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Homogenization of Richards' equation of van Genuchten–Mualem model ☆

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ABSTRACT

This paper is devoted to the homogenization of Richards' equation of van Genuchten–Mualem model, which is a nonlinear degenerate parabolic differential equation. It is usually used to model the motion of saturated–unsaturated water flow in porous media. We firstly apply the Kirchhoff transformation to the equation and obtain a simpler equivalent equation with a linear oscillated diffusion term. Then under the real assumption for van Genuchten–Mualem model, we obtain the homogenized equation based on the two-scale convergence theory. Some results on the first order corrector are also presented.

1. The physical introduction

Richards' equation [5,15–17] is usually accepted to describe the motion of saturated–unsaturated water flow in porous media. It is a nonlinear degenerate parabolic differential equation. Denoting $\varOmega \subset R^n$ $(n \geqslant 1)$ as the domain occupied by the porous medium with Lipschitz boundary and $(0,T]$ $(0 < T < +\infty)$ as the time interval. The continuity condition combined with Darcy's law leads to the following multi-scale Richards' equation

$$\partial_t \theta - \operatorname{div}\bigl(K^\epsilon(x,\theta)(\nabla p + \vec{e}_z)\bigr) = 0, \quad (x,t) \in \varOmega \times (0,T], \tag{1.1}$$

where $\epsilon > 0$ is a small parameter that signifies explicitly the multi-scale nature of medium, θ is the saturation, p is the pressure, K^ϵ denotes the permeability of the medium and $\vec{e}_z$ is the unit vector which points against the gravitational direction. Based on experimental results, different retention curves relating the permeability and the pressure to the saturation have been proposed in the literature [7,8,15,16,22]. According to the most popular van Genuchten–Mualem model [22,16], the constitutive relations are as follows

$$\begin{cases} \theta(p) = \begin{cases} \theta_r + \frac{\theta_s - \theta_r}{(1+|\alpha p|^n)^m} & \text{for } p \leqslant 0, \\ \theta_s & \text{for } p > 0, \end{cases} \\ K^\epsilon(x,\theta) = K_s^\epsilon(x) K_r(\varTheta) = K_s^\epsilon(x) \cdot \varTheta^{\frac{1}{2}}\bigl[1 - \bigl(1 - \varTheta^{\frac{1}{m}}\bigr)^m\bigr]^2, \\ \varTheta = \frac{\theta - \theta_r}{\theta_s - \theta_r}, \quad 0 \leqslant \varTheta \leqslant 1, \end{cases} \tag{1.2}$$

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here α, $m \in (0,1)$ and $n = \frac{1}{1-m}$ are the parameters of porous media, θ_s is the fluid content at saturation, θ_r is the residual fluid content. K_s^ϵ, K_r are the absolute and relative permeability respectively.

Without the saturated region, i.e., $\text{meas}\{\theta = \theta_s\} = 0$, Richards' equation (1.1) can also be expressed in terms of the saturation Θ as

$$\partial_t \Theta - \operatorname{div}\big(D^\epsilon(x,\Theta)\nabla\Theta + K_s^\epsilon(x)K_r(\Theta)\cdot \vec{e}_z\big) = 0, \quad (x,t) \in \Omega \times (0,T], \tag{1.3}$$

with the moisture diffusivity D defined by

$$D^\epsilon(x,\Theta) = -K^\epsilon(x,\theta)\frac{\partial p}{\partial \theta} = \frac{(1-m)K_s^\epsilon(x)}{\alpha m(\theta_s - \theta_r)}\Theta^{\frac{1}{2}}\big[(1-\Theta)^m + (1-\Theta)^{-m} - 2\big]. \tag{1.4}$$

In practical problems, due to complex heterogeneity of natural media and scarcity of the available field data, the coefficients K^ϵ and D^ϵ maybe oscillate rapidly with large contrast. In the above model, the multi-scale nature of the problem comes from the heterogeneity of K_s^ϵ, i.e., the coefficients D^ϵ and K^ϵ may be formulated as $D^\epsilon(x,s) = D(\frac{x}{\epsilon}, s)$ and $K^\epsilon(x,s) = K(\frac{x}{\epsilon}, s) = K_s(\frac{x}{\epsilon})K_r(s)$. On the other hand, noticing the expression of D^ϵ, one can find that Eq. (1.3) is degenerated by the fact that for $\Theta = 0$ the moisture diffusivity D^ϵ vanishes, while as Θ tends to 1, D^ϵ goes to infinity. In this paper, we will devote to the homogenization of multi-scale Richards' equation (1.3) of van Genuchten–Mualem model.

To this kind of multi-scale problem, it is impossible to account explicitly for the spatial variability at fine scale because of the computational resource limitations in realistic situation. In recent decades, there exists a vast literature on the upscaling or homogenization techniques for various multi-scale problems that lump the small-scale details of the medium into a few representative macroscopic parameters or effective parameters on a coarse scale which preserve the larger-scale behavior of the medium and are more appropriate for reservoir simulations; see, e.g., [23,4,6,9–11,14,19–21,24,25,17,27,18,28]. Of those papers, the information on the homogenization of immiscible two-phase flow and transport in porous media can be found in [4,6,25]; [28] dealt with the homogenization of the two-phase Stefan problem; in [11], the authors considered the homogenization of nonlinear degenerate evolution equations using the method of two-scale convergence. Here we would review some papers concerning on some degenerate problems which were similar to our problem. For example, in [19,24], the authors considered the homogenization of multi-scale degenerate problem

$$\partial_t u^\epsilon - \nabla\cdot a\left(\frac{x}{\epsilon}, \frac{t}{\epsilon}, u^\epsilon, \nabla u^\epsilon\right) = f,$$

under the frame of weighted Sobolev space theory, where the operator $a(y,s,\mu,\xi)$ satisfied the following degenerate condition

$$\Lambda_1(x)|\xi|^2 \leqslant a(y,s,\mu,\xi)\xi \leqslant \Lambda_2(x)|\xi|^2, \tag{1.5}$$

where Λ_i $(i = 1,2)$ can vanish or be infinity at some points and belongs to some special weighted Sobolev space.

In [23,18,20], the authors considered the homogenization of a nonlinear degenerate parabolic equations as

$$\partial_t b\big(u^\epsilon\big) - \nabla\cdot a\left(\frac{x}{\epsilon}, \frac{t}{\epsilon}, u^\epsilon, \nabla u^\epsilon\right) = f,$$

under the degenerate assumption that there exists some constant $\tau > 0$, such that

$$\big|b(s_1) - b(s_2)\big| \geqslant C(\delta,R)|s_1 - s_2|^\tau, \quad \forall s_1, s_2 \in [-R,R] \text{ with } 0 < \delta < |s_1|, \tag{1.6}$$

while the second order term is not degenerate.

The degenerate conditions (1.5) and (1.6) don't hold for the Richards' equation of van Genuchten–Mualem model. So far, the multi-scale Richards' equation (1.1) had been considered in [10,17]. In [10], the authors considered a non-degenerate settings for $\theta(s)$ and $K(\cdot,s)$ such as Gardner model [15]. An upscaling procedure was proposed and detailed numerical analysis was presented. In [17], the author assumed that

$$0 < K_{min} \leqslant K(y,s) \leqslant K_{max} \quad \text{and} \quad \theta'(s) \text{ vanishes at some points}, \tag{1.7}$$

to obtain the homogenized equation and furthermore assumed that $\theta'(s) > 0$ to get some results on the corrector. However for van Genuchten–Mualem model, $K(y,s)$ can approach zero because the relative permeability K_r depending only on saturation can approach zero in completely dry area ($\Theta = 0$) and the strictly monotonicity of saturation $\theta(s)$ in (1.2) is not valid either.

Combining the all above, the homogenization of the Richards' equation (1.3) under the real assumptions for van Genuchten–Mualem model is still open so far. However, the techniques needed to attack this problem are prepared well in the above literatures.

From the formulation (1.4), the key observation is that D^ϵ is separable, i.e., $D^\epsilon = D_s(\frac{x}{\epsilon})D_r(\Theta)$ with $D_s = K_s^\epsilon(x)$ being the absolute permeability of the medium (all the multi-scale nature is included in this term and it is non-degenerate) and $D_r = -K_r(\Theta)\frac{\partial p}{\partial \theta} = \frac{(1-m)}{\alpha m(\theta_s - \theta_r)}\Theta^{\frac{1}{2}}[(1-\Theta)^m + (1-\Theta)^{-m} - 2]$ being the relative diffusivity (this term is nonlinear and degenerate but no multi-scale information is explicitly included). Here, we point out that the diffusion tensor is not always separable in other physical model. Moreover, in many practical applications so-called block heterogeneities are encountered, implying a special conditions at the interface between two homogeneous blocks, as presented in [27,12]. These are beyond the scope of the paper.

Considering multi-scale nature of the problem and using the separated formulation of D^ϵ, (1.3) can be rewritten as

$$\partial_t \Theta - \operatorname{div}\left(D_s\left(\frac{x}{\epsilon}\right)D_r(\Theta)\nabla\Theta + K_s\left(\frac{x}{\epsilon}\right)K_r(\Theta)\cdot\vec{e}_z\right) = 0, \tag{1.8}$$

where the degeneration comes from the fact that the relative diffusivity D_r vanishes for $\Theta = 0$, while as Θ tends to 1, D_r goes to infinity.

Firstly, define the Kirchhoff transformation [3] as

$$\psi : [0,1] \to R^+, \qquad \Theta \mapsto \int_0^\Theta D_r(s)\,ds. \tag{1.9}$$

Since $D_r(s)$ is positive except for the point $s = 0$, this transformation is invertible. Denote

$$u^\epsilon = \psi(\Theta), \qquad \Theta = \psi^{-1}\left(u^\epsilon\right) \doteq b\left(u^\epsilon\right). \tag{1.10}$$

Applying the transformation (1.9) to (1.8), we have

$$\partial_t b\left(u^\epsilon\right) - \operatorname{div}\left(D_s\left(\frac{x}{\epsilon}\right)\nabla u^\epsilon + K_s\left(\frac{x}{\epsilon}\right)K_r\left(b\left(u^\epsilon\right)\right)\cdot\vec{e}_z\right) = 0, \tag{1.11}$$

here $b(s)$ is strictly increasing with $b'(0^+) = +\infty$ and $b'(+\infty) = 0$ and the diffusion term becomes linear. However, the problem (1.11) remains to be degenerated, leading to lacking of regularity for its solutions. Fortunately, in the form as (1.11), most conditions in (1.7) are valid. So mainly along the line of [17] and [3], we can derive the homogenized equation for Richards' equation of van Genuchten–Mualem model.

From now on, C will denote a generic positive constant which is independent of ϵ. In the homogenization procedure of Eq. (1.11), the main task is to prove the strongly convergence of solution sequence $\{u^\epsilon\}$ of (1.11) in $L^p(\Omega \times (0,T))$ (for some p), which cannot be followed directly from the fact $\|\nabla u^\epsilon\|_{L^2(\Omega\times(0,T))} \leqslant C$ and $\|\partial_t b(u^\epsilon)\|_{L^2(0,T;H^{-1}(\Omega))} \leqslant C$ due to the degeneration of $b(\cdot)$. To obtain the results on the corrector, the main difficulty is to handle the convergence of term $\int_\Omega\int_0^T \partial_t b(u^\epsilon)u^\epsilon\,dx\,dt$, which is also not obvious. In order to overcome these difficulties, we make some careful estimates and apply the compactness method based on the ideas of [3,17]. To derive the homogenization and corrector results, we do not suppose any L^∞-bounds on the solution u^ϵ and $b(u^\epsilon)$ of (1.11), which is needed in [23,20].

The layout of the paper is as follows. In Section 2, we give some assumptions on the coefficients of equation and provide the main result. In Section 3, we prove the a priori estimate and review the two-scale convergence results. These results are the basis of our theory. In Section 4, we prove our main theorems.

2. Problem setting and main results

Here we work in a slightly more general context which includes equation (1.11). Our aim is to establish the homogenization theory for the following problem

$$\begin{cases} \partial_t b(u^\epsilon) - \nabla\cdot\left(A(\frac{x}{\epsilon})\nabla u^\epsilon + \mathbf{g}(\frac{x}{\epsilon}, b(u^\epsilon))\right) = f(\frac{x}{\epsilon}, b(u^\epsilon)) & \text{in } \Omega_T, \\ u^\epsilon(x,t) = 0 & \text{on } \partial\Omega\times(0,T), \\ u^\epsilon(x,0) = u_0 & \text{in } \Omega. \end{cases} \tag{2.1}$$

Notations. $\Omega_T = \Omega\times(0,T)$, $V = L^2(0,T;H_0^1(\Omega))$, $Y = (0,1)^n$ and $H^1_{per}(Y)$ be the space of elements of $H^1(Y)$ having the same trace on opposite face of $Y = (0,1)^n$. $(\cdot,\cdot)$ stands for the inner product on L^2 and $\langle\cdot,\cdot\rangle$ for the duality pairing between H_0^1 and H^{-1}. For other function spaces we refer to [1].

Based on the van Genuchten–Mualem model, the following assumptions are supposed.

Assumptions.

- (H1) b is a strictly increasing and continuous function in R and $b'(s) \in [0+\infty]$, $s \in R$. b also satisfies

$$\exists L, M \in R, \qquad \left|b(s)\right| \leqslant L|s| + M. \tag{2.2}$$

Define the Legendre transform Ψ of the primitive of b by

$$\Psi : R \to [0,+\infty], \qquad s \to \sup_{z\in R}\left(zs - \int_0^s b(\tau)\,d\tau\right).$$

It is superlinear in the sense of (2.3) and admits the representation

$$\forall z \in R, \quad B(z) := \Psi\big(b(z)\big) = zb(z) - \int_0^z b(\tau)\,d\tau,$$
$$\forall \delta > 0,\ \exists C_\delta < +\infty, \quad |s| \leqslant \delta\Psi(s) + C_\delta, \quad \forall s \in R. \tag{2.3}$$

- (H2) The symmetrical tensor $A = (A(y))_{ij}$ $(1 \leqslant i, j \leqslant n)$ is continuous, periodic in y and satisfies

$$\exists 0 < \lambda_1 \leqslant \lambda_2, \quad \lambda_1|\xi|^2 \leqslant A(y)\xi \cdot \xi \leqslant \lambda_2|\xi|^2, \quad \forall \xi \in R^n.$$

- (H3) $\mathbf{g} = (g(y,s))_i$ $(1 \leqslant i \leqslant n) : R^n \times R \to R^n$ and $f(y,s) : R^n \times R \to R$ are continuous in s, periodic in y and fulfill the following identities

$$\exists k > 0 \quad \text{such that} \quad \big|\mathbf{g}(y,s)\big| \leqslant k, \quad \forall (y,s) \in R^n \times R \to R, \tag{2.4}$$
$$\exists \alpha, \beta \in (0,1] \quad \text{such that} \quad \big|\mathbf{g}(y,s_1) - \mathbf{g}(y,s_2)\big| \leqslant C|s_1 - s_2|^\alpha, \tag{2.5}$$
$$\big|f(y,s_1) - f(y,s_2)\big| \leqslant C|s_1 - s_2|^\beta, \tag{2.6}$$
$$\big|f(y,s_1)\big| \leqslant C, \quad \forall y \in R^n,\ s_1, s_2 \in R \tag{2.7}$$

and $u_0 \in L^\infty(\Omega)$.

Existence, uniqueness and regularity of the solution for the above problem is studied in several papers (see, for example, [3,13,29] and references therein). In the present paper, we only focus on the asymptotic behavior of the problem (2.1) as $\epsilon \to 0$. A definition of the weak solution for the problem (2.1) is as follows:

Definition 2.1. Fix $\epsilon > 0$, we say that $u^\epsilon \in V$ is a weak solution of problem (2.1), if it satisfies the following two identities:

1. $b(u^\epsilon) \in L^2(\Omega_T)$ and $\partial_t b(u^\epsilon) \in L^2(0,T;H^{-1}(\Omega))$ with

$$\int_0^T \langle \partial_t b(u^\epsilon), \varphi\rangle\,dt + \int_{\Omega_T} \big(b(u^\epsilon) - b(u_0)\big)\partial_t\varphi\,dx\,dt = 0, \tag{2.8}$$

for every $\varphi \in V \cap H^1(0,T;L^2(\Omega))$ with $\varphi(T) = 0$.

2. For all $\varphi \in V$,

$$\int_0^T \langle \partial_t b(u^\epsilon), \varphi\rangle\,dt + \int_{\Omega_T} \left(A\left(\frac{x}{\epsilon}\right)\nabla u^\epsilon \nabla\varphi + \mathbf{g}\left(\frac{x}{\epsilon}, b(u^\epsilon)\right)\cdot\nabla\varphi\right)dx\,dt = \int_{\Omega_T} f\varphi\,dx\,dt. \tag{2.9}$$

In order to express our main results conveniently, the following definition of the two-scale convergence [2] is needed.

Definition 2.2. Let $1 < q < \infty$. A sequence of function $v_\epsilon \in L^q(\Omega)$ is said to two-scale converge to a function $v \in L^q(\Omega \times Y)$ (denoted by $v_\epsilon \twoheadrightarrow v$), if

$$\int_\Omega v^\epsilon \psi\left(x, \frac{x}{\epsilon}\right)dx \to \int_\Omega \int_Y v(x,y)\psi(x,y)\,dy\,dx \quad (\epsilon \to 0), \tag{2.10}$$

for all $\psi \in L^{q^*}(\Omega; C_{per}(Y))$ $(q^* = \frac{q}{q-1})$.

In the rest of the section, the main results of this paper are summarized in the following theorems.

Theorem 2.3. *Suppose that* (H1)–(H3) *hold and u^ϵ is the sequence of weak solutions of problem* (2.1)*, then there exist $u \in V$ and $U(x,y,t) \in L^2(\Omega_T; H^1_{per}(Y))$ such that as $\epsilon \to 0$, one has*

$$u^\epsilon \to u \quad \text{strongly in } L^q\big(0,T;L^2(\Omega)\big), \text{ for some } q \in [1,2), \tag{2.11}$$
$$\nabla u^\epsilon \twoheadrightarrow \nabla_x u(x,t) + \nabla_y U(x,y,t), \tag{2.12}$$

where u also satisfies the following problem

$$\begin{cases} \partial_t b(u) - \nabla \cdot (A^* \nabla u + \mathbf{g}^*(b(u))) = f^*(b(u)) & \text{in } \Omega_T, \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0 & \text{in } \Omega. \end{cases} \tag{2.13}$$

Furthermore, if $u \in C(0,T;C^1(\Omega))$ and $U \in C(\Omega_T; C^1_{per}(Y))$, then

$$\nabla u^\epsilon \to \nabla u + \nabla_y U \quad \text{strongly in } L^2(\Omega_T). \tag{2.14}$$

The homogenized coefficients are defined as

$$A^*_{ij} = \int_Y \left(A_{ij}(y) + A_{ik}(y) \frac{\partial \Lambda^j}{\partial y_k} \right) dy, \tag{2.15}$$

$$\mathbf{g}^*(s) = \int_Y \big(A(y) \nabla_y \eta + \mathbf{g}(y,s) \big)\, dy, \tag{2.16}$$

$$f^*(s) = \int_Y f(y,s)\, dy, \tag{2.17}$$

where $\Lambda^i, \eta(\cdot,s) \in H^1_{per}(Y)$ are periodic solutions of the following cell problems respectively

$$\nabla_y \cdot \big(A(y) (\vec{e}_i + \nabla_y \Lambda^i) \big) = 0 \quad \text{in } Y \quad \text{and} \quad \int_Y \Lambda^i\, dy = 0, \tag{2.18}$$

$$-\nabla_y \cdot \big(A(y) \nabla_y \eta(y,s) \big) = \nabla_y \cdot \big(\mathbf{g}(y,s) \big) \quad \text{in } Y \quad \text{and} \quad \int_Y \eta\, dy = 0. \tag{2.19}$$

This theorem will be proved later in Section 4. As a special application of the theorem, we have the homogenization results on Richards' equation of van Genuchten–Mualem model.

Theorem 2.4. *Suppose $K_s(y)$ is uniformly bounded. Then, under the constitution relations of van Genuchten–Mualem, for Richards' equation (1.2)–(1.4) subject to the Dirichlet boundary condition $\Theta = \Theta_D(x,t) \in [0,1]$ ($\Theta_D(x,t)$ is sufficiently smooth) and initial condition $\Theta = \Theta_0(x) \in [0,1]$, the homogenized model is as follows*

$$\begin{aligned} &\partial_t \Theta^* - \nabla \cdot \big(K_s^* D_r(\Theta^*) \nabla \Theta^* + K_r(\Theta^*) K_s^*(x) \vec{e}_z \big) = 0, \quad \text{in } \Omega_T, \\ &\Theta^*(x,t) = \Theta_D(x,t), \quad x \in \partial\Omega; \qquad \Theta^*(x,0) = \Theta_0(x), \quad x \in \Omega, \end{aligned} \tag{2.20}$$

where K_s^* is a tensor defined as

$$(K_s^*)_{ij} = \int_Y \vec{e}_i K_s(y) \cdot (\vec{e}_j + \nabla_y \Lambda^j)\, dy,$$

where Λ^j is the periodic solutions of the following cell problems

$$\nabla_y \cdot \big(K_s(y) (\vec{e}_j + \nabla_y \Lambda^j) \big) = 0 \quad \text{in } Y \quad \text{and} \quad \int_Y \Lambda^j\, dy = 0.$$

Furthermore, we can introduce the pressure p^* and the homogenized Richards' equation can also be written in a saturation–pressure form as follows

$$\partial_t \theta^* - \nabla \cdot \big(K^*(\theta^*) (\nabla p^* + \vec{e}_z) \big) = 0 \quad \text{in } \Omega_T \tag{2.21}$$

with the same constitution relationship as in (1.2)

$$\theta^*(p^*) = \begin{cases} \theta_r + \frac{\theta_s - \theta_r}{(1 + |\alpha p^*|^n)^m} & \text{for } p^* \leqslant 0, \\ \theta_s & \text{for } p^* > 0, \end{cases} \tag{2.22}$$

where $K^*(\theta^*) = K_s^* K_r(\Theta^*)$ and we still have the relationship $D_r(\Theta^*) = -K_r(\theta^*) \frac{\partial p^*}{\partial \theta^*}$.

Proof. Firstly, by the maximum principle, we get that the solution Θ of Richards' equation (1.2)–(1.4) with boundary and initial conditions is bounded, i.e. $\Theta \in [0,1]$ (see [26]). It is easy to verify that $b(\cdot)$, $D_s(y)$ and $K(y,\cdot)$ in (1.11) satisfy the assumptions (H1)–(H3) under the relations of (1.2) and (1.4). Along the ideas of [20] and [23] to treat the nonhomogeneous boundary condition, (2.20) can be obtained by Theorem 2.3 and inverse transformation (1.9). Noting that $D_s^\epsilon = K_s^\epsilon$, the rest of the proof can be completed by a straight calculation. □

3. Preliminaries

In this section we will do some preliminary work. Firstly, we obtain some a priori bounds for the weak solution of (2.1). Secondly, we recall the main results concerning the method of two-scale convergence which will be used in the proof of Theorem 2.3.

Lemma 3.5. *Assume $B(u^0) \in L^1(\Omega)$ and (H1)–(H3) hold. If u^ϵ is the solution of problem (2.1) then*

$$\left\|\partial_t b\left(u^\epsilon\right)\right\|_{L^2(0,T;H^{-1}(\Omega))} + \left\|B\left(u^\epsilon\right)\right\|_{L^\infty(0,T;L^1(\Omega))} + \left\|\nabla u^\epsilon\right\|_{L^2(\Omega_T)} \leqslant C. \tag{3.1}$$

Proof. Setting $\varphi = u^\epsilon$ in the formulation (2.9), we get

$$\int_0^T \left\langle \partial_t b\left(u^\epsilon\right), u^\epsilon\right\rangle = \int_\Omega \left(B\left(u^\epsilon(T)\right) - B\left(u^0\right)\right),$$

$$\int_\Omega B\left(u^\epsilon(T)\right) + \int_{\Omega_T} A\left(\frac{x}{\epsilon}\right)\nabla u^\epsilon \cdot \nabla u^\epsilon = \int_\Omega B\left(u^0\right) - \int_{\Omega_T} \mathbf{g}\left(\frac{x}{\epsilon}, b\left(u^\epsilon\right)\right) \cdot \nabla u^\epsilon + \int_{\Omega_T} f u^\epsilon. \tag{3.2}$$

Noticing that the first term in (3.2) is nonnegative, we get from assumptions (H2) that

$$\lambda_1 \left\|\nabla u^\epsilon\right\|^2_{L^2(\Omega_T)} \leqslant \int_\Omega B\left(u^0\right) - \int_{\Omega_T} \mathbf{g}\left(\frac{x}{\epsilon}, b\left(u^\epsilon\right)\right) \cdot \nabla u^\epsilon + \int_{\Omega_T} f u^\epsilon. \tag{3.3}$$

Using (2.4) and (2.7), a straight calculation gives

$$\left|-\int_{\Omega_T} \mathbf{g}\left(\frac{x}{\epsilon}, b\left(u^\epsilon\right)\right) \cdot \nabla u^\epsilon\right| \leqslant \frac{\lambda_1}{4}\left\|\nabla u^\epsilon\right\|^2_{L^2(\Omega_T)} + C,$$

$$\left|\int_{\Omega_T} f\left(\frac{x}{\epsilon}, b\left(u^\epsilon\right)\right) u^\epsilon\right| \leqslant \frac{\lambda_1}{4}\left\|\nabla u^\epsilon\right\|^2_{L^2(\Omega_T)} + C.$$

The above two inequalities combine with (3.3) we get $\|\nabla u^\epsilon\|_{L^2(\Omega_T)} \leqslant C$.

Combining this result, assumptions (H2) and (H3), we have

$$\left\|\partial_t b\left(u^\epsilon\right)\right\|_{L^2(0,T;H^{-1}(\Omega))} \leqslant C. \tag{3.4}$$

Then from Lemma 1.5 [3], it follows that

$$\left\|B\left(u^\epsilon\right)\right\|_{L^\infty(0,T;L^1(\Omega))} \leqslant C. \qquad \square$$

Lemma 3.6. *Let u^ϵ be the solution of problem (2.1) and assume (H1)–(H3) hold, then up to a subsequence of u^ϵ (still denote by u^ϵ), there exists a function u such that*

$$u^\epsilon \to u \quad \text{weakly in } V, \tag{3.5}$$

$$u^\epsilon \to u \quad \text{strongly in } L^q\left(0,T;L^2(\Omega)\right), \text{ for some } 1 \leqslant q < 2, \tag{3.6}$$

$$b\left(u^\epsilon\right) \to b(u) \quad \text{strongly in } L^1(\Omega_T), \tag{3.7}$$

$$\partial_t b\left(u^\epsilon\right) \to \partial_t b(u) \quad \text{weakly in } L^2\left(0,T;H^{-1}(\Omega)\right), \tag{3.8}$$

$$B\left(u^\epsilon\right) \to B(u) \quad \text{strongly in } L^1(\Omega_T). \tag{3.9}$$

Proof. From Lemma 3.5, there exist functions u and w such that

$$u^\epsilon \to u \quad \text{weakly in } V,$$

$$\partial_t b\left(u^\epsilon\right) \to w \quad \text{weakly in } L^2\left(0,T;H^{-1}(\Omega)\right). \tag{3.10}$$

In the following, we will use some ideas from [3,17] to prove (3.7) and (3.9). Firstly, we will verify that $b(u^\epsilon) \to b(u)$ strongly in $L^1(\Omega_T)$. Let a small number $h > 0$ be given, we can choose $\varphi_1 = h^{-1}\int_{t-h}^t u^\epsilon(x,\tau)\eta(\tau)\,d\tau$ and $\varphi_2 =$

$h^{-1}\int_{t-h}^{t} u^{\epsilon}(x,\tau+h)\eta(\tau)\,d\tau$ as test function in (2.8) and (2.9) respectively, where $\eta(\tau)=1, \tau\in(0,T-h)$ and $\eta(\tau)=0$ elsewhere. Using integration transformation, we get

$$\int_0^T\int_\Omega \partial_t b^{\epsilon}\big(x,u(t)\big)(\varphi_2-\varphi_1)\,dx\,dt$$

$$=-\int_0^T\int_\Omega b\big(u^{\epsilon}(x,t)\big)(\partial_t\varphi_2-\partial_t\varphi_1)\,dx\,dt=\frac{1}{h}\int_0^{T-h}\int_\Omega\big[b\big(u^{\epsilon}(x,t+h)\big)-b\big(u^{\epsilon}(x,t)\big)\big]\big(u^{\epsilon}(x,t+h)-u^{\epsilon}(x,t)\big)\,dx\,dt$$

$$=\int_{\Omega_T}\left[A\left(\frac{x}{\epsilon}\right)\nabla u^{\epsilon}+\mathbf{g}\left(\frac{x}{\epsilon},b\big(u^{\epsilon}\big)\right)\right]\cdot(\nabla\varphi_1-\nabla\varphi_2)\,dx\,dt+\int_{\Omega_T} f\big(b\big(u^{\epsilon}\big)\big)(\varphi_2-\varphi_1)\,dx\,dt.$$

Moreover, we can get that $\|\varphi_1\|_{L^2(0,T;H_0^1(\Omega))}$ and $\|\varphi_2\|_{L^2(0,T;H_0^1(\Omega))}$ are bounded, see [17]. In fact, by Jensen's inequality, we have

$$\int_{\Omega_T}|\nabla\varphi_1|^2\,dx\,dt\leqslant\frac{1}{h}\int_{\Omega_T}\int_{t-h}^{t}\big|\nabla u^{\epsilon}(x,\tau)\big|^2\,d\tau\,dx\,dt\leqslant C\big\|\nabla u^{\epsilon}\big\|_{L^2(\Omega_T)}.$$

Thus, by (H2), (H3) and Lemma 3.5, we have

$$\frac{1}{h}\int_0^{T-h}\int_\Omega\big[b\big(u^{\epsilon}(x,t+h)\big)-b\big(u^{\epsilon}(x,t)\big)\big]\big(u^{\epsilon}(x,t+h)-u^{\epsilon}(x,t)\big)\,dx\,dt\leqslant C.$$

The above inequality and (3.1) imply that $b(u^{\epsilon})\to b(u)$ strongly in $L^1(\Omega_T)$ ([3], Lemma 1.9), which together with (3.10) yields (3.8).

Secondly, we will show that $u^{\epsilon}\to u$ strongly in $L^1(\Omega_T)$. Define the strictly convex function $h(s)=\int_0^s b(\tau)\,d\tau$ and observe that

$$b\big(u^{\epsilon}\big)\big(u^{\epsilon}-u\big)\geqslant h\big(u^{\epsilon}\big)-h(u)\quad\text{a.e. in }\Omega_T.$$

Recall the assumption (H1) $|b(u^{\epsilon})|\leqslant L|u^{\epsilon}|+M$, we have $b(u^{\epsilon})\in L^2(\Omega_T)$. This result together with (3.7) and (3.8) implies that $b(u^{\epsilon})$ strongly convergence to $b(u)$ in $L^2(0,T;H^{-1}(\Omega))$. Thus, we obtain

$$0\leftarrow\int_{\Omega_T} b\big(u^{\epsilon}\big)\big(u^{\epsilon}-u\big)\geqslant\int_{\Omega_T} h\big(u^{\epsilon}\big)-h(u)\doteq G\big(u^{\epsilon}\big)-G(u),$$

then we arrive at $\limsup_{\epsilon\to0}G(u^{\epsilon})\leqslant G(u)$. Lemma 4.2 in [17] implies that $u^{\epsilon}\to u$, strongly in $L^1(\Omega_T)$. Combining this fact with (3.5), we get (3.6). The proof of (3.9) is similar to the above, refer to [17]. □

The following facts about two-scale convergence [2] will be needed below.

Theorem 3.7. *If v^{ϵ} and ∇v^{ϵ} are bounded in $L^q(\Omega)$, then there exist $v\in W^{1,q}(\Omega)$ and $V\in L^q(\Omega,W_{per}^{1,q}(Y))$ such that, up to a subsequence, v^{ϵ} and ∇v^{ϵ} converge in two-scale in $W^{1,q}(\Omega)$,*

$$v^{\epsilon}\twoheadrightarrow v(x),\qquad\nabla v^{\epsilon}\twoheadrightarrow\nabla_x v(x)+\nabla_y V(x,y).$$

Using the two-scale convergence theory, it follows that [2,23,17]

Lemma 3.8. *Under the assumptions of Lemma 3.6, there exist a function $U\in L^2(\Omega_T;H_{per}^1(Y))$ and a subsequence of u^{ϵ} (still denoted by u^{ϵ}) such that*

$$\nabla u^{\epsilon}\twoheadrightarrow\nabla_x u(x,t)+\nabla_y U(x,y,t).\tag{3.11}$$

Furthermore, the pair (u,U) satisfies the following two-scale homogenized problem

$$\int_0^T \langle \partial_t b(u), \phi \rangle dt + \int_{\Omega_T} \int_Y A(y)\big(\nabla_x u(x) + \nabla_y U(x, y, t)\big) \cdot \big(\nabla_x \phi + \nabla_y \Phi(x, y, t)\big)\, dy\, dx\, dt$$
$$+ \int_{\Omega_T} \int_Y \mathbf{g}\big(y, b(u)\big) \cdot \big(\nabla_x \phi + \nabla_y \Phi(x, y, t)\big)\, dy\, dx\, dt = \int_{\Omega_T} \int_Y f\big(y, b(u)\big)\phi\, dy\, dx\, dt \tag{3.12}$$

for all $\phi \in C_0^\infty(\Omega_T)$ *and* $\Phi \in C_0^\infty(\Omega_T; C_{per}^\infty(Y))$.

4. Proof of Theorem 2.3

In this section, we will complete the proof of Theorem 2.3.

Proof. First, (2.11) and (2.12) have been proved in Lemma 3.6 and 3.8 respectively.

Setting $\phi = 0$ in (3.12), we have

$$\int_{\Omega_T} \int_Y A(y)\big(\nabla_x u(x, t) + \nabla_y U(x, y, t)\big) \cdot \nabla_y \Phi(x, y, t)\, dy\, dx\, dt = -\int_{\Omega_T} \int_Y \mathbf{g}\big(y, b(u)\big) \cdot \nabla_y \Phi(x, y, t)\, dy\, dx\, dt. \tag{4.1}$$

Noticing that u is independent of variable y and Φ is arbitrary, we may determine, up to a constant, that

$$U = \nabla u \cdot \vec{\Lambda}(x, y, t) + \eta(x, y, t), \qquad \vec{\Lambda}(x, y, t) = \big(\Lambda^1, \ldots, \Lambda^n\big)'. \tag{4.2}$$

And it is easy to verify that Λ^i and η satisfy the problem (2.18) and (2.19) respectively.

Let $\Phi = 0$ in (3.12), we obtain

$$\int_0^T \langle \partial_t b(u), \phi \rangle dt + \int_{\Omega_T} \int_Y A(y)\big(\nabla_x u(x, t) + \nabla_y U(x, y, t)\big) \cdot \nabla_x \phi\, dy\, dx\, dt$$
$$+ \int_{\Omega_T} \int_Y \mathbf{g}\big(y, b(u)\big) \cdot \nabla_x \phi\, dy\, dx\, dt = \int_{\Omega_T} \int_Y f\big(y, b(u)\big) \cdot \phi\, dy\, dx\, dt. \tag{4.3}$$

Using the expression of U, we get

$$\int_0^T \langle \partial_t b(u), \phi \rangle dt + \int_{\Omega_T} \int_Y A(y)(I_{d\times d} + \nabla_y \vec{\Lambda})\, dy \nabla_x u(x, t) \cdot \nabla_x \phi\, dx\, dt$$
$$+ \int_{\Omega_T} \int_Y \big(A(y)\nabla_y \eta + \mathbf{g}\big(y, b(u)\big)\big)\, dy \cdot \nabla_x \phi\, dx\, dt = \int_{\Omega_T} \int_Y f\big(y, b(u)\big) \cdot \phi\, dy\, dx\, dt.$$

Noting the formulation of (2.15)–(2.17), the above equality is just the weak form of the homogenized problem (2.13).

What's left is to prove the strong convergence (2.14) for ∇u^ϵ. To this purpose, we first have

$$\begin{aligned}
\lambda_1 \big\|\nabla u^\epsilon - \nabla u - \nabla_y U\big\|^2_{L^2(\Omega_T)} &\leqslant \int_{\Omega_T} A\left(\frac{x}{\epsilon}\right)\big(\nabla u^\epsilon - \nabla u - \nabla_y U\big) \cdot \big(\nabla u^\epsilon - \nabla u - \nabla_y U\big) \\
&= \int_{\Omega_T} A\left(\frac{x}{\epsilon}\right)\nabla u^\epsilon \cdot \nabla u^\epsilon - 2\int_{\Omega_T} A\left(\frac{x}{\epsilon}\right)\nabla u^\epsilon \cdot (\nabla u + \nabla_y U) \\
&\quad + \int_{\Omega_T} A\left(\frac{x}{\epsilon}\right)(\nabla u + \nabla_y U) \cdot (\nabla u + \nabla_y U) \doteq I_1 + I_2 + I_3,
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= -\int_0^T \langle \partial_t b(u^\epsilon), u^\epsilon \rangle - \int_{\Omega_T} \mathbf{g}\left(\frac{x}{\epsilon}, b(u^\epsilon)\right) \cdot \nabla u^\epsilon + \int_{\Omega_T} f\left(\frac{x}{\epsilon}, b(u^\epsilon)\right) u^\epsilon \\
&= -\int_\Omega \big(B\big(u^\epsilon(T)\big) - B\big(u^\epsilon(0)\big)\big) - \int_{\Omega_T} \mathbf{g}\left(\frac{x}{\epsilon}, b(u^\epsilon)\right) \cdot \nabla u^\epsilon + \int_{\Omega_T} f\left(\frac{x}{\epsilon}, b(u^\epsilon)\right) u^\epsilon.
\end{aligned}$$

Let's rewrite

$$\int_{\Omega_T} \mathbf{g}\left(\frac{x}{\epsilon}, b(u^\epsilon)\right) \cdot \nabla u^\epsilon = \int_{\Omega_T} \mathbf{g}\left(\frac{x}{\epsilon}, b(u)\right) \cdot \nabla u^\epsilon + \int_{\Omega_T} \left(\mathbf{g}\left(\frac{x}{\epsilon}, b(u)\right) - \mathbf{g}\left(\frac{x}{\epsilon}, b(u^\epsilon)\right)\right) \cdot \nabla u^\epsilon.$$

Then from (2.4) and (2.6), a straight calculation gives that

$$\begin{aligned}\left|\int_{\Omega_T} \left(\mathbf{g}\left(\frac{x}{\epsilon}, b(u)\right) - \mathbf{g}\left(\frac{x}{\epsilon}, b(u^\epsilon)\right)\right) \cdot \nabla u^\epsilon\right| &\leqslant C \int_{\Omega_T} \left|b(u) - b(u^\epsilon)\right|^{\frac{\alpha}{2}} \left|\nabla u^\epsilon\right| \\ &\leqslant C \left\|b(u) - b(u^\epsilon)\right\|_{L^1(\Omega_T)}^{\frac{\alpha}{2}} \left\|\nabla u^\epsilon\right\|_{L^2(\Omega_T)}. \end{aligned} \tag{4.4}$$

Thus, taking $\mathbf{g}(\frac{x}{\epsilon}, b(u))$ as a test function and using (3.11) and (3.7), the two-scale convergence of ∇u^ϵ implies that, as $\epsilon \to 0$,

$$\int_{\Omega_T} \mathbf{g}\left(\frac{x}{\epsilon}, b(u^\epsilon)\right) \cdot \nabla u^\epsilon \to \int_{\Omega_T} \int_Y \mathbf{g}\big(y, b(u)\big) \cdot (\nabla u + \nabla_y U).$$

Similarly, we have

$$\int_{\Omega_T} f\left(\frac{x}{\epsilon}, b(u^\epsilon)\right) \cdot u^\epsilon \to \int_{\Omega_T} \int_Y f\big(y, b(u)\big) \cdot u. \tag{4.5}$$

By (3.9), we deduce

$$\begin{aligned} I_1 &\to -\int_{\Omega} \big(B(u(T)) - B(u(0))\big) - \int_{\Omega_T} \int_Y \mathbf{g}\big(y, b(u)\big) \cdot (\nabla u + \nabla_y U) + \int_{\Omega_T} \int_Y f\big(y, b(u)\big) \cdot u \\ &= -\int_{\Omega_T} \partial_t b(u) \cdot u - \int_{\Omega_T} \int_Y \mathbf{g}\big(y, b(u)\big) \cdot (\nabla u + \nabla_y U) + \int_{\Omega_T} \int_Y f\big(y, b(u)\big) \cdot u. \end{aligned}$$

Selecting a proper function as the test function in I_2, I_3 and using Theorem 3.7, we have

$$I_2 + I_3 \to -\int_{\Omega_T} \int_Y A(y)(\nabla u + \nabla_y U) \cdot (\nabla u + \nabla_y U).$$

Combining the above with (3.12), we get $I_1 + I_2 + I_3 \to 0$. Thus we finish the proof. □

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