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Existence and global asymptotic behavior of positive solutions for nonlinear problems on the half-line

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ABSTRACT

In this paper, we aim at studying the existence, uniqueness and the exact asymptotic behavior of positive solutions to the following boundary value problem

$$\begin{cases} \dfrac{1}{A}(Au')' + a(t)u^\sigma = 0, \quad t \in (0,\infty), \\ \lim\limits_{t\to 0^+} u(t) = 0, \quad \lim\limits_{t\to\infty} \dfrac{u(t)}{\rho(t)} = 0, \end{cases}$$

where $\sigma < 1$, A is a continuous function on $[0,\infty)$, positive and differentiable on $(0,\infty)$ such that $\frac{1}{A}$ is integrable on $[0,1]$ and $\int_0^\infty \frac{1}{A(t)}\,dt = \infty$. Here $\rho(t) = \int_0^t \frac{1}{A(s)}\,ds$, for $t \geqslant 0$ and a is a nonnegative continuous function that is required to satisfy some assumptions related to the Karamata classes of regularly varying functions. Our arguments are based on monotonicity methods.

1. Introduction

In [15], Zhao considered the following problem

$$\begin{cases} u'' + \varphi(.,u) = 0, \quad \text{on } (0,\infty), \\ u > 0, \quad \text{on } (0,\infty), \\ \lim\limits_{t\to 0^+} u(t) = 0, \end{cases} \tag{1.1}$$

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where φ is a measurable function on $(0,\infty)\times(0,\infty)$, dominated by a convex positive function. Then he showed that there exists $b>0$ such that for each $\mu\in(0,b]$, there exists a positive continuous solution u of (1.1) satisfying $\lim_{t\to\infty}\frac{u(t)}{t}=\mu$.

On the other hand, in [10], Mâagli and Masmoudi generalized the result of Zhao to the more general boundary value problem

$$\begin{cases} \frac{1}{A}\big(Au'\big)'+f\big(.,u,Au'\big)=0, & \text{on } (0,\infty),\\ u>0, & \text{on } (0,\infty),\\ \lim\limits_{t\to 0^+}u(t)=0, \end{cases} \tag{1.2}$$

where A is a positive and differentiable function on $(0,\infty)$ and f is a measurable function on $(0,\infty)\times(0,\infty)\times(0,\infty)$, which may change sign and is dominated by a regular function. Then they proved the existence of a constant $b>0$ such that for each $\mu\in(0,b]$, problem (1.2) has a continuous solution u satisfying $\lim_{t\to\infty}\frac{u(t)}{\rho(t)}=\mu$, where $\rho(t)=\int_0^t\frac{1}{A(s)}\,ds$, for $t\geqslant 0$.

Note also that various existence results for this type of equations have appeared in the literature (see [1–15] and the references therein).

In this paper, we aim at studying the existence, uniqueness and the exact asymptotic behavior of positive solution to the following boundary value problem

$$\begin{cases} \frac{1}{A}\big(Au'\big)'+a(t)u^{\sigma}=0, & t\in(0,\infty),\\ u>0, & \text{on } (0,\infty),\\ \lim\limits_{t\to 0^+}u(t)=0, & \lim\limits_{t\to\infty}\frac{u(t)}{\rho(t)}=0, \end{cases} \tag{1.3}$$

where $\sigma<1$, A is a continuous function on $[0,\infty)$, positive and differentiable on $(0,\infty)$. We also assume that $\frac{1}{A}$ is integrable on $[0,1]$ and $\int_0^\infty\frac{1}{A(t)}\,dt=\infty$. The function ρ is defined by $\rho(t)=\int_0^t\frac{1}{A(s)}\,ds$, for $t\geqslant 0$.

The nonnegative potential function a is required to be continuous on $(0,\infty)$ that may be singular at 0 or unbounded near ∞ and satisfying some conditions related to the Karamata classes $\mathcal{K}$ and $\mathcal{K}^\infty$ (see Definitions 1.1 and 1.2 below).

For the case $\sigma<0$, the existence and the uniqueness of a positive continuous bounded solution to problem (1.3) is proved in [2, Theorem 2], under the condition that a is a positive continuous function on $(0,\infty)$ satisfying

$$\int_0^\infty A(s)\min\big(1,\rho(s)\big)a(s)\,ds<\infty. \tag{1.4}$$

Also some estimates for such solution are given. Thus, it is interesting to know the exact asymptotic behavior and to extend the study of (1.3) to $0\leqslant\sigma<1$.

Throughout this paper and without loss of generality, we assume that $\int_0^1\frac{1}{A(t)}\,dt=1$.

To state our result, we need some notations. We first introduce the following Karamata classes of regularly varying functions.

Definition 1.1. The class $\mathcal{K}$ is the set of all Karamata functions L defined on $(0,\eta]$ by

$$L(t):=c\exp\left(\int_t^\eta\frac{z(s)}{s}\,ds\right),$$

for some $\eta>1$ and where $c>0$ and $z\in C([0,\eta])$ such that $z(0)=0$.

Definition 1.2. The class $\mathcal{K}^\infty$ is the set of all Karamata functions L defined on $[1,\infty)$ by

$$L(t) := c\exp\left(\int_1^t \frac{z(s)}{s}\,ds\right),$$

where $c > 0$ and $z \in C([1,\infty))$ such that $\lim_{t\to\infty} z(t) = 0$.

It is easy to verify the following.

Remark 1.3. (i) A function L is in $\mathcal{K}$ if and only if L is a positive function in $C^1((0,\eta])$, for some $\eta > 1$, such that $\lim_{t\to 0^+} \frac{tL'(t)}{L(t)} = 0$.

(ii) A function L is in $\mathcal{K}^\infty$ if and only if L is a positive function in $C^1([1,\infty))$ such that $\lim_{t\to\infty} \frac{tL'(t)}{L(t)} = 0$.

Remark 1.4. (See [3].) Let L be a function in $\mathcal{K}^\infty$, then there exists $m \geqslant 0$ such that for every $\beta > 0$ and $t \geqslant 1$ we have

$$(1+\beta)^{-m}L(t) \leqslant L(\beta+t) \leqslant (1+\beta)^m L(t).$$

As a typical example of function belonging to the class $\mathcal{K}$ (see [11,13]), we quote

$$L(t) = \prod_{k=1}^m \left(\log_k\left(\frac{\omega}{t}\right)\right)^{\xi_k},$$

where ξ_k are real numbers, $\log_k x = \log\circ\log\circ\cdots\log x$ (k times) and ω is a sufficiently large positive real number such that L is defined and positive on $(0,\eta]$, for some $\eta > 1$.

In the sequel, we denote by $B^+((0,\infty))$ the set of nonnegative Borel measurable functions in $(0,\infty)$ and by $C_0([0,\infty))$ the set of continuous functions v on $[0,\infty)$ such that $\lim_{t\to\infty} v(t) = 0$. It is easy to see that $C_0([0,\infty))$ is a Banach space with the uniform norm $\|v\|_\infty = \sup_{t>0}|v(t)|$.

For two nonnegative functions f and g defined on a set S, the notation $f(t) \approx g(t)$, $t \in S$ means that there exists $c > 0$ such that $\frac{1}{c}f(t) \leqslant g(t) \leqslant cf(t)$, for all $t \in S$.

Furthermore, we denote by $G(t,s) = A(s)\min(\rho(t),\rho(s))$ the Green's function of the operator $u \mapsto -\frac{1}{A}(Au')'$ on $(0,\infty)$ with the Dirichlet conditions $\lim_{t\to 0^+} u(t) = 0$ and $\lim_{t\to\infty} \frac{u(t)}{\rho(t)} = 0$.

For $f \in B^+((0,\infty))$, we put

$$Vf(t) = \int_0^\infty G(t,s)f(s)\,dt, \quad \text{for } t > 0.$$

We point out that if the map $s \to A(s)\min(1,\rho(s))f(s)$ is continuous and integrable on $(0,\infty)$, then Vf is the solution of the boundary value problem

$$\begin{cases} -\frac{1}{A}\big(Au'\big)' = f, & \text{in } (0,\infty), \\ \lim\limits_{t\to 0^+} u(t) = 0, \\ \lim\limits_{t\to\infty} \dfrac{u(t)}{\rho(t)} = 0. \end{cases} \tag{1.5}$$

Throughout this paper we assume that the function a is nonnegative on $(0,\infty)$ and satisfies the following condition:

(H) $a \in C((0,\infty))$ such that

$$a(t) \approx \frac{1}{(A(t))^2}\big(\rho(t)\big)^{-\lambda}\big(1+\rho(t)\big)^{\lambda-\mu} L_1\big(\min\big(1,\rho(t)\big)\big) L_2\big(\max\big(1,\rho(t)\big)\big), \quad t>0,$$

where $\lambda \leqslant 2$, $\mu \geqslant 1$, $L_1 \in \mathcal{K}$ defined on $(0,\eta]$, for some $\eta > 1$ and $L_2 \in \mathcal{K}^\infty$ satisfying

$$\int_0^\eta \frac{L_1(s)}{s^{\lambda-1}}\,ds < \infty \quad \text{and} \quad \int_1^\infty \frac{L_2(s)}{s^{\mu}}\,ds < \infty. \tag{1.6}$$

In what follows, we put

$$\nu = \min\left(1, \frac{2-\lambda}{1-\sigma}\right), \qquad \zeta = -\min\left(1, \frac{\mu-1}{1-\sigma}\right) \tag{1.7}$$

and we define the function θ on $(0,\infty)$ by

$$\theta(t) = \big(\rho(t)\big)^{\nu}\big(1+\big(\rho(t)\big)\big)^{\zeta-\nu}\big(\widetilde{L}_1\big(\min\big(1,\rho(t)\big)\big)\big)^{\frac{1}{1-\sigma}}\big(\widetilde{L}_2\big(\max\big(1,\rho(t)\big)\big)\big)^{\frac{1}{1-\sigma}}, \tag{1.8}$$

where for $t \in (0,\eta)$,

$$\widetilde{L}_1(t) = \begin{cases} \int_0^t \frac{L_1(s)}{s}\,ds & \text{if } \lambda = 2, \\ L_1(t) & \text{if } 1+\sigma < \lambda < 2, \\ \int_t^\eta \frac{L_1(s)}{s}\,ds & \text{if } \lambda = 1+\sigma, \\ 1 & \text{if } \lambda < 1+\sigma, \end{cases} \tag{1.9}$$

and for $t \geqslant 1$

$$\widetilde{L}_2(t) = \begin{cases} \int_t^\infty \frac{L_2(s)}{s}\,ds & \text{if } \mu = 1, \\ L_2(t) & \text{if } 1 < \mu < 2-\sigma, \\ \int_1^{t+1} \frac{L_2(s)}{s}\,ds & \text{if } \mu = 2-\sigma, \\ 1 & \text{if } \mu > 2-\sigma. \end{cases} \tag{1.10}$$

Our main result is the following.

Theorem 1.5. *Let $\sigma < 1$ and assume* (H). *Then problem* (1.3) *has a unique positive continuous solution u satisfying for $t \in (0,\infty)$*

$$u(t) \approx \theta(t). \tag{1.11}$$

The content of this paper is organized as follows. In Section 2, we present some fundamental properties of the two Karamata classes of regularly varying functions $\mathcal{K}$ and $\mathcal{K}^\infty$ and we establish sharp estimates on some potential functions. In Section 3, exploiting the results of the previous section and using monotonicity methods, we prove Theorem 1.5.

2. Sharp estimates on the potential of some Karamata functions

We collect in this paragraph some properties of functions belonging to the Karamata class $\mathcal{K}$ (resp. $\mathcal{K}^\infty$) and we give estimates on some potential functions.

Proposition 2.1. *(See [11,13].)*

(i) *Let $L_1, L_2 \in \mathcal{K}$ (resp. $\mathcal{K}^\infty$) and $p \in \mathbb{R}$. Then the functions*

$$L_1 + L_2,\ L_1 L_2 \text{ and } L_1^p \text{ belong to the class } \mathcal{K} \ \big(\text{resp. } \mathcal{K}^\infty\big).$$

(ii) *Let L be a function in $\mathcal{K}$ (resp. $\mathcal{K}^\infty$) and $\varepsilon > 0$. Then we have*

$$\lim_{t \to 0^+} t^\varepsilon L(t) = 0 \quad \Big(\text{resp. } \lim_{t \to \infty} t^{-\varepsilon} L(t) = 0\Big).$$

Theorem 2.2. *(See [11,13].)*

(I) *Let $\gamma \in \mathbb{R}$ and L be a function in $\mathcal{K}$ defined on $(0, \eta]$. We have:*
   (i) *If $\gamma < -1$, then $\int_0^\eta s^\gamma L(s)\,ds$ diverges and $\int_t^\eta s^\gamma L(s)\,ds \underset{t \to 0^+}{\sim} -\frac{t^{\gamma+1} L(t)}{\gamma+1}$.*
   (ii) *If $\gamma > -1$, then $\int_0^\eta s^\gamma L(s)\,ds$ converges and $\int_0^t s^\gamma L(s)\,ds \underset{t \to 0^+}{\sim} \frac{t^{\gamma+1} L(t)}{\gamma+1}$.*

(II) *Let $\gamma \in \mathbb{R}$ and L be a function in $\mathcal{K}^\infty$. We have:*
   (i) *If $\gamma > -1$, then $\int_1^\infty s^\gamma L(s)\,ds$ diverges and $\int_1^t s^\gamma L(s)\,ds \underset{t \to \infty}{\sim} \frac{t^{\gamma+1} L(t)}{\gamma+1}$.*
   (ii) *If $\gamma < -1$, then $\int_1^\infty s^\gamma L(s)\,ds$ converges and $\int_t^\infty s^\gamma L(s)\,ds \underset{t \to \infty}{\sim} -\frac{t^{\gamma+1} L(t)}{\gamma+1}$.*

The proof of the next lemma can be found in [4] (see also [9]).

Lemma 2.3. *Let L be a function in $\mathcal{K}$ defined on $(0, \eta]$. Then we have*

$$\lim_{t \to 0^+} \frac{L(t)}{\int_t^\eta \frac{L(s)}{s}\,ds} = 0.$$

In particular

$$t \to \int_t^\eta \frac{L(s)}{s}\,ds \in \mathcal{K}.$$

If further $\int_0^\eta \frac{L(s)}{s}\,ds$ converges, then we have $\lim_{t \to 0^+} \frac{L(t)}{\int_0^t \frac{L(s)}{s}\,ds} = 0$.

In particular

$$t \to \int_0^t \frac{L(s)}{s}\,ds \in \mathcal{K}.$$

In the next lemma, we have the following properties related to the class $\mathcal{K}^\infty$. For the proof we refer to [3].

Lemma 2.4. *Let L be a function in $\mathcal{K}^\infty$. Then we have*

$$\lim_{t \to \infty} \frac{L(t)}{\int_1^t \frac{L(s)}{s}\,ds} = 0.$$

In particular

$$t \to \int_1^{t+1} \frac{L(s)}{s}\,ds \in \mathcal{K}^\infty.$$

If further $\int_1^\infty \frac{L(s)}{s}\,ds$ *converges, then we have*

$$\lim_{t\to\infty}\frac{L(t)}{\int_t^\infty \frac{L(s)}{s}\,ds}=0.$$

In particular

$$t\to\int_t^\infty \frac{L(s)}{s}\,ds\in\mathcal{K}^\infty.$$

Now, we put

$$b(t)=\frac{1}{(A(t))^2}\big(\rho(t)\big)^{-\beta}\big(1+\rho(t)\big)^{\beta-\gamma}L_3\big(\min\big(1,\rho(t)\big)\big)L_4\big(\max\big(1,\rho(t)\big)\big),\quad t>0, \tag{2.1}$$

where $L_3\in\mathcal{K}$ and $L_4\in\mathcal{K}^\infty$. We aim at giving sharp estimates on the potential function $Vb(t)$.

Proposition 2.5. *Assume that* $L_3\in\mathcal{K}$ *defined on* $(0,\eta]$, *for some* $\eta>1$ *and* $L_4\in\mathcal{K}^\infty$. *Let* $\beta\leqslant 2$ *and* $\gamma\geqslant 1$ *such that*

$$\int_0^\eta s^{1-\beta}L_3(s)\,ds<\infty\quad\text{and}\quad\int_1^\infty s^{-\gamma}L_4(s)\,ds<\infty. \tag{2.2}$$

Then for $t>0$

$$Vb(t)\approx\psi_\beta\big(\min\big(1,\rho(t)\big)\big)\phi_\gamma\big(\max\big(1,\rho(t)\big)\big),$$

where for $t\in(0,1]$,

$$\psi_\beta(t)=\begin{cases}\int_0^t \frac{L_3(s)}{s}\,ds & \text{if } \beta=2,\\ t^{2-\beta}L_3(t) & \text{if } 1<\beta<2,\\ t\int_t^\eta \frac{L_3(s)}{s}\,ds & \text{if } \beta=1,\\ t & \text{if } \beta<1,\end{cases}$$

and for $t\geqslant 1$

$$\phi_\gamma(t)=\begin{cases}t\int_t^\infty \frac{L_4(s)}{s}\,ds & \text{if } \gamma=1,\\ t^{2-\gamma}L_4(t) & \text{if } 1<\gamma<2,\\ \int_1^{t+1} \frac{L_4(s)}{s}\,ds & \text{if } \gamma=2,\\ 1 & \text{if } \gamma>2.\end{cases}$$

Proof. For $t>0$, we have

$$\begin{aligned}Vb(t)&=\int_0^\infty \frac{\min(\rho(t),\rho(s))}{A(s)}\big(\rho(s)\big)^{-\beta}\big(1+\rho(s)\big)^{\beta-\gamma}L_3\big(\min\big(1,\rho(s)\big)\big)L_4\big(\max\big(1,\rho(s)\big)\big)\,ds\\&=\int_0^\infty \min\big(\rho(t),\xi\big)\xi^{-\beta}(1+\xi)^{\beta-\gamma}L_3\big(\min(1,\xi)\big)L_4\big(\max(1,\xi)\big)\,d\xi\\&=F\big(\rho(t)\big),\end{aligned}$$

where

$$\begin{aligned}
F(r) &:= \int_0^\infty \min(r,\xi)\xi^{-\beta}(1+\xi)^{\beta-\gamma}L_3\big(\min(1,\xi)\big)L_4\big(\max(1,\xi)\big)\,d\xi \\
&\approx \int_0^\eta \min(r,\xi)\xi^{-\beta}L_3(\xi)\,d\xi + \int_\eta^\infty \min(r,\xi)\xi^{-\gamma}L_4(\xi)\,d\xi \\
&= I(r)+J(r).
\end{aligned}$$

Case 1: Assume that $0<r\leqslant 1$.

By using (2.2), we deduce that

$$J(r)\approx r. \tag{2.3}$$

On the other hand,

$$\begin{aligned}
I(r) &= \int_0^r \xi^{1-\beta}L_3(\xi)\,d\xi + r\int_r^\eta \xi^{-\beta}L_3(\xi)\,d\xi \\
&= I_1(r)+I_2(r).
\end{aligned}$$

Using Theorem 2.2 and hypothesis (2.2), we deduce that

$$I_1(r)\approx \begin{cases} r^{2-\beta}L_3(r) & \text{if } \beta<2, \\ \int_0^r \frac{L_3(\xi)}{\xi}\,d\xi & \text{if } \beta=2, \end{cases}$$

and

$$I_2(r)\approx \begin{cases} r^{2-\beta}L_3(r) & \text{if } 1<\beta\leqslant 2, \\ r\int_r^\eta \xi^{-\beta}L_3(\xi)\,d\xi & \text{if } \beta\leqslant 1. \end{cases}$$

Hence, it follows by Lemma 2.3, Proposition 2.1 and hypothesis (2.2) that

$$I(r)\approx \begin{cases} \int_0^r \frac{L_3(\xi)}{\xi}\,d\xi & \text{if } \beta=2, \\ r^{2-\beta}L_3(r) & \text{if } 1<\beta<2, \\ r\int_r^\eta \frac{L_3(\xi)}{\xi}\,d\xi & \text{if } \beta=1, \\ r & \text{if } \beta<1. \end{cases} \tag{2.4}$$

Combining (2.3), (2.4) and using Proposition 2.1 and hypothesis (2.2), we deduce that for $0<r\leqslant 1$,

$$F(r)\approx \psi_\beta(r). \tag{2.5}$$

Case 2: Assume that $r>\eta+1$.

By using (2.2), we deduce that

$$I(r)\approx 1. \tag{2.6}$$

On the other hand,

$$J(r) = \int_{\eta}^{r} \xi^{1-\gamma} L_4(\xi)\, d\xi + r \int_{r}^{\infty} \xi^{-\gamma} L_4(\xi)\, d\xi$$
$$= J_1(r) + J_2(r).$$

Using again Theorem 2.2 and hypothesis (2.2), we deduce that

$$J_1(r) \approx \begin{cases} r^{2-\gamma} L_4(r) & \text{if } 1 \leqslant \gamma < 2, \\ \int_{\eta}^{r} \xi^{1-\gamma} L_4(\xi)\, d\xi & \text{if } \gamma \geqslant 2, \end{cases}$$

and

$$J_2(r) \approx \begin{cases} r^{2-\gamma} L_4(r) & \text{if } \gamma > 1, \\ r \int_{r}^{\infty} \frac{L_4(\xi)}{\xi}\, d\xi & \text{if } \gamma = 1. \end{cases}$$

Hence, it follows from Lemma 2.4 and hypothesis (2.2) that

$$J(r) \approx \begin{cases} r \int_{r}^{\infty} \frac{L_4(\xi)}{\xi}\, d\xi & \text{if } \gamma = 1, \\ r^{2-\gamma} L_4(r) & \text{if } 1 < \gamma < 2, \\ \int_{\eta}^{r} \frac{L_4(\xi)}{\xi}\, d\xi & \text{if } \gamma = 2, \\ 1 & \text{if } \gamma > 2. \end{cases} \tag{2.7}$$

Combining (2.6), (2.7) and using Proposition 2.1, hypothesis (2.2) and Remark 1.4, we deduce that for $r > \eta + 1$,

$$F(r) \approx \begin{cases} r \int_{r}^{\infty} \frac{L_4(\xi)}{\xi}\, d\xi & \text{if } \gamma = 1, \\ r^{2-\gamma} L_4(r) & \text{if } 1 < \gamma < 2, \\ \int_{\eta}^{r} \frac{L_4(\xi)}{\xi}\, d\xi & \text{if } \gamma = 2, \\ 1 & \text{if } \gamma > 2, \end{cases}$$
$$\approx \phi_\gamma(r). \tag{2.8}$$

Now since the functions $r \to F(r)$ and $r \to \phi_\gamma(r)$ are positive and continuous on $[1, \eta + 1]$, we deduce that for $r \in [1, \eta + 1]$,

$$F(r) \approx \phi_\gamma(r). \tag{2.9}$$

Finally, using (2.5), (2.8) and (2.9), we obtain the required result. □

3. Proof of the main result

The next results will play a crucial role in the proof of Theorem 1.5.

Lemma 3.1. *Assume that the function a satisfies* (H) *and put $\omega(t) = a(t)\theta^\sigma(t)$ for $t > 0$. Then we have for $t \in (0, \infty)$*

$$V\omega(t) \approx \theta(t).$$

Proof. We recall that

$$\nu = \min\left(1, \frac{2-\lambda}{1-\sigma}\right), \qquad \zeta = -\min\left(1, \frac{\mu-1}{1-\sigma}\right)$$

and

$$\theta(t) = \left(\rho(t)\right)^{\nu}\left(1+\left(\rho(t)\right)\right)^{\zeta-\nu}\left(\widetilde{L}_1\left(\min\left(1,\rho(t)\right)\right)\right)^{\frac{1}{1-\sigma}}\left(\widetilde{L}_2\left(\max\left(1,\rho(t)\right)\right)\right)^{\frac{1}{1-\sigma}},$$

where for $t \in (0,1]$,

$$\widetilde{L}_1(t) = \begin{cases} \int_0^t \frac{L_1(s)}{s}\,ds & \text{if } \lambda = 2, \\ L_1(t) & \text{if } 1+\sigma < \lambda < 2, \\ \int_t^{\eta} \frac{L_1(s)}{s}\,ds & \text{if } \lambda = 1+\sigma, \\ 1 & \text{if } \lambda < 1+\sigma, \end{cases}$$

and for $t \geqslant 1$

$$\widetilde{L}_2(t) = \begin{cases} \int_t^{\infty} \frac{L_2(s)}{s}\,ds & \text{if } \mu = 1, \\ L_2(t) & \text{if } 1 < \mu < 2-\sigma, \\ \int_1^{t+1} \frac{L_2(s)}{s}\,ds & \text{if } \mu = 2-\sigma, \\ 1 & \text{if } \mu > 2-\sigma. \end{cases}$$

For $t > 0$, we have

$$\begin{aligned} \omega(t) \approx {} & \frac{1}{(A(t))^2}\left(\rho(t)\right)^{-\lambda+\nu\sigma}\left(1+\rho(t)\right)^{\lambda-\mu+(\zeta-\nu)\sigma} \\ & \times L_1\left(\min\left(1,\rho(t)\right)\right)\left(\widetilde{L}_1\left(\min\left(1,\rho(t)\right)\right)\right)^{\frac{\sigma}{1-\sigma}} L_2\left(\max\left(1,\rho(t)\right)\right)\left(\widetilde{L}_2\left(\max\left(1,\rho(t)\right)\right)\right)^{\frac{\sigma}{1-\sigma}}. \end{aligned}$$

Using Proposition 2.5 with $\beta = \lambda - \nu\sigma$ and $\gamma = \mu - \zeta\sigma$, $L_3(t) = L_1(t)(\widetilde{L}_1(t))^{\frac{\sigma}{1-\sigma}}$ and $L_4(t) = L_2(t)(\widetilde{L}_2(t))^{\frac{\sigma}{1-\sigma}}$, we obtain for $t \in (0,1]$

$$\begin{aligned} V\omega(t) &\approx \begin{cases} \int_0^{\rho(t)} \frac{L_1(s)}{s}(\int_0^s \frac{L_1(r)}{r}\,dr)^{\frac{\sigma}{1-\sigma}}\,ds & \text{if } \lambda = 2, \\ (\rho(t))^{\frac{2-\lambda}{1-\sigma}} L_1(\rho(t))(L_1(\rho(t)))^{\frac{\sigma}{1-\sigma}} & \text{if } 1+\sigma < \lambda < 2, \\ \rho(t)\int_{\rho(t)}^{\eta} \frac{L_1(s)}{s}(\int_s^{\eta} \frac{L_1(r)}{r}\,dr)^{\frac{\sigma}{1-\sigma}}\,ds & \text{if } \lambda = 1+\sigma, \\ \rho(t) & \text{if } \lambda < 1+\sigma, \end{cases} \\ &\approx \begin{cases} (\int_0^{\rho(t)} \frac{L_1(s)}{s}\,ds)^{\frac{1}{1-\sigma}} & \text{if } \lambda = 2, \\ (\rho(t))^{\frac{2-\lambda}{1-\sigma}}(L_1(\rho(t)))^{\frac{1}{1-\sigma}} & \text{if } 1+\sigma < \lambda < 2, \\ \rho(t)(\int_{\rho(t)}^{\eta} \frac{L_1(s)}{s}\,ds)^{\frac{1}{1-\sigma}} & \text{if } \lambda = 1+\sigma, \\ \rho(t) & \text{if } \lambda < 1+\sigma, \end{cases} \\ &\approx \theta(t). \end{aligned}$$

On the other hand, using again Proposition 2.5 and Remark 1.4, we get for $t \geqslant 1$,

$$V\omega(t) \approx \begin{cases} \rho(t)\int_{\rho(t)}^{\infty}\frac{L_2(s)}{s}(\int_s^{\infty}\frac{L_2(r)}{r}\,dr)^{\frac{\sigma}{1-\sigma}}\,ds & \text{if } \mu = 1, \\ (\rho(t))^{\zeta}L_2(\rho(t))(L_2(\rho(t)))^{\frac{\sigma}{1-\sigma}} & \text{if } 1 < \mu < 2-\sigma, \\ \int_1^{\rho(t)+1}\frac{L_2(s)}{s}(\int_1^{s+1}\frac{L_2(r)}{r}\,dr)^{\frac{\sigma}{1-\sigma}}\,ds & \text{if } \mu = 2-\sigma, \\ 1 & \text{if } \mu > 2-\sigma, \end{cases}$$

$$\approx \begin{cases} \rho(t)(\int_{\rho(t)}^{\infty}\frac{L_2(s)}{s}\,ds)^{\frac{1}{1-\sigma}} & \text{if } \mu = 1, \\ (\rho(t))^{\zeta}(L_2(\rho(t)))^{\frac{1}{1-\sigma}} & \text{if } 1 < \mu < 2-\sigma, \\ (\int_1^{\rho(t)+1}\frac{L_2(s)}{s}\,ds)^{\frac{1}{1-\sigma}} & \text{if } \mu = 2-\sigma, \\ 1 & \text{if } \mu > 2-\sigma, \end{cases}$$

$$\approx \theta(t).$$

This completes the proof. □

Lemma 3.2. *(See [2].) Let $a \geqslant 0$ and $u \in C^1((a,\infty))$ be a function satisfying*

$$\begin{cases} -\frac{1}{A}(Au')' \geqslant 0, & \text{in } (a,\infty), \\ \lim_{t\to a^+} u(t) = 0 \quad \text{and} \quad \lim_{t\to\infty}\frac{u(t)}{\rho(t)} = 0. \end{cases} \tag{3.1}$$

Then u is nondecreasing and nonnegative function on (a,∞).

Proof. Since by (3.1) the function $t \mapsto A(t)u'(t)$ is nonincreasing on (a,∞), then we have $\lim_{t\to a^+} A(t) \times u'(t) := l_0 \in \overline{\mathbb{R}}$ and $\lim_{t\to+\infty} A(t)u'(t) := l \in [-\infty,+\infty)$.

We claim that $l = 0$. To prove this, we have the following cases:

Case 1: If $l = -\infty$, then there exists $b > a$, such that

$$\forall s > b, \quad u'(s) < -\frac{1}{A(s)}.$$

Integrating this, we get

$$\forall t > b, \quad \frac{u(t)}{\rho(t)} - \frac{u(b)}{\rho(t)} < -\left(1 - \frac{\rho(b)}{\rho(t)}\right). \tag{3.2}$$

Now since $\lim_{x\to\infty}\frac{u(t)}{\rho(t)} = 0$ and $\lim_{t\to\infty}\rho(t) = \infty$, then by taking $t \to \infty$ in (3.2), we obtain a contradiction.

Case 2: Suppose that $l \in \mathbb{R}$. Then $\forall\varepsilon > 0$, there exists $b > a$, such that

$$\forall s > b, \quad \frac{l-\varepsilon}{A(s)} < u'(s) < \frac{l+\varepsilon}{A(s)}.$$

Integrating this, we get

$$\forall t > b, \quad (l-\varepsilon)\left(1 - \frac{\rho(b)}{\rho(t)}\right) < \frac{u(t)}{\rho(t)} - \frac{u(b)}{\rho(t)} < (l+\varepsilon)\left(1 - \frac{\rho(b)}{\rho(t)}\right).$$

So as in Case 1 by letting $t \to \infty$, we obtain

$$\forall\varepsilon > 0, \quad |l| \leqslant \varepsilon.$$

That is $l = 0$. Hence, by the monotonicity of $t \mapsto A(t)u'(t)$, we deduce that u is nondecreasing on (a, ∞) with $\lim_{t\to a^+} u(t) = 0$, which implies that u is a nonnegative function on (a, ∞). □

Proposition 3.3. *Assume that $\sigma < 0$. Let u, v be two positive functions belonging to $C^1((0, \infty))$ such that*

$$\begin{cases} -\dfrac{1}{A}\big(Au'\big)' \leqslant a(t)u^{\sigma}, & \text{in } (0, \infty), \\ \lim\limits_{t\to 0^+} u(t) = 0, \\ \lim\limits_{t\to\infty} \dfrac{u(t)}{\rho(t)} = 0, \end{cases} \tag{3.3}$$

and

$$\begin{cases} -\dfrac{1}{A}\big(Av'\big)' \geqslant a(t)v^{\sigma}, & \text{in } (0, \infty), \\ \lim\limits_{t\to 0^+} v(t) = 0, \\ \lim\limits_{t\to\infty} \dfrac{v(t)}{\rho(t)} = 0. \end{cases} \tag{3.4}$$

Then $u \leqslant v$.

Proof. Let $\widetilde{\omega}(t) := u(t) - v(t)$, for $t > 0$. Assume that there exists $t_0 \in (0, \infty)$ such that $\widetilde{\omega}(t_0) > 0$. Then there exists a maximal interval $(a, b) \subset (0, \infty)$ containing t_0 such that

$$\widetilde{\omega}(t) > 0, \quad \text{for each } t \in (a, b). \tag{3.5}$$

This implies that $\widetilde{\omega}(a) = 0$ and $\widetilde{\omega}(b) = 0$ if $b < \infty$.

So we have the following cases:

Case 1: If $b = \infty$, then we have

$$\begin{cases} -\dfrac{1}{A}\big(A\widetilde{\omega}'\big)' \leqslant 0, & \text{in } (a, \infty), \\ \lim\limits_{t\to a^+} \widetilde{\omega}(t) = 0, \\ \lim\limits_{t\to\infty} \dfrac{\widetilde{\omega}(t)}{\rho(t)} = 0. \end{cases}$$

From Lemma 3.2, we deduce that $\widetilde{\omega}$ is nonincreasing on $[a, \infty)$ with $\widetilde{\omega}(a) = 0$. This is a contradiction with the fact that $\widetilde{\omega}(t) > 0$, for each $t \in (a, \infty)$.

Case 2: If $b < \infty$, then we have

$$\begin{cases} -\dfrac{1}{A}\big(A\widetilde{\omega}'\big)' \leqslant 0, & \text{in } (a, b), \\ \widetilde{\omega}(a) = 0, \\ \widetilde{\omega}(b) = 0. \end{cases}$$

This implies that the function $t \to A(t)\widetilde{\omega}'(t)$ is nondecreasing on (a, b). In particular there exists $\lim_{t\to b^-} A(t)\widetilde{\omega}'(t) := l \in \overline{\mathbb{R}}$.

On the other hand, since $\widetilde{\omega}(b)=0$ and $\widetilde{\omega}\in C^1(0,\infty)$, it follows by (3.5) that $\lim_{t\to b^-}\frac{\widetilde{\omega}(t)}{t-b}=\widetilde{\omega}'(b)\leqslant 0$. So $\lim_{t\to b^-}A(t)\widetilde{\omega}'(t)\leqslant 0$.

Hence, for each $t\in(a,b)$, $A(t)\widetilde{\omega}'(t)\leqslant 0$ and $\widetilde{\omega}(a)=\widetilde{\omega}(b)=0$. This yields a contradiction. □

Proof of Theorem 1.5. From Lemma 3.1, there exists $M>1$ such that for each $t>0$

$$\frac{1}{M}\theta(t)\leqslant V\omega(t)\leqslant M\theta(t), \tag{3.6}$$

where $\omega(t)=a(t)\theta^\sigma(t)$.

On the other hand, using hypothesis (H) and Theorem 2.2, we verify that

$$\int_0^\infty A(s)\min\bigl(1,\rho(s)\bigr)\omega(s)\,ds<\infty. \tag{3.7}$$

We will discuss the following two cases.

Case 1: $\sigma<0$. It is obvious to see that if a satisfies hypothesis (H), then a verifies condition (1.4). This implies from [2, Theorem 2], that problem (1.3) has a unique positive solution u. We claim that u satisfies (1.11). Indeed, from (3.7) and (1.5), we deduce that the function $v(t):=V\omega(t)$ is a solution of

$$-\frac{1}{A}\bigl(Av'\bigr)'=\omega,\quad \text{in } (0,\infty). \tag{3.8}$$

Now using (3.6), (3.8), we verify by a simple computation that the functions $M^{\frac{\sigma}{1-\sigma}}v$ and $M^{\frac{-\sigma}{1-\sigma}}v$ satisfy respectively (3.3) and (3.4). Hence by Proposition 3.3, we obtain that

$$M^{\frac{\sigma}{1-\sigma}}v\leqslant u\leqslant M^{\frac{-\sigma}{1-\sigma}}v,$$

which proves that u satisfies (1.11).

Case 2: $0\leqslant\sigma<1$. Put $c_0=M^{\frac{1}{1-\sigma}}$, where the constant M is given in (3.6) and let

$$\Lambda=\left\{v\in C_0\bigl([0,\infty)\bigr):\ \frac{\theta(t)}{c_0(1+\rho(t))}\leqslant v(t)\leqslant\frac{c_0\theta(t)}{1+\rho(t)},\ t>0\right\}.$$

Clearly, the function $t\to\frac{\theta(t)}{1+\rho(t)}\in C_0([0,\infty))$ and so Λ is not empty.

We define the operator T on Λ by

$$Tv(t)=\frac{1}{1+\rho(t)}\int_0^\infty G(t,s)a(s)\bigl(1+\rho(s)\bigr)^\sigma v^\sigma(s)\,ds. \tag{3.9}$$

We shall prove that T has a fixed point in Λ.

First observe that for this choice of c_0, we have for all $v\in\Lambda$ and $t>0$

$$Tv(t)\leqslant\frac{c_0\theta(t)}{1+\rho(t)}\quad\text{and}\quad Tv(t)\geqslant\frac{\theta(t)}{c_0(1+\rho(t))}. \tag{3.10}$$

On the other hand, for all $t,s>0$, we have

$$\frac{G(t,s)}{1+\rho(t)}\leqslant A(s)\min\bigl(1,\rho(s)\bigr). \tag{3.11}$$

Since for each $s > 0$, the function $t \to \frac{G(t,s)}{1+\rho(t)}$ is in $C_0([0,\infty))$, we deduce by using (3.11) and (3.7) that $T(\Lambda) \subset C_0([0,\infty))$. Therefore, $T(\Lambda) \subset \Lambda$.

Now, let $(v_k)_k$ be a sequence of functions in $C_0([0,\infty))$ defined by

$$v_0 = \frac{\theta(t)}{c_0(1+\rho(t))} \quad \text{and} \quad v_{k+1} = Tv_k, \quad \text{for } k \in \mathbb{N}.$$

Since for $0 \leqslant \sigma < 1$, the operator T is nondecreasing and $T(\Lambda) \subset \Lambda$, we deduce that

$$v_0 \leqslant v_1 \leqslant v_2 \leqslant \cdots \leqslant v_k \leqslant v_{k+1} \leqslant \frac{c_0\theta(t)}{1+\rho(t)}.$$

Hence, by the convergence monotone theorem, the sequence $(v_k)_k$ converges to a function v satisfying

$$\frac{\theta(t)}{c_0(1+\rho(t))} \leqslant v \leqslant \frac{c_0\theta(t)}{1+\rho(t)} \quad \text{and} \quad v(t) = \frac{1}{1+\rho(t)} \int_0^\infty G(t,s)a(s)\big(1+\rho(s)\big)^\sigma v^\sigma(s)\,ds.$$

By similar argument as above, we prove that v is a continuous function on $[0,\infty)$.

Put $u(t) = (1+\rho(t))v(t)$. Then u is a positive continuous function satisfying

$$u = V\big(au^\sigma\big).$$

Since the function $s \to A(s)\min(1,\rho(s))a(s)u^\sigma(s)$ is continuous and integrable on $(0,\infty)$, then it follows that u is a solution of problem (1.3).

Finally, it remains to prove that u is the unique positive continuous solution satisfying (1.11). To this end, assume that problem (1.3) has two positive continuous solutions u, v satisfying (1.11). Then there exists a constant $m > 1$ such that

$$\frac{1}{m} \leqslant \frac{u}{v} \leqslant m.$$

This implies that the set

$$J = \left\{m \geqslant 1\colon \ \frac{1}{m} \leqslant \frac{u}{v} \leqslant m\right\}$$

is not empty. Let $c = \inf J$. Then $c \geqslant 1$ and we have $\frac{1}{c}v \leqslant u \leqslant cv$. It follows that $u^\sigma \leqslant c^\sigma v^\sigma$ and that the function $w := c^\sigma v - u$ satisfies

$$\begin{cases} -\dfrac{1}{A}\big(Aw'\big)' = a\big(c^\sigma v^\sigma - u^\sigma\big) \geqslant 0, \\ \displaystyle\lim_{t\to 0^+} w(t) = 0, \\ \displaystyle\lim_{t\to\infty} \frac{w(t)}{\rho(t)} = 0. \end{cases}$$

This implies by Lemma 3.2 that the function $w = c^\sigma v - u$ is nonnegative. By symmetry, we have also $v \leqslant c^\sigma u$. Hence $c^\sigma \in J$ and $c \leqslant c^\sigma$. Since $0 \leqslant \sigma < 1$, then $c = 1$ and therefore $u = v$. □

Example 3.4. Let $\sigma < 1$ and a be a positive continuous function on $(0,\infty)$ such that

$$a(t) \approx \frac{1}{(A(t))^2}\big(\rho(t)\big)^{-\lambda}\big(1+\rho(t)\big)^{\lambda-\mu}\log\left(\frac{2}{\min(1,\rho(t))}\right), \quad t > 0,$$

where $\lambda < 2$ and $\mu > 1$. Then by Theorem 1.5, problem (1.3) has a unique positive continuous solution u satisfying for $t > 0$,

$$u(t) \approx \big(\rho(t)\big)^{\nu}\big(1+\rho(t)\big)^{\zeta-\nu}\big(\widetilde{L}_1\big(\min\big(1,\rho(t)\big)\big)\big)^{\frac{1}{1-\sigma}}\big(\widetilde{L}_2\big(\max\big(1,\rho(t)\big)\big)\big)^{\frac{1}{1-\sigma}},$$

where $\nu = \min(1, \frac{2-\lambda}{1-\sigma})$, $\zeta = -\min(1, \frac{\mu-1}{1-\sigma})$,

$$\widetilde{L}_1(t) = \begin{cases} \log(\frac{2}{t}) & \text{if } 1+\sigma < \lambda < 2, \\ (\log(\frac{2}{t}))^2 & \text{if } \lambda = 1+\sigma, \\ 1 & \text{if } \lambda < 1+\sigma, \end{cases}$$

and

$$\widetilde{L}_2(t) = \begin{cases} \log(1+t) & \text{if } \mu = 2-\sigma, \\ 1 & \text{if } \mu \neq 2-\sigma. \end{cases}$$

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