

2. Preliminaries

In this section, first we recall various types of comparison lemmas which are often used throughout this paper. Secondly we collect some fundamental facts concerning zeros of solutions of one dimensional parabolic equations. Finally we introduce stationary solutions.

2.1. Comparison lemmas

Here we consider a general form of one dimensional nonlinear parabolic equations.

$$\begin{cases} w_t = w_{xx} + f(x, t, w), & (x, t) \in I \times (0, T), \\ \partial_\nu w(0, t) = g_0(t, w(0, t)), & t \in (0, T), \\ \partial_\nu w(1, t) = g_1(t, w(1, t)), & t \in (0, T), \\ w(x, 0) = w_0(x), & x \in I, \end{cases} \tag{3}$$

where $I = (0, 1)$ or $\mathbb{R}_+$. For the case $I = \mathbb{R}_+$, a boundary condition on $x = 1$ is not imposed. Here $f(x, t, w)$ and $g_i(t, w)$ $(i = 0, 1)$ are assumed to be

$$f, f_w \in C(\bar{I} \times [0, T) \times \mathbb{R}), \qquad g_i, \partial_w g_i \in C([0, T) \times \mathbb{R}) \quad (i = 0, 1). \tag{4}$$

Furthermore a solution $w(x,t)$ is assumed to be

$$w \in C(\bar{I} \times [0,T)) \cap C^{2,1}(\bar{I} \times (0,T)), \qquad w \in L^{\infty}(I \times (0, T-\epsilon)) \quad \text{for any } \epsilon > 0. \tag{5}$$

To define a super(sub)-solution, we put

$$\mathcal{L}w = w_t - w_{xx} - f(x,t,w), \qquad \mathcal{B}_0 w = \partial_\nu w - g_0(t,w), \qquad \mathcal{B}_1 w = \partial_\nu w - g_1(t,w).$$

We call $w(x,t)$ a super(sub)-solution of (3) if $w(x,t)$ satisfies (5) and

$$\begin{aligned} &\mathcal{L}w(x,t) \geq 0\ (\leq 0) \quad \text{in } I \times (0,T), \\ &\mathcal{B}_0 w(0,t) \geq 0\ (\leq 0) \quad \text{for } t \in (0,T), \qquad \mathcal{B}_1 w(1,t) \geq 0\ (\leq 0) \quad \text{for } t \in (0,T). \end{aligned}$$

Lemma 2.1. *$I = (0,1)$ or $I = \mathbb{R}_+$. Assume that $f(x,t,w)$ and $g_i(t,w)$ $(i = 0,1)$ satisfy (4). Let $w^{(1)}(x,t)$ $(w^{(2)}(x,t))$ be a sub-solution (super-solution) of (3) with (5). Then if $w_0^{(1)}(x) \leq w_0^{(2)}(x)$ for $x \in I$, then it holds that $w^{(1)}(x,t) \leq w^{(2)}(x,t)$ for $(x,t) \in I \times (0,T)$.*

Proof. Since this lemma seems not to be standard, for the convenience of reader, we provide the complete proof. First we consider the case $I = (0,1)$. We set $W(x,t) = (w^{(1)}(x,t) - w^{(2)}(x,t))_+$, where $w_+ = \max\{w, 0\}$. Then by assumptions, for any $\epsilon > 0$ there exists $c_i > 0$ $(i = 1,2,3)$ such that for $x \in I$ and $t \in (0, T-\epsilon)$

$$\begin{aligned} &\left|f(x,t,w^{(1)}) - f(x,t,w^{(2)})\right| W \leq c_1 W^2, \\ &\left|g_0(t,w^{(1)}) - g_0(t,w^{(2)})\right| W \leq c_2 W^2, \qquad \left|g_1(t,w^{(1)}) - g_1(t,w^{(2)})\right| W \leq c_3 W^2. \end{aligned}$$

Therefore by applying a trace inequality: $2(W(0)^2 + W(1)^2) \leq \int_0^1 W_x(x)^2 dx + c\int_0^1 W(x)^2 dx$, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^1 |W(x,t)|^2 dx &\leq -\int_0^1 |W_x(x,t)|^2 dx + c_1 \int_0^1 W(x,t)^2 dx + c_2 W(0,t)^2 + c_3 W(1,t)^2 \\ &\leq -\frac{1}{2}\int_0^1 |W_x(x,t)|^2 dx + c\int_0^1 W(x,t)^2 dx \quad \text{for } t \in (0, T-\epsilon). \end{aligned}$$

For the case $I = (0,1)$, the regularity assumption (5) implies $W \in C([0,T); L^2(0,1))$. Therefore since $W(x,0) \equiv 0$, by the Gronwall inequality, we obtain the conclusion. Next we consider the case $I = \mathbb{R}_+$. Let $u^{(1)}(x,t) = e^{-\sqrt{1+x^2}} w^{(1)}(x,t)$ and $u^{(2)}(x,t) = e^{-\sqrt{1+x^2}} w^{(1)}(x,t)$. Then we see that for $(x,t) \in \mathbb{R}_+ \times (0,T)$

$$\begin{aligned} &u_t^{(1)} \leq u_{xx}^{(1)} + a_1(x) u_x^{(1)} + a_2(x) u^{(1)} + e^{-\sqrt{1+x^2}} f(x,t,w^{(1)}), \\ &u_t^{(2)} \geq u_{xx}^{(2)} + a_1(x) u_x^{(2)} + a_2(x) u^{(2)} + e^{-\sqrt{1+x^2}} f(x,t,w^{(2)}), \end{aligned}$$

where $a_i(x) \in BC(\mathbb{R}_+)$ $(i = 1,2,3)$. Furthermore boundary conditions are given by

$$\begin{aligned} &\partial_\nu u^{(1)}(0,t) \leq e^{-\sqrt{1+x^2}} g_0(t, w^{(1)}(0,t)), \quad t > 0, \\ &\partial_\nu u^{(2)}(0,t) \geq e^{-\sqrt{1+x^2}} g_0(t, w^{(2)}(0,t)), \quad t > 0. \end{aligned}$$

By definition of $u^{(i)}(x,t)$ $(i = 1,2)$, we find that $u^{(i)} \in C([0,T); L^2(\mathbb{R}_+))$ $(i = 1,2)$. Furthermore assumptions (4) and (5) imply $|f(x,t,w^{(1)}) - f(x,t,w^{(2)})| \leq c_\epsilon |w^{(1)} - w^{(2)}|$ for $t \in (0, T-\epsilon)$ with any $\epsilon > 0$. Therefore the rest of the proof follows from the case $I = (0,1)$. □

We recall more standard comparison lemmas than the previous one.

Lemma 2.2. *$I = (0,1)$ or $I = \mathbb{R}_+$. Assume (4). Let $w^{(i)}(x,t)$ $(i = 1,2)$ be two functions satisfying (5) and*

$$\begin{aligned} &\mathcal{L}w^{(1)}(x,t) \leq 0 \quad \text{in } I \times (0,T), \qquad \mathcal{L}w^{(2)}(x,t) \geq 0 \quad \text{in } I \times (0,T), \\ &w^{(1)}(0,t) \leq w^{(2)}(0,t) \quad \text{for } t \in (0,T), \qquad w^{(1)}(1,t) \leq w^{(2)}(1,t) \quad \text{for } t \in (0,T). \end{aligned} \tag{6}$$

For the case $I = \mathbb{R}_+$, a boundary condition on $x = 1$ is not imposed. Then if $w_0^{(1)}(x) \leq w_0^{(2)}(x)$ for $x \in I$, then it holds that $w^{(1)}(x,t) \leq w^{(2)}(x,t)$ for $(x,t) \in I \times (0,T)$.

Proof. Since this is standard, we omit the proof. □

Next consider the case where f depends on w_x.

$$w_t = w_{xx} + f(x, t, w, w_x), \quad (x, t) \in I \times (0, T).$$

In this case, $f(x, t, w, p)$ and $g_i(t, w)$ $(i = 0, 1)$ are assumed to be

$$f, f_w, f_{w_x} \in C(\bar{I} \times [0, T) \times \mathbb{R}^2), \qquad g_i, \partial_w g_i \in C([0, T) \times \mathbb{R}) \quad (i = 0, 1). \tag{7}$$

Furthermore a solution $w(x, t)$ is assumed to be

$$w \in C(\bar{I} \times [0, T)) \cap C^{2,1}(\bar{I} \times (0, T)), \qquad w, w_x \in L^\infty(I \times (0, T - \epsilon)) \quad \text{for any } \epsilon > 0. \tag{8}$$

Lemma 2.3. *$I = (0, 1)$ or $I = \mathbb{R}_+$. We assume (7). Let $w^{(i)}(x, t)$ $(i = 1, 2)$ be two functions satisfying (8) and (6) with replaced $f(x, t, w)$ by $f(x, t, w, w_x)$. Then if $w_0^{(1)}(x) \le w_0^{(2)}(x)$ for $x \in I$, then it holds that $w^{(1)}(x, t) \le w^{(2)}(x, t)$ for $(x, t) \in I \times (0, T)$.*

Proof. Let $W(x, t)$ be as in Lemma 2.1. Additional assumptions $f_{w_x} \in C(\bar{I} \times [0, T) \times \mathbb{R}^2)$ and $w_x \in L^\infty(I \times (0, T - \epsilon))$ are used to obtain

$$\left| f(x, t, w^{(1)}, w_x^{(1)}) - f(x, t, w^{(2)}, w_x^{(2)}) \right| W \le c_\epsilon \left(W + |W_x| \right) W, \quad t \in (0, T - \epsilon).$$

Therefore the rest of proof follows from the same argument as in that of Lemma 2.1. □

Finally we recall a comparison lemma for a moving domain. Here we consider a moving domain O defined by

$$O = \{(x, t) \in \mathbb{R}_+ \times (0, T); 0 < x < z(t)\},$$

where $z(t)$ is a continuous function on $[0, T]$ and $z(t) > 0$ for $t \in [0, T)$.

Lemma 2.4. *Assume (4). Let $w^{(i)}(x, t) \in BC(\bar{O}) \cap C^{2,1}(O)$ $(i = 1, 2)$ be two functions satisfying*

$$\begin{aligned}
&\mathcal{L} w^{(1)} \le 0 \quad \text{in } O, \qquad \mathcal{L} w^{(2)} \ge 0 \quad \text{in } O,\\
&\mathcal{B}_0 w^{(1)}(0, t) \le 0 \quad \text{for } t \in (0, T), \qquad \mathcal{B}_0 w^{(2)}(0, t) \ge 0 \quad \text{for } t \in (0, T),\\
&w^{(1)}(z(t), t) \le w^{(2)}(z(t), t) \quad \text{for } t \in (0, T).
\end{aligned}$$

Then if $w_0^{(1)}(x) \le w_0^{(2)}(x)$ for $x \in (0, z(0))$, then it holds that $w^{(1)}(x, t) \le w^{(2)}(x, t)$ in O.

Proof. This lemma follows from Lemma 6.12 in [6]. □

2.2. Zeros of solutions of one dimensional parabolic equations

Consider the following one dimensional parabolic equations:

$$\begin{cases} u_t = u_{xx} + c(x, t)u, & (x, t) \in (-1, 1) \times (0, T). \\ u_x(\pm 1, t) = c_\pm(t) u(\pm 1, t), & t \in (0, T). \end{cases} \tag{9}$$

Here we assume that $c(x, t) \in C([-1, 1] \times [0, T])$ and $c_\pm(t) \in C([0, T])$. Let $\mathcal{N}(t)$ be the number of zeros of $u(\cdot, t)$ on $[0, 1]$. The following lemma is one of variants of results in [8].

Lemma 2.5 (*Theorem 6.15 in [6]*)**.** *Let $u(x, t)$ be a classical solution of (9) and $0 \le t_1 < t_2 < T$. If $\mathcal{N}(t_1) < \infty$, then it holds that $\mathcal{N}(t_2) \le \mathcal{N}(t_1)$.*

As a consequence of Theorems C and D in [1], the following result holds.

Lemma 2.6. *Let $u(x, t) \not\equiv 0$ be a classical solution of (9). Then it holds that $\mathcal{N}(t) < \infty$ for $t > 0$.*

Next we consider the same equation defined on $\mathbb{R}_+$:

$$\begin{cases} u_t = u_{xx} + c(x, t)u, & (x, t) \in \mathbb{R}_+ \times (0, T). \\ u_x(0, t) = c_0(t) u(0, t), & t \in (0, T), \end{cases} \tag{10}$$

where $c(x, t) \in C(\overline{\mathbb{R}}_+ \times [0, T])$ and $c_0(t) \in C([0, T])$. We denote by $\mathcal{N}(t)$ the number of zeros of $u(\cdot, t)$ on $\overline{\mathbb{R}}_+$. Then we can show the following lemma by the same way as in the proof of Theorem 6.15 in [6].

Lemma 2.7. *Let $u(x, t)$ be a classical solution of (10) and $0 \le t_1 < t_2 < T$. If $\mathcal{N}(t_1) < \infty$, then it holds that $\mathcal{N}(t_2) \le \mathcal{N}(t_1)$.*

2.3. Stationary solutions

Consider the stationary problem of (1):

$$\begin{cases} \phi'' = q\phi^{2q-1} & \text{in } \mathbb{R}_+, \\ \phi' = -\phi^q & \text{on } \{0\}. \end{cases} \tag{11}$$

To construct solutions of (11), we consider the following ODE problem:

$$\begin{cases} \phi'' = q\phi^{2q-1} & \text{in } \mathbb{R}_+ \\ \phi(0) = \alpha > 0, & \phi'(0) = -\alpha^q. \end{cases} \tag{12}$$

We denote by $\phi_\alpha(x)$ the unique solution of (12). Then for any $\alpha > 0$, $\phi_\alpha(x)$ gives a solution of (11) and is explicitly expressed by

$$\phi_\alpha(x) = \left((q-1)x + \alpha^{-(q-1)}\right)^{-1/(q-1)}.$$

For the case $\alpha = \infty$, we define

$$\phi_\infty(x) = (q-1)^{-1/(q-1)} x^{-1/(q-1)}.$$

Then $\phi_\infty(x)$ turns out to be a singular solution of (11) satisfying $\phi_\infty(0) = \infty$ and $\phi_\infty(x) = \lim_{\alpha\to\infty} \phi_\alpha(x)$. Moreover by the explicit formula of $\phi_\alpha(x)$, we see that these stationary solutions are completely ordered:

$$\phi_{\alpha_1}(x) < \phi_{\alpha_2}(x) \quad \text{if } \alpha_1 < \alpha_2.$$

3. Instability of stationary solutions

In this section, we consider the case $I = \mathbb{R}_+$.

$$\begin{cases} u_t = u_{xx} - qu^{2q-1}, & (x,t) \in \mathbb{R}_+ \times (0,T), \\ \partial_\nu u = u^q, & x = 0, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}_+. \end{cases} \tag{13}$$

Throughout this section we always assume

$$u_0 \in BC(\overline{R}_+), \qquad u_0(x) \geq 0. \tag{14}$$

Then (13) admits a unique classical solution $u(x,t)$, that is

$$u \in BC(\overline{\mathbb{R}}_+ \times [0, T-\epsilon)) \cap C^{2,1}(\overline{\mathbb{R}}_+ \times (0,T)) \quad \text{for any } \epsilon > 0,$$

where $T \in (0, \infty]$ is the maximal existence time.

3.1. Instability from below

Theorem 3.1. *Fix $\alpha \in (0,\infty)$. Let $u_0(x) \in BC(\overline{\mathbb{R}}_+)$ satisfy $u_0(x) \leq \phi_\alpha(x)$ ($u_0(x) \not\equiv \phi_\alpha(x)$), and $u(x,t)$ be a classical solution of (13). Then there exists $\delta > 0$ such that for any $R > 0$ there exist $t_0 > 0$ such that*

$$u(x,t) \leq \left(\phi_\alpha(x)^{-(q-1)} + \delta\right)^{-1/(q-1)}, \quad (x,t) \in (0,R) \times (t_0, \infty).$$

We introduce a new unknown function $v(x,t) = u(x,t)^{-(q-1)}$. Then $v(x,t)$ satisfies

$$\begin{cases} v_t = v_{xx} + \dfrac{q}{(q-1)v}\left((q-1)^2 - v_x^2\right), & (x,t) \in \mathbb{R}_+ \times (0,\infty), \\ v_x(0,t) = (q-1), & t \in (0,\infty), \\ v(x,0) = v_0(x) := u_0(x)^{-(q-1)}, & x \in \mathbb{R}_+. \end{cases} \tag{15}$$

We set

$$\psi_\alpha(x) = \phi_\alpha(x)^{-(q-1)} = (q-1)x + \alpha^{-(q-1)} \quad (\alpha > 0).$$

It is clear that $\psi_\alpha(x)$ gives a stationary solution of (15). First we assume that the initial data $v_0(x)$ satisfies the following conditions.

(A1) $v_0(x)$ is smooth enough and satisfies $v_0(0) > 0$ and $v_0'(0) = (q-1)$,
(A2) there exists $\beta > 0$ such that $0 \leq v_0(x) \leq \psi_\beta(x)$ for $x \in \mathbb{R}_+$,
(A3) $0 \leq v_0'(x) \leq (q-1)$ for $x \in \mathbb{R}_+$,
(A4) there exists $R_0 > 0$ such that $v_0'(x) = (q-1)$ for $x \in (R_0, \infty)$.

Lemma 3.1. *Let $u(x,t)$ be a classical solution of (13) and set $v(x,t)=u(x,t)^{-(q-1)}$. If $v_0(x)$ satisfies* (A1) *and* (A3)–(A4)*, then $v(x,t)$ satisfies $0 \le v_x(x,t) \le (q-1)$.*

Proof. Let $R > R_0$. Consider the following approximate equations:

$$\begin{cases} u_t = u_{xx} - qu^{2q-1}, & (x,t) \in (0,R) \times (0,T), \\ u_x = -u^q, & (x,t) \in \{0,R\} \times (0,T), \\ u(x,0) = u_0(x) := v_0(x)^{-1/(q-1)}, & x \in (0,R). \end{cases} \tag{16}$$

By (A1) and (A3), it is clear that $v_0(x)$ is strictly positive in $(0,R)$. Hence $u_0(x)$ is strictly positive and smooth. Furthermore since $R > R_0$, by (A1) and (A4), $u_0(x)$ satisfies the compatibility conditions $u_0' = -u_0^q$ on $x \in \{0,R\}$. Therefore there exists a unique solution $u_R(x,t) \in C^{2,1}([0,R] \times [0,T)) \cap C^\infty([0,R] \times (0,T))$ of (16), where $T \in (0,\infty]$ is the maximal existence time. First we claim that $u(x,t)$ is strictly positive in $(0,R) \times (0,T)$. Since $u_0(x)$ is strictly positive in $(0,R)$, there exists $\alpha_1 > 0$ such that $u_0(x) > \phi_{\alpha_1}(x)$ for $x \in (0,R)$. Therefore since $\phi_{\alpha_1}(x)$ is a stationary solution of (16), by Lemma 2.1, we find that $u_R(x,t) \ge \phi_{\alpha_1}(x)$ for $(x,t) \in (0,R) \times (0,T)$, which assures the claim. We put $v_R(x,t) = u_R(x,t)^{-(q-1)}$. Then by the positivity of $u_R(x,t)$, we see that $v_R(x,t) \in C^{2,1}([0,R] \times [0,T)) \cap C^\infty([0,R] \times (0,T))$ and it satisfies

$$\begin{cases} v_t = v_{xx} + \dfrac{q}{(q-1)v}\left((q-1)^2 - v_x^2\right), & (x,t) \in (0,R) \times (0,T), \\ v_x(0,t) = v_x(R,t) = (q-1), & t \in (0,T), \\ v(x,0) = v_0(x), & x \in (0,R). \end{cases}$$

Now we claim that

$$0 \le \partial_x v_R(x,t) \le (q-1), \quad (x,t) \in (0,R) \times (0,T). \tag{17}$$

Set $w_R(x,t) = \partial_x v_R(x,t)$. Then $w_R(x,t)$ satisfies

$$\begin{cases} w_t = w_{xx} - \dfrac{2q}{(q-1)v_R} w w_x - \dfrac{qw}{(q-1)v_R^2}\left((q-1)^2 - w^2\right), & (x,t) \in (0,R) \times (0,T), \\ w(0,t) = w(R,t) = (q-1), & t \in (0,T), \\ w(x,0) = w_0(x) := \partial_x v_0(x), & x \in (0,R). \end{cases}$$

Since $v_R(x,t) \in C^{2,1}([0,R] \times [0,T)) \cap C^\infty([0,R] \times (0,T))$, it is clear that $w_R(x,t) \in C^{1,0}([0,R] \times [0,T)) \cap C^\infty([0,R] \times (0,T))$. Therefore by Lemma 2.3 with (A3), we obtain

$$0 \le w_R(x,t) \le (q-1), \quad (x,t) \in (0,R) \times (0,T),$$

which assures the claim. To derive a priori estimates for $u_R(x,t)$, we construct a suitable super-solution. We choose a smooth function $U_0(x)$ satisfying $U_0(x) \ge u_0(x)$ in $[0,1]$, $U_0'(0) = -U_0(0)^q$ and $U_0'(1) = 0$. Let $U(x,t)$ be a unique solution of

$$\begin{cases} U_t = U_{xx} - qU^{2q-1}, & (x,t) \in (0,1) \times (0,\infty), \\ U_x(0,t) = U(0,t)^q, U_x(1,t) = 0, & t \in (0,\infty), \\ U(x,0) = U_0(x), & x \in (0,1). \end{cases} \tag{18}$$

Then by Theorem 4.7 in [3], $U(x,t)$ is globally defined. From (17), we note that

$$0 \le -\partial_x u_R(x,t) \le u_R(x,t)^q, \quad (x,t) \in (0,R) \times (0,T).$$

Hence it follows that $u_R(0,t) = \|u_R(t)\|_{L^\infty(0,R)}$ for $t \in (0,T)$. Therefore by $\partial_x u_R(1,t) \le 0$, applying Lemma 2.1, we see that $u_R(x,t)$ is globally defined and $u_R(x,t) \le U(x,t)$ for $(x,t) \in (0,1) \times (0,\infty)$. Furthermore it holds that $\partial_x u_R(x,t) \le 0$ for $(x,t) \in \mathbb{R}_+ \times (0,\infty)$. Therefore we get $\|u_R(t)\|_{L^\infty(0,R)} = u_R(0,t) \le U(0,t)$. By a parabolic regularity theory, there exist a sequence $\{R_i\}_{i=1}^\infty$ and a limiting function $\bar{u}(x,t)$ such that $R_i \to \infty$ and

$$\lim_{i\to\infty} u_{R_i}(x,t) = \bar{u}(x,t) \quad \text{in } C_{\mathrm{loc}}(\overline{\mathbb{R}}_+ \times [0,\infty)).$$

Then $\bar{u}(x,t)$ is a classical solution of (13) satisfying $\bar{u}(x,t) \le U(x,t)$ in $(0,1) \times (0,\infty)$ and $0 \le -\partial_x \bar{u}(x,t) \le \bar{u}(x,t)^q$. Therefore by a unique solvability of (13), it holds that $\bar{u}(x,t) \equiv u(x,t)$. Thus the proof is completed. □

Here we assume that $v_0(x)$ satisfies $v_0 \ge \psi_\alpha$ ($v_0 \not\equiv \psi_\alpha$) and (A1)–(A4). Put $u_0(x) = v_0(x)^{-1/(q-1)}$. Then it is verified that $u_0(x)$ is smooth and $\phi_\beta(x) \le u_0(x) \le \phi_\alpha(x)$. Therefore by Lemma 2.1, there exists a unique solution $u(x,t) \in C^{2,1}(\overline{\mathbb{R}}_+ \times [0,\infty))$ of (13) satisfying $\phi_\beta(x) \le u(x,t) \le \phi_\alpha(x)$ for $(x,t) \in \mathbb{R}_+ \times (0,\infty)$. Let $v(x,t) = u(x,t)^{-(q-1)}$. Then by the positivity of $u(x,t)$, we find that $v(x,t) \in C^{2,1}(\overline{\mathbb{R}}_+ \times [0,\infty))$ and it satisfies

$$\psi_\alpha(x) \le v(x,t) \le \psi_\beta(x), \quad (x,t) \in \mathbb{R}_+ \times (0,\infty). \tag{19}$$

To discuss the stability of the stationary solution $\psi_\alpha(x)$ of (15), we set

$$w(x,t) = v(x,t) - \psi_\alpha(x).$$

Since $v_x(x,t) = w_x(x,t) + (q-1)$, we see that

$$\begin{cases} w_t = w_{xx} - \dfrac{q\,(w_x + 2(q-1))}{(q-1)v} w_x, & (x,t) \in \mathbb{R}_+ \times (0,\infty), \\ w_x(0,t) = 0, & t \in (0,\infty), \\ w(x,0) = w_0(x) := v_0(x) - \psi_\alpha(x), & x \in \mathbb{R}_+. \end{cases} \tag{20}$$

Since $\psi_\alpha(x) = (q-1)x + \alpha^{-(q-1)}$, from $v_0 \geq \psi_\alpha$ and (A2)–(A4), we see that

$$w_0 \geq 0, \qquad w_0, w_0', w_0'' \in BC(\overline{\mathbb{R}}_+). \tag{21}$$

Furthermore by Lemma 3.1, it follows that

$$-(q-1) \leq w_x(x,t) \leq 0, \quad (x,t) \in \mathbb{R}_+ \times (0,\infty). \tag{22}$$

Therefore from (19) and (22), we obtain

$$\frac{q}{\psi_\beta} \leq \frac{q(w_x + 2(q-1))}{(q-1)v} \leq \frac{2q}{\psi_\alpha}. \tag{23}$$

To construct a sub-solution of (20), we consider the following problem:

$$\begin{cases} W_t = W_{xx} - \dfrac{q}{\psi_\beta(x)} W_x, & (x,t) \in \mathbb{R}_+ \times (0,\infty), \\ W_x(0,t) = 0, & t \in (0,\infty), \\ W(x,0) = w_0(x), & x \in \mathbb{R}_+. \end{cases} \tag{24}$$

Lemma 3.2. *Let $W(x,t)$ be a bounded classical solution of (24). If $v_0(x)$ satisfies* (A1)–(A4), *then it holds that*

$$w(x,t) \geq W(x,t), \quad (x,t) \in \mathbb{R}_+ \times (0,\infty).$$

Proof. From (22) and (23), we see that

$$-\frac{q(w_x + 2(q-1))}{(q-1)v} w_x \geq -\frac{2q}{\psi_\beta} w_x.$$

Therefore we get

$$\begin{cases} (w-W)_t \geq (w-W)_{xx} - \dfrac{q}{\psi_\beta(x)} (w-W)_x, & (x,t) \in \mathbb{R}_+ \times (0,\infty), \\ (w-W)_x(0,t) = 0, & t \in (0,\infty), \\ (w-W)(x,0) \equiv 0, & x \in \mathbb{R}_+. \end{cases}$$

By $\psi_\alpha(x) = (q-1)x + \alpha^{-(q-1)}$ and (19), we find that $w(x,t) \in L^\infty(\mathbb{R}_+ \times (0,\infty))$. Therefore since $\psi_\beta(x)^{-1} \in L^\infty(\mathbb{R}_+)$, a comparison lemma implies $w(x,t) \geq W(x,t)$ for $(x,t) \in \mathbb{R}_+ \times (0,\infty)$, which completes the proof. □

Lemma 3.3. *Let $W(x,t)$ be as in Lemma 3.2. Then if v_0 satisfies* (A1)–(A4), *there exists $\kappa \in \mathbb{R}$ such that $W(\cdot,t) \to \kappa$ in $C_{\mathrm{loc}}(\overline{\mathbb{R}}_+)$ as $t \to \infty$. Moreover κ is characterized by*

$$\kappa = \left(\int_0^\infty \theta_\beta(x)dx \right)^{-1} \left(\int_0^\infty w_0(x)\theta_\beta(x)dx \right), \tag{25}$$

where $\theta_\beta(x) = (x+b)^{-q/(q-1)}$, $b = \beta^{-(q-1)}/(q-1)$.

Proof. By (21) and $w_0'(0) = 0$, we see that $W(x,t) \in C^{2,1}(\overline{\mathbb{R}}_+ \times [0,\infty))$. First we derive a priori estimates for $W(x,t)$, $W_t(x,t)$, $W_x(x,t)$. A comparison argument implies

$$\sup_{t \geq 0} \|W(t)\|_\infty \leq \|W(0)\|_\infty = \|w_0\|_\infty.$$

Hence by the assumption $w_0 \in BC^2(\overline{\mathbb{R}}_+)$, $W(x,t)$ is uniformly bounded on $\mathbb{R}_+ \times (0,\infty)$. To derive a estimate for $W_t(x,t)$, differentiating (24) with respect to t and applying a comparison lemma, we obtain

$$\sup_{t \geq 0} \|W_t(t)\|_\infty \leq \|W_t(0)\|_\infty \leq \|w_0''\|_\infty + q\beta^{q-1}\|w_0'\|_\infty.$$

Therefore by the assumption $w_0 \in BC^2(\overline{\mathbb{R}}_+)$, a boundedness of $W_t(x,t)$ is derived. For simplicity, we put $I_z = (z, z+1)$. Then since $w(x,t)$ satisfies (24), we see that

$$\|W_{xx}(t)\|_{L^2(I_z)} \le \|W_t(t)\|_{L^2(I_z)} + q\beta^{q-1}\|W_x(t)\|_{L^2(I_z)}, \quad t > 0.$$

Hence by interpolation inequalities, there exists $c > 0$ independent of $z > 0$ such that

$$\|W_{xx}(t)\|_{L^2(I_z)} \le c\left(\|W_t(t)\|_{L^2(I_z)} + \|W(t)\|_{L^2(I_z)}\right), \quad t > 0.$$

As a consequence, by using the Sobolev inequality with $n = 1$, we get

$$\begin{aligned} \|W_x(t)\|_{L^\infty(I_z)} &\le c\left(\|W_{xx}(t)\|_{L^2(I_z)} + \|W(t)\|_{L^2(I_z)}\right) \\ &\le c\left(\|W_t(t)\|_{L^2(I_z)} + \|W(t)\|_{L^2(I_z)}\right), \quad t > 0. \end{aligned}$$

Therefore since $W(x,t)$ and $W_t(x,t)$ are uniformly bounded on $\mathbb{R}_+ \times (0,\infty)$, a boundedness of $W_x(x,t)$ is derived. By the explicit expression of $\psi_\beta(x)$, the right-hand side of (24) is rewritten by

$$W_{xx} - \frac{q}{\psi_\beta(x)}W_x = W_{xx} - \frac{\gamma}{x+b}W_x = \frac{1}{\theta_\beta(x)}\left(\theta_\beta(x)W_x\right)_x,$$

where $b = \beta^{-(q-1)}/(q-1)$, $\gamma = q/(q-1)$ and $\theta_\beta(x) = (x+b)^{-\gamma}$. Since $W(x,t), W_x(x,t) \in L^\infty(\mathbb{R}_+ \times (0,\infty))$ and $\theta_\beta(x) \in L^1(\mathbb{R}_+)$, we see that

$$\partial_t \int_0^\infty W(x,t)\theta_\beta(x)dx = \int_0^\infty \left(\theta_\beta(x)W_x(x,t)\right)_x dx = 0,$$

which implies

$$\int_0^\infty W(x,t)\theta_\beta(x)dx = \int_0^\infty w_0(x)\theta_\beta(x)dx. \tag{26}$$

Furthermore by a standard way, we get

$$\frac{d}{dt}\int_0^\infty |W_x(x,t)|^2\theta_\beta(x)dx = -2\int_0^\infty |W_t(x,t)|^2\theta_\beta(x)dx. \tag{27}$$

Integrating over $(0,\infty)$, we obtain

$$\int_0^\infty dt \int_0^\infty |W_t(x,t)|^2\theta_\beta(x)dx \le \frac{1}{2}\int_0^\infty |\partial_x w_0(x)|^2\theta_\beta(x)dx.$$

Hence since $\partial_x w_0 \in BC(\overline{\mathbb{R}}_+)$ and $\theta_\beta \in L^1(\mathbb{R}_+)$, there exists a sequence $\{t_k\}_{k=1}^\infty$ such that $t_k \to \infty$ and

$$\int_0^\infty |W_t(x,t_k)|^2\theta_\beta(x)dx \to 0.$$

By a parabolic regularity theory, there exist a limiting function $W_*(x) \in BC^2(\overline{\mathbb{R}}_+)$ and a subsequence $\{t_k\}_{k=1}^\infty$, which is denoted by the same symbol such that

$$\lim_{k\to\infty} W(\cdot,t_k) = W_* \quad \text{in } C^2_{\mathrm{loc}}(\overline{\mathbb{R}}_+).$$

Then $W_*(x)$ is a bounded stationary solution of (24). Hence $W_*(x)$ must be a constant, which is denoted by κ. Applying Lebesgue's dominant convergence lemma to (26), we obtain

$$\kappa\int_0^\infty \theta_\beta(x)dx = \int_0^\infty w_0(x)\theta_\beta(x)dx.$$

Furthermore since $W_x \in L^\infty(\mathbb{R}_+ \times (0,\infty))$, by Lebesgue's dominant convergence lemma, we obtain

$$\int_0^\infty |W_x(x,t_k)|^2\theta_\beta(x)dx \to 0.$$

Since $\int_0^\infty W_x(x,t)^2\theta_\beta(x)dx$ is decreasing with respect to t by (27), we conclude that

$$\int_0^\infty |W_x(x,t)|^2\theta_\beta(x)dx \to 0.$$

Therefore we obtain $W_x(\cdot,t) \to 0$ in $C_{\mathrm{loc}}(\overline{\mathbb{R}}_+)$ as $t \to \infty$. Now we claim that $W(\cdot,t) \to \kappa$ in $C_{\mathrm{loc}}(\overline{\mathbb{R}}_+)$ as $t \to \infty$. Let $\{\tau_k\}_{k\in\mathbb{N}}$ be any sequence such that $\tau_k \to \infty$ as $k \to \infty$. Then from $W \in L^\infty(\mathbb{R}_+ \times (0,\infty))$ and $W_x(\cdot,t) \to 0$ in $C_{\mathrm{loc}}(\overline{\mathbb{R}}_+)$ as $t \to \infty$, there exist $\kappa' \in \mathbb{R}$ and a subsequence $\{\tau_k\}_{k\in\mathbb{N}}$, which is denoted by the same symbol such that $W(\cdot,\tau_k) \to \kappa'$ in $C_{\rm loc}(\overline{\mathbb{R}}_+)$ as $k \to \infty$. We again apply Lebesgue's dominant convergence lemma in (26) to obtain

$$\kappa' \int_0^\infty \theta_\beta(x)dx = \int_0^\infty w_0(x)\theta_\beta(x)dx.$$

Hence it follows that $\kappa' = \kappa$. Therefore we obtain $W(\cdot,\tau_k) \to \kappa$ in $C_{\rm loc}(\overline{\mathbb{R}}_+)$ as $k \to \infty$. Since a sequence $\{\tau_k\}_{k\in\mathbb{N}}$ is arbitrary, we conclude that $W(\cdot,t) \to \kappa$ in $C_{\rm loc}(\overline{\mathbb{R}}_+)$ as $t \to \infty$, which shows the claim. Thus the proof is completed. $\square$

Combining Lemmas 3.2 and 3.3, we can show the instabilities of stationary solutions of (15), which is stated as follows.

Proposition 3.1. *Let $\alpha \in (0,\infty)$ and $u(x,t)$, $v(x,t)$ be as in Lemma 3.1. If $v_0(x)$ satisfies $v_0(x) \ge \psi_\alpha(x)$ ($v_0(x) \not\equiv \psi_\alpha(x)$), then there exists $\delta > 0$ such that for any $R > 0$ there exists $t_0 > 0$ such that $v(x,t) - \psi_\alpha(x) \ge \delta$ for $(x,t) \in (0,R) \times (t_0,\infty)$.*

Proof. For any initial data $v_0(x)$ satisfying $v_0(x) \ge \psi_\alpha(x)$ ($v_0(x) \not\equiv \psi_\alpha(x)$), we can choose a function $\xi_0(x)$ satisfying (A1)–(A4), $\psi_\alpha(x) \le \xi_0(x) \le v_0(x)$ and $\xi_0(x) \not\equiv \psi_\alpha(x)$. Let $\xi(x,t)$ be a unique solution of (15) with the initial data $\xi_0(x)$. Then by Lemma 2.1, it holds that $v(x,t) \ge \xi(x,t)$. Furthermore let $W(x,t)$ be a bounded classical solution of (24) with $w_0(x) = \xi_0(x) - \psi_\alpha(x)$. Then by Lemmas 3.2 and 3.3, we verify that $\xi(x,t) - \psi_\alpha(x) \ge W(x,t)$ and

$$\lim_{t\to\infty} W(\cdot,t) \to \kappa \quad \text{in } C_{\rm loc}(\overline{\mathbb{R}}_+),$$

where κ is characterized by (25). It is clear that $\kappa > 0$. Therefore for any $R > 0$ there exists $t_0 > 0$ such that $v(x,t) \ge \psi_\alpha(x) + \kappa/2$ for $x \in (0,R)$ and $t > t_0$, which completes the proof. $\square$

Theorem 3.1 follows from Proposition 3.1.

3.2. Instability from above

Theorem 3.2. *Fix $\alpha \in (0,\infty)$. Let $u_0(x) \in BC(\overline{\mathbb{R}}_+)$ satisfy $u_0(x) \ge \phi_\alpha(x)(u_0(x) \not\equiv \phi_\alpha(x))$, and $u(x,t)$ be a classical solution of (13). Then there exists $\delta > 0$ such that for any $R > 0$ there exist $t_0 > 0$ such that*

$$u(x,t) \ge \left(\phi_\alpha(x)^{-(q-1)} - \delta\right)^{-1/(q-1)}, \quad (x,t) \in (0,R) \times (t_0,\infty).$$

The proof of Theorem 3.2 is almost same as in the proof of Theorem 3.1. So we omit the proof of all lemmas and a proposition stated below. We assume the following conditions instead of (A1)–(A4):

(a1) $v_0(x)$ is smooth enough and satisfies $v_0'(0) = (q-1)$,
(a2) $v_0'(x) \ge (q-1)$ for $x \in \mathbb{R}_+$,
(a3) There exists $R_0 > 0$ such that $v_0'(x) = (q-1)$ for $x \in (R_0,\infty)$.

Lemma 3.4. *Let $u(x,t)$ be a classical solution of (13) and set $v(x,t) = u(x,t)^{-(q-1)}$. If $v_0(x)$ satisfies* (a1)–(a3), *then $v(x,t)$ satisfies $v_x(x,t) \ge (q-1)$.*

Let $v_0(x)$ satisfy $0 \le v_0(x) \le \psi_\alpha(x)$ and (a1)–(a3). Set $w(x,t) = v(x,t) - \psi_\alpha(x)$. From Lemma 3.4, it follows that $w_x(x,t) \ge 0$. Hence the second term on the right-hand side of (20) is estimated by

$$\frac{q(w_x + 2(q-1))}{(q-1)v} \ge \frac{2q}{\psi_\alpha}.$$

Therefore $w(x,t)$ satisfies

$$\begin{cases} w_t \le w_{xx} - \dfrac{2q}{\psi_\alpha} w_x, & (x,t) \in \mathbb{R}_+ \times (0,\infty), \\ w_x(0,t) = 0, & t \in (0,\infty), \\ w(x,0) = w_0(x) := v_0(x) - \psi_\alpha(x), & x \in \mathbb{R}_+. \end{cases}$$

Here we note that

$$w_0 \le 0, \quad w_0, w_0', w_0'' \in BC(\overline{\mathbb{R}}_+).$$

By the same manner, we consider the following problem:

$$\begin{cases} W_t = W_{xx} - \dfrac{2q}{\psi_\alpha} W_x, & (x,t) \in \mathbb{R}_+ \times (0,\infty), \\ W_x(0,t) = 0, & t \in (0,\infty), \\ W(x,0) = w_0(x), & x \in \mathbb{R}_+. \end{cases} \tag{28}$$

Lemma 3.5. *Let $W(x,t)$ be a bounded classical solution of (28). If $v_0(x)$ satisfies $0 \le v_0(x) \le \psi_\alpha(x)$ and (a1)–(a3), then it holds that*

$$w(x,t) \le W(x,t), \quad (x,t) \in \mathbb{R}_+ \times (0,\infty).$$

Lemma 3.6. *Let $W(x,t)$ be as in Lemma 3.5. Then if v_0 satisfies (a1)–(a3), there exists $\kappa \in \mathbb{R}$ such that $W(\cdot,t) \to \kappa$ in $C_{\mathrm{loc}}(\overline{\mathbb{R}}_+)$ as $t \to \infty$. Moreover κ is characterized by*

$$\kappa = \left(\int_0^\infty \hat{\theta}_\alpha(x)dx\right)^{-1}\left(\int_0^\infty w_0(x)\hat{\theta}_\alpha(x)dx\right),$$

where $\hat{\theta}_\alpha(x) = (x+b)^{-2q/(q-1)}$, $b = \alpha^{-(q-1)}/(q-1)$.

Proposition 3.2. *Let $\alpha \in (0,\infty)$ and $u(x,t)$, $v(x,t)$ be as in Lemma 3.4. If $v_0(x)$ satisfies $v_0(x) \ge \psi_\alpha(x)$ ($v_0(x) \not\equiv \psi_\alpha(x)$), then there exists $\delta > 0$ such that for any $R > 0$ there exists $t_0 > 0$ such that $v(x,t) - \psi_\alpha(x) \le -\delta$ for $(x,t) \in (0,R) \times (t_0,\infty)$.*

Theorem 3.2 follows from Proposition 3.2.

4. Boundedness of global solutions

Throughout this section, we always assume that $u_0(x)$ satisfies (14). Furthermore for simplicity of notations, we define

$$\psi_\infty(x) = (q-1)x, \qquad \psi_{-\Lambda}(x) = (q-1)x - \Lambda^{-(q-1)} \quad (\Lambda > 0)$$

and $r_\Lambda = \Lambda^{-(q-1)}/(q-1)$. Then it is verified that $\psi_{-\Lambda}(x) > 0$ for $x > r_\Lambda$.

Theorem 4.1. *Assume $u_0 \in BC(\overline{\mathbb{R}}_+)$. Let $u(x,t)$ be a classical solution of (13), and set $v(x,t) = u(x,t)^{-(q-1)}$ and $v_0(x) = u_0(x)^{-(q-1)}$. If there exists $\Lambda > 0$ such that $v_0(x) \ge \psi_{-\Lambda}(x)$ for $x > r_\Lambda$, then $u(x,t)$ is uniformly bounded on $\mathbb{R}_+ \times (0,\infty)$.*

First we rewrite a comparison lemma discussed in Section 2 in terms of $v(x,t)$. Put

$$\mathcal{L}_1 v = v_t - v_{xx} - \frac{q}{(q-1)v}\left((q-1)^2 - v_x^2\right).$$

Lemma 4.1. *Let $v^{(i)}(x,t) \in C(\overline{\mathbb{R}}_+ \times [0,T)) \cap C^{2,1}(\overline{\mathbb{R}}_+ \times (0,T))$ ($i = 1,2$) be two positive functions satisfying*

$$\inf_{(x,t)\in\mathbb{R}_+\times(0,T-\epsilon)} v^{(i)}(x,t) > 0 \quad \text{for any } \epsilon > 0 \ (i = 1,2).$$

Then if $v^{(i)}(x,t)$ ($i = 1,2$) satisfies

$$\begin{aligned}
&\mathcal{L}_1 v^{(1)} \ge 0 \quad \text{in } \mathbb{R}_+ \times (0,T), \qquad \mathcal{L}_1 v^{(2)} \le 0 \quad \text{in } \mathbb{R}_+ \times (0,T),\\
&v_x^{(1)}(0,t) = v_x^{(2)}(0,t) = (q-1) \quad \text{for } t \in (0,T),\\
&v_0^{(1)}(x) \ge v_0^{(2)}(x) \quad \text{for } x \in \mathbb{R}_+,
\end{aligned}$$

then it holds that $v^{(1)}(x,t) \ge v^{(2)}(x,t)$ for $(x,t) \in \mathbb{R}_+ \times (0,T)$.

Proof. Let $u^{(i)}(x,t) = v^{(i)}(x,t)^{-1/(q-1)}$ ($i = 1,2$). Then by the assumption, we find that $u^{(i)}(x,t) \in BC(\overline{\mathbb{R}}_+ \times [0,T)) \cap C^{2,1}(\overline{\mathbb{R}}_+ \times (0,T))$ ($i = 1,2$) and $u_0^{(1)}(x) \le u_0^{(2)}(x)$ for $x \in \mathbb{R}_+$. Furthermore we see that $u^{(1)}(x,t)$ is a sub-solution of (13) and $u^{(2)}(x,t)$ is a super-solution of (13). Therefore Lemma 2.1 implies $u^{(1)}(x,t) \le u^{(2)}(x,t)$ for $(x,t) \in \mathbb{R}_+ \times (0,T)$, which completes the proof. □

A global solvability of a solution of (13) is a consequence of Theorem 4.7 in [3].

Lemma 4.2. *Every positive classical solution of (13) is globally defined. Moreover for any $x_0 > 0$ there exists $M > 0$ such that $u(x,t) \le M$ for $(x,t) \in (x_0,\infty) \times (0,\infty)$.*

Proof. Let $\bar{u}_0(x)$ be a nonincreasing smooth function satisfying $\bar{u}_0'(0) = \bar{u}_0(0)^q$ and $\bar{u}_0(x) \ge u_0(x)$. We denote by $\bar{u}(x,t)$ a unique classical solution of (13) with the initial data $\bar{u}_0(x)$ and denote by T its maximal existence time. Then by Lemma 2.1, it follows that $u(x,t) \le \bar{u}(x,t)$ for $(x,t) \in \mathbb{R}_+ \times (0,T)$. Furthermore by the same argument as in the proof of Lemma 3.1, we see that $\bar{u}_x(x,t) \le 0$ for $(x,t) \in \mathbb{R}_+ \times (0,T)$. We take a smooth function $U_0(x)$ satisfying $U_0(x) \ge \bar{u}_0(x)$ on $(0,1)$, $U_0'(0) = -U_0(0)^q$ and $U_0'(1) = 0$. Let $U(x,t)$ be a unique classical solution of (18). Then we see that $\bar{u}(x,t) \le U(x,t)$ in

$(0, 1) \times (0, T)$ (see proof of Lemma 3.1). From Theorem 4.7 in [3], we note that $U(x, t)$ is globally defined and converges to the unique positive solution of

$$\begin{cases} \Psi'' = q\Psi^{2q-1} & \text{in } (0, 1), \\ \Psi(0) = \infty, & \Psi'(1) = 0. \end{cases}$$

Hence for any $x_0 \in (0, 1)$ there exists $M > 0$ such that $0 \le U(x, t) \le M$ for $(x, t) \in (x_0, 1) \times (0, \infty)$. Since $\bar{u}(x, t) \le U(x, t)$ in $(0, 1) \times (0, T)$ and $\bar{u}_x(x, t) \le 0$ in $\mathbb{R}_+ \times (0, T)$, $\bar{u}(x, t)$ is global defined and satisfies

$$\sup_{x>1} \bar{u}(x, t) \le \sup_{x_0<x<1} \bar{u}(x, t) \le M, \quad t > 0.$$

Therefore from $u(x, t) \le \bar{u}(x, t)$, we obtain the conclusion. $\square$

Set

$$u(x, t) = \phi_{\alpha(t)}(x) + d(x, t), \quad \alpha(t) = u(0, t), \alpha_0 = \alpha(0). \tag{29}$$

Then $d(x, t)$ satisfies

$$\begin{cases} d_t + \partial_t\phi_{\alpha(t)} = d_{xx} + q\phi_{\alpha(t)}^{2q-1} - q(\phi_{\alpha(t)} + d)^{2q-1}, & (x, t) \in \mathbb{R}_+ \times (0, \infty), \\ d(0, t) = d_x(0, t) = 0, & t \in (0, \infty), \\ d(x, 0) = d_0(x) := u_0(x) - \phi_{\alpha_0}(x), & x \in \mathbb{R}_+. \end{cases}$$

Here we assume the following conditions on the initial data.

(D) $u_0(x) > \phi_\alpha(x)$ for $\alpha < \alpha_0$ and u_0 intersects with ϕ_α exactly one time for $\alpha > \alpha_0$, where $\alpha_0 > 0$ is a constant given in (29).

Lemma 4.3. *Assume the condition* (D). *Then $\alpha(t)$ defined in* (29) *is monotone increasing for $t > 0$ and $d(x, t) \ge 0$ for $(x, t) \in \mathbb{R}_+ \times (0, \infty)$.*

Proof. Set $w_\alpha(x, t) = u(x, t) - \phi_\alpha(x)$. Then $w_\alpha(x, t)$ satisfies

$$\begin{cases} w_t = w_{xx} + c(x, t)w, & (x, t) \in \mathbb{R}_+ \times (0, \infty), \\ w_x = c_0(t)w, & (x, t) \in \{0\} \times (0, \infty), \end{cases}$$

where $c(x, t) = -q(2q - 1)\int_0^1 (\theta u(x, t) + (1 - \theta)\phi_\alpha(x))^{2q-2}d\theta$ and $c_0(t) = q\int_0^1 (\theta u(0, t) + (1 - \theta)\phi_\alpha(0))^{q-1}d\theta$. Let $\mathcal{N}_\alpha(t)$ be the number of zeros of $w_\alpha(\cdot, t)$ on $\overline{\mathbb{R}}_+$. Here from Lemma 2.7, we note that $\mathcal{N}_\alpha(t)$ is nonincreasing. Hence by the condition (D), it holds that $\mathcal{N}_\alpha(t) \le 1$ for $t \ge 0$ if $\alpha > \alpha_0$. We define $\alpha^* = \sup_{t\in(0,\infty)} \alpha(t) \in (0, \infty]$ and $\tau_\alpha = \inf\{t > 0; u(0, t) = \alpha\}$ for $\alpha_0 < \alpha < \alpha^*$. By definition of τ_α, it follows that $\mathcal{N}_\alpha(\tau_\alpha) = 1$ and $w_\alpha(0, \tau_\alpha) = 0$. Since $\mathcal{N}_\alpha(t)$ is nonincreasing, it follows that $\mathcal{N}_\alpha(t) = 1$ for $t \in [0, \tau_\alpha]$. We denote by $z_\alpha(t)$ a zero of $w_\alpha(x, t)$ for $t \in [0, \tau_\alpha]$. Then it is known that $z_\alpha(t)$ is continuous on $[0, \tau_\alpha)$ and $\lim_{t\to\tau_\alpha} z_\alpha(t) = z_\alpha^* \in [0, \infty]$ (e.g. Lemma 2.7 in [2]). Here we claim that $z_\alpha^* = 0$. Suppose $z_\alpha^* \in (0, \infty]$. Then since $\lim_{t\to\tau_\alpha} z_\alpha(t) = z_\alpha^* \in (0, \infty]$, there exists $\delta > 0$ such that $w_\alpha(x, t) < 0$ for $(x, t) \in [0, \delta] \times [0, \tau_\alpha)$. Therefore by a strong maximum principle and Hopf's boundary lemma, we see that $w_\alpha(0, \tau_\alpha) < 0$. However this contradicts the assumption, which assures the claim. Therefore since $w_\alpha(x, t) < 0$ for $x \in (0, z_\alpha(t))$ and $w_\alpha(x, t) > 0$ for $x \in (z_\alpha(t), \infty)$, we obtain $w_\alpha(x, \tau_\alpha) \ge 0$ for $x \in \mathbb{R}_+$, which implies

$$u(x, \tau_\alpha) \ge \phi_\alpha(x), \quad x \in \mathbb{R}_+. \tag{30}$$

Therefore by Lemma 2.1, a strong maximum principle and Hopf's boundary lemma, we obtain

$$u(x, t) > \phi_\alpha(x), \quad (x, t) \in \mathbb{R}_+ \times (\tau_\alpha, \infty),$$
$$u(0, t) > \alpha, \quad t \in (\tau_\alpha, \infty).$$

Since $u(0, \tau_\alpha) = \alpha$ and $u(0, t) > \alpha$ for $t > \tau_\alpha$, $\alpha(t)$ is monotone increasing. Furthermore the map $\alpha \mapsto \tau_\alpha$ is monotone increasing and continuous. Therefore it holds that $\tau_\alpha|_{\alpha=\alpha(t)} = t$. By using this relation in (30), we obtain for $t > 0$

$$u(x, t) > \phi_{\alpha(t)}(x), \quad x \in \mathbb{R}_+,$$

which implies that $d(x, t) > 0$ for $t > 0$. $\square$

Lemma 4.4. *Assume the condition* (D). *If $d_0(x) \in L^q(\mathbb{R}_+)$, then $d(x, t)$ satisfies*

$$0 \le d(x, t) \le (4\pi t)^{-1/2q}\|d_0\|_q. \tag{31}$$

Proof. By Lemma 4.3, we note that $\alpha_t(t) \ge 0$. Hence it follows that $\partial_t\phi_{\alpha(t)} \ge 0$. As a consequence, we obtain from $d(x, t) \ge 0$

$$d_t \le d_{xx}, \quad (x, t) \in \mathbb{R}_+ \times (0, \infty).$$

Since $d(0, t) = 0$ for $t > 0$, applying a comparison lemma and heat semigroup estimates, we conclude (31). $\square$

We set

$$v(x,t) = \psi_{\alpha(t)}(x) - \eta(x,t).$$

Let u_0 satisfy the condition (D). Then by Lemma 4.3, we note that $d(x,t) \geq 0$ and $\alpha_t(t) \geq 0$. Hence we see that $\eta(x,t) \geq 0$ and

$$\begin{aligned} v(x,t) &= u(x,t)^{-(q-1)} = \big(\phi_{\alpha(t)}(x) + d(x,t)\big)^{-(q-1)} \\ &\geq \psi_{\alpha(t)} - (q-1)\phi_{\alpha(t)}(x)^{-q} d(x,t) \\ &\geq \psi_{\alpha(t)} - (q-1)\phi_{\alpha_0}(x)^{-q} d(x,t). \end{aligned}$$

Therefore we get

$$0 \leq \eta(x,t) \leq (q-1)\phi_{\alpha_0}(x)^{-q} d(x,t). \tag{32}$$

Here we additionally assume that there exists $\Lambda > 0$ such that $v_0(x) \geq \psi_{-\Lambda}(x)$ for $x > r_\Lambda$. Then we see that for $x > r_\Lambda$

$$\begin{aligned} d_0(x) &= u_0(x) - \phi_{\alpha_0}(x) \\ &\leq \psi_{-\Lambda}(x)^{-1/(q-1)} - \left((q-1)x + \alpha_0^{-(q-1)}\right)^{-1/(q-1)} \\ &= \left((q-1)x - \Lambda^{-(q-1)}\right)^{-1/(q-1)} - \left((q-1)x + \alpha_0^{-(q-1)}\right)^{-1/(q-1)} \\ &\leq \left(\frac{1}{q-1}\right)\left(\Lambda^{-(q-1)} + \alpha_0^{-(q-1)}\right)\left((q-1)x - \Lambda^{q-1}\right)^{-q/(q-1)}. \end{aligned}$$

Hence from $d_0(x) \geq 0$ and $d_0 \in C(\overline{\mathbb{R}}_+)$, it follows that $d_0 \in L^q(\mathbb{R}_+)$. Therefore from (32) and Lemma 4.4, there exists $c_0 > 0$ such that

$$0 \leq \eta(x,t) \leq c_0 \left((q-1)x + \alpha_0^{-(q-1)}\right)^{q/(q-1)} \|d_0\|_q t^{-1/2q}. \tag{33}$$

The following lemma plays a essential role in our argument, which is proved in the end of this section.

Lemma 4.5. *Let $u(x,t)$ be a classical solution of (13). Then for any $\epsilon > 0$ there exists $c_\epsilon > 0$ such that*

$$\|u(t)\|_\infty \leq c_\epsilon t^\epsilon, \quad t \geq 1.$$

Hence from this lemma, there exists $\nu > 0$ such that

$$\alpha(t)^{q-1} \leq \nu t^{1/4q}, \quad t \geq 1.$$

Then by (33), it holds that

$$\eta(x,t) \leq c_0 \nu \left((q-1)x + \alpha_0^{-(q-1)}\right)^{q/(q-1)} \|d_0\|_q \alpha(t)^{-(q-1)} t^{-1/4q}, \quad t \geq 1.$$

We choose $z(t)$ as follows:

$$\left((q-1)z(t) + \alpha_0^{-(q-1)}\right)^{q/(q-1)} = \frac{t^{1/4q}}{2c_0\nu\|d_0\|_q}.$$

Then by the choice of $z(t)$, it is verified that

$$\eta(x,t) \leq \frac{1}{2}\alpha(t)^{-(q-1)}, \quad x \in (0, z(t)), t \geq 1.$$

Hence since $v(x,t) = \psi_{\alpha(t)}(x) - \eta(x,t)$, we get

$$\begin{aligned} v(x,t) &\geq \psi_{\alpha(t)}(x) - \frac{1}{2}\alpha(t)^{-(q-1)} \geq (q-1)x + \frac{1}{2}\alpha(t)^{-(q-1)} \\ &= \psi_\infty(x) + \frac{1}{2}\alpha(t)^{-(q-1)}, \quad x \in (0, z(t)), t \geq 1. \end{aligned} \tag{34}$$

Moreover since $v_0(x) \geq \psi_{-\Lambda}(x)$ for $x > r_\Lambda$ and $\psi_{-\Lambda}(r_\Lambda) = \infty$, applying Lemma 4.1, we get

$$v(x,t) \geq \psi_{-\Lambda}(x), \quad (x,t) \in (r_\Lambda, \infty) \times (0, \infty). \tag{35}$$

Now we define

$$v_*(x,t) = \begin{cases} \psi_\infty(x) + 2^{-1}\alpha(t)^{-(q-1)} & \text{if } x \leq z(t)/2, \\ \psi_\infty(z(t)/2) + 2^{-1}\alpha(t)^{-(q-1)} & \text{if } z(t)/2 \leq x \leq y(t), \\ \psi_{-\Lambda}(x) & \text{if } x \geq y(t), \end{cases}$$

where $y(t)$ is given by $\psi_\infty(z(t)/2) + 2^{-1}\alpha(t)^{-(q-1)} = \psi_{-\Lambda}(y(t))$, which is equivalent to

$$y(t) = \frac{z(t)}{2} + \frac{1}{q-1}\left(2^{-1}\alpha(t)^{-(q-1)} + \Lambda^{-(q-1)}\right).$$

Since $z(t) = \infty$ as $t \to \infty$, there exists $t_0 \geq 1$ such that $y(t) < z(t)$ for $t \geq t_0$. Hence by (34) and (35), we verify that

$$v_*(x,t) < v(x,t), \quad x \in \mathbb{R}_+, t \geq t_0. \tag{36}$$

Lemma 4.6. *For any $\mu > 0$, there exists $t_1 \geq t_0$ such that*

$$\int_0^\infty (v_*(x,t_1) - \psi_\infty(x))\theta_\mu(x)dx > 0,$$

where $\theta_\mu(x) = (x+b)^{-q/(q-1)}$, $b = \mu^{-(q-1)}/(q-1)$.

Proof. By definition of $z(t)$, there exist $D_0 > 0$ and $\tau_0 \geq 1$ such that

$$z(t) \geq D_0 t^{(q-1)/4q^2}, \quad t \geq \tau_0. \tag{37}$$

Hence there exists $c_\mu > 0$ such that for $t \geq \tau_0$

$$\int_0^{z(t)/2} (v_*(x,t) - \psi_\infty(x))\theta_\mu(x)dx = \frac{\alpha(t)^{-(q-1)}}{2}\int_0^{z(t)/2}\theta_\mu(x)dx \geq c_\mu\alpha(t)^{-(q-1)}.$$

By construction of $v_*(x,t)$, we note that $\psi_{-\Lambda}(x) \leq v_*(x,t) \leq \psi_\infty(x) + 2^{-1}\alpha(t)^{-(q-1)}$. Therefore since $\alpha_t(t) \geq 0$, we see that

$$\begin{aligned}\int_{z(t)/2}^\infty |v_*(x,t) - \psi_\infty(x)|\theta_\mu(x)dx &= \left(\Lambda^{-(q-1)} + 2^{-1}\alpha(t)^{-(q-1)}\right)\int_{z(t)/2}^\infty \theta_\mu(x)dx \\ &\leq \left(\Lambda^{-(q-1)} + 2^{-1}\alpha_0^{-(q-1)}\right)\int_{z(t)/2}^\infty \theta_\mu(x)dx.\end{aligned}$$

Here we note that

$$\int_{z(t)/2}^\infty \theta_\mu(x)dx \leq \int_{z(t)/2}^\infty x^{-q/(q-1)}dx = (q-1)\left(\frac{z(t)}{2}\right)^{-1/(q-1)}.$$

Hence from (37), there exists $D_1 > 0$ such that for $t \geq \tau_0$

$$\int_{z(t)/2}^\infty |v_*(x,t) - \psi_\infty(x)|\theta_\mu(x)dx \leq D_1 t^{-1/4q^2}.$$

Thus we obtain for $t \geq \tau_0$

$$\int_0^\infty (v_*(x,t) - \psi_\infty(x))\theta_\mu dx \geq c_\mu\alpha(t)^{-(q-1)} - D_1 t^{-1/4q^2}.$$

By Lemma 4.5, we obtain the desired conclusion. □

Proof of Theorem 4.1. First we assume the condition (D). Applying Lemma 4.1 with the condition (D), we see that $v(x,t) < \psi_{\alpha_0}(x)$ for $(x,t) \in \mathbb{R}_+ \times (0,\infty)$. From Lemma 4.6 and (36), there exists $t_1 > 0$ such that $v_*(x,t_1) < v(x,t_1)$ for $x \in \mathbb{R}_+$ and

$$\int_0^\infty (v_*(x,t_1) - \psi_\infty(x))\theta_{\alpha_0}(s)dx > 0. \tag{38}$$

Therefore by (38), we can choose a smooth function $\xi(x)$ satisfying (A1)–(A4) with $\beta = \alpha_0$ in (A2), $\xi(x) \leq v_*(x,t_1)$ and

$$\int_0^\infty (\xi(x) - \psi_\infty(x))\theta_{\alpha_0}(x)dx > 0. \tag{39}$$

We denote by $\xi(x,t)$ the solution of (15) with the initial data $\xi(x)$. Since $\xi(x) < v(x,t_1)$ for $x \in \mathbb{R}_+$, by Lemma 4.1, we see that $\xi(x,t) \leq v(x,t+t_1)$. By Lemmas 3.2 and 3.3, there exists a function $W(x,t)$ satisfying $\xi(x,t) - \psi_\infty(x) \geq W(x,t)$ and $W(\cdot,t) \to \kappa$ in $C_{loc}(\overline{\mathbb{R}}_+)$ as $t \to \infty$. Since κ is characterized by (25) and (39) implies $\kappa > 0$. Hence there exists $t_0 > 0$ such that $\xi(0,t) \geq \kappa/2$ for $t \geq t_0$. Since $\xi_x(x,t) \geq 0$, we obtain $\xi(x,t) \geq \kappa/2$ for $(x,t) \in \mathbb{R}_+ \times (t_0,\infty)$. Therefore from $v(x,t+t_1) \geq \xi(x,t)$, we conclude that

$$v(x,t) \geq \kappa/2, \quad (x,t) \in \mathbb{R}_+ \times (t_0+t_1,\infty),$$

which implies a boundedness of $u(x,t)$. Next we consider a general initial data $u_0(x)$ satisfying $v_0(x) \geq \psi_{-\Lambda}(x)$ for some $\Lambda > 0$. Here we choose a function $\bar{u}_0(x)$ satisfying (D) and

$$\bar{u}_0(x) \geq u_0(x), \quad \bar{u}_0(x)^{-(q-1)} \geq \psi_{-\Lambda}(x).$$

We denote by $\bar{u}(x,t)$ the solution of (13) with a initial data $\bar{u}_0(x)$. Then by the above arguments, $\bar{u}(x,t)$ is uniformly bounded on $(x,t) \in \mathbb{R}_+ \times (0,\infty)$. Therefore from Lemma 2.1, we conclude that $u(x,t)$ is also uniformly bounded on $(x,t) \in \mathbb{R}_+ \times (0,\infty)$. Thus the proof is completed. □

Proof of Lemma 4.5. The proof of Lemma 4.5 is almost same as in the proof of Theorem 1.1 in [5]. For the convenience of the reader, we give the brief proof of this lemma. First we consider the case $u_x(x,t) \leq 0$ for $(x,t) \in \mathbb{R}_+ \times (0,\infty)$. Therefore it is clear that

$$u(0,t) = \|u(t)\|_\infty, \quad t > 0. \tag{40}$$

We define

$$U_\lambda(y,t) = \epsilon_\lambda(t)^{\frac{1}{q-1}} u(\epsilon_\lambda(t)y,t), \quad \epsilon_\lambda(t) = \lambda(1+t)^{-\gamma} \ (\gamma > 0).$$

Then $U_\lambda(y,t)$ satisfies

$$\begin{cases} \partial_t U_\lambda = \dfrac{1}{\epsilon_\lambda^2}(\partial_y^2 U_\lambda - qU_\lambda^{2q-1}) + \dfrac{\dot{\epsilon}_\lambda}{\epsilon_\lambda}\left(y\partial_y U_\lambda + \dfrac{1}{q-1}U_\lambda\right), & (y,t) \in \mathbb{R}_+ \times (0,\infty), \\ \partial_y U_\lambda(0,t) = -U_\lambda(0,t)^q, & t \in (0,\infty), \\ U_\lambda(y,0) = U_{\lambda 0}(y) := \lambda^{1/(q-1)} u_0(\lambda y), & y \in \mathbb{R}_+. \end{cases}$$

For a comparison argument, we define

$$\mathcal{L}_\lambda U = U_t - \frac{1}{\epsilon_\lambda^2}(U_{yy} - qU^{2q-1}) - \frac{\dot{\epsilon}_\lambda}{\epsilon_\lambda}\left(yU_y + \frac{1}{q-1}U\right).$$

Now we look for a super-solution $\bar{U}_\lambda(y,t)$ which has the following form:

$$\bar{U}_\lambda(y,t) = \phi(y) - \frac{1}{2}\epsilon_\lambda(t)\dot{\epsilon}_\lambda(t)g(y),$$

where $\phi(y) = \phi_\alpha(y)$ with $\alpha = (q-1)^{-1/(q-1)}$ and $g(y)$ is a unique solution of

$$\begin{cases} g'' - q(2q-1)\phi^{2q-2}g = \phi^q \quad \text{in } (0,\infty), \\ g(0) = g'(0) = 0. \end{cases}$$

Then $g(y)$ is given by

$$g(y) = A_1(1+y)^{(2q-1)/(q-1)} + A_2(1+y)^{-q/(q-1)} - A_3(1+y)^{(q-2)/(q-1)},$$

where

$$A_1 = \frac{(q-1)^{(q-2)/(q-1)}}{(q+1)(3q-1)}, \qquad A_2 = \frac{(q-1)^{-1/(q-1)}}{2(3q-1)}, \qquad A_3 = \frac{(q-1)^{-1/(q-1)}}{2(q+1)}.$$

By definition of $g(y)$, it is easily verified that

$$g(y), g_y(y) \geq 0, \quad y \in \mathbb{R}_+.$$

First we claim that there exists $\lambda_0 > 0$ such that for $\lambda \in (0,\lambda_0)$

$$\mathcal{L}_\lambda \bar{U}_\lambda \geq 0, \quad (y,t) \in \mathbb{R}_+ \times (0,\infty).$$

By the same calculations as in [5], for the case $q \geq 3/2$, we see that

$$\mathcal{L}_\lambda \bar{U}_\lambda \geq -\frac{g}{2}(\dot{\epsilon}_\lambda^2 + \epsilon_\lambda \ddot{\epsilon}_\lambda) + \frac{1}{2}\frac{|\dot{\epsilon}_\lambda|}{\epsilon_\lambda}\phi^q + c_q \dot{\epsilon}_\lambda^2 \phi^{2q-3} g^2,$$

where $c_q = q(q-1)(2q-1)/4$. Here we note that $\epsilon_\lambda(t) = \gamma^{-1}(1+t)|\dot{\epsilon}_\lambda(t)|$ and $\ddot{\epsilon}_\lambda(t) = (\gamma+1)(1+t)^{-1}|\dot{\epsilon}_\lambda(t)|$. Hence we obtain for the case $q \geq 3/2$

$$\begin{aligned} \mathcal{L}_\lambda \bar{U}_\lambda &\geq \frac{\dot{\epsilon}_\lambda^2}{2}\left(-(2+\gamma^{-1})g + \gamma^{-1}\lambda^{-2}(1+t)^{2\gamma+1}\phi^q + 2c_q\phi^{2q-3}g^2\right) \\ &\geq \frac{\dot{\epsilon}_\lambda^2}{2}\left(-(2+\gamma^{-1})g + \gamma^{-1}\lambda^{-2}\phi^q + 2c_q\phi^{2q-3}g^2\right). \end{aligned}$$

Since $g(y) \sim y^{(2q-1)/(q-1)}$ and $\phi(y)^{2q-3}g(y)^2 \sim y^{(2q+1)/(q-1)}$ for large $y > 0$, there exists $\lambda_0 > 0$ such that $\mathcal{L}_\lambda \bar{U}_\lambda \geq 0$ for $(y,t) \in \mathbb{R}_+ \times (0,\infty)$ and $\lambda \in (0,\lambda_0)$. Therefore the claim is proved for the case $q \geq 3/2$. For the case $1 < q < 3/2$, by the same way as in [5], there exists $d_0 > 0$ such that

$$\mathcal{L}_\lambda \bar{U}_\lambda \geq \begin{cases} -\dfrac{g}{2}\left(\dot{\epsilon}_\lambda^2 + \epsilon_\lambda \ddot{\epsilon}_\lambda\right) + \dfrac{1}{2}\dfrac{|\dot{\epsilon}_\lambda|}{\epsilon_\lambda}\phi^q + d_0 \dot{\epsilon}_\lambda^2 \phi^{2q-3} g^2 & \text{if } (y,t) \in Q_1, \\ -\dfrac{g}{2}\left(\dot{\epsilon}_\lambda^2 + \epsilon_\lambda \ddot{\epsilon}_\lambda\right) + \dfrac{1}{2}\dfrac{|\dot{\epsilon}_\lambda|}{\epsilon_\lambda}\phi^q + d_0 \epsilon_\lambda^{2q-3}|\dot{\epsilon}_\lambda|^{2q-1} g^{2q-1} & \text{if } (y,t) \in Q_2, \end{cases}$$

where $Q_1 = \{(y,t) \in \mathbb{R}_+ \times (0,\infty); g(y)\epsilon_\lambda(t)|\dot{\epsilon}_\lambda(t)| \leq 2\phi(y)\}$ and $Q_2 = \{(y,t) \in \mathbb{R}_+ \times (0,\infty); g(y)\epsilon_\lambda(t)|\dot{\epsilon}_\lambda(t)| > 2\phi(y)\}$. Hence the estimate for $(y,t) \in Q_1$ are reduced to the case $q \geq 3/2$. Consequently there exists $\lambda_0' > 0$ such that $\mathcal{L}_\lambda \bar{U}_\lambda \geq 0$ for $(y,t) \in Q_1$ and $\lambda \in (0,\lambda_0')$. While, by the same way as before, we get

$$\mathcal{L}_\lambda \bar{U}_\lambda \geq \frac{\dot{\epsilon}_\lambda^2}{2}\left(-(2+\gamma^{-1})g + \gamma^{-1}\lambda^{-2}\phi^q + 2d_0\epsilon_\lambda^{2q-3}|\dot{\epsilon}_\lambda|^{2q-3}g^{2q-1}\right), \quad (y,t) \in Q_2.$$

For the case $1 < q < 3/2$, from $\epsilon_\lambda(t) = \lambda(1+t)^{-\gamma}$, we see that $\epsilon_\lambda^{2q-3}|\dot{\epsilon}_\lambda|^{2q-3} \geq \gamma^{2q-3}\lambda^{2(2q-3)}$. Hence we obtain

$$\mathcal{L}_\lambda \bar{U}_\lambda \geq \frac{\dot{\epsilon}_\lambda^2}{2}\left(-(2+\gamma^{-1})g + \gamma^{-1}\lambda^{-2}\phi^q + 2d_0\gamma^{2q-3}\lambda^{2(2q-3)}g^{2q-1}\right), \quad (y,t) \in Q_2.$$

Therefor there exists $\lambda_0'' > 0$ such that $\mathcal{L}_\lambda \bar{U}_\lambda \geq 0$ for $(y,t) \in Q_2$ and $\lambda \in (0,\lambda_0'')$. Thus the claim is proved. Next we claim that there exists $\lambda_1 > 0$ such that for $\lambda \in (0,\lambda_1)$

$$\bar{U}_\lambda(y,0) \geq U(y,0), \quad y \in \mathbb{R}_+.$$

This claim follows from p. 145 in [5] without any modifications. Finally, we claim that for any $\gamma > 0$ there exists a continuous function $0 < Z(t) < \infty$ such that $\bar{U}_\lambda(Z(t),t) > U(Z(t),t)$ for $t \in (0,\infty)$. From Lemma 4.2, we note that $u(x,t) \leq M$ for $(x,t) \in (1,\infty) \times (0,\infty)$. Hence it holds that

$$U(y,t) \leq \lambda^{1/(q-1)}M, \quad y \geq \epsilon_\lambda(t)^{-1}. \tag{41}$$

By the explicit expression of $g(y)$, there exists $y_1 > 0$ such that

$$g(y) \geq \frac{A_1}{2}y^{(2q-1)/(q-1)}, \quad y \geq y_1.$$

Hence by definition of $\bar{U}_\lambda(y,t)$, we obtain

$$\bar{U}_\lambda(y,t) \geq \frac{A_1\gamma}{4}\lambda^2(1+t)^{-(2\gamma+1)}y^{(2q-1)/(q-1)}, \quad y \geq y_1.$$

Here we set $Z(t) = K(1+t)^{\nu(q-1)}$, where $\nu > 0$ and $K > y_1$ are constants chosen later. Then it is clear that $Z(t) \geq y_1$ for $t > 0$. Hence it holds that

$$\bar{U}_\lambda(Z(t),t) \geq \frac{A_1\gamma}{4}\lambda^2 K^{\frac{2q-1}{q-1}}(1+t)^{-(2\gamma+1)+\nu(2q-1)}.$$

Therefore for any $\lambda, \gamma > 0$ there exist $K_0 > y_1$ and $\nu_0 > 0$ such that for $t \geq 0$

$$\begin{aligned} Z(t) &= K_0(1+t)^{\nu_0(q-1)} \geq \lambda^{-1}(1+t)^{\gamma} = \epsilon_\lambda(t)^{-1}, \\ \bar{U}_\lambda(Z(t),t) &\geq \frac{A_1\gamma}{4}\lambda^2 K_0^{\frac{2q-1}{q-1}}(1+t)^{-(2\gamma+1)+\nu_0(2q-1)} \geq \lambda^{1/(q-1)}M. \end{aligned}$$

Then from (41), it follows that

$$\bar{U}_\lambda(Z(t),t) \geq U_\lambda(Z(t),t), \quad t \geq 0,$$

which assures the claim. Therefore since $\partial_\nu U_\lambda(0,t) = U_\lambda(0,t)^q$ for $t > 0$ and $\partial_\nu \bar{U}_\lambda(0,t) = \bar{U}_\lambda(0,t)^q$ for $t > 0$, applying Lemma 2.4 in $\mathcal{Q} := \{(y,t) \in \mathbb{R}_+ \times (0,\infty); 0 < y < Z(t)\}$, we conclude that

$$U_\lambda(y,t) \leq \bar{U}_\lambda(y,t), \quad (y,t) \in \mathcal{Q}.$$

Here we take $y = 0$, then we get from $g(0) = 0$

$$\begin{aligned} u(0,t) &\leq \epsilon(t)^{-1/(q-1)}\phi(0) \\ &= \lambda^{-1/(q-1)}(q-1)^{-1/(q-1)}(1+t)^{\gamma/(q-1)}, \quad t > 0. \end{aligned}$$

Thus by using (40), we obtain

$$\|u(t)\|_\infty \leq c(1+t)^{\gamma/(q-1)}, \quad t > 0.$$

Next we consider a general case, namely we do not assume (40). For any initial data $u_0 \in BC(\overline{\mathbb{R}}_+)$, we can choose a function $\bar{u}_0 \in BC^1(\overline{\mathbb{R}}_+)$ satisfying $\partial_x \bar{u}_0(x) \le 0$ for $x \in \mathbb{R}_+$ and $\bar{u}_0(x) \ge u_0(x)$ for $x \in \mathbb{R}_+$. Let $\bar{u}(x,t)$ be a classical solution of (1). Then Lemma 2.1 implies $u(x,t) \le \bar{u}(x,t)$ for $(x,t) \in \mathbb{R}_+ \times (0,\infty)$. Furthermore by the same argument as in the proof of Lemma 3.1, we verify that $\bar{u}_x(x,t) \le 0$ for $(x,t) \in \mathbb{R}_+ \times (0,\infty)$. Therefore by the previous argument, $\bar{u}(x,t)$ satisfies

$$\|\bar{u}(t)\| \le c_\gamma (1+t)^\gamma, \quad t > 0$$

for any $\gamma > 0$. Thus the proof of Lemma 4.5 is completed. □

5. Large time behavior of global solutions

In this section, we study the large time behavior of positive bounded global solutions. Let $u(x,t)$ be a solution of (1) and define the ω-limit set of u by

$$\omega(u) = \{\xi \in BC(\overline{\mathbb{R}}_+);\, u(\cdot, t_k) \to \xi \text{ in } C_{\mathrm{loc}}(\overline{\mathbb{R}}_+) \text{ for some sequence } t_k \to \infty\}.$$

The purpose of this section is to investigate the ω-limit set of positive bounded global solutions. To state our result, we set

$$X = \{u \in L^{2q}(\mathbb{R}_+);\, u_x \in L^2(\mathbb{R}_+)\}.$$

Throughout this section, we assume (14) and use the same notations as in Section 4. Then our result is stated as follows.

Theorem 5.1. *Assume that $u_0 \in X \cap BC(\overline{\mathbb{R}}_+)$ and there exists $\Lambda > 0$ such that $v_0(x) \ge \psi_{-\Lambda}(x)$ for $x > r_\Lambda$. Then it holds that $\omega(u) = \{0\}$ or $\omega(u) = \{\phi_\alpha\}$ for some $\alpha > 0$.*

Proof. Define the energy functional $E(u)$ by

$$E(u) = \int_0^\infty \left(u_x^2 + u^{2q}\right) dx - \frac{1}{q+1} u(0)^{q+1}, \quad u \in X.$$

Since $u_0 \in X$, by a standard argument, there exists a unique solution $u \in C([0,\infty); X)$ of (1) such that

$$E(u(t_2)) - E(u(t_1)) = -\int_{t_1}^{t_2} \int_0^\infty u_t(x,t)^2 dxdt, \quad t_1 < t_2.$$

From Theorem 4.1, $u(x,t)$ is uniformly bounded on $\mathbb{R}_+ \times (0,\infty)$. Hence since $E(u(t))$ is nonincreasing, $E_\infty = \lim_{t\to\infty} E(u(t))$ exists. Therefore we get

$$\int_0^\infty \int_0^\infty u_t(x,t)^2 dxdt = -E_\infty + E(u_0) < \infty.$$

This implies

$$\lim_{t\to\infty} \int_t^\infty d\tau \int_0^\infty u_t(x,\tau)^2 dx = 0. \tag{42}$$

To obtain the regularity of $u_t(x,t)$, we differentiate (13) with respect to t. Then we see that $z(x,t) = u_t(x,t)$ satisfies $z_t = z_{xx} - q(2q-1)u^{2q-2}z$ for $(x,t) \in \mathbb{R}_+ \times (0,\infty)$ and $\partial_\nu z(0,t) = qu(0,t)^{q-1}z(0,t)$ for $t > 0$. Therefore since $u(x,t)$ is uniformly bounded on $\mathbb{R}_+ \times (0,\infty)$, by a parabolic regularity theory with (42), we get

$$\lim_{t\to 0} u_t(x,t) \to 0 \quad \text{in locally uniformly on } \overline{\mathbb{R}}_+. \tag{43}$$

Set

$$\alpha_1 = \liminf_{t\to\infty} u(0,t), \qquad \alpha_2 = \limsup_{t\to\infty} u(0,t).$$

By virtue of (43), if $\alpha_1 = \alpha_2 := \alpha$, we find that

$$\omega(u) = \{\phi_\alpha\} \quad \text{if } \alpha > 0, \qquad \omega(u) = \{0\} \quad \text{if } \alpha = 0,$$

which completes the proof. Therefore it is sufficient to show $\alpha_1 = \alpha_2$. To derive a contradiction, we suppose that $\alpha_1 < \alpha_2$. We put $\alpha_3 = (\alpha_1 + \alpha_2) > 0$. Then by definition of α_1 and α_2, there exists a sequence $\{t_k\}_{k\in\mathbb{N}}$ $(t_k \to \infty)$ such that $u(0,t_k) = \alpha_3$ for $k \in \mathbb{N}$. Therefore by (43), we see that $u(x,t_k) \to \phi_{\alpha_3}(x)$ in $C_{\mathrm{loc}}(\overline{\mathbb{R}}_+)$ as $k \to \infty$, which implies

$$\lim_{k\to\infty} |v(x,t_k) - \psi_{\alpha_3}(x)| = 0 \quad \text{in locally uniformly on } \overline{\mathbb{R}}_+. \tag{44}$$

We set $\gamma = (\alpha_3 + \alpha_2)/2$ and

$$v_R(x) = \begin{cases} \psi_\gamma(x) & \text{if } x < R, \\ \psi_\gamma(R) & \text{if } R \le x \le \rho_R, \\ \psi_{-\Lambda}(x) & \text{if } x > \rho_R, \end{cases}$$

where ρ_R is a unique root of $\psi_\gamma(R) = \psi_{-\Lambda}(\rho_R)$, which is given by $\rho_R = R + (\gamma^{-(q-1)} + \Lambda^{-(q-1)})/(q-1)$. Since $v_0(x) \ge \psi_{-\Lambda}(x)$ for $x > r_\Lambda$, by Lemma 4.1, we see that $v(x,t) \ge \psi_{-\Lambda}(x)$ for $(x,t) \in (r_\Lambda, \infty) \times (0,\infty)$. Therefore by (44) and $\psi_{\alpha_3}(x) - \psi_\gamma(x) = \alpha_3^{-(q-1)} - \gamma^{-(q-1)} > 0$, for any $R > 0$ there exists $k_0 \in \mathbb{N}$ such that for $k \ge k_0$

$$v(x, t_k) \ge v_R(x), \quad x \in \mathbb{R}_+. \tag{45}$$

By the same way as in the proof of Lemma 4.6, we can show that there exists $R_0 > 0$ such that

$$\int_0^\infty (v_{R_0}(x) - \psi_{\alpha_2}(x))\theta_\gamma(x)dx > 0,$$

where $\theta_\gamma(x) = (x+b)^{-q/(q-1)}$, $b = \gamma^{-(q-1)}/(q-1)$. Then by (45), there exists $k_1 \in \mathbb{N}$ such that

$$v(x, t_{k_1}) \ge v_{R_0}(x), \quad x \in \mathbb{R}_+.$$

Furthermore there exists a smooth function $\xi(x)$ satisfying (A1)–(A4) with $\beta = \gamma$ in (A2), $\xi(x) < v_{R_0}(x)$ and

$$\int_0^\infty (\xi(x) - \psi_{\alpha_2}(x))\theta_\gamma(x)dx > 0. \tag{46}$$

Let $\xi(x,t)$ be a solution of (15) with the initial data $\xi(x)$. Since $\xi(x) < v(x, t_{k_1})$, Lemma 4.1 shows that $\xi(x,t) \le v(x, t + t_{k_1})$ for $(x,t) \in \mathbb{R}_+ \times (0,\infty)$. Then by the same way as in the proof of Theorem 4.1, we see that there exists a function $W(x,t)$ satisfying $\xi(x,t) - \psi_{\alpha_2}(x) \ge W(x,t)$ and $W(x,t) \to \kappa$ in $C_{\mathrm{loc}}(\overline{\mathbb{R}_+})$ as $t \to \infty$. Since κ is characterized by (25), it follows from (46) that $\kappa > 0$. Therefore there exists $t_0 > 0$ such that $\xi(0,t) \ge \psi_{\alpha_2}(0) + \kappa/2$ for $t \ge t_0$. Since $v(0,t) \ge \xi(0,t)$ for $t \ge 0$, we deduce that

$$u(0,t) = v(0,t)^{-1/(q-1)} \le \left(\alpha_2^{-(q-1)} + \kappa/2\right)^{-1/(q-1)}, \quad t \ge t_0.$$

However this contradicts the definition of α_2, which completes the proof. □

6. Sign changing solutions in a bounded interval

In this section, we study the large time behavior of sign changing solutions of (1) with $I = (-1, 1)$.

$$\begin{cases} u_t = u_{xx} - q|u|^{2q-2}u, & (x,t) \in (-1,1) \times (0,T), \\ \partial_\nu u = |u|^{q-1}u, & (x,t) \in \{-1,1\} \times (0,T), \\ u(x,0) = u_0(x), & x \in (-1,1). \end{cases} \tag{47}$$

As is stated in Introduction, the large time behavior of positive solutions of (47) is completely understood. Every positive solution converges to the positive singular solutions $\Psi(x)$ as $t \to \infty$. Here we provide a complete classification of the large time behavior of sign changing solutions of (47), which is stated as follows.

Theorem 6.1. *Let $u_0 \in C([-1,1])$ and $u(x,t)$ be a classical solution of (47). Then $u(x,t)$ converges to either $\pm\Psi(x)$ uniformly on any compact set in $(-1,1)$ or zero uniformly on $[-1,1]$ as $t \to \infty$.*

Remark 6.1. Theorem 6.1 gives a negative answer to the question given in Introduction: "Are there solutions which converge to the sign changing singular solutions $\pm\Psi_s(x)$ as $t \to \infty$"?

Proof. We denote by $\mathcal{N}(t)$ the number of zeros of $u(\cdot,t)$. By Lemmas 2.5 and 2.6, we see that $\mathcal{N}(t) < \infty$ for $t > 0$ and $\mathcal{N}(t)$ is a nonincreasing function. Therefore there exist $t_0 > 0$ and $\mathcal{N}_\infty \in \mathbb{N} \cup \{0\}$ such that

$$\mathcal{N}_\infty = \mathcal{N}(t), \quad t \ge t_0.$$

Let $-1 < z_1(t) < \cdots < z_{\mathcal{N}_\infty}(t) \le 1$ be zeros of $u(\cdot,t)$ for $t \ge t_0$. Then the large time behavior of solutions are classified in terms of $\mathcal{N}_\infty$.

$$\text{(I) } \mathcal{N}_\infty = 0, \qquad \text{(II) } \mathcal{N}_\infty \ge 1.$$

For the case (I), since the solution $u(x,t)$ is positive or negative, Theorem 4.7 in [3] implies that the solution $u(x,t)$ converges to the positive singular solution $\Psi(x)$ or the negative singular solution $-\Psi(x)$ as $t \to \infty$. Next we consider the case (II). Set $O = \{(x,t) \in (-1,1) \times (t_0,\infty);\ -1 < x < z_1(t)\}$. Here we recall that $z_1(t)$ is continuous on $[t_0,\infty)$ (e.g. Lemma 2.7 in [2]).

Now we fix a nonnegative function $U_0(x) \in C_c^\infty(\overline{\mathbb{R}}_+)$ satisfying $U_0(x+1) \ge |u(x,t_0)|$ for $x \in (-1,1)$. Let $U(x,t)$ be a solution of (1) ($I = \mathbb{R}_+$) with the initial data $U_0(x)$. Since $U_0(x)$ has compact support, there exists $\Lambda > 0$ such that $U_0(x) \le \psi_{-\Lambda}(x)^{-1/(q-1)}$ for $x > r_\Lambda$. Therefore by Theorem 4.1, $U(x,t)$ is uniformly bounded on $\mathbb{R}_+ \times (0,\infty)$. Here we put $\hat{U}(x,t) = U(x+1, t-t_0)$ and $\hat{U}_0(x) = U_0(x+1)$. Then it is easily verified that $\hat{U}(x,t)$ satisfies $\hat{U}_t = \hat{U}_{xx} - q\hat{U}^{2q-1}$ for $(x,t) \in (-1,\infty) \times (t_0,\infty)$ and $\partial_\nu \hat{U}(-1,t) = \hat{U}(-1,t)^q$ for $t > t_0$. Furthermore by definition of O, we see that

$$u(z(t),t) = 0 \quad \text{for } t > t_0, \qquad \hat{U}(z(t),t) > 0 \quad \text{for } t > t_0.$$

Therefore since $|u(x,t_0)| \le \hat{U}_0(x)$ for $x \in (0, z(t_0))$ and

$$\partial_\nu u(-1,t) = u(-1,t)^q \quad \text{for } t > t_0, \qquad \partial_\nu \hat{U}(-1,t) = \hat{U}(-1,t)^q \quad \text{for } t > t_0,$$

applying Lemma 2.4 in O, we obtain $|u(x,t)| \le \hat{U}(x,t)$ in O. Hence $|u(-1,t)|$ is uniformly bounded for $t \in (0,\infty)$. By the same way, we obtain a boundedness of $|u(1,t)|$. We set

$$m_0 = \sup_{t \in (t_0,\infty)} (|u(1,t)| + |u(-1,t)|) + \sup_{x \in (-1,1)} |u(x,t_0)|,$$

then $\bar{U}(x) \equiv m_0$ becomes a super-solution. Hence Lemma 2.2 implies $|u(x,t)| \le \bar{U}(x)$ for $(x,t) \in (-1,1) \times (t_0,\infty)$. Therefore $u(x,t)$ is uniformly bounded on $\mathbb{R}_+ \times (0,\infty)$. Since there are no nontrivial bounded stationary solutions of (47), the solution $u(x,t)$ converges to zero uniformly on $x \in [0,1]$ as $t \to \infty$, which completes the proof. □

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