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On generalized stable and related laws

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ABSTRACT

A probabilistic characterization is given for Schneider's generalized stable laws. They belong to the larger family of laws which are invariant under length-biasing followed by a random beta scaling. Questions of infinite divisibility and self-decomposability are pursued. Moment determinacy of reciprocal generalized stable laws are investigated.

1. Introduction

Janson [10] says that the law $L(X)$ of a positive random variable X has moments of gamma type if its moment function has the form

$$M(t) := E\left(X^t\right) = CD^t \frac{\prod_{j=1}^{J} \Gamma(a_j t + b_j)}{\prod_{k=1}^{K} \Gamma(a'_k t + b'_k)}$$

for some integers $J, K \geqslant 0$ and real constants $C, D > 0$, a_j, b_j, a'_k, b'_k. He shows by example that many common and not so common laws have moments of gamma type. Janson [11] is a supplement with further examples. Janson [10, Theorem 5.4] shows that $L(X)$ has an infinitely differentiable density function if $\sum_{j=1}^{J} |a_j| - \sum_{k=1}^{K} |a'_k| > 0$ which condition will hold for cases occurring in this paper. The case where $a_j = a'_k = 1$ and $b_j, b'_k > 0$ are called Dufresne laws; see Chamayou [4] and references therein.

The positive stable laws comprise an important subclass of laws with moments of gamma type. Let $0 < \alpha < 1$ and S_α be a random variable having the standard stable law with density function $\sigma_\alpha(x)$, meaning that its Laplace–Stieltjes transform is

$$\widehat{\sigma}_\alpha(\theta) = E\left(e^{-\theta S_\alpha}\right) = e^{-\theta^\alpha}. \tag{1.1}$$

It is known that (Shanbhag and Sreehari [27, Corollary 1])

$$E\left(S_\alpha^{-t}\right) = \frac{\Gamma(1 + t/\alpha)}{\Gamma(1 + t)} \quad (t > -\alpha). \tag{1.2}$$

See Janson [10, Example 3.10] for the 'obvious' simple derivation using (1.1). So $a_1 = -\alpha^{-1}$ and $-a'_1 = b_1 = b'_1 = 1$ for stable(α). It follows too from (1.1) that

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$$-\frac{d}{d\theta}\widehat{\sigma}_\alpha(\theta) = \alpha\theta^{\alpha-1}\widehat{\sigma}_\alpha(\theta), \tag{1.3}$$

equivalently, that

$$y\sigma_\alpha(y) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^y (y-v)^{-\alpha}\sigma_\alpha(v)\,dv. \tag{1.4}$$

The stable(α) law is an infinitely divisible law (abbreviated infdiv law with definitions recalled in Section 2) whose Lévy measure has support $[0,\infty)$ and density $\ell_\alpha(y) = (\alpha/\Gamma(1-\alpha))y^{-\alpha-1}$. So (1.4) asserts that the y-tilt of σ_α equals the convolution of σ_α and the y-tilt of ℓ_α. A general form of this relation characterizes any positive infdiv law; see (9.3) below or Steutel and van Harn [29, Theorem 4.17].

Infdiv laws in general have a central importance in the limit theory of sums of independent random variables and stochastic modelling. Consequently they have been extended in various ways. Lévy's semistable laws comprise the earliest such extension. The contemporary approach to these is through finding all characteristic function solutions of functional equations satisfied by the characteristic function of symmetric stable laws. See Ramachandran and Lau [22, Chapter 3]. It follows from this approach that semistable laws coincide with those infinitely divisible laws whose Lévy measure is formed from a multiplicatively periodic modulation of a stable Lévy measure. Extending still further to almost periodic modulation gives so-called pseudostable laws; see Uchaikin and Zolotarev [31, p. 191]. Semistable laws are generalized in another direction (Ramachandran and Lau [22]) by starting from more general functional equations.

Modifying stable Lévy measures arises in other contexts. For example, the exponentially tilted density function $\sigma_{\alpha,c}(x) \propto e^{-cx}\sigma_\alpha(x)$ $(x>0)$ corresponds to the infinitely divisible law whose Lévy measure has the density proportional to $e^{-cx}x^{-\alpha-1}$. This family of laws includes the inverse Gaussian ($\alpha = \frac{1}{2}$), and they were introduced for modelling survival data (Hougaard [9]). Similar two-sided exponential thinnings of the Lévy density of a general stable law yields the so-called truncated, or tempered, stable laws advocated by some for modelling financial returns. See Cont and Tankov [6, pp. 110, 119] and Pakes [17].

Our goal in this paper is exploring the generalization of (1.4) obtained by replacing the tilting factor y on the left-hand side by the general power y^m where $m>0$. More specifically, is there a (positive) law $L(X)$ having a density function f solving the integral equation

$$y^m f(y) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^y (y-v)^{-\alpha} f(v)\,dv.$$

Schneider [26] restricts $m = 1, 2, \ldots$ in which case a positive solution for the integral equation is equivalent to showing there is a completely monotone solution $\widehat{f}$ of the differential equation

$$\left(-\frac{d}{d\theta}\right)^m \widehat{f}(\theta) = \alpha\theta^{\alpha-1}\widehat{f}(\theta). \tag{1.5}$$

Schneider [25] gives two reasons for pursuing this extension. The first is to show that σ_α is the first member of a sequence of density functions (indexed by m) which have an 'explicit' form in terms of Fox H-functions, i.e., inverses of Mellin transforms (1.1); see Mathai [14]. This relation is not mentioned in the main essay Janson [10] but it is in the supplement. Second, the reciprocal generalized stable laws in the case $m = 2$ arise in connection with particle transport along the one-dimensional lattice; see Bernasconi et al. [1].

The proof method in Schneider [25] is by way of an analytical treatment of the integral equation satisfied by $g(x) = x^{-2}f(x^{-1})$, i.e., ignoring an essentially arbitrary normalization factor,

$$g(x) = x^{m+\alpha-2}\int_x^\infty (z-x)^{-\alpha}z^\alpha g(z)\,dz. \tag{1.6}$$

This equation with $m = 2$ is essentially (27) in Bernasconi et al. [1]. Schneider shows that the Mellin transform of g has the form (1.1) with $J = m$ and $K = 1$. There is more detail in Schneider [26] where he explains (somewhat obscurely) that (1.5) has a unique completely monotone solution. Defining $a = m+\alpha-1$, a positive law $L(X)$ whose density function satisfies (1.6) is called a reciprocal generalized stable law with parameters a and m, abbreviated r-gstable(a,m), and the law of $Y = X^{-1}$ is a generalized stable law (in the sense of Schneider), abbreviated gstable(a,m). It transpires that these laws are well-defined if $-(m-1) < \alpha < 1$. Schneider [26] shows that gstable(a,m) laws arise as limit laws of a certain infinite dimensional non-linear system.

Questions which are left open include:

- Is there a more direct probabilistic construction?
- What is the relation, if any, with the stable(α) law?
- Are generalized stable laws infdiv, or even self-decomposable?

We provide some direct answers starting from results in Pakes [16] about characterizing probability laws by invariance under length-biasing (or size-biasing) followed by an independent random contraction, i.e., multiplication by an independent factor $B \in (0,1)$. The salient facts are recalled in Section 2, as well as some about infinite divisibility and self-decomposability. Proposition 2.1 relates a typology of spectrally-positive self-decomposable laws to their background driving Lévy processes.

The particular case where the contraction factor B has a beta(a,b) law is discussed in Section 3. If $r > 0$ denotes the order of length-biasing, then there is a unique scale family $\mathcal{L}(a,b,r)$ of laws having the above invariance property. The scaling can be chosen so that $L(X)$ has a density function $g(x)$ satisfying (3.2) below. Theorems 3.1 and 3.2 assert some infinite divisibility properties.

In general X has at least two infinite product representations; compare (3.7) and Lemma 3.2. Certain parameter combinations yield a finite product representation. Two examples are exhibited and in both cases $L(X)$ and $L(X^{-1})$ are infdiv.

Section 4 is devoted to a third finite product reduction giving the gstable(a,m) laws. We begin by explaining how the case $r = a$ and $m := a + b > 0$ yields laws whose Laplace–Stieltjes transforms solve the fractional-order version of (1.5), thus locating our results in the wider context of fractional-order dynamics and statistics. See Klafter et al. [12] or Uchaikin and Sibatov [30] for an overview and references to this area of investigation. The further restriction $m = 1, 2, \ldots$ yields the gstable(a,m) laws (Definition 4.1 and Theorems 4.1 and 4.3). In the case that $0 < a \leqslant 1$ Theorem 4.2 exhibits X as a finite product with S_a as one factor, thus legitimizing the nomenclature in this case.

Some infinite divisibility and self-decomposability results are obtained in Section 5. In particular, the gstable(a,m) is a generalized gamma convolution (defined in Section 5) if $0 < a \leqslant 1$ and $m = 2, 3, \ldots$. Infinite divisibility is an open question if $1 < a < m$.

Some further self-decomposability properties are established in Section 6. Janson [10, Remark 11.2] observes that there exists no general theory of infinite divisibility laws with moments of gamma type. Our results add to the corpus of specific cases. Representations for the density functions of generalized stable laws and related Laplace transforms are obtained for the physically significant case $m = 2$ in Section 7.

If $a_j, a'_k > 0$ in (1.1), then $L(X)$ has finite moments of all positive orders. Janson [10, Remark 11.1] raises the question of whether $L(X)$ is uniquely determined by its moment sequence. He exhibits examples showing that non-uniqueness can occur. Reciprocal generalized stable laws have finite moments of all positive integer orders. Section 8 gives results which completely answer the moment problem for these and the related laws identified as Cases 1 and 2 in Section 3.

Proofs of many of the results are confined to Section 9. There is some minor duplication of notation between sections, but no confusion should result.

2. Preliminaries on characterization by length-biasing

Let $X \geqslant 0$ have the distribution function F and suppose that $E(X^r) < \infty$ for a positive constant r. Define the moment function of X by $M(t) = E(X^t)$ for all real t for which this function takes finite values. Finally, let $r_X = \sup\{x \geqslant 0\colon\ F(x) < 1\}$ be the right-extremity of the law $L(X)$ of X, and similarly for other random variables. The length-bias operator of order $r \geqslant 0$ by definition maps F to the distribution function

$$\widehat{F}_r(x) = \frac{1}{M(r)} \int_0^x z^r\, dF(z) \quad (x \geqslant 0).$$

A random variable having this law is denoted by $\widehat{X}_r$, and it is stochastically larger than X. See Lemma 2.1 in Pakes [16] for this and other properties of the length-bias operator.

Let B be a random variable such that $P(0 \leqslant B \leqslant 1) = 1$. We may ask whether there is a law represented by X such that

$$X \stackrel{L}{=} B\widehat{X}_r, \tag{2.1}$$

where $\stackrel{L}{=}$ denotes equality in law and the factors on the right-hand side are independent. If $P(B = 0) < 1$ and $r_B \leqslant 1$ then the answer is 'Yes', and solutions are unique up to scaling if and only if $r_B = 1$. We always assume the last condition. Solutions satisfy $r_X = \infty$ if and only if $P(B = 1) = 0$. Finally, if X satisfies (2.1), then $M(t) < \infty$ for all positive t.

A solution X of (2.1) can be represented as an infinite product of random variables as follows. Note that we assume without further comment that factors in products of random variables are independent. Let $N(t) = E(B^t)$ denote the moment function of B. If X satisfies (2.1) and $\mathcal{M} = M(r)$, then M satisfies the functional equation

$$M(t) = N(t)\frac{M(r+t)}{M(r)}$$

whose solution has an infinite product form implying the representation

$$X \stackrel{L}{=} \mathcal{M}^{1/r} \prod_{n=0}^{\infty} \left(\frac{N(rn)}{N(r(n+1))}\right)^{1/r} \widehat{B}_{rn}. \tag{2.2}$$

It follows that if B has an absolutely continuous law, then so does X. In addition it can be written in the limiting form

$$X \stackrel{L}{=} \mathcal{M}^{1/r} \lim_{n\to\infty} \big(N(rn)\big)^{-1/r} \prod_{j=0}^{n-1} \widehat{B}_{jr}. \tag{2.3}$$

In summary, *any* non-degenerate law for B such that $r_B = 1$ yields a unique scale family solution for X, and we write $L(X) \in \mathcal{L}(B, r)$ to denote inclusion in the type of law so defined. This gives characterizations of many common laws and we refer to Pakes [16] and references there for examples.

We will be concerned with the case where $B = e^{-V}$ and V has a positive infinitely divisible (infdiv) law. This means that $N(t) = E(e^{-tV}) = e^{-\psi(t)}$, where the cumulant function ψ has the canonical form

$$\psi(t) = \int_0^\infty \big(1 - e^{-tx}\big)\, \nu(dx), \tag{2.4}$$

and the Lévy measure ν satisfies $\int_0^\infty (x \wedge 1)\, \nu(dx) < \infty$. Note that our assumption $r_B = 1$ implies that there is no drift term.

The corresponding representation for solutions of (2.1) is $X = e^{-W}$ where W has a spectrally-positive infdiv law. This means that $M(t) = e^{-\xi(t)}$ where

$$\xi(t) = At + \int_0^\infty \big(1 - e^{-tx} - txe^{-x}\big)\, \Pi(dx), \tag{2.5}$$

and the Lévy measure Π satisfies $\int_0^\infty (x^2 \wedge 1)\, \Pi(dx) < \infty$. The drift constant A is arbitrary because it corresponds to the scaling constant e^{-A} for X. We state the following result which extends Theorem 5.1 in Pakes [16] by dropping an implied restriction that $\int_0^1 x\, \Pi(dx) < \infty$. Its proof is the same as in Pakes [16].

Theorem 2.1. *Let ν and Π be Lévy measures, as in (2.4) and (2.5), respectively, and related by*

$$\Pi(dx) = \big(1 - e^{-rx}\big)^{-1}\, \nu(dx), \tag{2.6}$$

equivalently,

$$\psi(t) = \xi(t) + \xi(r) - \xi(r+t). \tag{2.7}$$

Then any pair of the following statements implies the third:

(a) $X \cong B\widehat{X}_r$;
(b) $B = e^{-V}$ *and V has the positive infdiv law with cumulant function (2.7); and*
(c) $X = e^{-W}$ *where W has a spectrally-positive infdiv law whose Lévy measure is Π as in (2.6).*

Note that the proof that (a) and (b) implies (c) in Pakes [16] gives the specific drift rate

$$A = -r^{-1} \int_0^\infty \big(1 - e^{-rx} - rxe^{-x}\big)\, \Pi(dx),$$

i.e., $M(r) = 1$.

We end this section by recalling a classification of infdiv laws. Suppose that $L(\mathcal{X})$ is an arbitrary spectrally-positive infdiv law with Lévy measure Λ supported in $(0, \infty)$ (so $\int_0^\infty (x^2 \wedge 1)\, \Lambda(dx) < \infty$), and with no Gaussian component. We say that $L(\mathcal{X})$ is of: Type 0 if Λ is a finite measure, Type 1 if Λ is infinite but $\int_0^1 x\, \Lambda(dx) < \infty$, and Type 2 otherwise. This typology reflects the sample path behaviour of the corresponding Lévy process (which will be said to have the same type): a Type 0 process is compound Poisson, and otherwise sample paths have jump times which are dense in the positive reals. Paths of a Type 1 process have bounded variation in bounded intervals and Type 2 process paths have unbounded variation in bounded intervals. An infdiv law whose cumulant function has the form (2.4) cannot be of Type 2; the corresponding Lévy process is a subordinator, i.e., its paths are non-decreasing. A Type 2 infdiv law is two-sided. See Sato [23, p. 65] for these concepts, but note that he uses A, B and C for the more usual 0, 1 and 2.

A spectrally-positive infdiv law $L(\mathcal{X})$ with Lévy density $\ell(x)$ is self-decomposable if and only if $x\ell(x)$ is non-increasing on the positive real line (Steutel and van Harn [29, p. 277]). We say too that a random variable is self-decomposable if its law has this property. In addition, there is a Lévy process $(D_s\colon\ s \geqslant 0)$ called the background driving Lévy process (BDLP), such that $\mathcal{X}$ has the stochastic integral representation

$$\mathcal{X} = \int_0^\infty e^{-s}\, dD_s.$$

The significance of this result is that $L(\mathcal{X})$ is the limiting law of the Ornstein–Uhlenbeck process defined by $d\mathcal{X}_s = -\mathcal{X}_s\, ds + dD_s$. See Sato [23, §17] and the penultimate paragraph on p. 426.

The Lévy measure N_b of the BDLP is given by

$$N_b(x,\infty) = x\ell(x) \quad \text{if } x > 0, \tag{2.8}$$

and, since we limit ourselves to spectrally-positive laws, $N_b(-\infty, 0) = 0$, i.e., the BDLP also is spectrally positive.

The following result relates the typologies of $L(\mathcal{X})$ and its BDLP.

Proposition 2.1.

(a) *The BDLP is Type* 2 *if and only if* $L(\mathcal{X})$ *is Type* 2.
(b) *If* $L(\mathcal{X})$ *is Type* 1, *then the BDLP is Type* 0 *if and only if*

$$\lim_{x\to 0} x\ell(x) = N_b(0,\infty) < \infty. \tag{2.9}$$

Proof. It follows from (2.8) that

$$\int_0^1 x\ell(x)\, dx = N_b(1,\infty) + \int_0^1 \int_x^{1+} N_b(dy)\, dx = N_b(1) + \int_0^{1+} y\, N_b(dy),$$

and (a) follows. This identity implies that if $L(\mathcal{X})$ is Type 1, then the BDLP is Type 0 or Type 1 and, from (2.8), the former occurs if and only if (2.9) holds. $\square$

3. Beta scaling

Let beta(a,b) denote the beta law with parameters $a, b > 0$, i.e., its density function is

$$h(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)} \quad (0 < x < 1)$$

and $B(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the beta function. The moment function of this law is

$$N(t) = \frac{B(a+t,b)}{B(a,b)} = \frac{\Gamma(a+t)}{\Gamma(a+b+t)} \cdot \frac{\Gamma(a+b)}{\Gamma(a)}. \tag{3.1}$$

The right-hand side is identically equal to unity if $b = 0$, and hence we can consistently define the beta$(a,0)$ law as the point mass at unity, i.e., $P(B=1) = 1$.

We use notation such as $Z \cong (\cdot)$ to mean that the random variable has the law specified by $(\cdot)$ which can be a named law or a distribution function. The following result is easily proved.

Lemma 3.1. *If* $B \cong \text{beta}(a,b)$ *and* $Z \cong H$, *then the density function of the product* BZ *is*

$$p(x) = \frac{x^{a-1}}{B(a,b)} \int_x^\infty (z-x)^{b-1} z^{-a-b+1}\, dH(z).$$

We now assume that $B \cong \text{beta}(a,b)$, in which case we write $\mathcal{L}(a,b,r)$ to denote the scale family of laws $L(X)$ which satisfy (2.1). Since X has a density function, denoted by g, we can apply Lemma 3.1 to the product in (2.1) to obtain the equivalent density function identity

$$g(x) = \frac{x^{a-1}}{B(a,b)M(r)} \int_x^\infty (z-x)^{b-1} z^{r-a-b+1} g(z)\, dz. \tag{3.2}$$

Scaling so that $M(r)B(a,b)=1$ shows that Schneider's equation (1.6) is the particular case of (3.2) with $b=1-\alpha$ and $a=r=m+\alpha-1$. We consider this case in the next section.

Theorem 2.1 is applicable to the beta scaling case. This follows from the Malmstén integral representation of the gamma function (Whittaker and Watson [32, p. 249]) which we express as

$$\log\frac{\Gamma(c+\beta t)}{\Gamma(c)}=-\int_0^\infty\left[\frac{1-e^{-xt}}{1-e^{-x/\beta}}e^{-cx/\beta}-\beta te^{-x/\beta}\right]\frac{dx}{x}. \tag{3.3}$$

For later reference, let γ_c denote a random variable having the gamma law with parameter c, i.e., its density function is $x^{c-1}e^{-x}/\Gamma(c)$ $(x>0)$. The moment function of γ_c is

$$E(\gamma_c^t)=\frac{\Gamma(c+t)}{\Gamma(c)}. \tag{3.4}$$

Using (3.3) to compute the right-hand side of (3.1) shows that $V=-\log B$ has a positive infdiv law with the Lévy measure in (2.4) having the density

$$n(x)=\frac{e^{-ax}(1-e^{-bx})}{x(1-e^{-x})}\quad(x>0), \tag{3.5}$$

a result first found by Shanbhag et al. [28, Remark 2]. This gives the following corollary of Theorem 2.1.

Theorem 3.1. *If $L(X)\in\mathcal{L}(a,b,r)$, then $W=-\log X$ has a spectrally-positive Type 2 infdiv law with arbitrary drift parameter and whose Lévy measure has the density*

$$\pi_W(x)=\frac{e^{-ax}(1-e^{-bx})}{x(1-e^{-x})(1-e^{-rx})}. \tag{3.6}$$

It can be asked whether $L(X)$ itself is infdiv. A sufficient condition can be inferred from Theorem 3.1 (with proof in Section 9) as follows.

Theorem 3.2. *Suppose that $L(X)\in\mathcal{L}(a,b,r)$. Then $L(X)$ is infdiv if $a\leqslant 2$ and $b\geqslant r$, and then it is of Type 1. If $b<r$, then $L(X)$ is not infdiv.*

The law $L(X)=\text{gamma}(a)$ is infdiv for all a and it satisfies $X\cong B_{a,b}\widehat{X}_r$ with $b=r$, showing that the condition $a\leqslant 2$ in the theorem is not a necessary one. Hence an open question is what can be said when $a>2$ and $b<r$?

It follows from (3.1) that $\widehat{B}_r\cong\text{beta}(a+r,b)$, so we see from (2.2) that X is an infinite product of scaled beta random variables. The scaling constants in (2.2) can be simplified:

$$\frac{N(rn)}{N(r(n+1))}=\frac{\Gamma(a+rn)}{\Gamma(r+a+rn)}\cdot\frac{\Gamma(r+a+b+rn)}{\Gamma(a+b+rn)}=\left(\frac{a+b+rn}{a+rn}\right)^r\left(1+O\left(n^{-2}\right)\right),$$

where the right-hand term comes from a known asymptotic expansion for quotients of gamma functions. Thus we can write

$$X\cong\mathcal{M}\prod_{n=0}^\infty\frac{a+b+rn}{a+rn}B_{a+rn,b}, \tag{3.7}$$

where $B_{a,b}\cong\text{beta}(a,b)$, and $\mathcal{M}$ is an arbitrary scaling constant. The presence of the beta random factors can be inferred from expanding (3.6) as

$$x\pi_W(x)=\frac{1-e^{-bx}}{1-e^{-x}}\sum_{n=0}^\infty e^{-(a+rn)x}$$

and reference to (3.3).

This insight can be used to show that although the solution $L(X)$ is unique up to scaling, the representation (3.7) is not unique. This could be anticipated from the existence of distributional identities of the sort discussed e.g. by Chamayou [4]. A different representation for X is obtained by expanding the factor $(1-e^{-x})^{-1}$ in (3.6), with the following result. Its proof is in Section 9.

Lemma 3.2. *If $L(X)\in\mathcal{L}(a,b,r)$, then the representation (3.7) can be expressed as*

$$X\stackrel{L}{=}\mathcal{M}\left[\prod_{n=0}^\infty\frac{a+b+n}{a+n}B_{(a+n)/r,b/r}\right]^{1/r},$$

where $\mathcal{M}=M(r)$.

Filling the gap in Theorem 3.2 is partially resolved by seeking parameter combinations, if any, which give a simple finite representation for $M(t)$ and/or X. This is fruitful because the resulting laws have moments of gamma type. We consider two cases in this section. Pakes [16, p. 298] pursues finite term reductions of $M(t)$ by manipulating product identities for gamma functions. We show here a simpler approach using the Lévy density (3.6).

Case 1. Suppose m is a positive integer and $b = mr$. Then

$$\pi_W(x) = \left[x\left(1 - e^{-x}\right)\right]^{-1} \sum_{j=0}^{m-1} e^{-(a+jr)x}.$$

It follows from (3.3) and (3.4) that

$$X \stackrel{L}{=} \mathcal{M} \prod_{j=0}^{m-1} \gamma_{a+jr}.$$

The case $r = 1$ occurs as Theorem 4.3 in Pakes [16]. The logarithm of a gamma random variable has a self-decomposable law, and hence $\log X$ has a self-decomposable law without the restrictions of Theorem 3.2. In addition, $L(X)$ belongs to the Bondesson class $\mathcal{B}$ of laws (Bondesson [3, pp. 68, 79]), i.e., it has a density $f(x)$ which is hyperbolically completely monotone (meaning that for each $u > 0$, the product $f(uv)f(u/v)$ is a completely monotone function of $v + v^{-1}$). This class of laws has the property that X and X^{-1} are self-decomposable.

Case 2. If $b = m$, a positive integer, then

$$\pi_W(x) = \left[x\left(1 - e^{-x}\right)\right]^{-1} \sum_{j=0}^{m-1} e^{-(a+j)x}$$

whence

$$X \stackrel{L}{=} \mathcal{M} \prod_{j=0}^{m-1} (\gamma_{(a+j)/r})^{1/r}.$$

This was derived using different means by Pakes and Khattree [19, Theorem 6.1]. Again $\log X$ has a self-decomposable law, and $L(X)$ and $L(X^{-1})$ belong to $\mathcal{B}$ if $r \leqslant 1$ (Bondesson [3, p. 85]).

4. Generalized stable laws

There is much activity modelling physical systems in terms of fractional-order differential equations. See Klafter et al. [12] and Uchaikin and Sibatov [30] for recent accounts and references. In particular, the time-dependent density function of a stable process solves an evolution equation in which the spatial derivative has fractional order. The Laplace–Stieltjes transform of the law $L(Y)$ derived from Theorem 3.1 satisfies a fractional-order differential equation generalizing (1.5).

Define the order $\alpha \in [0, 1]$ Weyl integral of the function $\phi(\theta)$ by

$$(I_\alpha \phi)(\theta) = \frac{1}{\Gamma(\alpha)} \int_\theta^\infty (y - \theta)^{\alpha-1} \phi(y)\, dy,$$

assumed to be finite. For a constant $a > 0$ choose the positive integer ν_a such that $\nu_a - 1 < a \leqslant \nu_a$. Define the order-$a$ fractional derivative of ϕ by

$$D^a \phi(\theta) = \left(I_{\nu_a - a} \phi^{(\nu_a)}\right)(\theta).$$

It is easily checked that

$$D^a e^{-\theta y} = (-1)^{\nu_a} y^a e^{-\theta y},$$

agreeing with the usual case when $a = \nu_a$.

It follows that the order-a derivative of the Laplace transform $\widehat{f}(\theta)$ is

$$D^a \widehat{f}(\theta) = (-1)^{\nu_a} \int_0^\infty y^a e^{-\theta y} f(y)\, dy.$$

A routine calculation using (3.7) shows that the density function $f(y)$ of $Y = X^{-1}$ solves

$$B(a,b)M(r)y^{a+b}f(y) = \int_0^y (y-v)^{b-1}v^{a-r}f(v)\,dv,$$

and hence $\widehat{f}$ satisfies

$$B(a,b)M(r)(-1)^{\nu_{b+r}}D^{b+r}\widehat{f}(\theta) = \Gamma(b)\theta^{-b}D^{a-r}\widehat{f}(\theta).$$

Apart from a constant multiplier, this takes the form (1.5) if $a = r$ and $a + b = m > 0$ and it reduces precisely to (1.3) if $a = r = \alpha < 1$ and $b = 1 - \alpha$.

We now focus on the case $m = 1, 2, \ldots$.

Definition 4.1. The law-type $L(Y)$ is a generalized stable law if $\mathcal{M} = 1$, $r = a > 0$ and $b \geqslant 0$ is such that $a + b = m$, a positive integer. We write $Y \cong \text{gstable}(a,m)$. This choice corresponds to Schneider's representation in (1.6) if $a = m + \alpha - 1$ and $b = 1 - \alpha = m - a$. Since $0 < a < m$, the allowable range for α is $-(m-1) < \alpha < 1$. If $m \geqslant 2$, then this extends the range $0 < \alpha < 1$ examined by Schneider [25,26].

In particular $L(X)$ has the reciprocal law, r-gstable(a,m), it solves

$$X \stackrel{L}{=} B_{a,m-a}\widehat{X}_a, \tag{4.1}$$

and it has moments of gamma type.

Theorem 4.1. *Let $0 < a < m$. The* r-gstable(a,m) *law has the finite product moment function*

$$M(t) = \frac{\Gamma(1+t/a)}{\Gamma(1+t)} \prod_{j=1}^{m-1} \frac{\Gamma((j+t)/a)}{\Gamma(j/a)}. \tag{4.2}$$

and the product is interpreted as unity if $m = 1$ (in which case $\alpha = a \leqslant 1$). In particular

$$E(X) = a^{-1}\Gamma(m/a) = m^{-1}\Gamma(1 + m/a). \tag{4.3}$$

Proof. The proof follows as for the two above cases by decomposing π_W in the form

$$\begin{aligned} x\pi_W(x) &= \frac{1-e^{-mx}}{(1-e^{-x})(1-e^{-ax})} - \frac{1}{1-e^{-x}} \\ &= \frac{e^{-ax}}{1-e^{-ax}} - \frac{e^{-x}}{1-e^{-x}} + \sum_{j=1}^{m-1}\frac{e^{-jx}}{1-e^{-ax}}, \end{aligned}$$

where the sum is understood to be zero if $m = 1$. The product (4.2) telescopes if $t = 1$. Observe that $a = m$ if $\alpha = 1$, in which case $\pi_W(x)$ is identically zero and X is almost surely constant-valued. $\square$

The solution of (4.1) in the case $a = 1$ and $m \geqslant 2$ occurs as Theorem 4.3 in Pakes [16], and the solution $X = \mathcal{M}S_a^{-1}$ for the case $m = 1$ and $a < 1$ is an alternative and cleaner characterization of the reciprocal stable law given as Theorem 4 in Pakes [16].

Referring to (1.2) and (3.4) we obtain the following explicit representation for $L(Y)$ in the case that $a \leqslant 1$.

Theorem 4.2. *If $0 < a \leqslant 1$ and $m = 1, 2, \ldots$ and $Y \cong$ gstable(a,m), then*

$$Y \stackrel{L}{=} S_a \prod_{j=1}^{m-1} \gamma_{j/a}^{-1/a}, \tag{4.4}$$

where the product is interpreted as unity if $m = 1$.

Theorem 4.2 holds in a more general sense if $0 < a \leqslant 1$ and $m = a + b > 1$ is not an integer. This can be seen by supposing that $X \stackrel{d}{=} S_a^{-1}T$. It follows from (1.3) and (3.1) that the moment function version of (4.1) implies that the moment function M_T of T satisfies $M_T(t) = B(1+t, m-1)M_T(a+t)/M_T(a)$, i.e., $T = B_{1,m-1}\widehat{T}_a$. It follows from (3.7) that

$$T = \prod_{n=0}^{\infty} \frac{m+an}{1+an} B_{1+an,m-1}.$$

No useful simplification appears possible without more assumptions.

If $a > 1$, then the quotient factor in (4.2) is *not* the moment function of any random variable. Indeed its *reciprocal* is the moment function of $S_{1/a}^{-1/a}$. However, the following generalization of Schneider's key result is evident from Theorem 4.1.

Theorem 4.3. *If $m = 1, 2, \ldots$ and $0 < a < m$, i.e., $-(m-1) < \alpha < 1$, then (1.5) has a unique completely monotone solution which is the Laplace–Stieltjes transform of a scaled version of $Y = X^{-1}$ whose moment function is $M(-t)$ where M is given by (4.2).*

We end this section with the following result about existence of moments and the right-hand tail behaviour of $Y = X^{-1}$ where $L(X) \in \mathcal{L}(a, b, r)$. Schneider [25] obtains the asymptotic form of $g(x)$ as $x \to 0$, and this may be used to determine the right-hand tail behaviour of generalized stable laws, although he does not do this. His proof uses the facts that the moment function M is the Mellin transform of $xg(x)$ and that the Mellin inverse of the right-hand side of (4.2) is a Fox H-function, which has a power series expansion. Clearly his methodology can be applied to Cases 1 and 2 in Section 3 since the moment function there is a finite product of gamma functions. The following result covering all cases uses a much simpler argument.

Theorem 4.4. *Suppose that $Y = X^{-1}$, where $L(X) \in \mathcal{L}(a, b, r)$. Then $E(Y^\nu) < \infty$ if and only if $\nu < a$ and*

$$P(Y > y) \sim Ky^{-a} \quad (y \to \infty) \tag{4.5}$$

where

$$K = \frac{M(r-a)}{aB(a,b)M(r)}.$$

Proof. The moment function form of (2.1) is

$$M(t) = N(t)\frac{M(t+r)}{M(r)},$$

and since $M(t)$ is finite if $t > 0$, it follows that it is finite for $t < 0$ such that the right-hand side is finite. But $N(t) < \infty$ if and only if $t > -a$, and this implies the first assertion. In addition, $M(r-a) < \infty$, so it follows that the integral in (3.2) converges as $x \downarrow 0$ to $M(r-a)$, and hence $g(x) \sim aKx^{a-1}$, implying the tail estimate (4.5). □

This result implies that the gstable(a, m) law has finite moments of integer order up to $m - 1$, but not larger. In this case, $K^{-1} = aB(a, m-a)M(a)$.

5. Infinite divisibility

We begin with the following result about infinite divisibility of r-gstable(a, m) laws.

Theorem 5.1.

(a) *The* r-gstable(a, m) *law is infdiv if:*
  (i) *$a \leqslant \frac{1}{2}$ and $m \geqslant 1$; or*
  (ii) *$\frac{1}{2} < a \leqslant 1$ and $m \geqslant 2$; or if*
  (iii) *$1 < a \leqslant 2$ and $m \geqslant 4$.*

(b) *The* r-gstable(a, m) *law is* not *infdiv if $a > m/2$, $m = 1, 2, \ldots$.*

Proof. Theorem 3.2 implies assertion (a). Proving assertion (b) uses the fact that if $L(X)$ is infdiv, then there is a positive constant c such that $P(X > x) \geqslant \exp(-cx \log x)$ if $x \gg 1$; see Steutel and van Harn [29, p. 115]. The right-hand tail estimate of the r-gstable(a, m) density function is derived by Schneider [25, p. 215]. Integration by parts yields the tail estimate

$$P(X > x) \sim Ax^B \exp\bigl(-kx^\tau\bigr),$$

where A, B and k are positive constants, and $\tau = a/(m-a)$. Clearly $\tau > 1$ if and only if $a > m/2$, thus giving a tail so thin that it violates the above necessary condition for infinite divisibility, and assertion (b) follows. □

It follows that the r-gstable(a, m) law is not infdiv if $m = 2$ and $a > \frac{1}{2}m = 1$, or if $m = 3$ and $a > 3/2$. The case $m = 3$ and $1 < a \leqslant 3/2$ is undecided. Finally, the r-gstable(a, m) law is *not* infdiv if $m \geqslant 2$ and $a \in (m-1, m)$. This is the parameter range used by Schneider for his definition of gstable laws.

The r-gstable$(a, 1)$ law is just the reciprocal stable(a) law, which Theorem 5.1 asserts is infdiv if and only if $a \leqslant \frac{1}{2}$. This assertion is a particular case of the following general result for powers of a stable law.

Lemma 5.1. *If $0 < a < 1$ and $\beta > 0$, then $L(S_a^{-\beta})$ is infdiv if and only if $\beta \geqslant a/(1-a)$.*

Proof. The direct (if) assertion follows from Theorem 1 in Shanbhag et al. [28] which implies that if $\beta \geqslant a/(1-a)$, then $L(S_a^{-\beta})$ is an exponential mixture, and hence it is infdiv. These authors remark that $L(S_a^{-\beta})$ need not be infdiv if $\beta < a/(1-a)$. In fact, under this condition, a Chernoff inequality leads to the tail estimate

$$P\big(S_a^{-\beta} > x\big) \leqslant \exp\big(-Kx^{a/\beta(1-a)}\big),$$

which implies that the tail is too thin for infinite divisibility. □

Next, we ask whether the gstable(a, m) laws are infdiv? An incomplete answer is that if $a \in (0, 1]$ and $m \geqslant 2$, then they are self-decomposable, an outcome which generalizes the stable(α) case where $m = 1$ and $\alpha = a$. Proving this is quite simple, but it needs some preliminary notions. Suppose that $Z \geqslant 0$ is a random variable independent of S_a. With reference to (4.4), the random variable

$$Y = S_a Z^{1/a} \tag{5.1}$$

inherits infdiv properties from Z. For example, if $L(Z)$ is infdiv then so is $L(Y)$. To see this, recall that the Laplace–Stieltjes transform has the form $\exp(-C_Z(\theta))$ where the cumulant function C_Z is a Bernstein function, meaning that

$$C_Z(\theta) = \int_0^\infty \big(1 - e^{-\theta x}\big) L(dx),$$

where L is a Lévy measure. The cumulant function of Y is $C_Z(\theta^a)$, a composition of two Bernstein functions, and hence it is Bernstein too; see Corollary 3.7(iii) in Schilling et al. [24]. Again, if $L(Z)$ is self-decomposable, then so is $L(Y)$. This follows easily using the Laplace–Stieltjes transform version of the autoregression representation of Z. The closure result we need asserts that if $L(Z)$ is a generalized gamma convolution (GGC), then so is $L(Y)$.

The law $L(Z)$ is a GGC if Z is a sum of independent gamma random variables (allowing differing shape parameters) or it is the weak limit of such sums. Such laws are self-decomposable and they are precisely the positive infdiv laws whose Lévy measures have a density $\ell(x)$ such that $x\ell(x)$ is completely monotone:

$$x\ell(x) = \int_{(0,\infty)} e^{-xv}\, \tau(dv), \tag{5.2}$$

where the so-called Thorin measure τ satisfies

$$\int_{(0,1)} |\log v|\, \tau(dv) < \infty \quad \text{and} \quad \int_{[1,\infty)} v^{-1}\, \tau(dv) < \infty. \tag{5.3}$$

See Bondesson [3], Steutel and van Harn [29], or Schilling et al. [24] for these facts. The above GGC closure result is proved in Bondesson [3, p. 41] using complex variable theory, and Steutel and van Harn [29, p. 363] give a real analytic proof.

A more probabilistic approach yields the closure assertion along with a formula for the Thorin measure of $L(Y)$ showing that it is absolutely continuous. See Section 9 for the proof of the following assertion.

Theorem 5.2. *Suppose that Y and Z are related by (5.1). If $L(Z)$ is a GGC law with the Lévy density (5.2), then $L(Y)$ is a GGC law whose Lévy density λ is given by*

$$y\lambda(y) = \int_0^\infty e^{-yz} u(z)\, dz,$$

where

$$u(z) = a \int_{(0,\infty)} f_R\big(zv^{-1/a}\big) v^{-1/a}\, \tau(dv) \tag{5.4}$$

is the density of a Thorin measure. Here

$$f_R(y) = \frac{\sin(\pi a)}{\pi} \frac{y^{a-1}}{1 + 2y^a \cos(\pi a) + y^{2a}} \quad (y > 0), \tag{5.5}$$

is the density function of $R = S'_a/S_a$ where S'_a is an independent copy of S_a.

We mention, as an aside, that the density function (5.5) occurs often in discussions about laws associated with the Mittag-Leffler function

$$E_a(x)=\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(1+na)},$$

an obvious generalization of the exponential function; $E_1(x)=e^x$. Indeed the Laplace–Stieltjes transform of R is

$$\widehat{f}_R(\theta)=E\big(e^{-\theta R}\big)=E\big(e^{-(\theta/S_a)^a}\big)=\sum_{n=0}^{\infty}\frac{(-\theta)^n}{n!}E\big(S_a^{-na}\big)=E_a\big(-\theta^a\big),$$

upon using the moment identity (1.2). The assertion usually found about (5.5) is a reference to tables showing the Laplace transform relation of f_R to $\widehat{f}_R$ which makes plain that f_R really is a density function. Alternatively the form of f_R is derived from a Mellin transform table. See Kotz et al. [13, p. 205] for references. The connection to the quotient R is seldom mentioned, although it is plainly implicit in Exercise 4.21 in Chaumont and Yor [5]. More specifically, they ask for the density function of R^a which, from (5.5), is proportional to $(1+2z\cos(\pi a)+z^2)^{-1}$, and to prove it, they suggest evaluating a certain integral using residue calculus. A simple direct proof of (5.5) is offered in Section 9.

We now state a principle result of the paper.

Theorem 5.3. *If $0<a\leqslant 1$ and $m=2,3,\dots,$ then the* gstable(a,m) *law is a GGC and hence self-decomposable.*

Proof. It follows from Theorem 4.2 that Y has the form (5.1) where $Z=\prod_{j=1}^{m-1}\gamma_{j/a}$ implying that $L(Z)\in\mathcal{B}$ (defined in Section 3); see Bondesson [3, p. 85]. Hence $L(Z)$ is a GGC, and the assertion follows as explained above. □

A question is whether $L(Y)\in\mathcal{B}$? This would follow if $L(S_a)\in\mathcal{B}$ because $\mathcal{B}$ is closed under independent products. The inclusion $L(S_a)\in\mathcal{B}$ is known if a^{-1} is a positive integer and it is open otherwise (Bondesson [3, p. 85]). Bondesson conjectures that $L(S_a)\in\mathcal{B}$ if $a\leqslant\frac{1}{2}$. Finally, since the above infdiv laws are positive laws with density functions, they are of Type 1.

The major open question is whether $L(Y)$ is infdiv if $m\geqslant 2$ and $1<a<m$. This includes Schneider's parameter range $0<\alpha<1$.

6. Self-decomposability

In this section we present results about self-decomposability and the BDLP typology of some laws discussed above. We note first that the dichotomy in assertion (b) of Proposition 2.1 reflects qualitatively different behaviours of the density function $h(x)$ of $L(\mathcal{X})$ in a neighbourhood of $x=0$. The following result makes this precise and it is related to Theorem 53.6 in Sato [23]. In particular, the typology of the BDLP is determined by the behaviour at the origin of the distribution function H of $L(\mathcal{X})$. The proof is in Section 9.

Proposition 6.1. *Suppose that $L(\mathcal{X})$ is Type 0 or 1 and denote the limit (2.9) by ξ. If $\xi<\infty$ (i.e., $L(\mathcal{X})$ is Type 0), then H is regularly varying at the origin, specifically, there is a slowly varying function $\lambda(x)$ such that $H(x)=x^{\xi}\lambda(1/x)$. If $\xi=\infty$ (i.e., $L(\mathcal{X})$ is Type 1), then H is rapidly varying at the origin.*

Assertion (a) of the following corollary follows from the tail estimate (4.5) and Proposition 6.1.

Corollary 6.1.

(a) *The BDLP of any self-decomposable member of $\mathcal{L}(a,b,r)$ is compound Poisson.*
(b) *If $L(X)\in\mathcal{L}(a,b,r)$ and the law of $Y=X^{-1}$ is self-decomposable, then its BDLP has Type 1.*

Proof of (b). It follows from Proposition 2.1 that $L(Y)$ is Type 0 or 1. If it is the former, then Proposition 6.1 implies that the survivor function of $L(X)$ is regularly varying at infinity. This cannot be because $E(X^t)<\infty$ for all positive t. □

As in Section 3, we assume that $B\cong\text{beta}(a,b)$. Shanbhag et al. [28] show that the law of $V=-\log B$ is self-decomposable if either $b\geqslant 1$, or $b<1$ and $a+b\geqslant 1$. They assert that V need not be self-decomposable if $a+b<1$. The following result sharpens this, and the details (in Section 9) are a little easier than the proof in Shanbhag et al. [28].

Theorem 6.1. *Suppose that $B\cong\text{beta}(a,b)$. Then $V=-\log B$ has a self-decomposable law if and only if either $b\geqslant 1$, or $b<1$ and $a\geqslant\frac{1}{2}(1-b)$. In these cases the BDLP is Type 0.*

In the case $b<1$, Theorem 6.1 asserts that there is a critical value $\bar{a}=\frac{1}{2}(1-b)$ such that V is self-decomposable if and only if $a \geqslant \bar{a}$. Similar outcomes obtain for $W=-\log X$ in the following result.

Theorem 6.2. *Suppose that $L(X)\in\mathcal{L}(a,b,r)$.*

(i) *If $b\geqslant 1$ or $b\geqslant r$, then $W=-\log X$ is self-decomposable.*
(ii) *If $b<1$ and $b<r$, then there exists a positive number $\bar{a}<\frac{1}{2}(1-b)$ such that W is self-decomposable if and only if $a\geqslant\bar{a}$.*
(iii) *If $r=a$, then W is self-decomposable. In particular, if $m=1,2,\dots$ and $Y\cong\mathrm{gstable}(a,m)$, then $W=\log Y$ is self-decomposable. In each case, the BDLP is Type 2.*

7. The case $m=2$

In this section we set $m=2$ and exhibit a representation of the density function g and the Laplace transforms $\widehat{g}$ and $\widehat{f}$. Note that the density function f of $Y=X^{-1}$ is obtained from $f(y)=y^{-2}g(1/y)$. So we assume that $a+b=2$ and $r=a<2$, in which case (4.2) with $\mathcal{M}=1$ becomes

$$M(t)=\frac{\Gamma(1+t/a)}{\Gamma(1+t)}\cdot\frac{\Gamma((1+t)/a)}{\Gamma(1/a)}. \tag{7.1}$$

We consider first the sub-case $0<a\leqslant 1$ and then the general case $0<a<2$. Recall that $\sigma_a(x)$ denotes the density function of the positive stable(a) law.

Theorem 7.1. *Let $m=2$ and $0<a\leqslant 1$. The density function g has the exponential mixture form*

$$g(x)=E\big[Q\,e^{-Qx}\big], \tag{7.2}$$

where Q has the density function

$$q(s)=\big(\Gamma(1+1/a)\big)^{-1}s^{-1}E\big[\sigma_a(s/S_a)\big]. \tag{7.3}$$

The Laplace transform of the gstable$(a,2)$ *density function f is*

$$\widehat{f}(\theta)=\frac{2\sqrt{\theta}}{\Gamma(1/a)}K_{1/a}\big(2\theta^{a/2}\big), \tag{7.4}$$

where K_ν is a modified Bessel function, i.e., the Macdonald function.

The exponential mixture form (7.2) implies that the r-gstable$(a,2)$ law is infdiv if $a\leqslant 1$, in agreement with Theorem 5.1 (a) (i, ii). But note that the r-gstable$(a,2)$ law is not infdiv if $1<a\leqslant 2$.

Remark 7.1. It follows from (7.2) that

$$\widehat{g}(\theta)=E\left[\frac{Q}{Q+\theta}\right]=E\left[\int_0^\infty\frac{s}{s+\theta}s^{-1}\sigma_a(s/S_a)\,ds\right]=E\left[\frac{S_a}{S_aS'_a+\theta}\right].$$

The formal expansion of this quantity as a power series in θ has zero radius of convergence, as follows from Theorem 8.1.

Remark 7.2. Apart from a scale constant, (7.4) agrees with Eq. (3.22) in Schneider [25], whose derivation involves power series expansions of H-functions and Bessel functions.

Starting with a different probability interpretation of (7.1) gives the following more complicated result valid if $0<a<2$.

Theorem 7.2. *Let $m=2$ and $0<a<2$. Then (7.4) is valid and the density function of X is*

$$g(x)=\frac{a(a-2)}{\Gamma(2/a)}x^{a-1}E\big[S_\eta^{a-1}\big(1-4(xS_\eta)^a\big)_+^{1/a-3/2}\big]\quad(x\geqslant 0) \tag{7.5}$$

where $\eta=2/a$.

Recalling that $\eta=2/a$, we have the following product representation for the gstable$(a,2)$ law which shows that Y is a product of two random variables with fat right-hand tails.

Corollary 7.1. *Suppose that $L(Y) = \mathrm{gstable}(a, m)$ where $a < m = 2$. Then $Y \overset{L}{=} J_a V_\eta$, where J_a has the Pareto distribution function*

$$H_a(y) = \left(1 - 4y^{-a}\right)_+^{\zeta},$$

and V_η has the inverse length-biased stable distribution function

$$P(V_\eta \leqslant y) = \frac{E[S_\eta^{-1}; S_\eta \leqslant y]}{E[S_\eta^{-1}]} = \left(\Gamma(1 + 2/a)\right)^{-1} \int_0^y v^{-1}\sigma_\eta(v)\,dv.$$

We end this section by stating a power series representation for $\widehat{g}(\theta)$ thus giving explicit closed forms for all moments of the r-gstable$(a, 2)$ law with $1 \leqslant a < 2$.

Theorem 7.3. *Let $m = 2$ and $1 \leqslant a < 2$. Then*

$$\widehat{g}(\theta) = \frac{1}{\Gamma(1/a)} \sum_{n=0}^{\infty} (-\theta)^n \frac{\Gamma(1 + n/a)\Gamma((n + 1)/a)}{(n!)^2}, \tag{7.6}$$

which has a finite positive radius of convergence if $a = 1$, and it is entire if $1 < a < 2$.

8. Moment determinacy of r-gstable laws

Let $m = 1, 2, \ldots$ and $0 < a < m$. Knowing that the r-gstable(a, m) law has finite moments $M(n)$ of all orders, questions arise about conditions under which the power series form of its moment generating function has a positive radius of convergence, and when these moments uniquely determine the law.

To address the first question, recall that if $\{c_n\}$ is a sequence of non-negative numbers such that $0 \leqslant R^{-1} = \lim_{n\to\infty} c_n^{1/n} \leqslant \infty$ exists, then R is the radius of convergence of the power series $\sum_{n\geqslant 0} c_n\theta^n$. Secondly, noting that $(\Gamma(x))^{1/x} \sim x/e$ and $\Gamma(b + x) \sim x^b\Gamma(x)$ (b a real constant), both as $x \to \infty$, we have if $\rho > 0$ that

$$\left(\Gamma(b + \rho n)\right)^{1/n} \sim (\rho n/e)^\rho. \tag{8.1}$$

It follows from (4.2) that

$$\left(M(n)\right)^{1/n} \sim \mathcal{M}(ae)^{-m/a} n^{(m/a)-1} \quad (n \to \infty). \tag{8.2}$$

This estimate, together with $(n!)^{1/n} \sim n/e$, settles the radius of convergence question.

Theorem 8.1. *Let $m = 1, 2, \ldots$ and $0 < a < m$. Then the power series form of the moment generating function of the* r-gstable(a, m) *law has radius of convergence*

$$R_a = \begin{cases} \infty & \text{if } \frac{1}{2}m < a < m, \\ \frac{m^2}{4\mathcal{M}} & \text{if } a = \frac{1}{2}m, \text{ and} \\ 0 & \text{if } 0 < a < \frac{1}{2}m. \end{cases}$$

We will say that a probability law supported in $[0, \infty)$ is determinate (in the Stieltjes sense) if it has finite moments of all positive integer orders and if there is no other such law having the same sequence of moments. Among the many reviews of this matter we cite Pakes et al. [20]. Before stating our result we must recall some derived parameters from Janson [10, pp. 8–10]. For our purposes we need only report for the r-gstable(a, m) law that, using his notation, $\rho_+ = \infty$,

$$\gamma' = \gamma = \frac{m}{a} - 1 > 0, \quad \text{and} \quad \delta = \frac{m - 1}{2}\gamma.$$

He shows (his Theorem 6.1) that the density function g has the generalized Weibull asymptotic form

$$g(x) \sim Cx^{c_1 - 1}\exp\left(-c_2 x^{1/\gamma}\right) \quad (x \to \infty) \tag{8.3}$$

where C and c_2 are positive constants and

$$c_1 = \frac{\delta + \frac{1}{2}}{\gamma} = \frac{m - 1}{2} + \frac{a}{2(m - a)} > 0.$$

The following result shows that γ is the critical parameter determining whether or not the r-gstable(a, m) is determinate.

Theorem 8.2. *If $m = 1, 2, \dots$ and $0 < a < m$, then the* r-gstable(a, m) *law is determinate if and only if*

$$\frac{m}{3} \leqslant a < m.$$

This result would follow directly from the form of (8.3) and Theorem 2 in Pakes et al. [20], except that there is a small but vital oversight in its short proof. These authors are concerned with density functions supported in $[0, \infty)$ and having the generalized form

$$\tilde{g}(x) = K_Q^{-1} Q(x) \exp\bigl(-cx^{1/\gamma}\bigr), \tag{8.4}$$

where $c, \gamma > 0$ and Q satisfies the following constraints: there are constants $k \geqslant 1$, $0 < \xi \leqslant 1$ and $x' \geqslant 0$ such that if $x \geqslant x'$, then

$$0 < Q(x) \leqslant \max\bigl(x^{\xi-1}, x^k\bigr). \tag{8.5}$$

The proof of Theorem 2 in Pakes et al. [20] is simply the application of Krein's condition to prove that $\tilde{g}$ is indeterminate if $\gamma > 2$ and it asserts that $Q(x) = O(\log x)$ if $x \geqslant x'$. It is clear that there must be some restriction on the rate at which $Q(x)$ can approach zero as $x \to \infty$ to avoid it dominating the Weibull factor in (8.4).

Thus Theorem 2 in Pakes et al. [20] should be expressed as follows.

Theorem 8.3. *If $\tilde{g}$ is a density function having the form (8.4) and (8.5) and if, in addition, there are positive constants ζ, k_1 such that $Q(x) > k_1 x^{-\zeta}$ if $x \geqslant x'$, then a law with density $\tilde{g}$ is determinate if and only if $\gamma \leqslant 2$.*

Theorem 8.2 follows directly from Theorem 8.3. We note in passing that inspection of the proof of Theorem 2 in Pakes et al. [20] makes it clear that if $\gamma > 2$, then $\tilde{g}$ is indeterminate under a much weaker additional condition, for example, if $\tilde{g}(x) \propto \exp[-cx^{1/\gamma}(1 + o(1))]$ as $x \to \infty$.

Referring to Theorem 8.2, and recalling that if $0 < a < 1$ then $L(S_a^{-1})$ is the r-gstable$(a, 1)$ law, we see that $L(S_a^{-1})$ is determinate if and only if $1/3 \leqslant a < 1$. From Theorem 8.1, its moment generating function has positive radius of convergence if and only if $\frac{1}{2} \leqslant a < 1$. Similarly, the r-gstable$(a, 2)$ law is determinate if and only if $2/3 \leqslant a < 2$.

The finite product laws $L(X)$ identified as Cases 1 and 2 in Section 3 and the r-gstable(a, m) laws have the general representation

$$X = S_a^{-b} \prod_{j=1}^{m} \gamma_{a_j}^{\rho_j}, \tag{8.6}$$

where $0 < a \leqslant 1$, $0 \leqslant b < \infty$ and the a_j's and ρ_j's are positive constants. This includes examples arising from quantum mechanics as examined by Penson et al. [21]. They consider for positive integer ρ the sequences: (i) $s'_n = (2\rho n)!$; (ii) $s''_n = ((\rho n)!)^2$; (iii) $s'''_n = ((\rho n)!)^3$; and (iv) $s'_n s''_n$. Corresponding density functions are identified using Mellin transform techniques with the result that, by finding suitable Stieltjes-classes (Pakes [18] and references therein), (i) and (ii) are uniquely determining if $\rho = 1$, and not so if $\rho = 2, 3, \dots$.

Assuming only that ρ is a positive constant, it is clear that the sequences (i) to (iii) are the moment sequences of $\varepsilon^{2\rho}$, $(\varepsilon_1\varepsilon_2)^\rho$ and $(\varepsilon_1\varepsilon_2\varepsilon_3)^\rho$, respectively, where the ε's are independent and have the standard exponential law. Ostrovska and Stoyanov [15] consider the moment problem for the related products $\prod_{j=1}^m \varepsilon_j$, where the factors are independent and standard exponential.

Referring to (8.3) the Janson parameters are

$$\gamma = b/a + \sum_{j=1}^{m} \rho_j \quad \text{and} \quad \delta = \sum_{j=1}^{m} a_j - \frac{1}{2}m.$$

The next result embracing all these cases follows from (8.1) and Theorem 8.3.

Theorem 8.4. *The radius of convergence of the moment generating function of the law $L(X)$ defined by (8.6) is*

$$R = \begin{cases} \infty & \text{if } \gamma < 1, \\ e^2 \prod_{j=1}^m \rho_j^{-\rho_j} & \text{if } \gamma = 1, \text{ and} \\ 0 & \text{if } \gamma > 1. \end{cases}$$

In addition, $L(X)$ is determinate if and only if $\gamma \leqslant 2$.

Note that these outcomes are independent of the shape parameters a_j. In particular, $b=0$ and $\gamma=m$ for Case 1 (in Section 3), so the radius of convergence is finite and positive if $m=1$ and it is zero if $m \geqslant 2$. However $L(X)$ is determinate if $m=2$, and indeterminate if $m \geqslant 3$. We have $b=0$ and $\gamma=m/r$ for Case 2, so $L(X)$ is determinate if and only if $r \geqslant \frac{1}{2}m$, $m=1,2,\ldots$. The sequences (i) and (ii) above are determining if and only if $\rho \leqslant 1$. Penson et al. [21] did not pursue the moment problem for sequences (iii) and (iv), but it is clear that $\gamma=3\rho$ for (iii) and $\gamma=4\rho$ for (iv), so these sequences are determining if and only if $0<\rho \leqslant 2/3$ and $0<\rho \leqslant \frac{1}{2}$, respectively. Finally, if all $\rho_j=\rho$ in Theorem 8.4, then the critical value is $2/m$, and $L(X)$ is indeterminate if and only if $\rho > 2/m$.

9. Proofs

Proof of Theorem 3.2. It suffices to show that $X \cong Z\gamma_2$ because mixtures of the gamma(2) law are infdiv, and the restriction on the range of a is because there are mixtures of the gamma(c) law with $c>2$ which are not infdiv. See Steutel and van Harn [29, pp. 346, 409] for these facts. We approach this by choosing a constant $c>0$ and seek conditions ensuring the existence of a random variable Z such that the product representation $X \cong Z\gamma_c$ holds. This is the case if and only if $M(t)\Gamma(c)/\Gamma(c+t)$ is a moment function. However, (3.6) and (3.3) imply that this condition holds if the function $q(x)$ defined by

$$xq(x)=x\pi_W(x)-\frac{e^{-cx}}{1-e^{-x}}$$

is non-negative for $x \geqslant 0$. This is equivalent to the condition

$$\frac{e^{-ax}(1-e^{-bx})}{1-e^{-rx}}-e^{-cx} \geqslant 0 \quad (x>0).$$

Letting $x \to 0$ shows that $b \geqslant r$ is a necessary condition for infinite divisibility. Conversely, this condition implies that $e^{-bx} \leqslant e^{-rx}$, all $x \geqslant 0$. Hence $q(x) \geqslant 0$ if also $a \leqslant c$. Finally, it is obvious that $\int_0^1 x^2 q(x)\,dx<\infty$, and hence the factorization holds with $-\log Z$ having the spectrally-positive infdiv law with Lévy density q. The infdiv assertion of Theorem 3.2 follows by choosing $c=2$. To see that $L(X)$ is Type 1, just note that it is a positive law and hence not Type 2. In addition $E(\exp(-\theta X))=E[(1+\theta Z)^{-2}] \to 0$ as $\theta \to \infty$, so $L(X)$ is not Type 0. □

Proof of Lemma 3.2. This is best shown by exploiting an unexpected symmetry which is revealed by allowing the contraction factor B in (2.1) to be a power of a beta random variable. Specify positive constants α, β, ρ, r, and consider the relation

$$X \stackrel{L}{=} B^{1/\rho}\widehat{X}_r \quad \text{where } B \cong \text{beta}(\alpha/\rho, \beta/\rho). \tag{9.1}$$

With $W=\log X$ it is clear from (3.3) and Theorem 2.1 that the Lévy density π_W of $L(W)$ is given by

$$x\pi_W(x)=\frac{e^{-\alpha x}(1-e^{-\beta x})}{(1-e^{-\rho x})(1-e^{-rx})}.$$

This is symmetric in ρ and r, so interchanging these parameters in (9.1) yields the equivalent relation

$$X \stackrel{L}{=} D^{1/r}\widehat{X}_\rho \quad \text{where } D \cong \text{beta}(\alpha/r, \beta/r).$$

Let

$$c_n(\rho,r)=\frac{N_B(rn/\rho)}{N_B(r(n+1)/\rho)}=\frac{\Gamma((\alpha+rn)/\rho)}{\Gamma((\alpha+\beta+rn)/\rho)}\cdot\frac{\Gamma((\alpha+\beta+r+rn)/\rho)}{\Gamma((\alpha+r+rn)/\rho)}.$$

It follows from (2.2) and (3.1) that the moment function M of X with scaling chosen so that $M(r)=1$ is

$$M(t)=\prod_{n=0}^{\infty}\bigl(c_n(\rho,r)\bigr)^{t/r}\frac{\Gamma((\alpha+rn+t)/\rho)}{\Gamma((\alpha+rn)/\rho)}\cdot\frac{\Gamma((\alpha+\beta+rn)/\rho)}{\Gamma((\alpha+\beta+rn+t)/\rho)}.$$

Interchanging ρ and r yields the equivalent form

$$M(t)=\mathcal{M}^t\prod_{n=0}^{\infty}\bigl(c_n(r,\rho)\bigr)^{t/\rho}\frac{\Gamma((\alpha+\rho n+t)/r)}{\Gamma((\alpha+\rho n)/r)}\cdot\frac{\Gamma((\alpha+\beta+\rho n)/r)}{\Gamma((\alpha+\beta+\rho n+t)/r)},$$

where

$$\mathcal{M}=\prod_{n=0}^{\infty}\bigl(c_n(r,\rho)\bigr)^{-1/\gamma}\left(\frac{\alpha+\beta+\rho n}{\alpha+\rho n}\right)^{1/r},$$

since $M(r)=1$. Hence

$$M(t)=\prod_{n=0}^{\infty}\left(\frac{\alpha+\beta+\rho n}{\alpha+\rho n}\right)^{t/r}\frac{B((\alpha+\rho n+t)/r,\beta/r)}{B((\alpha+\rho n)/r,\beta/r)},$$

and this implies the representation

$$X\stackrel{L}{=}\left[\prod_{n=0}^{\infty}\frac{\alpha+\beta+\rho n}{\alpha+\rho n}B_{(\alpha+\rho n)/r,\beta/r}\right]^{1/r}.$$

Setting $\rho=1$, $a=\alpha$ and $b=\beta$ in this development yields the assertion. □

The analytical origin of the above symmetry relation appears to derive from power transformation identities satisfied by Meijer-G functions. To understand this, note that the Laplace–Stieltjes transform $\widehat{f}(\theta)$ of X^{-1} is related to the moment function M of X through the Mellin transform relation

$$\Gamma(t)M(t)=\int_0^{\infty}\theta^{t-1}\widehat{f}(\theta)\,d\theta.$$

It follows from (2.3) and (3.1) that $\widehat{f}$ can be represented as the limit of a sequence of scaled Meijer-G functions. See Mathai [14, p. 60] for facts about these functions, and in particular their specification as the inverse Mellin transform of products and quotients of gamma functions. However, it is much easier to derive the symmetry relation by inspecting the form of the Lévy density, as above.

Proof of Theorem 5.2. From (5.2) the cumulant function of $L(Y)$ is

$$C_Y(\theta)=\int_0^{\infty}\bigl(1-\exp\bigl(-x\theta^a\bigr)\bigr)\ell(x)\,dx=\int_0^{\infty}E\bigl[1-\exp\bigl(-x^{1/a}\theta S_a\bigr)\bigr]\ell(x)\,dx$$
$$=a\int_0^{\infty}\bigl(1-e^{-y\theta}\bigr)y^{a-1}E\bigl[S_a^{-a}\ell\bigl((y/S_a)^a\bigr)\bigr]\,dy.$$

It follows that $L(Y)$ is infdiv with a Lévy density $\lambda(y)$ satisfying

$$y\lambda(y)=aE\bigl[(y/S_a)^a\ell\bigl((y/S_a)^a\bigr)\bigr]=aE\left[\int_{(0,\infty)}\bigl(\exp\bigl(-(y/S_a)^a v\bigr)\bigr)\,\tau(dv)\right].$$

As a function of y, the exponential term in the integrand is the Laplace–Stieltjes transform of a scaled positive stable(a) random variable S_a', say. Hence

$$y\lambda(y)=a\int_{(0,\infty)}E\bigl[\exp\bigl(-yRv^{1/a}\bigr)\bigr]\,\tau(dv).\tag{9.2}$$

Thus $y\lambda(y)$ is the Laplace transform of the density $u(z)$ as specified by (5.4). To verify that u is the density of a Thorin measure we begin by showing that $I:=\int_0^1\log z^{-1}u(z)\,dz$ is finite. Write $I=I_1+I_2$ where

$$I_1=a\int_{[1,\infty)}v^{-1/a}\int_0^1\log z^{-1}f_R\bigl(zv^{-1/a}\bigr)\,dz\,\tau(dv).$$

We repeatedly use the bound

$$f_R(y)\leqslant K\bigl(y^{a-1}\wedge y^{-a-1}\bigr),$$

which is evident from (5.5) and where K denotes the normalization constant.

We have $zv^{-1/a}\leqslant 1$ in the inner integral and hence

$$I_1\leqslant aK\int_{[1,\infty)}v^{-1}\int_0^1 z^{a-1}\log z^{-1}\,dz\,\tau(dv)<\infty$$

since the inner integral equals a^{-2}.

Since $zv^{-1/a} \leqslant 1$ if and only if $z \leqslant v^{1/a}$, we obtain the estimate

$$I_2 \leqslant aK \int_{(0,1)} v^{-1/a} \left[\int_0^{v^{1/a}} v^{-1+1/a} z^{a-1} \log z^{-1}\, dz + \int_{v^{1/a}}^1 v^{1+1/a} z^{-a-1} \log z^{-1}\, dz \right] \tau(dv).$$

Integration by parts yields

$$I_{21} := -a \int_0^{v^{1/a}} z^{a-1} \log z\, dz = -\int_0^{v^{1/a}} \log z\, dz^a = (v/a)\bigl(1 + \log v^{-1}\bigr)$$

and

$$I_{22} := -a \int_{v^{1/a}}^1 z^{-a-1} \log z\, dz = \int_{v^{1/a}}^1 \log z\, dz^{-a} \leqslant (av)^{-1} \log v^{-1}.$$

Hence

$$I_2 \leqslant Ka^{-1} \int_{(0,1)} \bigl(1 + 2\log v^{-1}\bigr)\, \tau(dv) < \infty,$$

whence $I < \infty$.

Next, write $J := \int_1^\infty z^{-1} u(z)\, dz = J_1 + J_2$ where

$$J_1 = a \int_{(1,\infty)} v^{-1/a} \int_1^\infty z^{-1} f_R\bigl(zv^{-1/a}\bigr)\, dz\, \tau(dv).$$

Proceeding as for I_1 we have

$$J_1 \leqslant aK \int_{[1,\infty)} v^{-1} \left[\int_1^{v^{1/a}} z^{a-2}\, dz + v \int_{v^{1/a}}^\infty z^{-a-2}\, dz \right] \tau(dv).$$

If $a < 1$ then the first inner integral is bounded above by $(1-a)^{-1}$. So computing the second inner integral gives the bound

$$J_1 \leqslant const. \int_{[1,\infty)} v^{-1}\, \tau(dv) + const. \int_{[1,\infty)} v^{-1/a}\, \tau(dv) < \infty.$$

Finally,

$$J_2 = aK \int_{(0,1)} v^{-1/a} \int_1^\infty z^{-1} f_R\bigl(zv^{-1/a}\bigr)\, dz\, \tau(dv) \leqslant aK \int_{(0,1)} v \int_1^\infty z^{-a-2}\, dz\, \tau(dv) < \infty.$$

Hence $J < \infty$ and we conclude that $u(z)$ is indeed a Thorin density. It follows that $L(Y)$ is a GGC, as asserted, and hence it is self-decomposable.

A direct, if formal, proof of (5.5) starts with the identity

$$f_R(x) = \int_0^\infty y\sigma_a(xy)\sigma_a(y)\, dy = E\bigl[S_a \sigma_a(xS_a)\bigr],$$

recalling that σ_a is the density function of S_a. Substitute the power series form of σ_a written as (see p. 583 in [7])

$$\sigma_a(x) = (\pi x)^{-1} \Im \sum_{n=1}^\infty \frac{\Gamma(1+an)}{n!} \bigl(-x^a e^{-i\pi an}\bigr)^n,$$

and note that $E(S_a^{-an}) = n!/\Gamma(1+an)$. This yields a geometric series with sum (5.5). □

Proof of Proposition 6.1. The densities h and ℓ are related by (Steutel and van Harn [29, p. 95])

$$xh(x) = \int_0^x y\ell(y)h(x-y)\,dy. \tag{9.3}$$

So if ξ is finite then for $\epsilon \in (0,\xi)$ there exists $x(\epsilon)$ such that $\xi - \epsilon < y\ell(y) \leqslant \xi$ if $0 < y \leqslant x(\epsilon)$. Hence, if $x \in (0, x(\epsilon)]$, then it follows from (9.3) that $(\xi-\epsilon)H(x) \leqslant xh(x) \leqslant \xi H(x)$, implying that

$$\lim_{x\to 0} \frac{xh(x)}{H(x)} = \xi. \tag{9.4}$$

However, self-decomposable laws are unimodal so h is monotone in a neighbourhood of the origin; see Steutel and van Harn [29, p. 235]. Regular variation of H follows from (9.4) and Lamperti's theorem (Bingham et al. [2, p. 59]).

If $\xi = \infty$, then since $x\ell(x)$ is non-increasing, it follows from (9.3) that $xh(x) \geqslant x\ell(x)\int_0^x h(x-y)$, i.e.,

$$x\ell(x) \leqslant \frac{xh(x)}{H(x)}.$$

Hence the right-hand side $\to \infty$ as $x \to 0$, and altering details of the proof of Lamperti's theorem will show that H is rapidly varying at the origin, i.e., $\lim_{x\to 0} H(cx)/H(x) = 0$ if $0 < c < 1$. □

Proof of Theorem 6.1. Referring to (3.5) let $k(x) = xn(x)$ and note that the self-decomposability property holds if and only if k is non-increasing. We have $k(0) = b > 0$, and if $\phi(u) = k(x)$, where $u = e^{-x}$, then

$$-\frac{k'(x)}{k(x)} = \frac{\phi'(u)}{\phi(u)} = \frac{q(u)}{u(1-u)(1-u^b)},$$

where

$$q(u) = a(1-u)\bigl(1-u^b\bigr) - u - bu^b - (1-b)u^{1+b}.$$

If $b \geqslant 1$ and $0 \leqslant u < 1$, then

$$q(u) > u - bu^b + (b-1)u^{1+b} = (u+b)\bigl(1-u^b\bigr) > 0.$$

If $b < 1$ and $a \geqslant 1-b$, then

$$q(u) > 1 - b + bu > 0.$$

So in both cases $-k'(x) > 0$, i.e., V is self-decomposable.

Suppose that $b < 1$ and $a + b < 1$. We have $q(0) = a$, $q'(0) = -\infty$, and $q(1) = q'(1) = 0$. Algebra yields

$$q''(u) = \bigl[a(1-b) + au(1+b) + b(1-b) - \bigl(1-b^2\bigr)\bigr]bu^{b-2}.$$

It follows that $q''(1) = (2a - (1-b))b$, and that $q''(u) = 0$ has a unique solution $\bar{u}$ and

$$1 - \bar{u} = \frac{1-b-2a}{1-b^2-a(1+b)}.$$

The denominator is positive, so q has no point of inflection in $(0,1)$ if $a \geqslant \frac{1}{2}(1-b)$ in which case $q''(1) \geqslant 0$. It follows that q is convex decreasing in $(0,1)$, and hence takes only positive values. So again V is self-decomposable.

A calculation using l'Hôpital's rule shows that $k'(0) = \frac{1}{2}(1-b) - a$, so if $a < \frac{1}{2}(1-b)$ then k cannot be monotone, hence V is not self-decomposable. Finally, it follows from (3.6) that $\lim_{x\to 0} xn(x) = b$, hence Proposition 6.1 implies that the BDLP is compound Poisson with jump rate b. □

Proof of Theorem 6.2. Referring to (3.6), let $K(x) = x\pi_W(x)$. Clearly

$$r(x) := -\frac{K'(x)}{K(x)} = -\frac{k'(x)}{k(x)} + \frac{re^{-rx}}{1-e^{-rx}}.$$

It follows from the proof of Theorem 6.1 that $K'(x) < 0$ for all positive x if $b \geqslant 1$.

For the case $b \geqslant r$ it suffices to note that

$$\frac{re^{-rx}}{1-e^{-rx}} = \frac{r}{e^{rx}-1} = \left(\sum_{j\geqslant 1} \frac{r^{j-1}x^j}{j!}\right)^{-1}$$

is a decreasing function of r. Hence this function exceeds $be^{-bx}/(1-e^{-bx})$, so again $K'(x) < 0$, and (i) follows.

Assume now that $b < 1$ and $b < r$. Some algebra gives the identity

$$r(x) = a + \frac{1}{e^x - 1} + \frac{r}{e^{rx} - 1} - \frac{b}{e^{bx} - 1}, \tag{9.5}$$

implying that

$$r(x) = a - e^{-bx}\big(1 + o(1)\big) \quad (x \to \infty),$$

and hence that $\delta(x) := r(x) - a < 0$ if $x > x'$ where x' is a positive number depending on a. In addition $\delta(x) \to 0$ as $x \to \infty$, and $\delta(x) \to \infty$ as $x \downarrow 0$. We show that $\delta(x)$ has a negative global minimum as follows.

Let $w = e^{bx}$, $\beta = 1/b$ and $\rho = r/b$. Then $\beta, \rho > 1$ and

$$\phi(w) := \frac{\delta(x)}{b} = \frac{\beta}{w^\beta - 1} + \frac{\rho}{w^\rho - 1} - \frac{1}{w - 1} \quad (w > 1).$$

The condition $\phi(w) = 0$ is equivalent to

$$\psi(w) := \frac{\beta(w-1)}{w^\beta - 1} + \frac{\rho(w-1)}{w^\rho - 1} = 1. \tag{9.6}$$

We show that this equation has a unique solution in $(1, \infty)$.

To see this, let $c > 1$ and $\psi_c(w) = c(w-1)/(w^c - 1)$. Then $\psi_c(1+) = 1$, $\psi_c(w) \to 0$ as $w \to \infty$ and

$$\frac{\psi_c'(w)}{\psi_c(w)} = \frac{w^c - 1 - cw^{c-1}(w-1)}{(w-1)(w^c - 1)}.$$

The numerator vanishes if $w = 1$, and the mean value theorem shows it equals $(w-1)(c\zeta^{c-1} - cw^{c-1})$, where $1 < \zeta < w$. Since $c - 1 > 0$, it follows that $\psi_c'(w) < 0$ in $(1, \infty)$. We conclude that, since $\psi(1) = 2$, (9.6) has a unique positive solution, and hence that there is a number $\widetilde{x} > 0$ such that $\delta(x) > 0$ in $(0, \widetilde{x})$ and $\delta(x) < 0$ in $(\widetilde{x}, \infty)$. Hence $\delta(x)$ has a global minimum at $\bar{x} \in (\widetilde{x}, \infty)$, and $\bar{a} := -\delta(\bar{x}) > 0$. It follows that $r(x) \geqslant 0$ for all positive x if and only if $a \geqslant \bar{a}$, and hence that W is self-decomposable if and only if this condition holds. The upper bound for $\bar{a}$ follows from Theorem 6.1 since $\bar{a} = \frac{1}{2}(1 - b)$ if the r-dependent term in (9.5) is absent. This completes the proof of (ii).

The assertion (iii) follows by setting $z = e^x$ in (9.5) and observing that

$$r(x) = \frac{q(z)}{(z^a - 1)(z^b - 1)} + \frac{1}{z - 1},$$

where we now define

$$q(z) = az^{a+b} - (a+b)z^a + b.$$

But $q(1) = 0$ and $q'(z) = a(a+b)z^{a-1}(z^b - 1) > 0$ if $z > 1$. It follows that $K'(x) < 0$ if $x > 0$. $\square$

The proof for (ii) can be extended to show that $\bar{x}$ is the unique critical point of δ, and hence the shape of its graph resembles that of a Morse potential function.

Proof of Theorem 7.1. It follows from (4.4) that $X \cong S_a^{-1}\gamma_{1/a}^{1/a}$. An elementary calculation gives the density function of X as

$$\begin{aligned} g(x) &= \frac{1}{\Gamma(1 + 1/a)} \int_0^\infty v e^{-(xv)^a} \sigma_a(v)\, dv \\ &= \frac{1}{\Gamma(1 + 1/a)} E\big[S_a e^{-(xS_a)^a}\big] = \frac{1}{\Gamma(1 + 1/a)} E\big[S_a e^{-xS_a S_a'}\big], \end{aligned} \tag{9.7}$$

where $S_a' \cong \text{stable}(a)$ and is independent of S_a. Observe that the function $E[(S_a')^{-1} e^{-xS_a S_a'}]$ $(x > 0)$ is completely monotone and that it has the value $\Gamma(1 + 1/a)$ at $x = 0$. Hence there is a non-negative random variable Q such that

$$E\big(e^{-Qx}\big) = \big(\Gamma(1 + 1/a)\big)^{-1} E\big[\big(S_a'\big)^{-1} e^{-xS_a S_a'}\big].$$

Differentiating the right-hand side yields the right-hand side of (9.7), whence (7.2). Noting that

$$E\big[\big(S_a'\big)^{-1} e^{-xS_a S_a'}\big] = \int_0^\infty v^{-1} E\big[e^{-xvS_a}\big] \sigma_a(v)\, dv,$$

the change of variable $y = vS_a$ and inversion of the Laplace–Stieltjes transform yields

$$P(Q \leqslant s) = \int_0^s y^{-1} E\big[\sigma_a(y/S_a)\big]\,dy.$$

Thus Q has the density function (7.3).

To prove (7.4) we note that $Y \cong S_a \gamma_{1/a}^{-1/a}$ and hence

$$\widehat{f}(\theta) = E\big[e^{-\theta^a/\gamma_{1/a}}\big] = \big(\Gamma(1/a)\big)^{-1} \int_0^\infty z^{(1/a)-1} e^{-t^a/z}\,dz.$$

Setting $c = 2\theta^{1/a}$, we observe that the integral equals $2(c/2)^{a/2} K_{-1/a}(c)$, and the assertion follows from the symmetry property $K_{-\nu}(\cdot) = K_\nu(\cdot)$. $\square$

Proof of Theorem 7.2. Write (7.1) as

$$\Gamma(1+t)M(t) = \frac{\Gamma(1+t/a)\Gamma((1+t)/a)}{\Gamma(1/a)} \tag{9.8}$$

and note that this can be expressed as

$$\varepsilon X \stackrel{L}{=} \big(\varepsilon' \gamma_{1/a}\big)^{1/a}$$

where ε and ε' have the standard exponential law. The density function of the left-hand side is

$$p(z) = \int_0^\infty x^{-1} e^{-z/x} g(x)\,dx = \int_0^\infty e^{-zy} y f(y)\,dy,$$

i.e., the Laplace transform of $yf(y)$.

On the other hand, it can be shown using entry 6.561 #16 in Gradshteyn and Ryzhik [8] that

$$\int_0^\infty x^{t+\frac{1}{2}(a-1)} K_{1-1/a}\big(2x^{a/2}\big)\,dx = (2a)^{-1} \Gamma(1+t/a)\Gamma\big((1+t)/a\big),$$

and hence the inverse Mellin transform version of (9.8) is

$$p(z) = \frac{2z^{\frac{1}{2}(a-1)}}{\Gamma(1+1/a)} K_{1-1/a}\big(2z^{a/2}\big). \tag{9.9}$$

Next, observe that $\widehat{f}(\theta) = \int_\theta^\infty p(z)\,dz$, and hence computing the integral using (9.9) and the substitution $z = v\theta$ yields

$$\widehat{f}(\theta) = \frac{4\theta^{(a+1)/2}}{\Gamma(1/a)} \int_1^\infty v^{1/a} K_{1-1/a}\big(2\theta^{a/2} v\big)\,dv.$$

Setting $\nu = 1 - 1/a$ and $c = 2\theta^{a/2}$, the last integral equals

$$\int_1^\infty v^{1-\nu} K_\nu(cv)\,dv = \int_1^\infty v^{1-\nu} K_{-\nu}(cv)\,dv = c^{-1} K_{1-\nu}(c),$$

where we have used the entries 6.561 #8 and #16 in Gradshteyn and Ryzhik [8], and (7.4) follows.

To prove (7.5) we compute $f(y)$ by expressing (7.4) as a Laplace transform. Let $\nu = 1 - 1/a$ and observing that $K_\nu(z) = K_{-\nu}(z)$, entry 8.432 #3 in Gradshteyn and Ryzhik [8] gives the representation

$$K_\nu(z) = \frac{\sqrt{\pi}(z/2)^{-\nu}}{\Gamma(\frac{1}{2} - \nu)} \int_1^\infty \big(t^2 - 1\big)^{-\nu - \frac{1}{2}} e^{-zt}\,dt,$$

valid if the integral converges, i.e., if $a < 2$. It follows that

$$p(z)=\frac{2\sqrt{\pi}}{\Gamma(1+1/a)\Gamma(1/a-\frac{1}{2})}\int_0^\infty (t^2-1)_+^{1/a-3/2}\left[\exp\left(-2tz^{a/2}\right)\right]dt, \tag{9.10}$$

where $(\cdot)_+$ denotes the positive part of $(\cdot)$.

The duplication formula for gamma functions can be expressed as

$$\Gamma(2n/a)=\pi^{-\frac{1}{2}}2^{(2n/a)-1}\Gamma(n/a)\Gamma\left(\frac{1}{2}+n/a\right).$$

Taking $n=1$ yields

$$C:=\frac{2\sqrt{\pi}}{\Gamma(1+1/a)\Gamma(1/a-\frac{1}{2})}=\frac{2^{(2/a)-1}(2-a)}{\Gamma(2/a)}. \tag{9.11}$$

The exponential factor in the integrand at (9.10) equals $E[\exp(-zS_\eta(2t)^{2/a})]$, where $\eta=a/2<1$. The exponential term in this expectation is the Laplace–Stieltjes transform of the point mass $\delta_{((2t)^{a/2}S_\eta)}(dy)$. The corresponding distribution function equals unity if and only if $t\leqslant\frac{1}{2}(y/S_\eta)^{a/2}$. Hence inversion yields

$$\int_0^y vf(v)\,dv=CE\left[\int_0^{\frac{1}{2}(y/S_\eta)^{a/2}}(t^2-1)_+^{1/a-3/2}\,dt\right].$$

Differentiation with respect to y and further reduction together with (9.11) leads to the representation

$$f(y)=\frac{a(2-a)}{\Gamma(2/a)}y^{-a-1}E\left[S_\eta^{a-1}\left(1-4(S_\eta/y)^a\right)_+^{1/a-3/2}\right]. \tag{9.12}$$

The assertion follows since $g(x)=x^{-2}f(x^{-1})$. □

Proof of Corollary 7.1. Integrating (9.12) yields the distribution function of Y,

$$F(y)=\left(\Gamma(1+2/a)\right)^{-1}E\left[S_\eta^{-1}\left(1-4(S_\eta/y)^a\right)_+^{\zeta}\right], \tag{9.13}$$

where $\zeta=a^{-1}-\frac{1}{2}$. If J_a and V_η are independent random variables with laws as specified in the assertion, and noting that the support of H_a is $[2^{2/a},\infty)$ and that $E(V_\eta)=(E(S_\eta^{-1}))^{-1}=(\Gamma(1+2/a))^{-1}$, it follows that (9.13) can be expressed as

$$F(y)=E\left[H_a(y/V_\eta)\right]=P(J_aV_\eta\leqslant y),$$

and the assertion follows. Note that the random scaling V_η has also the effect of randomly shifting the support of J_a. □

Proof of Theorem 7.3. Let $\phi=4^{1/a}S_\eta$ in (7.5). Compute the Laplace transform

$$\begin{aligned}\int_0^\infty x^{a-1}\left(1-(\phi x)^a\right)_+^{\zeta-1}e^{-\theta x}\,dx&=a^{-1}\phi^{-a}\int_0^1(1-v)^{\zeta-1}\left(\exp\left(-(\theta/\phi)v^a\right)\right)dv\\&=a^{-1}\phi^{-a}\sum_{n=0}^\infty\frac{(-\theta/\phi)^n}{n!}B(1+n/a,\zeta).\end{aligned}$$

The interchange of integration and summation is valid because $B(1+n/a,\zeta)\sim(a/n)^\zeta$, and hence the power series is almost surely uniformly convergent for $|\theta/\phi|\leqslant R$ and any finite positive value of R. A standard argument based on the monotone convergence and dominated convergence theorems shows that the expected value of this Laplace transform can be computed term by term, giving

$$\widehat{g}(\theta)=\frac{2-a}{4\Gamma(2/a)}\sum_{n=0}^\infty\frac{(-2^{-2/a}\theta)^n}{n!}B(1+n/a,\zeta)E\left(S_\eta^{-n-1}\right).$$

Since $E(S_\eta^{-n-1})=\Gamma(1+2(n+1)/a)/\Gamma(n+2)$ and $(2-a)\Gamma(\zeta)=2a\Gamma(\frac{1}{2}+1/a)$, the duplication formula can be used to show that

$$(2-a)B(1+n/a)E\left(S_\eta^{-n-1}\right)=2^{2+2n/a}\frac{\Gamma(2/a)}{\Gamma(1/a)}\cdot\frac{\Gamma(1+n/a)\Gamma((n+1)/a)}{n!},$$

and (7.6) follows. The convergence assertions are implied by Theorem 8.1. □

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