



Positive and sign-changing solutions of a Schrödinger–Poisson system involving a critical nonlinearity



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ABSTRACT

In this paper, we consider the Schrödinger–Poisson system

$$\begin{cases} -\Delta u + u + l(x)\phi u = k(x)|u|^4u + \mu h(x)u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = l(x)u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where μ is a positive constant and the nonlinear growth of $|u|^4u$ reaches the Sobolev critical exponent since $2^* = 6$ for three spatial dimensions. We prove the existence of at least a pair of fixed sign solutions and a pair of sign-changing solutions in $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ under some suitable conditions on the nonnegative functions l , k and h , but not requiring any symmetry properties on them.

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1. Introduction

In this paper, we study the existence and multiplicity of fixed sign and sign-changing solutions of the following nonlinear Schrödinger–Poisson system

$$\begin{cases} -\Delta u + u + l(x)\phi u = k(x)|u|^4u + \mu h(x)u & \text{in } \mathbb{R}^3, \\ -\Delta \phi = l(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \quad (1.1)$$

where l , k and h are nonnegative functions, μ is a positive constant, and the nonlinear growth of $|u|^4u$ reaches the Sobolev critical exponent since the critical exponent $2^* = 6$ in three spatial dimensions, which is why we call critical nonlinearity in the title.

As we shall see in Section 2, system (1.1) can be easily reduced into a nonlinear Schrödinger equation with a nonlocal term. Briefly, the Poisson equation is solved by using the Lax–Milgram theorem, so, for every u in $H^1(\mathbb{R}^3)$, a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ is obtained such that $-\Delta \phi = l(x)u^2$ and that, inserted into the first equation of (1.1), gives

$$-\Delta u + u + l(x)\phi_u u = k(x)|u|^4u + \mu h(x)u \quad \text{in } \mathbb{R}^3. \quad (1.2)$$

Our main purpose in this work is to study the existence of the sign-changing solutions to Eq. (1.2).

In general, finding a sign-changing solution of an equation is much more difficult than finding a mere solution. Although there were several abstract theories or methods to study sign-changing solutions, they are only applicable to some specific situations. For example, when a problem involves a small parameter, usually Lyapunov–Schmidt reduction procedure [1] can be used to find sign-changing solutions. In [4], Bartsch established an abstract critical point theory in partially order

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Hilbert spaces by virtue of critical groups and studied superlinear problems. In [27], one kind of Ljusternik–Schnirelman theory was established to study sign-changing critical points of an even functional. Some linking type theorems were also obtained in partially ordered Hilbert spaces. The methods and abstract critical point theory of [4,5,27] involved the dense Banach space $C(\Omega)$ (Ω is a smooth bounded domain) of continuous functions in the Hilbert space $H_0^1(\Omega)$, in which the cone has nonempty interior and this framework requires strong hypotheses such as boundedness of the domain. In [31], Schechter et al. established relationships between sign-changing critical point theorems and the linking type theorem of Schechter and the saddle point theorem of Rabinowitz, and applied them to study sign-changing solutions for the nonlinear Schrödinger equation with jumping or oscillating nonlinearities and of double resonance.

It seems that all the methods mentioned above cannot be applied directly to Eq. (1.2), which is considered in the whole space \mathbb{R}^3 with nonlocal term. Our methods used here involve neither the Palais–Smale sequence nor the Ekeland variational principle. Our idea is inspired by Hirano et al. [21], but the procedure is a little simpler than that in [21]. With the help of several lemmas (see Section 3), we get at least a pair of fixed sign solutions as well as a pair of sign-changing solutions.

The general problem

$$\begin{cases} -\Delta u + u + l(x)\phi u = f(x, u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi = l(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \quad (1.3)$$

is called the Schrödinger–Poisson system which is obtained while looking for the existence of standing waves for the nonlinear Schrödinger equation interacting with an unknown electrostatic field. In this context the nonlinear term f simulates, as usual, the interaction between many particles, while the solution ϕ of the Poisson equation plays the role of a potential determined by the charge of the wave function itself. For more and detailed physical background of system (1.3), we refer the readers to [6,7] and the references therein.

Many mathematicians have been devoted to the study of system (1.3) with various nonlinearities $f(x, u)$. There are lots of works in the literature not only on the subcritical cases such as [2,11–13,15–17,24–26,28,30,32,35,36,39], and on the supercritical cases like [3], but also on the critical cases like [3,14,20,38,40]. However, all these results are about existence, multiplicity and behavior of positive solutions to system (1.3). Only little information about sign-changing solutions for system (1.3) is known. Recently, Ianni [23] used a dynamical approach together with a limit procedure to study the existence of infinitely many radially symmetric sign-changing solutions of system (1.3) in the case of $f(x, u) = |u|^{p-2}u$ ($4 \leq p < 6$) and function $l(x) \equiv 1$. To our best knowledge, we have not seen any results related to sign-changing solutions to system (1.1).

In the present paper, we assume the following hypotheses (H):

- (H_l) $l \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, $l(x) \geq 0$ for any $x \in \mathbb{R}^3$ and $l \not\equiv 0$;
- (H_{k₁}) $k(x) \geq 0$ for any $x \in \mathbb{R}^3$;
- (H_{k₂}) there exist $x_0 \in \mathbb{R}^3$, $\delta_1 > 0$ and $\rho_1 > 0$ such that $k(x_0) = \max_{\mathbb{R}^3} k(x)$ and $|k(x) - k(x_0)| \leq \delta_1 |x - x_0|^\alpha$ for $|x - x_0| < \rho_1$ with $1 \leq \alpha < 3$;
- (H_{h₁}) $h \in L^{3/2}(\mathbb{R}^3)$ and $h(x) \geq 0$ for any $x \in \mathbb{R}^3$;
- (H_{h₂}) there are $\delta_2 > 0$ and $\rho_2 > 0$ such that $h(x) \geq \delta_2 |x - x_0|^{-\beta}$ for $|x - x_0| < \rho_2$, where x_0 is given by (H_{k₂});
- (H_{h_μ}) $0 < \mu < \bar{\mu}$, where $\bar{\mu}$ is defined by

$$\bar{\mu} := \mu_h = \inf_{u \in H^1(\mathbb{R}^3) \setminus \{0\}} \left\{ \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx : \int_{\mathbb{R}^3} h(x) |u|^2 dx = 1 \right\}.$$

Remark 1. From the above assumptions, we have the following three remarks.

- (1) The hypotheses (H_{k₁}) and (H_{k₂}) mean that $k \in L^\infty(\mathbb{R}^3)$.
- (2) Lemma 2.2 (iii) shows that $\bar{\mu}$ is achieved.

The main results of the present paper read as follows.

Theorem 1.1. Assume that the hypotheses (H) hold with $1 < \beta < 3$. Then system (1.1) has at least one positive solution (ψ_1, ϕ_{ψ_1}) in $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$.

Theorem 1.2. Assume that the hypotheses (H) hold with $\frac{3}{2} < \beta < 3$. Then system (1.1) has at least one sign-changing solution (ψ_2, ϕ_{ψ_2}) in $H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$.

We say that a solution (u, ϕ) is positive if $u > 0$, $\phi > 0$ for any $x \in \mathbb{R}^3$ and we call a solution (u, ϕ) sign-changing if u is sign-changing since ϕ is always nonnegative.

Hereafter we use the following notations. $H^1(\mathbb{R}^3)$ is the usual Hilbert space endowed with the norm $\|u\|^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + |u|^2) dx$ and the standard inner product. $D^{1,2}(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\|_{D^{1,2}}^2 = \int_{\mathbb{R}^3} |\nabla u|^2 dx$. $H^{-1}(\mathbb{R}^3)$ denotes the dual space of $H^1(\mathbb{R}^3)$. $L^p(\mathbb{R}^3)$ ($1 \leq p \leq +\infty$) is a Lebesgue space with the norm denoted by $\|u\|_p$. For any $\rho > 0$ and for any $x \in \mathbb{R}^3$, $B_\rho(x)$ denotes the ball of radius ρ centered at x . C denotes various

positive constants, which may vary from line to line. \mathcal{S} denotes the best Sobolev constant for the embedding of $D^{1,2}(\mathbb{R}^3)$ in $L^6(\mathbb{R}^3)$ defined by

$$\mathcal{S} = \inf_{u \in D^{1,2}(\mathbb{R}^3) \setminus \{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^3} |u|^6 dx\right)^{1/3}}.$$

The remainder of this paper is organized as follows. In Section 2, we give some useful preliminaries. In Section 3 we study the existence of a positive solution of (1.1), where we not only prove Theorem 1.1 but also prove several lemmas which pave the way for getting sign-changing solutions. Then Section 4 is devoted to proving Theorem 1.2.

2. Preliminaries

In this section, our aim is to give some useful preliminary lemmas. First of all, let us study the variational setting of the problem. In fact, for any $u \in H^1(\mathbb{R}^3)$, denote $L_u(v)$ the linear functional in $D^{1,2}(\mathbb{R}^3)$ by

$$L_u(v) = \int_{\mathbb{R}^3} l(x)u^2 v dx.$$

One may deduce from the hypothesis (H_l) , Hölder and Sobolev inequalities that

$$|L_u(v)| \leq \|l\|_{\infty} \|u\|_{12/5}^2 \|v\|_6 \leq C \|l\|_{\infty} \|u\|_{12/5}^2 \|v\|_{D^{1,2}}. \quad (2.4)$$

Hence, for any $u \in H^1(\mathbb{R}^3)$, the Lax–Milgram theorem implies that there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} \nabla \phi_u \nabla v = \int_{\mathbb{R}^3} l(x)u^2 v dx \quad \text{for any } v \in D^{1,2}(\mathbb{R}^3),$$

i.e., ϕ_u is the weak solution of $-\Delta \phi = l(x)u^2$. Moreover it holds that

$$\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{l(y)u^2(y)}{|x-y|} dy.$$

Clearly $\phi_u(x) \geq 0$ for any $x \in \mathbb{R}^3$. We also, in particular, have that

$$\|\phi_u\|_{D^{1,2}}^2 = \int_{\mathbb{R}^3} |\nabla \phi_u|^2 dx = \int_{\mathbb{R}^3} l(x)\phi_u u^2 dx. \quad (2.5)$$

Using (2.4) and (2.5), we obtain that

$$\|\phi_u\|_6 \leq C \|\phi_u\|_{D^{1,2}} \leq C \|u\|_{12/5}^2 \leq C \|u\|^2. \quad (2.6)$$

Then we arrive at

$$\int_{\mathbb{R}^3} l(x)\phi_u(x)u^2(x)dx \leq C \|u\|^4. \quad (2.7)$$

Denote

$$F(u) = \int_{\mathbb{R}^3} l(x)\phi_u(x)u^2(x)dx \quad (2.8)$$

and introduce the following Euler functional of the problem (1.2)

$$I(u) = \frac{1}{2} \|u\|^2 + \frac{1}{4} F(u) - \int_{\mathbb{R}^3} \left(\frac{1}{6} k(x)|u|^6 + \frac{\mu}{2} h(x)u^2 \right) dx. \quad (2.9)$$

Then (2.7) implies that $F : H^1(\mathbb{R}^3) \rightarrow \mathbb{R}$ is well defined. And by [22, Lemma 2.2] we know that the functional $F \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$ and hence the functional $I \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$. Moreover,

$$\langle I'(u), v \rangle = \int_{\mathbb{R}^3} (\nabla u \nabla v + uv) dx + \int_{\mathbb{R}^3} l(x)\phi_u uv dx - \int_{\mathbb{R}^3} (k(x)|u|^4 uv + \mu h(x)uv) dx$$

for any $v \in H^1(\mathbb{R}^3)$ and if u is a critical point of I in $H^1(\mathbb{R}^3)$, then (u, ϕ_u) is a solution of system (1.1). Noting that ϕ_u is always nonnegative, in particular, if $u > 0$, then $\phi_u > 0$, therefore, to find the positive and sign-changing solutions of system (1.1), it suffices to study the positive and sign-changing critical points of I in $H^1(\mathbb{R}^3)$, respectively.

Lemma 2.1 is a direct conclusion of [37, Lemma 2.13].

Lemma 2.1. Assume that the hypothesis (H_{h_1}) holds. Then the functionals $\psi_h : u \in H^1(\mathbb{R}^3) \mapsto \int_{\mathbb{R}^3} h(x)u^2 dx$ is weakly continuous. For each $v \in H^1(\mathbb{R}^3)$, $\psi_h : u \in H^1(\mathbb{R}^3) \mapsto \int_{\mathbb{R}^3} h(x)uv dx$ is weakly continuous.

Lemma 2.2. Assume that the hypotheses (H_I) and (H_{h_1}) hold. Then the following statements are valid.

- (i) F is a weakly continuous functional.
- (ii) If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then $F(u_n - u) = F(u_n) - F(u) + o(1)$.
- (iii) The following infimum $\bar{\mu}$ is achieved

$$\bar{\mu} := \mu_h = \inf_{u \in H^1(\mathbb{R}^3) \setminus \{0\}} \left\{ \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx : \int_{\mathbb{R}^3} h(x) u^2 dx = 1 \right\}. \quad (2.10)$$

Proof. The proofs of (i) and (iii) are the same as [22, Lemmas 2.3, 2.5], respectively. And it follows from (i) that (ii) holds. \square

Lemma 2.3. If the hypotheses (H_I) , (H_{k_1}) , (H_{h_1}) and (H_μ) hold, then $I(0) = 0$ and

- (I₁) there are constants $\rho, \alpha_0 > 0$ such that $I|_{\partial B_\rho} \geq \alpha_0$;
- (I₂) for every $u_0 \in H^1(\mathbb{R}^3)$ with $\|u_0\| = 1$ and $\text{meas}(\text{supp}(ku_0)) > 0$, there exists $\rho_* > 0$ such that $I(tu_0) < 0$ for any $t > \rho_*$.

Proof. It is clear that $I(0) = 0$. It follows from Lemma 2.2 (iii) and the Sobolev inequality that

$$I(u) \geq \frac{1}{2} \|u\|^2 - C \|u\|^6 - \frac{\mu}{2\bar{\mu}} \|u\|^2 = \|u\|^2 \left(\frac{1}{2} - \frac{\mu}{2\bar{\mu}} - C \|u\|^4 \right).$$

Set $\rho = \|u\|$ small enough such that $C\rho^4 \leq \frac{1}{4} \left(1 - \frac{\mu}{\bar{\mu}} \right)$. Hence we have

$$I(u) \geq \frac{1}{4} \left(1 - \frac{\mu}{\bar{\mu}} \right) \rho^2. \quad (2.11)$$

Choosing $\alpha_0 = \frac{1}{4} \left(1 - \frac{\mu}{\bar{\mu}} \right) \rho^2$, we get the statement (I₁).

Let $u = tu_0$, with $\|u_0\| = 1$. By (2.7) and the assumptions (H_{h_1}) and (H_μ) , we have that

$$I(u) = I(tu_0) \leq t^6 \left(\frac{1}{2t^4} \|u_0\|^2 + \frac{C}{4t^2} \|u_0\|^4 - \frac{1}{6} \int_{\mathbb{R}^3} k(x) |u_0|^6 dx \right).$$

Let $\rho_* > 0$ be fixed such that for all $t > \rho_*$ we have $\frac{1}{2t^4} \|u_0\|^2 + \frac{C}{4t^2} \|u_0\|^4 - \frac{1}{6} \int_{\mathbb{R}^3} k(x) |u_0|^6 dx < 0$. Then (I₂) follows. \square

Next, we prove an important lemma, by which we analyze the behavior of the Nehari set \mathcal{N} defined by

$$\mathcal{N} := \{u \in H^1(\mathbb{R}^3) \setminus \{0\} : G(u) = 0\}, \quad \text{where } G(u) = \langle I'(u), u \rangle.$$

Lemma 2.4. Suppose that the hypotheses (H_I) and (H_μ) hold. Then we have the following conclusions.

- (1) For every $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, there exists a unique $t_u \equiv t(u) > 0$ such that $t_u u \in \mathcal{N}$.
- (2) If $\langle I'(u), u \rangle < 0$, then $0 < t_u < 1$; if $\langle I'(u), u \rangle > 0$, then $t_u > 1$.
- (3) t_u is a continuous functional in $H^1(\mathbb{R}^3)$ with respect to u .
- (4) If $\|u\| \rightarrow 0$, then $t_u \rightarrow +\infty$.

Proof. (1) For every $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, define $g(t) = I(tu)$ and

$$f(t) = \|u\|^2 + t^2 F(u) - t^4 \int_{\mathbb{R}^3} k(x) |u|^6 dx - \mu \int_{\mathbb{R}^3} h(x) |u|^2 dx.$$

Then we have $g'(t) = tf'(t)$. By the definition of \mathcal{N} , for $t > 0$, we obtain that

$$g'(t) = \langle I'(tu), u \rangle = 0 \Leftrightarrow tu \in \mathcal{N}. \quad (2.12)$$

From the structure of the functional I , we know that $\sup_{t>0} g(t)$ is achieved at some $t_u = t(u) > 0$, and then $g'(t_u) = 0$. Hence, by (2.12), $t_u u \in \mathcal{N}$. It remains to prove that such t_u with $g'(t_u) = 0$ is unique, i.e.,

it is sufficient to prove that the solution of $f(t) = 0$ in $(0, +\infty)$ is unique.

In fact, from

$$f'(t) = 2tF(u) - 4t^3 \int_{\mathbb{R}^3} k(x) |u|^6 dx = 0,$$

we obtain a unique

$$t_u^* = \sqrt{\frac{F(u)}{2 \int_{\mathbb{R}^3} k(x) |u|^6 dx}} > 0 \quad (2.13)$$

such that $f'(t_u^*) = 0$ and $f'(t) > 0$ for any $t \in (0, t_u^*)$; $f'(t) < 0$ for any $t \in (t_u^*, +\infty)$. Moreover, since $0 < \mu < \bar{\mu}$, by Lemma 2.2 (iii), $f(0) = \|u\|^2 - \mu \int_{\mathbb{R}^3} h(x)|u|^2 dx > 0$. Therefore, from $f(t_u) = 0$, $t_u \in (t_u^*, +\infty)$ and so t_u must be unique. That is, for any $u \in H^1(\mathbb{R}^3)$, there exists a unique t_u satisfying

$$\|u\|^2 + t_u^2 F(u) - t_u^4 \int_{\mathbb{R}^3} k(x)|u|^6 dx - \mu \int_{\mathbb{R}^3} h(x)|u|^2 dx = 0. \quad (2.14)$$

This proves (1).

(2) If $\langle I'(u), u \rangle < 0$, using Lemma 2.2 (iii) and the assumption of $0 < \mu < \bar{\mu}$, we get that

$$\|u\|^2 + F(u) < \int_{\mathbb{R}^3} k(x)|u|^6 dx + \mu \int_{\mathbb{R}^3} h(x)|u|^2 dx \leq \int_{\mathbb{R}^3} k(x)|u|^6 dx + \frac{\mu}{\bar{\mu}} \|u\|^2,$$

which means that

$$F(u) < \int_{\mathbb{R}^3} k(x)|u|^6 dx. \quad (2.15)$$

It is deduced from (2.13) and (2.15) that $t_u^* < 1$. Moreover, we have that $f(1) = \langle I'(u), u \rangle < 0$. Then $t_u^* < t_u < 1$, because $f(t)$ decreases in $(t_u^*, +\infty)$ and $f(t_u) = 0$. Thus, in the case of $\langle I'(u), u \rangle < 0$, we have that $0 < t_u < 1$.

In the case of $\langle I'(u), u \rangle > 0$, we have $f(1) = \langle I'(u), u \rangle > 0$. If $t_u^* \geq 1$, we deduce that $t_u > t_u^* \geq 1$. Now we consider $t_u^* < 1$. Since $f(t)$ decreases in $(t_u^*, +\infty)$ and $f(1) > 0$, to be sure that $f(t_u) = 0$ for $t_u \in (t_u^*, +\infty)$, it must have $t_u > 1$. Therefore, in this case we have that $t_u > 1$. This finishes the proof of (2).

(3) Let $(u_n)_{n \in \mathbb{N}}$ be such that $u_n \rightarrow u$ in $H^1(\mathbb{R}^3)$. By (2.14) there exists a unique positive real sequence $(t_{u_n})_{n \in \mathbb{N}}$ satisfying

$$\|u_n\|^2 + t_{u_n}^2 F(u_n) - t_{u_n}^4 \int_{\mathbb{R}^3} k(x)|u_n|^6 dx - \mu \int_{\mathbb{R}^3} h(x)|u_n|^2 dx = 0, \quad (2.16)$$

which implies $(t_{u_n})_{n \in \mathbb{N}}$ is bounded in \mathbb{R} . Going if necessary to a subsequence, still denoted by $(t_{u_n})_{n \in \mathbb{N}}$, we may assume that there is $t_0 > 0$ such that $\lim_{n \rightarrow \infty} t_{u_n} = t_0$ and then as $n \rightarrow \infty$ passing to the limit in (2.16) we get that

$$g'(t_0) = t_0 \left(\|u\|^2 + t_0^2 F(u) - t_0^4 \int_{\mathbb{R}^3} k(x)|u|^6 dx - \mu \int_{\mathbb{R}^3} h(x)|u|^2 dx \right) = 0.$$

Hence it follows from (2.12) that $t_0 u \in \mathcal{N}$. According to the uniqueness of t_u , we arrive at $t_0 = t_u$, i.e., $\lim_{n \rightarrow \infty} t_{u_n} = t_u$. We have proved that t_u is continuous with respect to $u \in H^1(\mathbb{R}^3)$.

(4) When $\|u\| \rightarrow 0$, we claim that $t_u \rightarrow +\infty$. Otherwise, if $\|u\| \rightarrow 0$ and there exists $M > 0$ such that $|t_u| \leq M$, then by the Sobolev inequality and $k \in L^\infty(\mathbb{R}^3)$, we obtain that

$$t_u^4 \int_{\mathbb{R}^3} k(x)|u|^6 dx = o(\|u\|^2). \quad (2.17)$$

From (2.17) and Lemma 2.2 (iii) we deduce that

$$\|u\|^2 + t_u^2 F(u) - t_u^4 \int_{\mathbb{R}^3} k(x)|u|^6 dx - \mu \int_{\mathbb{R}^3} h(x)|u|^2 dx \geq \left(1 - \frac{\mu}{\bar{\mu}}\right) \|u\|^2 - o(\|u\|^2) > 0,$$

which contradicts (2.14). Hence $t_u \rightarrow +\infty$. This proves (4) of Lemma 2.4 and so we finish the proof of Lemma 2.4. \square

For any $u \in \mathcal{N}$, by (1) of Lemma 2.4, we have that $t_u = 1$. Moreover, by the proof (1) of Lemma 2.4, we have the following corollary.

Corollary 2.5. If $u \in \mathcal{N}$, then $\max_{t>0} I(tu) = I(u)$.

3. Existence of a positive solution

Lemma 3.1. Assume the hypotheses (H_1) , (H_{k_1}) , (H_{h_1}) and (H_μ) hold. Let $(u_n)_{n \in \mathbb{N}} \subset \mathcal{N}$ be such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$ and $I(u_n) \rightarrow d$, but any subsequence of $(u_n)_{n \in \mathbb{N}}$ does not converge strongly to u in $H^1(\mathbb{R}^3)$. Then one of the following conclusions holds:

- (1) $d \geq \frac{1}{3} \delta^{\frac{3}{2}} \|k\|_\infty^{-\frac{1}{2}}$ if $u = 0$;
- (2) $d > I(t_u u)$ if $u \neq 0$ and $\langle I'(u), u \rangle < 0$;
- (3) $d > \frac{1}{3} \delta^{\frac{3}{2}} \|k\|_\infty^{-\frac{1}{2}}$ if $u \neq 0$ and $\langle I'(u), u \rangle \geq 0$;

where t_u is defined as in Lemma 2.4.

Proof. We borrow an idea from Hirano et al. [21] to prove this lemma. From $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$ we have that $u_n - u \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$. Then by Lemmas 2.1 and 2.2 (i) we arrive at

$$\int_{\mathbb{R}^3} h(x)|u_n - u|^2 dx \rightarrow 0 \quad \text{and} \quad F(u_n - u) \rightarrow 0. \quad (3.18)$$

Going if necessary to a subsequence, we may assume that for some $a \geq 0$ and $b \geq 0$

$$\|u_n - u\|^2 \rightarrow a \quad \text{and} \quad \int_{\mathbb{R}^3} k(x)|u_n - u|^6 dx \rightarrow b. \quad (3.19)$$

Since any subsequence of $(u_n)_{n \in \mathbb{N}}$ does not converge strongly to u in $H^1(\mathbb{R}^3)$, one has $a \neq 0$. By the Brézis–Lieb Lemma [8], (3.18) and $(u_n)_{n \in \mathbb{N}} \subset \mathcal{N}$, we get that

$$d + o(1) = I(u_n) = I(u) + \frac{1}{2}\|u_n - u\|^2 - \frac{1}{6} \int_{\mathbb{R}^3} k(x)|u_n - u|^6 dx + o(1) \quad (3.20)$$

and

$$0 = \langle I'(u_n), u_n \rangle = \langle I'(u), u \rangle + \|u_n - u\|^2 - \int_{\mathbb{R}^3} k(x)|u_n - u|^6 dx + o(1). \quad (3.21)$$

Let $g(t) = I(tu)$, $\beta(t) = \frac{a}{2}t^2 - \frac{b}{6}t^6$ and $\gamma(t) = g(t) + \beta(t)$. It follows from (3.19) and (3.21) that $\gamma'(1) = g'(1) + \beta'(1) = 0$ and $t = 1$ is the only critical point of $\gamma(t)$ in $(0, +\infty)$, which implies that

$$\gamma \text{ achieves its maximum at } t = 1. \quad (3.22)$$

In the following, we will prove the three possibilities of the conclusions respectively.

(1) If $u = 0$, then from (3.19) and (3.21) one deduces that

$$0 = \langle I'(u_n), u_n \rangle = \|u_n\|^2 - \int_{\mathbb{R}^3} k(x)|u_n|^6 dx + o(1) \rightarrow a - b,$$

which implies that $a = b$ and $b \neq 0$, since $a \neq 0$. Using the Sobolev inequality, we have that

$$\|u_n\|^2 \geq \int_{\mathbb{R}^3} |\nabla u_n|^2 dx \geq \mathcal{S} \left(\int_{\mathbb{R}^3} |u_n|^6 dx \right)^{\frac{1}{3}}$$

and then

$$\int_{\mathbb{R}^3} k(x)|u_n|^6 dx \leq \|k\|_{\infty} \int_{\mathbb{R}^3} |u_n|^6 dx \leq \|k\|_{\infty} (\mathcal{S}^{-1}\|u_n\|^2)^3,$$

i.e., $b \leq \|k\|_{\infty} (\mathcal{S}^{-1}b)^3$. Therefore, by the fact that $b \neq 0$, we obtain

$$b \geq \mathcal{S}^{\frac{3}{2}} \|k\|_{\infty}^{-\frac{1}{2}}. \quad (3.23)$$

Thus, combining (3.19), (3.20), (3.23) with the assumption of $u = 0$, we get

$$d = \frac{1}{2} \lim_{n \rightarrow \infty} \|u_n - u\|^2 - \frac{1}{6} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} k(x)|u_n - u|^6 dx \geq \frac{1}{3} \mathcal{S}^{\frac{3}{2}} \|k\|_{\infty}^{-\frac{1}{2}}.$$

This proves the case (1).

(2) Now we consider the case that $u \neq 0$ and $\langle I'(u), u \rangle < 0$. In this case, by (3.21), we get that $a > b \geq 0$. Then we arrive at

$$\beta'(t) = at - bt^5 > bt(1 - t^4) \geq 0$$

for any $t \in (0, 1)$, which implies that β strictly increases in $(0, 1)$ and then

$$\beta(t) > \beta(0) = 0 \quad \text{for any } t \in (0, 1). \quad (3.24)$$

By applying Lemma 2.4 to the u with $\langle I'(u), u \rangle < 0$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}$ and $0 < t_u < 1$. Then by (3.24) we arrive at $\beta(t_u) > 0$. Therefore, combining (3.20) with (3.22), we get that

$$d = \gamma(1) > \gamma(t_u) = g(t_u) + \beta(t_u) > I(t_u u).$$

This proves the second statement.

(3) For the third case, we separate it into two steps. First, we consider that $u \neq 0$ and $\langle I'(u), u \rangle = 0$. Then from Corollary 2.5 and (I₁) of Lemma 2.3 we obtain that

$$I(u) = \max_{t>0} I(tu) > 0. \quad (3.25)$$

By (3.21) and the same process as in the proof of (3.23), we can deduce that

$$a = b \geq \delta^{\frac{3}{2}} \|k\|_{\infty}^{-\frac{1}{2}}. \quad (3.26)$$

Thus from (3.20), (3.25) and (3.26) we obtain that

$$d = \gamma(1) = I(u) + \frac{a}{2} - \frac{b}{6} > \frac{1}{3} \delta^{\frac{3}{2}} \|k\|_{\infty}^{-\frac{1}{2}}.$$

Next, we prove the case that $u \neq 0$ and $\langle I'(u), u \rangle > 0$. In this case (3.21) implies that $b > a$. So $b > a > 0$. Since $\beta'(t) = at - bt^5$, we get $t_u^{**} = \left(\frac{a}{b}\right)^{\frac{1}{4}} < 1$ such that $\beta'(t_u^{**}) = 0$. Note that

$$\beta'(t) \geq 0 \quad \text{for any } t \in (0, t_u^{**}) \quad \text{and} \quad \beta'(t) \leq 0 \quad \text{for any } t \in (t_u^{**}, \infty). \quad (3.27)$$

We get the maximum of β as follows

$$\max_{t>0} \beta(t) = \beta(t_u^{**}) = \frac{a^{\frac{3}{2}}}{3b^{\frac{1}{2}}}. \quad (3.28)$$

It is now deduced from

$$\int_{\mathbb{R}^3} k(x) |u_n - u|^6 dx \leq \|k\|_{\infty} \int_{\mathbb{R}^3} |u_n - u|^6 dx \leq \|k\|_{\infty} (\delta^{-1} \|u_n - u\|^2)^3$$

that

$$\frac{a^{\frac{3}{2}}}{b^{\frac{1}{2}}} \geq \delta^{\frac{3}{2}} \|k\|_{\infty}^{-\frac{1}{2}}. \quad (3.29)$$

Inserting (3.29) into (3.28), we get that

$$\beta(t_u^{**}) = \max_{t>0} \beta(t) \geq \frac{1}{3} \delta^{\frac{3}{2}} \|k\|_{\infty}^{-\frac{1}{2}}. \quad (3.30)$$

By Lemma 2.4 we know that $t_u > 1$. Hence $0 < t_u^{**} < 1 < t_u$. By the definition of t_u , we know that $I(t_u^{**}u) \geq 0$. Hence from (3.22) and (3.30) we obtain that

$$d = \gamma(1) > \gamma(t_u^{**}) = I(t_u^{**}u) + \beta(t_u^{**}) \geq \frac{1}{3} \delta^{\frac{3}{2}} \|k\|_{\infty}^{-\frac{1}{2}}.$$

This proves the third statement. In sum, we finish the proof of this lemma. \square

Next, we define the following minimization problem

$$c_1 = \inf_{u \in \mathcal{N}} I(u).$$

The following estimate to the minimum c_1 will be useful in what follows.

Lemma 3.2. Suppose the hypotheses (H) hold with $1 < \beta < 3$. Then

$$c_1 < \frac{1}{3} \delta^{\frac{3}{2}} \|k\|_{\infty}^{-\frac{1}{2}}.$$

Proof. The idea here is to find an element in \mathcal{N} such that the value of the functional I at this element is strictly less than $\frac{1}{3} \delta^{\frac{3}{2}} \|k\|_{\infty}^{-\frac{1}{2}}$. To construct this element, we need the extremal function u_{ε, x_0} of the embedding $D^{1,2}(\mathbb{R}^3)$ into $L^6(\mathbb{R}^3)$, where

$$u_{\varepsilon, x_0} = C \frac{\varepsilon^{\frac{1}{4}}}{(\varepsilon + |x - x_0|^2)^{\frac{1}{2}}}$$

and C is a normalizing constant and x_0 is given in (H_{k_2}) . Let $\varphi \in C_0^\infty(\mathbb{R}^3)$ be such that $0 \leq \varphi \leq 1$, $\varphi|_{B_{R_2}} \equiv 1$ and $\text{supp } \varphi \subset B_{2R_2}$ for some $R_2 > 0$. Set $v_\varepsilon = \varphi u_{\varepsilon, x_0}$ and then $v_\varepsilon \in H^1(\mathbb{R}^3)$ with $v_\varepsilon(x) \geq 0$ for any $x \in \mathbb{R}^3$. The following asymptotic estimates

hold for ε small enough (see [9]):

$$\|\nabla v_\varepsilon\|_2^2 = K_1 + O\left(\varepsilon^{\frac{1}{2}}\right), \quad \|v_\varepsilon\|_6^2 = K_2 + O(\varepsilon), \quad (3.31)$$

$$\|v_\varepsilon\|_s^s = \begin{cases} O(\varepsilon^{\frac{s}{4}}) & s \in [2, 3), \\ O\left(\varepsilon^{\frac{s}{4}} |\ln \varepsilon|\right) & s = 3, \\ O\left(\varepsilon^{\frac{6-s}{4}}\right) & s \in (3, 6), \end{cases} \quad (3.32)$$

with $\frac{K_1}{K_2} = \delta$. For this v_ε , by Lemma 2.4, we know that there exists a unique $t_{v_\varepsilon} > 0$ such that $t_{v_\varepsilon} v_\varepsilon \in \mathcal{N}$. Thus $c_1 \leq I(t_{v_\varepsilon} v_\varepsilon)$.

To prove $c_1 < \frac{1}{3} \delta^{\frac{3}{2}} \|k\|_\infty^{-\frac{1}{2}}$, it is enough to prove that

$$\max_{t>0} I(tv_\varepsilon) < \frac{1}{3} \delta^{\frac{3}{2}} \|k\|_\infty^{-\frac{1}{2}}. \quad (3.33)$$

Since $I(tv_\varepsilon) \rightarrow -\infty$ as $t \rightarrow \infty$, there exists $t_\varepsilon > 0$ such that $I(t_\varepsilon v_\varepsilon) = \max_{t>0} I(tv_\varepsilon)$. And by Lemma 2.3, $\max_{t>0} I(tv_\varepsilon) \geq \alpha_0 > 0$. Then we have $I(t_\varepsilon v_\varepsilon) \geq \alpha_0 > 0$. Thus from the continuity of I , we may assume that there exists some positive t_0 such that $t_\varepsilon \geq t_0 > 0$. Moreover, from $I(tv_\varepsilon) \rightarrow -\infty$ as $t \rightarrow \infty$ and $I(t_\varepsilon v_\varepsilon) \geq \alpha_0 > 0$, we get that there exists T_0 such that $t_\varepsilon \leq T_0$. Hence $t_0 \leq t_\varepsilon \leq T_0$. Let $I(t_\varepsilon v_\varepsilon) = A(\varepsilon) + B(\varepsilon) + C(\varepsilon)$, where

$$\begin{aligned} A(\varepsilon) &= \frac{t_\varepsilon^2}{2} \int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx - \frac{t_\varepsilon^6}{6} \int_{\mathbb{R}^3} k(x_0) |v_\varepsilon|^6 dx, \\ B(\varepsilon) &= \frac{t_\varepsilon^6}{6} \int_{\mathbb{R}^3} k(x_0) |v_\varepsilon|^6 dx - \frac{t_\varepsilon^6}{6} \int_{\mathbb{R}^3} k(x) |v_\varepsilon|^6 dx, \end{aligned}$$

and

$$C(\varepsilon) = \frac{t_\varepsilon^2}{2} \int_{\mathbb{R}^3} |v_\varepsilon|^2 dx + \frac{t_\varepsilon^4}{4} F(v_\varepsilon) - \frac{t_\varepsilon^2 \mu}{2} \int_{\mathbb{R}^3} h(x) |v_\varepsilon|^2 dx.$$

First, we claim

$$A(\varepsilon) \leq \frac{1}{3} \delta^{\frac{3}{2}} \|k\|_\infty^{-\frac{1}{2}} + C\varepsilon^{\frac{1}{2}}. \quad (3.34)$$

Indeed, let

$$z(t) = \frac{t^2}{2} \int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx - \frac{t^6}{6} \int_{\mathbb{R}^3} k(x_0) |v_\varepsilon|^6 dx.$$

It is easy to see that $z(t)$ achieves its maximum at T_ε with

$$T_\varepsilon = \left(\frac{\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx}{\int_{\mathbb{R}^3} k(x_0) |v_\varepsilon|^6 dx} \right)^{\frac{1}{4}}.$$

Therefore, from (3.31), we have that

$$z(T_\varepsilon) = \sup_{t \geq 0} z(t) = \frac{1}{3} \frac{\left(\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx \right)^{\frac{1}{2}}}{\left(\int_{\mathbb{R}^3} k(x_0) |v_\varepsilon|^6 dx \right)^{\frac{1}{2}}} = \frac{1}{3} \delta^{\frac{3}{2}} \|k\|_\infty^{-\frac{1}{2}} + O\left(\varepsilon^{\frac{1}{2}}\right). \quad (3.35)$$

This proves (3.34). Second, we claim that $B(\varepsilon) \leq C\varepsilon^{1/2}$. In fact, since $t_0 \leq t_\varepsilon \leq T_0$ and $k \in L^\infty(\mathbb{R}^3)$, by the definition of v_ε , (H_{k_2}) and using a change of variables, we obtain that for ε small enough

$$\begin{aligned} B(\varepsilon) &= \frac{t_\varepsilon^6}{6} \int_{\mathbb{R}^3} (k(x_0) - k(x)) |v_\varepsilon|^6 dx \\ &\leq C\delta_1 \int_{|x-x_0|<\rho_1} \frac{|x-x_0|^\alpha \varepsilon^{\frac{3}{2}}}{(\varepsilon + |x-x_0|^2)^3} dx + C \int_{|x-x_0|\geq\rho_1} \frac{\varepsilon^{\frac{3}{2}}}{(\varepsilon + |x-x_0|^2)^3} dx \\ &\leq C\delta_1 \varepsilon^{\frac{3}{2}} \int_0^{\rho_1} \frac{r^{2+\alpha}}{(\varepsilon + r^2)^3} dr + C\varepsilon^{\frac{3}{2}} \int_{\rho_1}^\infty r^{-4} dr \end{aligned}$$

$$\begin{aligned}
&= C\delta_1\varepsilon^{\frac{\alpha}{2}} \int_0^{\rho_1\varepsilon^{-\frac{1}{2}}} \frac{\rho^{2+\alpha}}{(1+\rho^2)^3} d\rho + C\rho_1^{-3}\varepsilon^{\frac{3}{2}} \\
&\leq C\delta_1\varepsilon^{\frac{\alpha}{2}} + C\varepsilon^{\frac{3}{2}} \leq C\varepsilon^{\frac{1}{2}}.
\end{aligned} \tag{3.36}$$

So we obtain $B(\varepsilon) \leq C\varepsilon^{1/2}$. Therefore, to finish the proof, it suffices to show that

$$\lim_{\varepsilon \rightarrow 0^+} \frac{C(\varepsilon)}{\varepsilon^{1/2}} = -\infty. \tag{3.37}$$

Actually, from the definition of v_ε , (H_{h_2}) and for any ε such that $0 < \varepsilon \leq \rho_2^2$, it follows that

$$\begin{aligned}
\int_{\mathbb{R}^3} h(x)|v_\varepsilon|^2 dx &\geq C\delta_2 \int_{|x-x_0|<\rho_2} \frac{|x-x_0|^{-\beta}\varepsilon^{\frac{1}{2}}}{\varepsilon + |x-x_0|^2} dx + \int_{|x-x_0|\geq\rho_2} h(x)|v_\varepsilon|^2 dx \\
&\geq C\delta_2\varepsilon^{\frac{1}{2}} \int_0^{\rho_2} \frac{r^2}{r^\beta(\varepsilon + r^2)} dr \\
&= C\delta_2\varepsilon^{1-\frac{\beta}{2}} \int_0^{\rho_2\varepsilon^{-\frac{1}{2}}} \frac{\rho^2}{\rho^\beta(1+\rho^2)} d\rho \\
&\geq C\delta_2\varepsilon^{1-\frac{\beta}{2}} \int_0^1 \frac{\rho^2}{2\rho^\beta} d\rho = C\varepsilon^{1-\frac{\beta}{2}}.
\end{aligned} \tag{3.38}$$

Thus, by the fact that $t_0 \leq t_\varepsilon \leq T_0$ and hypothesis (H_l) , we have that

$$\begin{aligned}
C(\varepsilon) &= \frac{t_\varepsilon^2}{2} \int_{\mathbb{R}^3} |v_\varepsilon|^2 dx + \frac{t_\varepsilon^4}{4} F(v_\varepsilon) - \frac{t_\varepsilon^2 \mu}{2} \int_{\mathbb{R}^3} h(x)|v_\varepsilon|^2 dx \\
&\leq C\|v_\varepsilon\|_2^2 + C\|v_\varepsilon\|_{12/5}^4 - \mu C\varepsilon^{1-\frac{\beta}{2}} \\
&\leq C\varepsilon^{\frac{1}{2}} + C\varepsilon - \mu C\varepsilon^{1-\frac{\beta}{2}}.
\end{aligned}$$

It is deduced from $1 < \beta < 3$ that for fixed μ we have that

$$\frac{C(\varepsilon)}{\varepsilon^{1/2}} \leq C + C\varepsilon^{\frac{1}{2}} - \mu C\varepsilon^{\frac{1}{2}-\frac{\beta}{2}} \rightarrow -\infty, \quad \text{as } \varepsilon \rightarrow 0.$$

Hence (3.37) holds. Then (3.33) follows and the proof is complete. \square

Theorem 3.3. Suppose that the hypotheses (H) hold with $1 < \beta < 3$. Then there exists a positive $\psi_1 \in \mathcal{N}$ such that $c_1 = I(\psi_1)$ and then ψ_1 is a positive critical point of the functional I in $H^1(\mathbb{R}^3)$.

Proof. By the definition of c_1 , we may assume that there exists $(v_n)_{n \in \mathbb{N}} \subset \mathcal{N}$ such that $I(v_n) \rightarrow c_1$ as $n \rightarrow \infty$. It is also known from Lemma 3.2 that

$$c_1 < \frac{1}{3} \mathcal{J}^{\frac{3}{2}} \|k\|_\infty^{-\frac{1}{2}}. \tag{3.39}$$

Since $(v_n)_{n \in \mathbb{N}} \subset \mathcal{N}$, we have that

$$\|v_n\|^2 + F(v_n) - \mu \int_{\mathbb{R}^3} h(x)v_n^2 dx = \int_{\mathbb{R}^3} k(x)|v_n|^6 dx. \tag{3.40}$$

Using (3.40) and Lemma 2.2 (iii), we get that

$$\begin{aligned}
c_1 + o(1) &= \frac{1}{2} \left(\|v_n\|^2 - \mu \int_{\mathbb{R}^3} h(x)v_n^2 dx \right) + \frac{1}{4} F(v_n) - \frac{1}{6} \int_{\mathbb{R}^3} k(x)|v_n|^6 dx \\
&= \frac{1}{N} \left(\|v_n\|^2 - \mu \int_{\mathbb{R}^3} h(x)v_n^2 dx \right) + \left(\frac{1}{4} - \frac{1}{6} \right) F(v_n) \\
&\geq \frac{1}{N} \left(1 - \frac{\mu}{\bar{\mu}} \right) \|v_n\|^2,
\end{aligned}$$

which implies $(v_n)_{n \in \mathbb{N}}$ is bounded in $H^1(\mathbb{R}^3)$, since $0 < \mu < \bar{\mu}$. Going if necessary to a subsequence, we may assume that $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^3)$. Suppose that any subsequence of $(v_n)_{n \in \mathbb{N}}$ does not converge strongly to v in $H^1(\mathbb{R}^3)$ and then by Lemma 3.1 we obtain one of the following three cases:

- (1) $c_1 \geq \frac{1}{3} \mathcal{J}^{\frac{3}{2}} \|k\|_{\infty}^{-\frac{1}{2}}$ if $v = 0$;
 (2) $c_1 > I(t_v v)$ if $v \neq 0$ and $\langle I'(v), v \rangle < 0$;
 (3) $c_1 > \frac{1}{3} \mathcal{J}^{\frac{3}{2}} \|k\|_{\infty}^{-\frac{1}{2}}$ if $v \neq 0$ and $\langle I'(v), v \rangle \geq 0$.

However, from (3.39) we know that both cases (1) and (3) do not occur. Moreover, from the definition of t_v , we know that $t_v v \in \mathcal{N}$. So $I(t_v v) \geq c_1$, and then $c_1 > I(t_v v) \geq c_1$ by (2), which is a contradiction. Hence the second case (2) is also impossible to happen. Therefore there must exist some subsequence of $(v_n)_{n \in \mathbb{N}}$ converging strongly to v in $H^1(\mathbb{R}^3)$. Going if necessary to a subsequence, we may assume that $v_n \rightarrow v$ in $H^1(\mathbb{R}^3)$. By the Sobolev inequality, Lemma 2.2 (iii) and (3.40), we arrive at

$$\left(1 - \frac{\mu}{\bar{\mu}}\right) \|v_n\|^2 \leq \int_{\mathbb{R}^3} k(x) |v_n|^6 dx \leq \|k\|_{\infty} (\mathcal{J}^{-1} \|v_n\|^2)^3. \quad (3.41)$$

Since $(v_n)_{n \in \mathbb{N}} \subset \mathcal{N}$, $\|v_n\| \neq 0$ for any $n \in \mathbb{N}$ and hence by (3.41) we have $\|v_n\| \geq C$ for some positive C , which depends on the constants $\mu, \bar{\mu}, \|k\|_{\infty}$ and \mathcal{J} . Therefore $v \neq 0$ and then $v \in \mathcal{N}$ and $I(v) = c_1$. By the Lagrange multiplier rule, there exists $\theta \in \mathbb{R}$ such that $I'(v) = \theta G'(v)$. Then

$$0 = \langle I'(v), v \rangle = \theta \left(2\|v\|^2 + 4F(v) - 6 \int_{\mathbb{R}^3} k(x) |v|^6 dx - 2\mu \int_{\mathbb{R}^3} h(x) |v|^2 dx \right),$$

which implies that

$$\theta \left(-4 \left(\|v\|^2 - \mu \int_{\mathbb{R}^3} h(x) |v|^2 dx \right) - 2F(v) \right) = 0$$

since $v \in \mathcal{N}$. Then $\theta = 0$ and hence v is a nontrivial critical point of the functional I in $H^1(\mathbb{R}^3)$.

Note that if $(v_n)_{n \in \mathbb{N}} \subset \mathcal{N}$ and $I(v_n) \rightarrow c_1$ as $n \rightarrow \infty$, then $(|v_n|)_{n \in \mathbb{N}} \subset \mathcal{N}$ and $I(|v_n|) \rightarrow c_1$ as $n \rightarrow \infty$. Hence we may assume that $v \geq 0$ in \mathbb{R}^3 . By standard arguments as in DiBenedetto [18] and Tolksdorf [33], we have that $v \in L^{\infty}(\mathbb{R}^3)$ and $v \in C_{loc}^{1,\omega}(\mathbb{R}^3)$ with $0 < \omega < 1$. Furthermore, by Harnack's inequality (see Trudinger [34]), $v(x) > 0$ for any $x \in \mathbb{R}^3$. Thus v is a positive critical point of I in $H^1(\mathbb{R}^3)$. We finish the proof of Theorem 3.3 by choosing $\psi_1 = v$. \square

4. Existence of sign-changing solutions

In this section, we will prove the existence of sign-changing solutions of (1.1). A function w is called sign-changing if $w^+ \neq 0$ and $w^- \neq 0$, where $w^+ = \max\{w, 0\}$ and $w^- = \max\{-w, 0\}$. Denote

$$\mathcal{N}_* = \{w = w^+ - w^- \in H^1(\mathbb{R}^3) : w^+ \in \mathcal{N}, w^- \in \mathcal{N}\}$$

and define

$$c_2 = \inf_{w \in \mathcal{N}_*} I(w).$$

We will prove that c_2 is achieved at some point ψ_2 and ψ_2 is a sign-changing critical point of the functional I in $H^1(\mathbb{R}^3)$.

Lemma 4.1. Suppose that the hypotheses (H) hold with $\frac{3}{2} < \beta < 3$. Then $c_2 < c_1 + \frac{1}{3} \mathcal{J}^{\frac{3}{2}} \|k\|_{\infty}^{-\frac{1}{2}}$.

Proof. We are going to find an element in \mathcal{N}_* such that the value of I at this element is strictly less than $c_1 + \frac{1}{3} \mathcal{J}^{\frac{3}{2}} \|k\|_{\infty}^{-\frac{1}{2}}$. Let ψ_1 be the positive critical point of I obtained in Theorem 3.3 and v_{ε} be constructed in Lemma 3.2.

First, we claim that there exist $a_0 > 0$ and $b_0 \in \mathbb{R}$ such that

$$a_0 \psi_1 + b_0 v_{\varepsilon} \in \mathcal{N}_*. \quad (4.42)$$

In fact, denote $\varphi(s) = \psi_1 + s v_{\varepsilon}$ with $s \in \mathbb{R}$, and define $s_1 \in [-\infty, +\infty)$ and $s_2 \in (-\infty, +\infty]$ by $s_1 = \inf\{s \in \mathbb{R} : \varphi^+(s) \neq 0\}$ and $s_2 = \sup\{s \in \mathbb{R} : \varphi^-(s) \neq 0\}$. We know $s_1 < s_2$, because $\varphi(s)$ is strictly increasing. Since $t(\varphi^+(s)) - t(\varphi^-(s)) \rightarrow +\infty$ as $s \rightarrow s_1 + 0$ and $t(\varphi^+(s)) - t(\varphi^-(s)) \rightarrow -\infty$ as $s \rightarrow s_2 - 0$ by (3) and (4) of Lemma 2.4, there exists $s_0 \in (s_1, s_2)$ such that $t(\varphi^+(s_0)) = t(\varphi^-(s_0))$ by (3) of Lemma 2.4. Thus

$$\begin{aligned} t(\varphi^+(s_0)) \varphi(s_0) &= t(\varphi^+(s_0)) (\varphi^+(s_0) - \varphi^-(s_0)) \\ &= t(\varphi^+(s_0)) \varphi^+(s_0) - t(\varphi^-(s_0)) \varphi^-(s_0) \in \mathcal{N}_*. \end{aligned}$$

By the definition of t_u , we have $t(\varphi^+(s)) > 0$, which implies that (4.42) holds.

Second, we claim that there is $\varepsilon > 0$ such that

$$\sup_{a>0, b \in \mathbb{R}} I(a \psi_1 + b v_{\varepsilon}) < c_1 + \frac{1}{3} \mathcal{J}^{\frac{3}{2}} \|k\|_{\infty}^{-\frac{1}{2}}. \quad (4.43)$$

In fact, it follows from (I_2) of [Lemma 2.3](#) that for any $a > 0$, $b \in \mathbb{R}$ such that $\|a\psi_1 + bv_\varepsilon\| > \rho_*$ we have $I(a\psi_1 + bv_\varepsilon) < 0$. Thus it suffices to consider the case that $\|a\psi_1 + bv_\varepsilon\| \leq \rho_*$, which means that it is sufficient to consider that a and b are contained in a bounded interval. Since ψ_1 is a solution of [\(1.2\)](#), it holds

$$\begin{aligned} & \int_{\mathbb{R}^3} (\nabla(a\psi_1)\nabla(bv_\varepsilon) + (a\psi_1)(bv_\varepsilon) - \mu h(x)(a\psi_1)(bv_\varepsilon)) dx \\ &= ab \left(\int_{\mathbb{R}^3} k(x)|\psi_1|^5 v_\varepsilon dx - \int_{\mathbb{R}^3} l(x)\phi_{\psi_1}\psi_1 v_\varepsilon dx \right). \end{aligned} \quad (4.44)$$

Let

$$\begin{aligned} g_{1,\varepsilon}(b) &= \frac{1}{2} \int_{\mathbb{R}^3} |\nabla(bv_\varepsilon)|^2 dx - \frac{1}{6} \int_{\mathbb{R}^3} k(x_0)|bv_\varepsilon|^6 dx, \\ g_{2,\varepsilon}(b) &= \frac{1}{6} \int_{\mathbb{R}^3} k(x_0)|bv_\varepsilon|^6 dx - \frac{1}{6} \int_{\mathbb{R}^3} k(x)|bv_\varepsilon|^6 dx, \\ g_{3,\varepsilon}(a, b) &= \frac{1}{4} F(a\psi_1 + bv_\varepsilon) - \frac{1}{4} F(a\psi_1) \end{aligned}$$

and

$$g_{4,\varepsilon}(a, b) = \frac{1}{6} \int_{\mathbb{R}^3} k(x) (|a\psi_1|^6 + |bv_\varepsilon|^6 - |a\psi_1 + bv_\varepsilon|^6) dx.$$

It follows from the Calculus Lemma [\[19\]](#) that

$$g_{4,\varepsilon}(a, b) \leq C \int_{\mathbb{R}^3} k(x) (|\psi_1|^5 v_\varepsilon + \psi_1 |v_\varepsilon|^5) dx. \quad (4.45)$$

Using [\(4.44\)](#), [\(4.45\)](#) and [Corollary 2.5](#), we deduce that

$$\begin{aligned} I(a\psi_1 + bv_\varepsilon) &= I(a\psi_1) + g_{1,\varepsilon}(b) + g_{2,\varepsilon}(b) + g_{3,\varepsilon}(a, b) + g_{4,\varepsilon}(a, b) \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^3} |bv_\varepsilon|^2 dx - \frac{\mu}{2} \int_{\mathbb{R}^3} h(x)|bv_\varepsilon|^2 dx \\ &\quad + \int_{\mathbb{R}^3} (\nabla(a\psi_1)\nabla(bv_\varepsilon) + (a\psi_1)(bv_\varepsilon) - \mu h(x)(a\psi_1)(bv_\varepsilon)) dx \\ &\leq I(\psi_1) + g_{1,\varepsilon}(b) + g_{2,\varepsilon}(b) + g_{3,\varepsilon}(a, b) + \frac{1}{2} \int_{\mathbb{R}^3} |bv_\varepsilon|^2 dx \\ &\quad + C \int_{\mathbb{R}^3} k(x)|v_\varepsilon|^5 \psi_1 dx + C \int_{\mathbb{R}^3} k(x)|\psi_1|^5 v_\varepsilon dx \\ &\quad + C \int_{\mathbb{R}^3} l(x)\phi_{\psi_1}\psi_1 v_\varepsilon dx - \frac{\mu}{2} \int_{\mathbb{R}^3} h(x)|bv_\varepsilon|^2 dx, \end{aligned} \quad (4.46)$$

By [\(3.35\)](#) we obtain that

$$\sup_{b \in \mathbb{R}} g_{1,\varepsilon}(b) = \frac{1}{3} \mathcal{J}^{\frac{3}{2}} \|k\|_\infty^{-\frac{1}{2}} + O\left(\varepsilon^{\frac{1}{2}}\right). \quad (4.47)$$

Since b is bounded and $1 \leq \alpha < 3$, we get from [\(3.36\)](#) that

$$g_{2,\varepsilon}(b) = \frac{1}{6} \int_{\mathbb{R}^3} k(x_0)|bv_\varepsilon|^6 dx - \frac{1}{6} \int_{\mathbb{R}^3} k(x)|bv_\varepsilon|^6 dx \leq C\varepsilon^{\frac{1}{2}}. \quad (4.48)$$

By [\(3.32\)](#) and the fact of $\psi_1 \in H^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, we obtain that

$$\int_{\mathbb{R}^3} k(x)|v_\varepsilon|^5 \psi_1 dx \leq \|k\|_\infty \|\psi_1\|_\infty \int_{\mathbb{R}^3} |v_\varepsilon|^5 dx \leq C\varepsilon^{\frac{1}{4}}, \quad (4.49)$$

and

$$\int_{\mathbb{R}^3} k(x)|\psi_1|^5 v_\varepsilon dx \leq \|k\|_\infty \|\psi_1^5\|_\infty \int_{\mathbb{R}^3} v_\varepsilon dx \leq C\varepsilon^{\frac{1}{4}}. \quad (4.50)$$

We claim

$$g_{3,\varepsilon}(a, b) \leq C\varepsilon^{\frac{1}{4}}. \quad (4.51)$$

Actually by calculation we arrive at

$$\begin{aligned} g_{3,\varepsilon}(a, b) &= \frac{1}{4} \int_{\mathbb{R}^3} l(x) (\phi_{a\psi_1+bv_\varepsilon} (a\psi_1 + bv_\varepsilon)^2 - \phi_{a\psi_1} (a\psi_1)^2) dx \\ &= ab \int_{\mathbb{R}^3} l(x) \phi_{a\psi_1} \psi_1 v_\varepsilon dx + ab \int_{\mathbb{R}^3} l(x) \phi_{bv_\varepsilon} \psi_1 v_\varepsilon dx \\ &\quad + \frac{1}{2} b^2 \int_{\mathbb{R}^3} l(x) \phi_{a\psi_1} (v_\varepsilon)^2 + \frac{1}{4} b^2 \int_{\mathbb{R}^3} l(x) \phi_{bv_\varepsilon} (v_\varepsilon)^2 dx \\ &\quad + a^2 b^2 \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{1}{|x-y|} l(y) \psi_1(y) v_\varepsilon(y) l(x) \psi_1(x) v_\varepsilon(x) dx dy. \end{aligned} \quad (4.52)$$

Using the Hölder inequality, (2.6), (3.31), (3.32) and the fact that a, b are bounded, we obtain that

$$\int_{\mathbb{R}^3} l(x) \phi_{a\psi_1} \psi_1 v_\varepsilon dx \leq \|l\|_\infty \|\phi_{a\psi_1}\|_6 \|\psi_1\|_{12/5} \|v_\varepsilon\|_{12/5} \leq C \|v_\varepsilon\|_{12/5} \leq C \varepsilon^{\frac{1}{4}}, \quad (4.53)$$

$$\int_{\mathbb{R}^3} l(x) \phi_{bv_\varepsilon} \psi_1 v_\varepsilon dx \leq \|l\|_\infty \|\phi_{bv_\varepsilon}\|_6 \|\psi_1\|_{12/5} \|v_\varepsilon\|_{12/5} \leq C \|v_\varepsilon\|_{12/5}^3 \leq C \varepsilon^{\frac{3}{4}}, \quad (4.54)$$

$$\int_{\mathbb{R}^3} l(x) \phi_{bv_\varepsilon} (v_\varepsilon)^2 dx \leq \|l\|_\infty \|\phi_{bv_\varepsilon}\|_6 \|v_\varepsilon\|_{12/5}^2 \leq C \|v_\varepsilon\|_{12/5}^4 \leq C \varepsilon, \quad (4.55)$$

and

$$\int_{\mathbb{R}^3} l(x) \phi_{a\psi_1} (v_\varepsilon)^2 dx \leq \|l\|_\infty \|\phi_{a\psi_1}\|_6 \|v_\varepsilon\|_{12/5}^2 \leq C \varepsilon^{\frac{1}{2}}. \quad (4.56)$$

Moreover, by Lemma [29, p. 31], it holds

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{l(y) \psi_1(y) v_\varepsilon(y) l(x) \psi_1(x) v_\varepsilon(x)}{|x-y|} dx dy \leq \left(\int_{\mathbb{R}^3} |l(x) \psi_1(x) v_\varepsilon(x)|^{\frac{6}{5}} dx \right)^{\frac{5}{3}} \leq C \|\psi_1\|_{\frac{12}{5}}^2 \|v_\varepsilon\|_{\frac{12}{5}}^2 \leq C \varepsilon^{\frac{1}{2}}. \quad (4.57)$$

It follows from (4.52)–(4.57) that the claim (4.51) holds. Hence combining (3.38) with (4.46)–(4.51), for $\frac{3}{2} < \beta < 3$, we obtain that

$$\begin{aligned} I(a\psi_1 + bv_\varepsilon) &\leq I(\psi_1) + \frac{1}{3} \delta^{\frac{3}{2}} \|k\|_\infty^{-\frac{1}{2}} - C \varepsilon^{1-\frac{\beta}{2}} + C \varepsilon^{\frac{1}{4}} \\ &< I(\psi_1) + \frac{1}{3} \delta^{\frac{3}{2}} \|k\|_\infty^{-\frac{1}{2}} = c_1 + \frac{1}{3} \delta^{\frac{3}{2}} \|k\|_\infty^{-\frac{1}{2}}, \end{aligned} \quad (4.58)$$

as $\varepsilon \rightarrow 0$. Hence the claim (4.43) follows. This proves Lemma 4.1. \square

Lemma 4.2. *If the hypotheses (H) hold with $\frac{3}{2} < \beta < 3$, then there exists ψ_2 in \mathcal{N}_* such that $I(\psi_2) = c_2$.*

Proof. From the definition of c_2 we may assume that there exists $(w_n)_{n \in \mathbb{N}} \subset \mathcal{N}_*$ such that $I(w_n) \rightarrow c_2$. And we may assume that there exist constants d_1 and d_2 such that $I(w_n^+) \rightarrow d_1$ and $I(w_n^-) \rightarrow d_2$ and $c_2 = d_1 + d_2$. By the definition of c_1 , w_n^+ and w_n^- , it holds that

$$d_1 \geq c_1 \quad \text{and} \quad d_2 \geq c_1. \quad (4.59)$$

Just as the proof of Theorem 3.3, there are positive constants C_1, C_2, C_3 and C_4 such that

$$C_1 \leq \|w_n^+\| \leq C_2 \quad \text{and} \quad C_3 \leq \|w_n^-\| \leq C_4. \quad (4.60)$$

Going if necessary to a subsequence, we may assume that $w_n^+ \rightharpoonup w^+$ and $w_n^- \rightharpoonup w^-$. If $w^+ = 0$ or $w^- = 0$, by (1) of Lemma 3.1 and (4.59), we obtain that

$$c_1 + \frac{1}{3} \delta^{\frac{3}{2}} \|k\|_\infty^{-\frac{1}{2}} \leq d_1 + d_2 = c_2,$$

which contradicts Lemma 4.1. Hence we may assume that $w^+ \neq 0$ and $w^- \neq 0$. Using Lemma 3.1, we get one of the following:

- (I₁) there is a subsequence of $(w_n^+)_{n \in \mathbb{N}}$ converging strongly to w^+ in $H^1(\mathbb{R}^3)$;
- (I₂) $d_1 > I(t_{w^+} w^+)$ if $w^+ \neq 0$ and $\langle I'(w^+), w^+ \rangle < 0$;
- (I₃) $d_1 > \frac{1}{3} \delta^{\frac{3}{2}} \|k\|_\infty^{-\frac{1}{2}}$ if $w^+ \neq 0$ and $\langle I'(w^+), w^+ \rangle \geq 0$;

and we also have one of the following:

- (II₁) there is a subsequence of $(w_n^-)_{n \in \mathbb{N}}$ converging strongly to w^- in $H^1(\mathbb{R}^3)$;
- (II₂) $d_2 > I(t_{w^-} w^-)$ if $w^- \neq 0$ and $\langle I'(w^-), w^- \rangle < 0$;
- (II₃) $d_2 > \frac{1}{3} \delta^{\frac{3}{2}} \|k\|_{\infty}^{-\frac{1}{2}}$ if $w^- \neq 0$ and $\langle I'(w^-), w^- \rangle \geq 0$.

We claim that only (I₁) and (II₁) happen. In fact, if the pair (I₁) and (II₂) or the pair (I₂) and (II₂) holds, then from $w^+ - t_{w^-} w^- \in \mathcal{N}_*$ or $t_{w^+} w^+ - t_{w^-} w^- \in \mathcal{N}_*$ respectively, we arrive at respectively

$$c_2 \leq I(w^+ - t_{w^-} w^-) = I(w^+) + I(t_{w^-} w^-) < d_1 + d_2 = c_2$$

or

$$c_2 \leq I(t_{w^+} w^+ - t_{w^-} w^-) = I(t_{w^+} w^+) + I(t_{w^-} w^-) < d_1 + d_2 = c_2.$$

Any one of the above two inequalities is not true. If the pair (I₁) and (II₃) or the pair (I₂) and (II₃), or the pair (I₃) and (II₃) occurs, then by Lemma 3.2 we obtain the following three possibilities:

$$c_1 + \frac{1}{3} \delta^{\frac{3}{2}} \|k\|_{\infty}^{-\frac{1}{2}} \leq I(w^+) + \frac{1}{3} \delta^{\frac{3}{2}} \|k\|_{\infty}^{-\frac{1}{2}} < d_1 + d_2 = c_2;$$

$$c_1 + \frac{1}{3} \delta^{\frac{3}{2}} \|k\|_{\infty}^{-\frac{1}{2}} \leq I(t_{w^+} w^+) + \frac{1}{3} \delta^{\frac{3}{2}} \|k\|_{\infty}^{-\frac{1}{2}} < d_1 + d_2 = c_2;$$

$$c_1 + \frac{1}{3} \delta^{\frac{3}{2}} \|k\|_{\infty}^{-\frac{1}{2}} \leq \frac{1}{3} \delta^{\frac{3}{2}} \|k\|_{\infty}^{-\frac{1}{2}} + \frac{1}{3} \delta^{\frac{3}{2}} \|k\|_{\infty}^{-\frac{1}{2}} < d_1 + d_2 = c_2,$$

while any one of the above three possibilities contradicts the conclusions of Lemma 4.1. Hence all the pairs (I₁) and (II₂), (I₂) and (II₂), (I₁) and (II₃), (I₂) and (II₃), (I₃) and (II₃) do not occur. The pairs (I₂) and (II₁), (I₃) and (II₁), (I₃) and (II₂) also do not happen by a similar proof. Since we have considered all the cases, we know that only the pair (I₁) and (II₁) holds. We may assume that $w_n^+ \rightarrow w^+$ strongly in $H^1(\mathbb{R}^3)$ and $w_n^- \rightarrow w^-$ strongly in $H^1(\mathbb{R}^3)$. From these and (4.60) we have that $w^+ \neq 0$ and $w^- \neq 0$. Set $\psi_2 = w^+ - w^-$. Then $\psi_2 \in \mathcal{N}_*$ and $I(\psi_2) = d_1 + d_2 = c_2$. This proves Lemma 4.2. \square

According to Lemma 4.2 we know that $\psi_2 \in \mathcal{N}_*$ is sign-changing and $I(\psi_2) = c_2$. Since \mathcal{N}_* usually is not a manifold, the Lagrange multiplier rule may not be applied. In order to show that ψ_2 is a critical point of the functional I in $H^1(\mathbb{R}^3)$, i.e., $I'(\psi_2) = 0$, we need an idea from [10,21].

Theorem 4.3. *If the hypotheses (H) hold with $\frac{3}{2} < \beta < 3$, then ψ_2 is a sign-changing critical point of the functional I in $H^1(\mathbb{R}^3)$.*

Proof. Suppose that ψ_2 is not a critical point of I , i.e., $I'(\psi_2) \neq 0$. For any $u \in \mathcal{N}$, we have that

$$\begin{aligned} \int_{\mathbb{R}^3} k(x)|u|^6 dx &= \|u\|^2 + F(u) - \mu \int_{\mathbb{R}^3} h(x)|u|^2 dx \\ &\geq \left(1 - \frac{\mu}{\tilde{\mu}}\right) \|u\|^2 + F(u) \geq F(u) \end{aligned}$$

and then

$$\begin{aligned} \langle G'(u), u \rangle &= 2 \left(\|u\|^2 - \mu \int_{\mathbb{R}^3} h(x)|u|^2 dx \right) + 4F(u) - 6 \int_{\mathbb{R}^3} k(x)|u|^6 dx \\ &= -4 \int_{\mathbb{R}^3} k(x)|u|^6 dx + 2F(u) \leq -2F(u) < 0. \end{aligned} \quad (4.61)$$

Therefore, for any $u \in \mathcal{N}$, $\|G'(u)\|_{H^{-1}} = \sup_{\|v\|=1} |\langle G'(u), v \rangle| \neq 0$. Set

$$\Phi(u) = I'(u) - \left\langle I'(u), \frac{G'(u)}{\|G'(u)\|} \right\rangle \frac{G'(u)}{\|G'(u)\|}, \quad u \in \mathcal{N} \quad (4.62)$$

Then we get that $\Phi(\psi_2) \neq 0$. In fact, if $\Phi(\psi_2) = 0$, then, by (4.61) and (4.62), it holds that

$$0 = \langle I'(\psi_2), \psi_2 \rangle = \left\langle I'(\psi_2), \frac{G'(\psi_2)}{\|G'(\psi_2)\|} \right\rangle \left\langle \frac{G'(\psi_2)}{\|G'(\psi_2)\|}, \psi_2 \right\rangle \neq 0,$$

which is a contradiction. Let $\delta \in (0, \min\{\|\psi_2^+\|, \|\psi_2^-\|\}/3)$ such that

$$\|\Phi(v) - \Phi(\psi_2)\| \leq \frac{1}{2} \|\Phi(\psi_2)\| \quad \text{for each } v \in \mathcal{N} \text{ with } \|v - \psi_2\| \leq 2\delta.$$

Let $\chi : \mathcal{N} \rightarrow [0, 1]$ be a Lipschitz mapping such that

$$\chi(v) = \begin{cases} 1, & \text{for } v \in \mathcal{N} \text{ with } \|v - \psi_2\| \leq \delta, \\ 0, & \text{for } v \in \mathcal{N} \text{ with } \|v - \psi_2\| \geq 2\delta. \end{cases}$$

Let $\eta : [0, s_0] \times \mathcal{N} \rightarrow \mathcal{N}$ be the solution of the differential equation

$$\eta(0, v) = v, \quad \frac{d\eta(s, v)}{ds} = -\chi(\eta(s, v))\Phi(\eta(s, v)), \quad \text{for } (s, v) \in [0, s_0] \times \mathcal{N},$$

where s_0 is a positive number. We set

$$r(\tau) = t((1 - \tau)\psi_2^+ + \tau\psi_2^-)((1 - \tau)\psi_2^+ + \tau\psi_2^-)$$

and $\sigma(\tau) = \eta(s_0, r(\tau))$, for $0 \leq \tau \leq 1$. If $\tau \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, we have

$$I(\sigma(\tau)) \leq I(r(\tau)) = I(r^+(\tau)) + I(r^-(\tau)) < I(\psi_2^+) + I(\psi_2^-) = I(\psi_2),$$

and $I(\sigma(\frac{1}{2})) = I(r(\frac{1}{2})) < I(\psi_2)$, i.e., $I(\sigma(\tau)) < I(\psi_2)$ for all $\tau \in (0, 1)$. Since $t(\sigma^+(\tau)) - t(\sigma^-(\tau)) \rightarrow -\infty$ as $\tau \rightarrow 0+0$ and $t(\sigma^+(\tau)) - t(\sigma^-(\tau)) \rightarrow +\infty$ as $\tau \rightarrow 1-0$, there exists $\tau_1 \in (0, 1)$ such that $t(\sigma^+(\tau_1)) = t(\sigma^-(\tau_1))$. Then we have $\sigma(\tau_1) \in \mathcal{N}_*$ and $I(\sigma(\tau_1)) < I(\psi_2)$, which is a contradiction. This proves [Theorem 4.3](#). \square

Remark 2. If (ψ, ϕ_ψ) is the solution of (1.1), then $(-\psi, \phi_{-\psi})$ is also its solution. Hence, by [Theorems 3.3](#) and [4.3](#), we know that (1.1) has at least one pair of fixed sign solutions and at least one pair of sign-changing solutions under the hypotheses (H) with $\frac{3}{2} < \beta < 3$.

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