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On one-parameter semigroups generated by commuting continuous injections

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ABSTRACT

The problem of the embeddability of two commuting continuous injections $f, g : I = (0, b] \to I$ in Abelian semigroups is discussed. We consider the case when there is no iteration semigroup in which f and g can be embedded. Explaining this phenomenon we modify the definition of an iteration semigroup introducing a new notion – a T-iteration semigroup of f and g, that is a family $\{f^t : I \to I,\ t \in T\}$ of continuous injections for which $f^u \circ f^v = f^{u+v}$, $u, v \in T$, such that $f = f^1$ and $g = f^s$ for an $s \in T$ and $s \notin \mathbb{Q}$, where $T \subsetneq \mathbb{R}^+$ is a dense semigroup which can be extended to a group. We determine a maximal semigroup of indices $\mathrm{Sem}(f, g) \subsetneq \mathbb{R}^+$ such that for every T-iteration semigroup $T \subset \mathrm{Sem}(f, g)$. We give also a construction of maximal T-iteration semigroups of f and g that is such semigroups for which $T = \mathrm{Sem}(f, g)$. We examine also some other Abelian semigroups of continuous functions containing f and g.

1. Introduction

We consider the problem of the embeddability of two commuting continuous injections in semi-flows called here iteration semigroups. The characterization of the embeddability of continuous commutable bijections in iteration groups is given in [12]. It turns out that, except some very regular particular case, omitting the assumption of surjectivity results in the lack of the embeddability in an iteration semigroup. In this paper such a case is considered. We explain this phenomenon and we construct the Abelian semigroups of mappings defined in the same interval I which substitute the iteration semigroups. The construction of the maximal Abelian subsemigroups containing settled two commuting mappings is given. To this end let us introduce the following notions.

Let I be an interval and let T be an additive dense subsemigroup of $\mathbb{R}^+$ such that $1 \in T$. A one parameter family $\mathcal{F} := \{f^t : I \to I,\ t \in T\}$ of continuous functions f^t such that $f^t \circ f^s = f^{t+s}$ for all $t, s \in T$ is said

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to be a T-*iteration semigroup*, however the semigroup T will be called a *support of* $\mathcal{F}$. We will also say that $\mathcal{F}$ is *supported by semigroup* T. Note that every T-iteration semigroup is Abelian.

If for every $x \in I$ the mapping $t \in T \to f^t(x)$ is an injection then a T-iteration semigroup is said to be *injective*. If $T = \mathbb{R}^+$ then T-iteration semigroup is said to be an *iteration semigroup* also called in the literature a *semiflow* (see [8]). Note that if f^1 is an injection then the remaining f^t are also injective.

If $f, g : I \to I$ are given functions and there exists a T-iteration semigroup $\{f^t : I \to I,\ t \in T\}$ such that $f^1 = f$ and $f^s = g$ for an $s \in T$ then we say that f and g are T-*embeddable*. If $T = \mathbb{R}^+$ then we will say shortly that f and g are *embeddable*.

A family of functions $\mathcal{A}$ is said to be *disjoint* whenever $f, g \in \mathcal{A}$ and $f(x) = g(x)$ for some x then $f = g$ (see [2]). Note that a T-iteration semigroup $\{f^t : I \to I,\ t \in T\}$ is disjoint if and only if, for every $t \in T$, f^t either has no fixed points or is the identity.

Denote here by $\mathbb{N}$ the set of natural numbers with 0. The mappings $f, g : I \to I$ are said to be *iteratively incommensurable* when for every $x \in I$ and every $n, m \in \mathbb{N}$ such that $n + m \neq 0$, $f^n(x) \neq g^m(x)$. In such a case the graphs of iterates are disjoint.

2. Preliminaries

Let $I = (0, b]$ be an interval. On given functions f and g we assume the general hypothesis:

(H) $f, g : I \to I$ are continuous, strictly increasing, $f \circ g = g \circ f$ and f, g are iteratively incommensurable.

Note that the assumption $I = (0, b]$ implies that f and g are not surjections, $f < id$ and $g < id$.

It is easily visible that for every $x \in I$ there exists a unique sequence $\{m_k(x)\}$ of positive integers such that $f^{m_k(x)+1}(x) \leqslant g^k(x) < f^{m_k(x)}(x)$. Moreover, there exists the finite limit

$$\lim_{k\to\infty} \frac{m_k(x)}{k} =: s(f, g),$$

and this limit does not depend on x (see [11]). This limit $s := s(f, g)$ is said to be the *iterative index of* f *and* g. Index $s \notin \mathbb{Q}$ if and only if f and g are iteratively incommensurable.

Assume that f and g satisfy (H). Define

$$\mathcal{N}_+(x) := \big\{(n, m) \in \mathbb{N} \times \mathbb{N}\colon\ f^n(x) \in g^m[I]\big\},$$
$$\mathcal{N}_-(x) := \big\{(n, m) \in \mathbb{N} \times \mathbb{N}\colon\ g^m(x) \in f^n[I]\big\}$$

and

$$C_+(x) := \big\{g^{-m} \circ f^n(x)\colon\ (n, m) \in \mathcal{N}_+(x)\big\},$$
$$C_-(x) := \big\{f^{-n} \circ g^m(x)\colon\ (n, m) \in \mathcal{N}_-(x)\big\}.$$

Put

$$L_{f,g} := C_+(x)^d,$$

where A^d means the set of all limit points of A. After [6] we quote

Proposition 1. *(See Theorem 1 in [6], Theorem 1 in [9].) The set $L_{f,g}$ does not depend on the choice of x. $C_-(x)^d = C_+(x)^d$ and $L_{f,g}$ is either a Cantor set, i.e. a perfect and nowhere dense set or $L_{f,g}$ is an interval.*

If we drop the assumption that $f(b) < b$ and assume that f and g are the surjections then we get the following.

Lemma 1. *(See Theorem 1 in [6].) If the surjections f, g satisfy* (H) *with $I = (0, b)$ then $L_{f,g} = \{g^{-m} \circ f^n(x): n, m \in \mathbb{N}\}^d = \{f^{-n} \circ g^m(x): n, m \in \mathbb{N}\}^d = \{g^m \circ f^n(x): n, m \in \mathbb{Z}\}^d$.*

Proposition 2. *(See Corollary 1 in [7].) If $\operatorname{Int} L_{f,g} \neq \emptyset$ then there exists a unique iteration semigroup continuous with respect to iterative parameter in which f and g are embeddable.*

In particular (see [7]), if f and g are diffeomorphisms on $(0, c)$ for some $c < b$ and the derivatives f', g' are of finite variation in $(0, c)$, then there exists a continuous iteration semigroup embedding f and g.

Proposition 3. *(See Corollary 1 in [7].) If $\operatorname{Int} L_{f,g} = \emptyset$ then f and g are not embeddable in any iteration semigroup.*

Now let us quote some useful fact from [3]. Denote by $B_i(J)$ the set of all increasing bijections from open interval J onto itself. A disjoint group $G \subset B_i(J)$ is said to be a *spoiled group* whenever $L(G) := \{f(x): f \in G\}^d$ for an $x \in J$ is a Cantor set. The set $L(G)$ does not depend on x (see Theorem 2a in [3]).

For $L \subset I$ being a Cantor set let L^{*-}, L^{*+} and L^{**} be the sets of all left-sided, right-sided and two-sided limit points of L, respectively.

Proposition 4. *(See Proposition 5 in [3].) If G is a spoiled subgroup of $B_i(I)$, $L = L(G)$ and $f \in G$, then $\overline{f}(L^{*-}) = L^{*-}$, $\overline{f}(L^{*+}) = L^{*+}$ and $\overline{f}(L^{**}) = L^{**}$, where $\overline{f}$ is a continuous extension of f onto $\operatorname{cl} I$.*

The important role in the next considerations is played by the following.

Proposition 5. *(See Theorem 2 in [6], Theorem 2 in [10].) Let f and g satisfy* (H)*, where $I = (0, b]$ or $I = (0, b)$. The system of Abel's equations*

$$\begin{cases} \varphi(f(x)) = \varphi(x) + 1 \\ \varphi(g(x)) = \varphi(x) + s(f, g) \end{cases}, \quad x \in I, \tag{1}$$

where φ is an unknown function, has a unique up to an additive constant continuous solution. This solution is decreasing. The solution is invertible if and only if $\operatorname{Int} L_{f,g} \neq \emptyset$. Moreover, the closure of each component of the set $I \setminus L_{f,g}$ is a maximal interval of constancy of φ.

Combining the selected parts of Lemma 2 and Theorem 4 in [5], Theorem 1 in [6], Theorem 6 in [3], Proposition 12 in [4] and Theorem 2 in [10] we get

Proposition 6. *Let f, g satisfy* (H) *on $I = (0, b]$. Then for every interval $J = (0, b')$, $b < b'$ and for every homeomorphic extension $\overline{f}$ of f onto J such that $\overline{f}(x) < x$, $x \in J$ and $\overline{f}(b') = b'$ there exists a unique homeomorphic extension $\bar{g}$ of g mappings $\operatorname{cl} J$ onto itself such that*

$$\bar{g} \circ \overline{f} = \overline{f} \circ \bar{g}.$$

$\overline{f}$, $\bar{g}$ satisfy (H) *on J, $s(\overline{f}, \bar{g}) = s(f, g)$ and $L_{\overline{f},\bar{g}} \cap [0, b) = L_{f,g} \setminus \{b\}$. Moreover, every continuous solution φ of system (1) has a unique continuous extension $\overline{\varphi}$ satisfying system*

$$\begin{cases} \overline{\varphi}(\overline{f}(x)) = \overline{\varphi}(x) + 1 \\ \overline{\varphi}(\bar{g}(x)) = \overline{\varphi}(x) + s(f, g) \end{cases}, \quad x \in J. \tag{2}$$

If $\operatorname{Int} L_{f,g} = \emptyset$ *then* $\overline{\varphi}[L_{\overline{f},\overline{g}}] = \mathbb{R}$*, the functions* $\overline{\varphi}_{|L^{**}_{\overline{f},\overline{g}}}$, $\overline{\varphi}_{|L^{*-}_{\overline{f},\overline{g}}}$ *and* $\overline{\varphi}_{|L^{*+}_{\overline{f},\overline{g}}}$ *are injective and the closure of each component of the set* $I \setminus L_{\overline{f},\overline{g}}$ *is a maximal interval of constancy of* $\overline{\varphi}$*.*

From the last statement we get

Corollary 1. *If* $\operatorname{Int} L_{f,g} = \emptyset$ *and* φ *is a continuous solution of system (1) then* φ *restricted to the set* $L^{**}_{f,g} \cup L^{*-}_{f,g}$ *and* φ *restricted to the set* $L^{**}_{f,g} \cup L^{*+}_{f,g}$ *are strictly decreasing.*

Let $G := \{\overline{f}^n \circ \overline{g}^m \colon\ n, m \in \mathbb{Z}\}$. This is a spoiled subgroup of $B_i(I)$. By Propositions 4 and 6 we get the following.

Corollary 2. *If* $\overline{f}$ *and* $\overline{g}$ *are the extensions of* f *and* g *defined as in Proposition 6 and* $\operatorname{Int} L_{f,g} = \emptyset$ *then*

$$\overline{f}\big[L^{**}_{\overline{f},\overline{g}}\big] = L^{**}_{\overline{f},\overline{g}} = \overline{g}\big[L^{**}_{\overline{f},\overline{g}}\big],$$
$$\overline{f}\big[L^{*-}_{\overline{f},\overline{g}}\big] = L^{*-}_{\overline{f},\overline{g}} = \overline{g}\big[L^{*-}_{\overline{f},\overline{g}}\big].$$

The aim of the paper is to consider the problem of T-embeddability in the case when $L_{f,g}$ is not an interval. As it was mentioned in Proposition 3, in such a case there is no iteration semigroup embedding f and g. This means that always $T \neq \mathbb{R}^+$.

As a matter of the above fact we have the lack of surjectivity. Under the surjectivity, with some additional conditions, f and g can be embedded even in an iteration group but nonmeasurable with respect to the iterative parameter (Theorem 2 in [12]). So, our aim is to construct a new notion, a modified iteration semigroup, with the restriction on time index, where f and g can be embedded without any changes in domains of theirs iterates. Thus we will consider T-iteration semigroups embedded f and g and we determine the maximal semigroups with this property. Obviously then $T \neq \mathbb{R}^+$. For this purpose let us introduce

Definition. A T-iteration semigroup such that $T \neq \mathbb{R}^+$ and T is dense in $\mathbb{R}^+$ is said to be a *refinement iteration semigroup.*

To determine the refinement T-iteration semigroups containing f and g such that $\operatorname{Int} L_{f,g} = \emptyset$ we define a special subsemigroup $\operatorname{Sem}(f,g)$ of $\mathbb{R}^+$ limiting the sets of indices T. We give a construction of refinement iteration semigroups supported by this maximal semigroup $\operatorname{Sem}(f,g)$. To do this we first determine some special simple semigroups generated by f and g^{-1} as well as by g and f^{-1} supported by $(\mathbb{Z}+s\mathbb{Z}) \cap \mathbb{R}^+$. Next we deal with a set-valued semigroups generated by f and g. Based on the properties of these semigroups we give the mentioned construction. We deal also with the structure of Abelian semigroups of continuous injections containing given functions.

3. Auxiliary results

Let us start with some useful lemmas

Lemma 2. *Let* φ *be a continuous solution of Abel's system of Eqs. (1). Then for every* $a \in (0,b]$

$$\varphi\big[L_{f,g} \cap (0,a]\big] = \big[\varphi(a), \infty\big).$$

Proof. Let $b < b'$ and $\overline{f}$ and $\overline{g}$ be the homeomorphic commuting extensions of f and g on $(0,b')$. Let $b \in L_{\overline{f},\overline{g}}$. Then, by Proposition 6, $L_{f,g} \cap (0,a] = L_{\overline{f},\overline{g}} \cap (0,a]$. Let $\overline{\varphi}$ be a continuous solution of the system of Abel's equations for $\overline{f}$ and $\overline{g}$ and $\varphi = \overline{\varphi}_{|I}$. We know, by Proposition 6, that $\overline{\varphi}$ is decreasing and $\overline{\varphi}[L_{\overline{f},\overline{g}} \cap (0,b')] = \mathbb{R}$.

Hence $\mathbb{R} = \varphi[L_{f,g} \cap (0,a]] \cup \overline{\varphi}[L_{\overline{f},\overline{g}} \cap (a,b')]$, $\varphi[L_{f,g} \cap (0,a]] \subset [\varphi(a),\infty)$ and $\overline{\varphi}[L_{\overline{f},\overline{g}} \cap (a,b')] \subset (-\infty,\varphi(a)]$. This relations imply that $\varphi[L_{f,g} \cap (0,a]] = [\varphi(a),\infty)$. If $b \notin L_{\overline{f},\overline{g}}$ then $\overline{\varphi}$ is constant in the interval $[c,b]$, where $c = \sup L_{f,g}$. Hence by the first part of the proof we get our equality. $\square$

Lemma 3. *The set $\{n - sm\colon\ (n,m) \in \mathcal{N}_+(x)\}$ for every $x \in I$ is dense in $\mathbb{R}^+$.*

Proof. Let $x \in I$ and φ be a continuous solution of (1). Since φ is decreasing, $\varphi(x) \in [\varphi(b),\infty)$. Fix $c \geqslant 0$ and put $y := \varphi(x) + c$. By Lemma 2 there exists $z \in L_{f,g}$ such that $\varphi(z) = y$. The definition of the set $L_{f,g}$ ensures the existence of a sequence $\{(n_k,m_k)\}$ with terms in $\mathcal{N}_+(x)$ such that $g^{-m_k} \circ f^{n_k}(x) \to z$, $k \to \infty$. The continuity of φ gives $\varphi(z) = \lim_{k\to\infty} \varphi(g^{-m_k} \circ f^{n_k}(x))$. From system (1) we get

$$\varphi\bigl(g^{-m_k} \circ f^{n_k}(x)\bigr) = \varphi(x) + n_k - sm_k.$$

Hence $\lim_{k\to\infty}(n_k - sm_k) = \varphi(z) - \varphi(x) = c$, what means that the set $\{n - sm\colon\ (n,m) \in \mathcal{N}_+(x)\}$ is dense in $\mathbb{R}^+$. $\square$

The same property has the set $\{n - sm\colon\ (n,m) \in \mathcal{N}_-(x)\}$.

Now we consider some particular but useful refinement iteration semigroups. Define

$$\mathcal{N}_+ := \bigl\{(n,m) \in \mathbb{N}\times\mathbb{N}\colon\ f^n \leqslant g^m\bigr\}, \qquad \mathcal{N}_- := \bigl\{(n,m) \in \mathbb{N}\times\mathbb{N}\colon\ g^m \leqslant f^n\bigr\}$$

and

$$G_f^+ := \bigl\{g^{-m} \circ f^n\colon\ (n,m) \in \mathcal{N}_+\bigr\},$$
$$G_g^- := \bigl\{f^{-n} \circ g^m\colon\ (n,m) \in \mathcal{N}_-\bigr\}.$$

Note that $\mathcal{N}_- = \mathcal{N}_-(b)$ and $\mathcal{N}_+ = \mathcal{N}_+(b)$. By Proposition 5 it is easy to see that

$$(n,m) \in \mathcal{N}_+ \quad \text{if and only if} \quad n - sm \geqslant 0$$

and

$$(n,m) \in \mathcal{N}_- \quad \text{if and only if} \quad n - sm \leqslant 0.$$

By Lemma 3 sets

$$V^+ := \bigl\{n - sm\colon\ (n,m) \in \mathcal{N}_+\bigr\}, \qquad V^- := \bigl\{sm - n\colon\ (n,m) \in \mathcal{N}_-\bigr\}, \tag{3}$$

are dense in $\mathbb{R}^+$, $V^+ \cap V^- = \{0\}$ and $V^+ + V^- = \mathbb{R}^+ \cap (\mathbb{Z} + s\mathbb{Z})$.

Putting $\overline{h^t} := g^{-m} \circ f^n$ for $t = n - sm$ and $\underline{h^t} := f^{-n} \circ g^m$ for $t = sm - n$ we can write $G_f^+ = \{\overline{h^t}\colon\ t \in V^+\}$ and $G_g^- = \{\underline{h^t}\colon\ t \in V^-\}$, i.e. G_f^+ is a V^+-iteration semigroup and G_g^- is a V^--iteration semigroup. We have the following.

Theorem 1. *G_f^+ and G_g^- are disjoint refinement iteration semigroups supported, respectively, by V^+ and V^-, $f \in G_f^+$ and $g \in G_g^-$. Semigroups G_f^+ and G_g^- have the only one common element, the identity function. Moreover, the functions from G_f^+ commute with the functions from G_g^- and*

$$G_{f,g} := \bigl\{h_1 \circ h_2\colon\ h_1 \in G_f^+,\ h_2 \in G_g^-\bigr\}$$

is a disjoint refinement semigroup containing f and g, supported by the semigroup $V := (\mathbb{Z} + s\mathbb{Z}) \cap \mathbb{R}^+$.

The proof is very technical but for convenience of the readers we present it below.

Proof. Take $h_1, h_2 \in G_f^+$. There exist $(n,m),(q,p) \in \mathcal{N}_+$ such that $h_1 = g^{-m} \circ f^n$ and $h_2 = g^{-p} \circ f^q$. By the definition of set $\mathcal{N}_+$ we get $f^n \leqslant g^m$ and $f^q \leqslant g^p$. Consequently, $f^n \circ f^q \leqslant g^m \circ f^q$ and $g^m \circ f^q \leqslant g^m \circ g^p$, and, furthermore, $f^n \circ f^q \leqslant g^m \circ g^p$, whence $(n+q, m+p) \in \mathcal{N}_+$ and, by the commutativity of f^n and g^{-p} on $g^p[I]$ and the inclusion $f^q[I] \subset g^p[I]$,

$$h_1 \circ h_2 = g^{-m} \circ f^n \circ g^{-p} \circ f^q = g^{-(m+p)} \circ f^{n+q} \in G_f^+.$$

Similarly $h_2 \circ h_1 = g^{-(m+p)} \circ f^{n+q}$. Reasoning for G_g^- is the same.

Now we show the disjointness of the set G_f^+. Let $h_1 = g^{-m} \circ f^n \in G_f^+$, $h_2 = g^{-p} \circ f^q \in G_f^+$. Assume that there exists an $x_0 \in I$ such that $h_1(x_0) = h_2(x_0)$. Then, for $m \geqslant p$, $f^n(x_0) = g^{m-p}(f^q(x_0))$ and for $m < p$, $g^{p-m} \circ f^n(x_0) = f^q(x_0)$. In the first case, for $n \geqslant q$, after the substitution $y_0 := f^q(x_0)$, we obtain $f^{n-q}(y_0) = g^{m-p}(y_0)$, what means that $n-q = m-p = 0$ and consequently, $h_1 = h_2$. For $n < q$ putting $y_0 := f^n(x_0)$ we get $y_0 = g^{m-p} \circ f^{q-n}(y_0)$, what contradicts to the condition $f(x) < x$ and $g(x) < x$. In the second case one can use similar argumentation. Again, reasoning for G_g^- is the same. To show that $f \in G_f^+$ and $g \in G_g^-$ it is enough to see that $(1,0) \in \mathcal{N}_+$ and $(0,1) \in \mathcal{N}_-$.

Suppose $g^{-m} \circ f^n = f^{-q} \circ g^p$ for some $(n,m) \in \mathcal{N}_+$ and $(q,p) \in \mathcal{N}_-$. Then $g^p[I] \subset f^q[I]$, $f^n[I] \subset g^m[I]$ and we get $f^{n+q} = f^q \circ f^n = f^q \circ g^m \circ f^{-q} \circ g^p = g^m \circ f^q \circ f^{-q} \circ g^p = g^{m+p}$ what, in a view of noncommensurability of f and g, gives $n+q = m+p = 0$ and, consequently, $n = q = m = p = 0$. Thus $G_f^+ \cap G_g^- = \{id\}$.

Take $h_1 = g^{-m} \circ f^n \in G_f^+$ and $h_2 = f^{-q} \circ g^p \in G_g^-$. Put $H_1 := h_1 \circ h_2 = (g^{-m} \circ f^n) \circ (f^{-q} \circ g^p)$ and $H_2 := h_2 \circ h_1 = (f^{-q} \circ g^p) \circ (g^{-m} \circ f^n)$. Note that there exists a δ such that $(0,\delta) \subset g^m[I]$ and all factors of H_1 and H_2 commute on $(0,\delta)$. Thus on the interval $(0,\delta)$ functions H_1 and H_2 coincide. Let $x \in I$. Taking the index i such that $f^i(x) \in (0,\delta)$ we get $f^i(g^{-m}(x)) = g^{-m}(f^i(x))$ and consequently $H_1(f^i(x)) = f^i(H_1(x))$ and $H_2(f^i(x)) = f^i(H_2(x))$. Since f^i is invertible, $H_1 = H_2$. Hence we infer that $G_{f,g}$ is a semigroup containing f and g. By Lemma 3, G_f^+, G_g^- and $G_{f,g}$ are the refinement semigroups supported, respectively, by V^+, V^- and $V^+ + V^- = \mathbb{R}^+ \cap (\mathbb{Z} + s\mathbb{Z})$. $\square$

Put $\mathcal{N}_+^* := \mathcal{N}_+ \cup (\mathbb{N} \times -\mathbb{N})$, $\mathcal{N}_-^* := \mathcal{N}_- \cup (-\mathbb{N} \times \mathbb{N})$ and define

$$G_{f,g}^+ := \{g^{-m} \circ f^n\colon\ (n,m) \in \mathcal{N}_+^*\}, \qquad G_{f,g}^- := \{f^{-n} \circ g^m\colon\ (n,m) \in \mathcal{N}_-^*\}.$$

Similarly as in Theorem 1, $G_{f,g}^+$, $G_{f,g}^-$ can be treated as iteration semigroups. Moreover, using the same technic of the proof as in Theorem 1 we obtain also the following.

Corollary 3. *$G_{f,g}^+$ and $G_{f,g}^-$ are disjoint refinement iteration semigroups, $G_f^+ \subset G_{f,g}^+$, $G_g^- \subset G_{f,g}^-$, $f, g \in G_{f,g}^+ \cap G_{f,g}^-$ and $G_{f,g} = G_{f,g}^+ \circ G_{f,g}^-$.*

Hence we infer

Remark 1. $G_{f,g}$ is not a minimal refinement iteration semigroup containing f and g.

4. Main results

Let f and g satisfy (H), $\operatorname{Int} L_{f,g} = \emptyset$ and φ be a continuous solution of Abel's system of Eqs. (1). Define (see [4])

$$\operatorname{Realm}(f,g) := \{h : I \to I,\ \exists c \in \mathbb{R},\ \forall x \in I,\ \varphi(h(x)) = \varphi(x) + c\}.$$

It is easy to verify the following.

Remark 2. $\mathrm{Realm}(f,g)$ with the operation of composition is a semigroup containing f and g.

The set $\mathrm{Realm}(f,g)$ does not depend on the choice of the solution φ (because φ is unique up to an additive constant). Since for every $h \in \mathrm{Realm}(f,g)$ the function $\varphi \circ h - \varphi$ is constant in I we can define the mapping $\mathrm{ind} : \mathrm{Realm}(f,g) \to \mathbb{R}$ by the formula

$$\mathrm{ind}\, h := \varphi \circ h - \varphi.$$

Remark 3. The function ind is a homomorphism mapping semigroup $\mathrm{Realm}(f,g)$ into $\mathbb{R}^+$.

Proof. Let $h \in \mathrm{Realm}(f,g)$. Since $h(b) \leqslant b$ and φ is decreasing $\mathrm{ind}\, h = \varphi(h(b)) - \varphi(b) \geqslant 0$. Let $h_1, h_2 \in \mathrm{Realm}(f,g)$. Then $\varphi + \mathrm{ind}\, h_1 \circ h_2 = \varphi \circ (h_1 \circ h_2) = \varphi \circ h_2 + \mathrm{ind}\, h_1 = \varphi + \mathrm{ind}\, h_2 + \mathrm{ind}\, h_1$. $\square$

Remark 4. If $h \in \mathrm{Realm}(f,g)$ is continuous then $\mathrm{ind}\, h = 0$ if and only if h has a fixed point.

Proof. If h has a fixed point x_0 then $\mathrm{ind}\, h = \varphi(h(x_0)) - \varphi(x_0) = 0$. Conversely, if $\mathrm{ind}\, h = 0$ then $\varphi(h(x)) = \varphi(x)$, for $x \in I$. We know, by Proposition 5, that φ is decreasing and each closure of component of the set $I \setminus L_{f,g}$ is a maximal interval of constancy of φ. Hence $h[\mathrm{cl}\, J] \subset \mathrm{cl}\, J$ for every component J of $I \setminus L_{f,g}$. Consequently h has a fixed point in each interval $\mathrm{cl}\, J$. $\square$

Remark 5. If $h \in \mathrm{Realm}(f,g)$ is continuous, strictly increasing, commutes with f and $\mathrm{ind}\, h > 0$, then $\mathrm{ind}\, h = s(f,h)$.

This is a consequence of Proposition 5 since the pair (f,h) satisfies (H).

Lemma 4. *If $h : I \to I$ is either continuous or monotonic and commutes with f and g then $h \in \mathrm{Realm}(f,g)$.*

Proof. Since $f, g \in \mathrm{Realm}(f,g)$ we have $\varphi \circ h \circ f = \varphi \circ f \circ h = \varphi \circ h + 1$ and $\varphi \circ h \circ g = \varphi \circ g \circ h = \varphi \circ h + s$. Putting $\psi = \varphi \circ h$ gives

$$\begin{cases} \psi\big(f(x)\big) = \psi(x) + 1 \\ \psi\big(g(x)\big) = \psi(x) + s \end{cases}, \quad x \in I,$$

hence ψ is a solution of system (1). If h is continuous then ψ is also continuous. If h is monotonic then ψ is also monotonic and consequently continuous except at most countable set. Since the set $L_{f,g}$ is uncountable ψ is continuous at least one point of $L_{f,g}$. The solution of (1) continuous at least one point of $L_{f,g}$ is unique up to an additive constant (see Theorem 2 in [6] and Corollary 2 in [10]), so $\psi = \varphi + c$, for a c, hence $\varphi \circ h = \varphi + c$. The proof is ended. $\square$

Let $C(I,I) := \{f : I \to I,\ f \text{ is continuous}\}$ and $M(I,I) := \{f : I \to I,\ f \text{ is monotonic}\}$. Lemma 4 implies the following.

Theorem 2. *If $\mathcal{A} \subset C(I,I)$ or $\mathcal{A} \subset M(I,I)$ is an Abelian semigroup and $f, g \in \mathcal{A}$, then $\mathcal{A} \subset \mathrm{Realm}(f,g)$.*

Directly by the above statement and Theorem 1 we get

Corollary 4. *If f and g are T-embeddable in $\{f^t,\ t \in T\}$ then $f^t \in \mathrm{Realm}(f,g)$ for every $t \in T$.*

Corollary 5. $G_{f,g} \subset \mathrm{Realm}(f,g)$.

Remark 6. $\operatorname{ind}[G_f^+] = V^+$, $\operatorname{ind}[G_g^-] = V^-$, where V^+ and V^- are given by (3), and $\operatorname{ind}[G_{f,g}] = V^+ + V^- = \{n - sm\colon n, m \in \mathbb{N}\} \cap \mathbb{R}^+$.

Proof. Let $(n,m) \in \mathcal{N}_+$. We have $n = \operatorname{ind} f^n = \operatorname{ind} g^m \circ (g^{-m} \circ f^n) = \operatorname{ind} g^m + \operatorname{ind} g^{-m} \circ f^n$. Since $\operatorname{ind} g^m = ms$ we obtain $\operatorname{ind} g^{-m} \circ f^n = n - sm$. Similarly $\operatorname{ind} f^{-n} \circ g^m = sm - n$. Since $\operatorname{ind}[G_f^+] := \{\operatorname{ind} g^{-m} \circ f^n,\ (n,m) \in \mathcal{N}_+\}$ we get the first statement. The second one is a consequence of Theorem 1. □

It is also easy to see that

$$\operatorname{ind}\bigl[G_{f,g}^+\bigr] = V^+ + M \quad \text{and} \quad \operatorname{ind}\bigl[G_{f,g}^-\bigr] = V^- + M,$$

where $M := \{n + sm\colon n, m \in \mathbb{N}\}$.

Now we move to semigroups of set-valued functions.

Let φ be a continuous solution of Abel's system of Eqs. (1). Define the following set-valued function

$$F^t(x) = \varphi^{-1}\bigl[t + \varphi(x)\bigr], \quad t \geqslant 0. \tag{4}$$

The values of F^t are either closed intervals or singletons. Denote $cc[I] := \{[c,d] \subset I\}$. We have

Theorem 3. *The family $\{F^t : I \to cc[I],\ t \geqslant 0\}$ is a set-valued iteration semigroup, that is*

$$F^u \circ F^v(x) = F^{u+v}(x), \quad u, v \geqslant 0,\ x \in I,$$

where

$$F^u \circ F^v(x) := \bigcup_{y \in F^v(x)} F^u(y),$$

such that $f(x) \in F^1(x)$ and $g(x) \in F^s(x)$ for $x \in I$, where $s = \operatorname{ind} g$.

Proof. Fix an $x \in I$. Let $z \in F^u \circ F^v(x)$. Then there exists a $y \in F^v(x)$ such that $z \in F^u(y)$, that is $\varphi(y) = v + \varphi(x)$ and $\varphi(z) = u + \varphi(y)$. Consequently, $\varphi(z) = u + v + \varphi(x)$, what means that $z \in F^{u+v}(x)$. To see the opposite inclusion let $z \in F^{u+v}(x)$. Then $\varphi(z) = u + v + \varphi(x)$. Take a $y \in F^v(x)$. We have $\varphi(y) = v + \varphi(x)$, and, what follows, $\varphi(z) = u + \varphi(y)$. Whence $z \in F^u(y)$ and, consequently, $z \in F^u \circ F^v(x)$. To show that $f(x) \in F^1(x)$ and $g(x) \in F^s(x)$ for $x \in I$ it is enough to use Abel's equations $\varphi(f(x)) = \varphi(x) + 1$ and $\varphi(g(x)) = \varphi(x) + s$, respectively. □

Lemma 3 allows us to introduce the following families of functions

$$\begin{aligned} f_-^t(x) &:= \sup\bigl\{g^{-m} \circ f^n(x)\colon n - sm > t,\ (n,m) \in \mathcal{N}_+(x)\bigr\}, \quad t \geqslant 0, \\ f_+^t(x) &:= \inf\bigl\{g^{-m} \circ f^n(x)\colon n - sm < t,\ (n,m) \in \mathcal{N}_+(x)\bigr\}, \quad t > 0 \end{aligned} \tag{5}$$

defined on I.

We have $f_-^t \leqslant f_+^t$ for $t > 0$. This is a simple consequence of the following implication (see Lemma 2 in [6])

$$n_1 - sm_1 < n_2 - sm_2 \quad \Rightarrow \quad g^{-m_2} \circ f^{n_2}(x) < g^{-m_1} \circ f^{n_1}(x) \tag{6}$$

for $(n_1, m_1), (n_2, m_2) \in \mathcal{N}_+(x)$.

We have also

$$f_+^t < id, \quad t > 0. \tag{7}$$

In fact, if $(n_1, m_1), (n_2, m_2) \in \mathcal{N}_+(x)$ and $n_1 - sm_1 \leqslant 0 < n_2 - sm_2 < t$ then $g^{-m_2} \circ f^{n_2}(x) \leqslant x < g^{-m_1} \circ f^{n_1}(x)$. Now, taking into account Lemma 3, we obtain that $\inf\{g^{-m} \circ f^n(x)\colon\ n - sm < t,\ (n,m) \in \mathcal{N}_+(x)\} = \inf\{g^{-m} \circ f^n(x)\colon\ 0 < n - sm < t,\ (n,m) \in \mathcal{N}_+(x)\} < x$ which proves (7).

If $n - sm > t$, $n, m \in \mathbb{N}$ then $n - sm > 0$ and, by Proposition 5, $f^n < g^m$. Thus $(n,m) \in \mathcal{N}_+(x)$ for every $x \in I$ and consequently

$$f_-^t = \sup\{g^{-m} \circ f^n\colon\ n - sm > t,\ n, m \in \mathbb{N}\}. \tag{8}$$

Similarly we get

$$f_+^t = \inf\{g^{-m} \circ f^n\colon\ 0 < n - sm < t,\ n, m \in \mathbb{N}\}. \tag{9}$$

In fact, $\{g^{-m} \circ f^n(x)\colon\ n - sm < t,\ (n,m) \in \mathcal{N}_+(x)\} = \{g^{-m} \circ f^n(x)\colon\ 0 < n - sm < t,\ n, m \in \mathbb{N}\} = \{g^{-m} \circ f^n(x)\colon\ n - sm \leqslant 0,\ (n,m) \in \mathcal{N}_+(x)\}$. If $0 < n - sm$ then $g^{-m} \circ f^n(x) < x$. If $n - sm \leqslant 0$ and $(n,m) \in \mathcal{N}_+(x)$ then $x \neq g^{-m} \circ f^n(x)$ which implies (9).

Let $\bar{f}$ and $\bar{g}$ be the homeomorphic extensions of f and g on an interval $J = (0, b') \supset (0, b] = I$ defined as in Proposition 6. Put

$$\overline{f_-^t} := \sup\{\bar{g}^{-m} \circ \bar{f}^n\colon\ n - sm > t,\ n, m \in \mathbb{N}\} \tag{10}$$

and

$$\overline{f_+^t} := \inf\{\bar{g}^{-m} \circ \bar{f}^n\colon\ 0 < n - sm < t,\ n, m \in \mathbb{N}\}. \tag{11}$$

Since $\bar{g}^{-m} \circ \bar{f}^n = g^m \circ f^n$ in $(0, b]$ for $n - sm > 0$ in a view of (8) and (9) we get

$$\overline{f_-^t}_{|I} = f_-^t \quad \text{and} \quad \overline{f_+^t}_{|I} = f_+^t. \tag{12}$$

Theorem 4. *Both of the families* $\{f_-^t\colon\ t \geqslant 0\}$ *and* $\{f_+^t\colon\ t > 0\}$ *are iteration semigroups of increasing functions.*

Proof. The mappings f_+^t are increasing as lower bounds of strictly increasing functions. In [13] (see Lemma 16) it is proved that $\overline{f_\pm^u} \circ \overline{f_\pm^v} = \overline{f_\pm^{u+v}}$ for $u, v \in \mathbb{R}^+$. Hence, directly by (12), we get also $f_\pm^u \circ f_\pm^v = f_\pm^{u+v}$, $u, v \in \mathbb{R}^+$. $\square$

If $L_{f,g}$ is an interval then $f_+^t = f_-^t$ for all $x \in I$ and $t \geqslant 0$ and these families build the continuous iteration semigroup $\{f^t := f_+^t = f_-^t,\ t \geqslant 0\}$, the same unique iteration semigroup in which f and g can be embedded.

Let further again $L_{f,g}$ be a Cantor set. We have

$$I \setminus L_{f,g} = \bigcup_{\alpha \in A} I_\alpha, \tag{13}$$

where I_α are open pairwise disjoint intervals and $A = (-\infty, \rho] \cap \mathbb{Q}$ for a $\rho > 0$ except at most one interval I_ρ as far as $\rho \in \mathbb{Q}$, in this case $I_\rho = (\varrho, b]$.

Lemma 5. *For every $x \in I$ and $t \geqslant 0$, $f_-^t(x) \in L_{f,g}$ and for every $t > 0$, $f_+^t(x) \in L_{f,g}$.*

Proof. Let $x \in I$ and $t \geqslant 0$. By Lemma 3 the set $S(x) := \{n - sm\colon (n,m) \in \mathcal{N}_+(x)\}$ is dense in $\mathbb{R}^+$, so there exists a decreasing sequence $\{n_k - sm_k\} \subset S(x)$ converging to t such that $f_-^t(x) = \lim_{k\to\infty} g^{-m_k} \circ f^{n_k}(x)$. By Lemma 2 in [6] the sequence $\{g^{-m_k} \circ f^{n_k}(x)\}$ is strictly increasing, so, by Proposition 1, $f_-^t(x) \in \{g^{-m_k} \circ f^{n_k}(x)\}^d \subset L_{f,g}$. For $f_+^t(x)$ one can use analogous argumentation. □

Taking into account (12), directly by Proposition 6 and Lemmas 11, 20, 26, 24 in [13] applied for the semigroups $\{\overline{f_\pm^t},\ t > 0\}$, we get the following non surjective version of these lemmas.

Lemma 6. *The functions $f_\pm^t$ are discontinuous, however, they are constant on every interval $\operatorname{cl} I_\alpha$. Moreover, for every $x \in I$ the mappings $t \to f_\pm^t(x)$ are strictly decreasing.*

Lemma 7. *If $f_-^t(x_0) = f_+^t(x_0) =: d_t$ for an $x_0 \in \operatorname{Int} I$ then $d_t \in L_{f,g}^{**}$. If $f_-^t(x_0) \neq f_+^t(x_0)$ for an $x_0 \in \operatorname{Int} I$ then $(f_-^t(x_0), f_+^t(x_0)) = I_\alpha$ for an $\alpha \in A$ and $\alpha < \rho$.*

We prove the following.

Lemma 8. *f_-^t and f_+^t are in $\operatorname{Realm}(f,g)$ and $\operatorname{ind} f_-^t = \operatorname{ind} f_+^t = t$ for $t > 0$.*

Proof. Fix $x \in I$ and $t > 0$. By Lemma 3, there exists an increasing sequence $\{n_k - sm_k\}$ such that $n_k - sm_k \to t$ and $\lim_{k\to\infty} g^{-m_k} \circ f^{n_k}(x) = f_+^t(x)$. By the continuity of φ we get

$$\varphi\bigl(f_+^t(x)\bigr) = \lim_{k\to\infty} \varphi\bigl(g^{-m_k} \circ f^{n_k}(x)\bigr) = \lim_{k\to\infty} \varphi(x) + n_k - sm_k = \varphi(x) + t.$$

The proof for f_-^t is analogous. □

By Remark 3 and Lemma 8 we get

Corollary 6. *The index function ind is an epimorphism of the semigroup $\operatorname{Realm}(f,g)$ onto $\mathbb{R}^+$.*

Theorem 5. *$f_-^t(x) = \inf F^t(x)$ and $f_+^t(x) = \sup F^t(x)$ for $x \in I$ and $t > 0$.*

Proof. By Proposition 6 the continuous solution φ of system (1) is weakly decreasing and the intervals $\operatorname{cl} I_\alpha$ from the decomposition (13) are the maximal intervals of constancy of φ. Fix an $x \in I$ such that $F^t(x)$ is not a singleton. Hence $F^t(x) = \varphi^{-1}[t + \varphi(x)] = \operatorname{cl} I_\alpha$ for an $\alpha \in A$. By Lemma 8 $\varphi(f_\pm^t(x)) = \varphi(x) + t$, so $f_-^t(x), f_+^t(x) \in F^t(x) = \operatorname{cl} I_\alpha$. Suppose $f_-^t(x) = f_+^t(x) =: d_t$, then, by Lemma 7, $d_t \in L_{f,g}^{**}$ but this is a contradiction since $L_{f,g}^{**} \cap \operatorname{cl} I_\alpha = \emptyset$. On the other hand, by Lemma 5, $f_-^t(x), f_+^t(x) \in L_{f,g}$. Since $f_-^t(x) \neq f_+^t(x)$ and $f_-^t(x), f_+^t(x) \in \operatorname{cl} I_\alpha$ Lemma 7 implies that $[f_-^t(x), f_+^t(x)] = \operatorname{cl} I_\alpha$. Thus we get $F^t(x) = \operatorname{cl} I_\alpha = [f_-^t(x), f_+^t(x)]$. If $F^t(x)$ is a singleton then $F^t(x) = f_-^t(x) = f_+^t(x)$ and the thesis is proved. □

Every function from an Abelian semigroup $\mathcal{A}$ containing f and g can be estimated by elements of families $\{f_-^t\}$ and $\{f_+^t\}$ defined by (5). Namely,

Theorem 6. *If $\mathcal{A}$ is an Abelian semigroup containing f and g then for every $h \in \mathcal{A}$ without fixed point*

$$f_-^{\operatorname{ind} h} \leqslant h \leqslant f_+^{\operatorname{ind} h}.$$

Proof. We know that $\mathcal{A} \subset \mathrm{Realm}(f,g)$. Let $h \in \mathcal{A}$ and put $t := \operatorname{ind} h$. Then $\varphi(h(x)) = \varphi(x) + t$ and, by Lemma 8, $\varphi(f_-^t(x)) = \varphi(x) + t$, $\varphi(f_+^t(x)) = \varphi(x) + t$. Thus $\varphi(h(x)) = \varphi(f_-^t(x)) = \varphi(f_+^t(x))$. Hence, by the last statement in Proposition 6, either $h(x) = f_-^t(x) = f_+^t(x)$ or $h(x), f_-^t(x), f_+^t(x) \in \operatorname{cl} I_\alpha$ for an $\alpha \in A$ and $\operatorname{cl} I_\alpha$ is a maximal interval of constancy of φ. By Lemma 19 from [13] $f_-^t(x)$ and $f_+^t(x)$ are the ends of I_α, so we get the inequality

$$f_-^t(x) \leqslant h(x) \leqslant f_+^t(x), \quad x \in I. \qquad \square$$

Now we prove some invariant properties of limit sets $L_{f,g}^{**}$, $L_{f,g}^{*-}$ and $L_{f,g}^{*+}$.

Lemma 9.

$$\begin{aligned} f\big[L_{f,g}^{**}\big] &= L_{f,g}^{**} \cap f[I], & g\big[L_{f,g}^{**}\big] &= L_{f,g}^{**} \cap g[I], \\ f\big[L_{f,g}^{*-}\big] &= L_{f,g}^{*-} \cap f[I], & g\big[L_{f,g}^{*-}\big] &= L_{f,g}^{*-} \cap g[I], \\ f\big[L_{f,g}^{*+}\big] &= L_{f,g}^{*+} \cap f[I], & g\big[L_{f,g}^{*+}\big] &= L_{f,g}^{*+} \cap g[I]. \end{aligned}$$

Proof. Let $\bar{f}, \bar{g} : I' \to I'$ be the functions defined in Proposition 6, where $I = [0,b] \subset [0,b'] = I'$ for a $b' > b$. By Theorem 1 from [6] it is known that $L_{f,g} \setminus \{b\} = L_{\bar{f},\bar{g}} \cap [0,b)$. Whence we infer that $L_{f,g}^{**} \setminus \{b\} = L_{\bar{f},\bar{g}}^{**} \cap [0,b)$ and $L_{f,g}^{*-} \setminus \{b\} = L_{\bar{f},\bar{g}}^{*-} \cap [0,b)$. Thus, by Corollary 2, $f[L_{f,g}^{**} \setminus \{b\}] = \bar{f}[L_{\bar{f},\bar{g}}^{**} \cap [0,b)] = \bar{f}[L_{\bar{f},\bar{g}}^{**}] \cap [0, \bar{f}(b)) = L_{\bar{f},\bar{g}}^{**} \cap \bar{f}[I] = L_{f,g}^{**} \cap f[I]$. If $b \notin L_{f,g}^{**}$ the assertion is obvious.

If $b \in L_{f,g}^{**}$ then $b \in L_{\bar{f},\bar{g}}^{**}$, $f(b) \in \bar{f}[L_{\bar{f},\bar{g}}^{**}] = L_{\bar{f},\bar{g}}^{**}$ and $f(b) < b$. Thus $f(b) \in L_{f,g}^{**}$. Hence $f[L_{f,g}^{**}] = f[L_{f,g}^{**} \setminus \{b\}] \cup \{f(b)\} = L_{f,g}^{**} \cap f[I] \cup \{f(b)\} = L_{f,g}^{**} \cap (0, f(b)] \cup \{f(b)\} = L_{f,g}^{**} \cap f[I]$.

For a function g reasoning is the same. To prove the remaining thesis one can use a similar argumentation. $\square$

Denote by $C^*(I,I)$ the subset of $C(I,I)$ of all injections.

Lemma 10. *If $h \in C^*(I,I)$ and h commute with f and g, then $h[L_{f,g}^{**}] = L_{f,g}^{**} \cap h[I]$, $h[L_{f,g}^{*-}] = L_{f,g}^{*-} \cap h[I]$, $h[L_{f,g}^{*+}] = L_{f,g}^{*+} \cap h[I]$ and for every interval I_α, $\alpha \in A$ from the decomposition (13) there exists a $\beta \in A$ such that $h[I_\alpha] = I_\beta \cap h[I]$.*

Proof. We show that h is iteratively incommensurable with f or with g. Let φ be a continuous solution of system (1). (i) If there exists $x_0 \in I$ and $n, m \geqslant 0$ such that $f^n(x_0) = h^m(x_0)$, then $\varphi(f^n(x_0)) = \varphi(h^m(x_0))$. The left hand side of the last equality is equal to $\varphi(x_0) + n$ and the right hand side is equal to $\varphi(x_0) + m \cdot \operatorname{ind} h$. Hence $\operatorname{ind} h = \frac{n}{m}$. (ii) If there exists $x_0 \in I$ and $n, m \geqslant 0$ such that $g^n(x_0) = h^m(x_0)$, then, analogously, we obtain that $\operatorname{ind} h = s\frac{n}{m}$.

This two cases cannot be satisfied simultaneously so either the pair (f,h) or the pair (g,h) satisfies the assumption (H). Hence, by Lemma 9 applied for the pair (f,h) or the pair (g,h), we get our assertion.

By the proved part we get that for every $\alpha \in A$, $h[I_\alpha] \subset h[I \setminus L_{f,g}] = h[I] \cap \bigcup_{\beta \in A} I_\beta$. Since $h[I_\alpha]$ is an interval it should be equal to one of the intervals $I_\beta \cap h[I]$. $\square$

Using the idea of the proof of Lemma 9, in a view of Proposition 6, we can justify

Lemma 11. *If φ is a continuous solution of system (1), then $\varphi_{|L_{f,g}^{**}}$ and $\varphi_{|L_{f,g}^{*-}}$ are invertible.*

Remark 7. If $\mathcal{H} \subset C^*(I,I)$ is a disjoint semigroup in which all functions commute with f and g then $\mathcal{H} \subset \mathrm{Realm}(f,g)$ and $\mathcal{H}$ is Abelian.

Proof. Let $h \in \mathcal{H}$ and φ be a continuous solution of system (1). Then $\varphi \circ h \circ f = \varphi \circ f \circ h = \varphi \circ h + 1$ and $\varphi \circ h \circ g = \varphi \circ g \circ h = \varphi \circ h + s$. By the uniqueness of continuous solution of system (1) $\varphi \circ h = \varphi + c$ for a $c \in \mathbb{R}$. Hence $h \in \operatorname{Realm}(f, g)$ and consequently $\mathcal{H} \subset \operatorname{Realm}(f, g)$.

Now let $h_1, h_2 \in \mathcal{H}$ and φ be a continuous solution of system (1). Then $\varphi \circ h_1 \circ h_2 = \varphi \circ h_2 + \operatorname{ind} h_1 = \varphi + \operatorname{ind} h_1 + \operatorname{ind} h_2 = \varphi \circ h_2 \circ h_1$. By the assumption $h_1 \circ h_2$ and $h_2 \circ h_1$ commute with f and g. Hence, by Lemma 10, these functions map the set $L^{**}_{f,g}$ into itself. Let $x_0 \in L^{**}_{f,g}$. Then $h_1(h_2(x_0)) \in L^{**}_{f,g}$ and $h_2(h_1(x_0)) \in L^{**}_{f,g}$. Since $\varphi(h_1(h_2(x_0))) = \varphi(h_2(h_1(x_0)))$, in a view of Lemma 11, $h_1(h_2(x_0)) = h_2(h_1(x_0))$. Hence $h_1 \circ h_2 = h_2 \circ h_1$ since $\mathcal{H}$ is a disjoint semigroup. $\square$

By Proposition 6 the continuous solution φ of the system (1) is constant on every interval $\operatorname{cl} I_\alpha$ from the decomposition (13). Thus

$$\varphi[I \setminus L_{f,g}] = \bigcup_{\alpha \in A} \varphi[\operatorname{cl} I_\alpha] = \varphi\big[L^{*-}_{f,g}\big] = \varphi\big[L^{*+}_{f,g}\big].$$

The solution φ is determined uniquely up to an additive constant thus we may assume that $\varphi(b) = 0$; in the case of bijection $\varphi(0) = \infty$. Put

$$K_{f,g} := \varphi[I \setminus L_{f,g}] = \big\{\varphi[I_\alpha]\colon\ \alpha \in A\big\}.$$

Thus the set $K_{f,g}$ is countable. Define

$$\operatorname{Sem}(f, g) := \big\{t \geqslant 0\colon\ K_{f,g} + t = K_{f,g} \cap [t, \infty)\big\}. \tag{14}$$

Note that set $\operatorname{Sem}(f, g)$ is uniquely determined by f and g.

Theorem 7. $\operatorname{Sem}(f, g)$ *is an additive semigroup, countable and dense in* $\mathbb{R}^+$.

Proof. Let $K_{f,g} + t = K_{f,g} \cap [t, \infty)$ and $K_{f,g} + s = K_{f,g} \cap [s, \infty)$. Then $K_{f,g} + t + s = K_{f,g} \cap [t, \infty) + s = (K_{f,g} + s) \cap [t + s, \infty) = K_{f,g} \cap [s, \infty) \cap [t + s, \infty) = K_{f,g} \cap [t + s, \infty)$ which means that $\operatorname{Sem}(f, g)$ is an additive semigroup. The set $\operatorname{Sem}(f, g)$ is countable because $K_{f,g}$ is countable.

Let us note that $K_{f,g} = \varphi[L^{*-}_{f,g}]$. By Corollary 5 we have for $(n, m) \in \mathcal{N}_+ = \mathcal{N}_+(b)$

$$\varphi\big(g^{-m}\big(f^n(x)\big)\big) = n - sm + \varphi(x), \quad x \in I \setminus \{0\},$$

so, by Lemma 10,

$$\begin{aligned}\varphi\big[L^{*-}_{f,g}\big] + n - sm &= \varphi\big[g^{-m} \circ f^n\big[L^{*-}_{f,g}\big]\big] = \varphi\big[L^{*-}_{f,g} \cap \big(0, g^{-m} \circ f^n(b)\big]\big] \\ &= \varphi\big[L^{*-}_{f,g}\big] \cap \varphi\big[\big(0, g^{-m} \circ f^n(b)\big]\big] = \varphi\big[L^{*-}_{f,g}\big] \cap [n - sm, \infty),\end{aligned}$$

which gives that $n - sm \in \operatorname{Sem}(f, g)$ for $(n, m) \in \mathcal{N}_+(b)$. By Lemma 3, we get the density of $\operatorname{Sem}(f, g)$ in $\mathbb{R}^+$. $\square$

Theorem 8. *For every Abelian semigroup* $\mathcal{A} \subset C^*(I, I)$ *containing* f *and* g, $\operatorname{ind}[\mathcal{A}] \subset \operatorname{Sem}(f, g)$.

Proof. Let $h \in \mathcal{A}$. Then h commutes with f and g. By Theorem 2 $h \in \operatorname{Realm}(f, g)$, that is $\varphi(h(x)) = \varphi(x) + \operatorname{ind} h$ for $x \in I$. Hence, by Lemma 10, $K_{f,g} \cap [\operatorname{ind} h, \infty) = \varphi[L^{*-}_{f,g}] \cap [\operatorname{ind} h, \infty) = \varphi[L^{*-}_{f,g} \cap h[I]] = \varphi[h[L^{*-}_{f,g}]] = \varphi[L^{*-}_{f,g}] + \operatorname{ind} h$, since $\varphi[h[I]] = [\operatorname{ind} h, \infty)$. Thus $\operatorname{ind} h \in \operatorname{Sem}(f, g)$. $\square$

Note that if $u, v \in C^*(I, I)$ and $u[I] \subset v[I]$ then $v^{-1} \circ u \in C^*(I, I)$. This simple property inspires the following.

Definition. An Abelian semigroup $\mathcal{A} \subset C^*(I, I)$ is said to be *saturated* if $v^{-1} \circ u \in \mathcal{A}$ for every $u, v \in \mathcal{A}$ such that $u[I] \subset v[I]$.

Each saturated semigroup can be extended to the local group of germs, that is, for every function $u \in \mathcal{A}$ its inverse u^{-1} is defined on a neighborhood of zero.

Similarly an additive semigroup $T \subset \mathbb{R}^+$ is said to be saturated if $t - s \in T$ for every $t, s \in T$ such that $s \leqslant t$.

Note that semigroup $\mathcal{A} = \{t + id,\ t \in T\}$ defined in $[0, \infty)$ is saturated if and only if T is saturated semigroup.

Theorem 9. *Let $T \subset \mathbb{R}^+$ be a semigroup. If f and g are T-embeddable and* $\operatorname{Int} L_{f,g} = \emptyset$ *then $T \neq \mathbb{R}^+$ and if additionally T is a saturated semigroup then T is dense and $T \subset \operatorname{Sem}(f, g)$. Moreover,* $\operatorname{ind} f^t = t$ *for $t \in T$.*

Proof. Let $\{f^t : I \to I,\ t \in T\}$ be a T-iteration semigroup such that $f^1 = f$ and $f^s = g$ for an $s \in T$. Put $\gamma(t) := \operatorname{ind} f^t$. By Theorem 8 $\gamma[T] \subset \operatorname{Sem}(f, g) \subset \mathbb{R}^+$ and, by Remark 3,

$$\gamma(u + v) = \gamma(u) + \gamma(v), \quad u, v \in T.$$

Suppose that $T = \mathbb{R}^+$. Then $\gamma(t) = t$, since $\gamma(1) = 1$ and the only additive function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is a linear function (see e.g. [1, p. 34]). In a consequence $\operatorname{Sem}(f, g) = \mathbb{R}^+$ but this is a contradiction.

Further assume that T is saturated semigroup. Put

$$\gamma_0(x) := \begin{cases} -\gamma(-x), & x \in -T \\ 0, & x = 0 \\ \gamma(x), & x \in T \end{cases}.$$

Note that γ_0 is additive and increasing since γ is nonnegative and its domain T is a saturated semigroup. Note that $u \notin \mathbb{Q}$ since f and g are iteratively incommensurable. Hence, in a view of Lemma 3, semigroup T is dense in $\mathbb{R}^+$. Define the following functions $\gamma^+, \gamma^- : \mathbb{R} \to \mathbb{R}$

$$\gamma^+(x) := \lim_{t \to x^+} \gamma_0(t), \qquad \gamma^-(x) := \lim_{t \to x^-} \gamma_0(t).$$

Obviously γ^+ and γ^- are additive and increasing, thus there exist a_+ and a_- such that $\gamma^+(x) = a_+ x$ and $\gamma^-(x) = a_- x$, for $x \in \mathbb{R}$ (see e.g. [1, p. 34]). Since $\gamma^-(x) \leqslant \gamma_0(x) \leqslant \gamma^+(x)$ for $x \in -T \cup \{0\} \cup T$ we have $a_- x \leqslant \gamma_0(x) \leqslant a_+ x$ for $x \in -T \cup \{0\} \cup T$. Putting $x = 1$ and $x = -1$ we get $a_- \leqslant 1 \leqslant a_+$ and $-a_- \leqslant -1 \leqslant -a_+$. Hence $a_- = a_+ = 1$ and $\gamma(x) = x$ for $x \in T$. $\square$

Conclusion. Theorems 9 and 8 explain the phenomenon that commuting functions f and g with $\operatorname{Int} L_{f,g} = \emptyset$ cannot be embeddable in any iteration semigroup but only in T-iteration semigroups for which the sets of iterative indices T are limited by the semigroup $\operatorname{Sem}(f, g)$.

In the rest of the note we shall show that there exist such the best T-iteration semigroups embedding f and g for which $T = \operatorname{Sem}(f, g)$.

Lemma 12. *If $\mathcal{A} \subset C^*(I, I)$ is an Abelian disjoint semigroup containing f and g then* $\operatorname{ind}_{|\mathcal{A}}$ *is a monomorphism.*

Proof. We show that homomorphism $\operatorname{ind}_{|\mathcal{A}}$ is invertible. By Theorem 2 $\mathcal{A} \subset \operatorname{Realm}(f, g)$. Let $h_1, h_2 \in \mathcal{A}$. If $\operatorname{ind} h_1 = \operatorname{ind} h_2$ then $\varphi(h_1(x)) = \varphi(h_2(x))$ for all $x \in I$. Take an $x_0 \in L^{**}_{f,g}$. By Lemma 10 $h_1(x_0) \in L^{**}_{f,g}$ and $h_2(x_0) \in L^{**}_{f,g}$ and, by Lemma 11, $h_1(x_0) = h_2(x_0)$. Hence $h_1 = h_2$ since $\mathcal{A}$ is a disjoint semigroup. $\square$

Theorem 10. *Every Abelian disjoint saturated semigroup $\mathcal{A} \subset C^*(I,I)$ such that $f,g \in \mathcal{A}$ with $\operatorname{Int} L_{f,g} = \emptyset$ is isomorphic with a countable and dense subsemigroup of* $\operatorname{Sem}(f,g)$.

Proof. Let us note that $G_{f,g} \subset \mathcal{A}$, since $\mathcal{A}$ is a saturated Abelian semigroup. Hence, by Theorem 1 and Theorem 8,

$$(\mathbb{Z} + s\mathbb{Z}) \cap \mathbb{R}^+ \subset \operatorname{ind}[G_{f,g}] \subset \operatorname{ind}[\mathcal{A}] \subset \operatorname{Sem}(f,g).$$

Thus, by Lemma 12, the semigroup $\mathcal{A}$ is isomorphic with the countable dense semigroup $\operatorname{ind}[\mathcal{A}]$. $\square$

Corollary 7. *Every Abelian saturated disjoint semigroup $\mathcal{A} \subset C^*(I,I)$ containing f and g with $\operatorname{Int} L_{f,g} = \emptyset$ is a refinement iteration semigroup.*

Proof. Define $T := \operatorname{ind}[\mathcal{A}]$ and

$$f^t := \operatorname{ind}^{-1}(t), \quad t \in T.$$

By Theorem 10 the family $\{f^t\colon t \in T\}$ is a refinement iteration semigroup such that $f^1 = f$ and $f^s = g$, where $s = s(f,g)$. $\square$

Let I_α, $\alpha \in A$ be the intervals defined by the decomposition (13) and $\{F^t : I \to cc[I],\ t \geqslant 0\}$ be the set-value iteration semigroup defined by (4). Now we prove a fundamental property of these semigroups, namely that semigroup $\operatorname{Sem}(f,g)$ is the maximal support, where the set-valued functions F^t do not degenerate on the intervals I_α, $\alpha \in A$, and moreover, they are surjective on this family of intervals. This means that $\{F^t : I \to cc[I],\ t \in \operatorname{Sem}(f,g)\}$ is the “best” set valued T-iteration semigroup.

Theorem 11. $\operatorname{Sem}(f,g)$ *is the set of all $t \geqslant 0$ such that for every $\alpha \in A$, $F^t[I_\alpha]$ is a proper interval and for every $I_\beta \subset F^t[I]$ there exists an I_α such that $\operatorname{cl} I_\beta = F^t[I_\alpha]$.*

Proof. Put $T := \{t \geqslant 0\colon$ for every $\alpha \in A$, $\operatorname{Int} F^t[I_\alpha] \neq \emptyset$ and for every $\beta \in A$ such that $I_\beta \subset F^t[I]$ there exists an $\alpha \in A$ such that $\operatorname{cl} I_\beta = F^t[I_\alpha]\}$.

Let $t \in T$ and $y \in K_{f,g}$. Then there exists an $\alpha \in A$ such that $\{y\} = \varphi[I_\alpha]$. By (4), $\varphi[F^t(x)] = \varphi(x) + t$ for $x \in \operatorname{Int} I$, so for every $\alpha \in A$, $\varphi[F^t[I_\alpha]] = \varphi[I_\alpha] + t$. Since $\operatorname{Int} F^t[I_\alpha] \neq \emptyset$ it follows, by Theorem 5 and Lemma 7, that there exists a $\beta \in A$ such that $F^t[I_\alpha] = \operatorname{cl} I_\beta$. Thus $\varphi[F^t[I_\alpha]] = \varphi[I_\beta] = \{z\}$ for a $z \in K_{f,g}$ and $z = y + t$. In a consequence, $K_{f,g} + t \subset K_{f,g}$. Since $K_{f,g} \subset [0,\infty)$ we get

$$K_{f,g} + t \subset K_{f,g} \cap [t,\infty).$$

Note that for $t \geqslant 0$, $\varphi[F^t[I]] = [t,\infty)$. In fact, by (4), $\varphi[F^t[I]] = \varphi[\varphi^{-1}[\varphi[I] + t]] = \varphi[(0,b]] + t = [t,\infty)$, since $\varphi(b) = 0$.

Now, let $x \in K_{f,g} \cap [t,\infty)$. Then there exists a $\beta \in A$ such that $\{x\} = \varphi[I_\beta]$ and $x \in \varphi[F^t[I]]$. Hence $\varphi[I_\beta] \in \varphi[F^t[I]]$ and, consequently, $I_\beta \subset F^t[I]$, since φ is monotonic and constant only in the intervals $\operatorname{cl} I_\omega$, $\omega \in A$. Since $t \in T$, $\operatorname{cl} I_\beta = F^t[I_\alpha]$ for an $\alpha \in A$. Hence $\{x\} = \varphi[I_\beta] = \varphi[F^t[I_\alpha]] = \varphi[I_\alpha] + t \in K_{f,g} + t$, so

$$K_{f,g} \cap [t,\infty) \subset K_{f,g} + t.$$

Thus $T \subset \operatorname{Sem}(f,g)$.

Now, let $t \in \operatorname{Sem}(f,g)$ and fix an $\alpha \in A$. Then $\varphi[I_\alpha] + t = \varphi[I_\beta]$ for a $\beta \in A$. On the other hand $\varphi[F^t[I_\alpha]] = \varphi[I_\alpha] + t = \varphi[I_\beta] = \varphi[\operatorname{cl} I_\beta]$. In a view of Theorem 5 and Lemma 6 $F^t[I_\alpha] = [f^t_-(x_0), f^t_+(x_0)]$ for an $x_0 \in I_\alpha$. Suppose $f_-^t(x_0) = f_+^t(x_0) =: d_t$. Then, by Lemma 7, $d_t \in L_{f,g}^{**}$. Obviously $\inf I_\beta =: u_\beta \in L_{f,g}^{*-}$ and $\varphi(u_\beta) = \varphi(d_t)$. In a view of Corollary 1 φ restricted to the set $L_{f,g}^{**} \cup L_{f,g}^{*-}$ is invertible. Thus $d_t = u_\alpha$ and, consequently, $L_{f,g}^{**} \cap L_{f,g}^{*-} \neq \emptyset$ but this is a contradiction. Hence $F^t[I_\alpha]$ is a proper interval.

Now, let $I_\beta \subset F^t[I]$. Then $\{x_\beta\} := \varphi[I_\beta] \in \varphi[F^t[I]] = [t, \infty)$, so $x_\beta \in K_{f,g} \cap [t, \infty) = K_{f,g} + t$. Hence there exists an $\alpha \in A$ such that $\varphi[I_\beta] = \{x_\beta\} = \varphi[I_\alpha] + t = \varphi[F^t[I_\alpha]]$. Since $F^t[I_\alpha]$ is a closed proper interval and φ is monotonic and constant only in the intervals $\operatorname{cl} I_\omega$, $\omega \in A$ we get the equality $\operatorname{cl} I_\beta = F^t[I_\alpha]$ and this means that $t \in T$, so $\operatorname{Sem}(f,g) \subset T$. $\square$

As a direct consequence of the above theorem and Lemmas 6 and 7 we get the following.

Corollary 8. *For every $x \in L_{f,g}^{**}$ and $t \in \operatorname{Sem}(f,g)$ the set $F^t(x)$ is a singleton i.e. $f_-^t(x) = f_+^t(x)$.*

As an application of Theorem 11 we get also the following.

Lemma 13. *Let $b \in L_{f,g}$. Then there exists a unique piecewise linear T-iteration semigroup $\{p^t\colon t \in \operatorname{Sem}(f,g)\}$ on I for which $p^t_{|\operatorname{cl} I_\alpha}$ for $\alpha \in A$ are the linear increasing functions such that $p^t[\operatorname{cl} I_\alpha] = F^t[\operatorname{cl} I_\alpha]$. Moreover, $p^t \in \operatorname{Realm}(f,g)$ and $\operatorname{ind} p^t = t$ for $t \in \operatorname{Sem}(f,g)$.*

Proof. For every $t \in \operatorname{Sem}(f,g)$ and $\alpha \in A$ denote by p^t_α the linear increasing bijection mapping $\operatorname{cl} I_\alpha$ onto the interval $F^t[\operatorname{cl} I_\alpha]$. All such the mappings are unique. Define

$$\widetilde{p}^t : \bigcup_{\alpha \in A} \operatorname{cl} I_\alpha \to \bigcup_{\alpha \in A} F^t[\operatorname{cl} I_\alpha]$$

such that $\widetilde{p}^t_{|\operatorname{cl} I_\alpha} = p^t_\alpha$. The functions $\widetilde{p}^t$ are strictly increasing and have a unique continuous extension p^t on I. In fact, the set $\bigcup_{\alpha \in A} I_\alpha$ is dense in I and, by Theorem 11, $\bigcup_{\alpha \in A} F^t[I_\alpha]$ is dense in the interval $F^t[I]$ and, in a consequence, $\lim_{u \to x-} \widetilde{p}^t(u) = \lim_{u \to x+} \widetilde{p}^t(u) =: p^t(x)$ for $x \in L_{f,g}^{**} = I \setminus \bigcup_{\alpha \in A} I_\alpha$. For every $t, s \in \operatorname{Sem}(f,g)$ and $\alpha \in A$ the functions $p^t \circ p^s$ and p^{t+s} are linear in $\operatorname{cl} I_\alpha$ and

$$p^t \circ p^s[\operatorname{cl} I_\alpha] = p^t\big[F^s[\operatorname{cl} I_\alpha]\big] = F^t\big[F^s[\operatorname{cl} I_\alpha]\big] = F^{t+s}[\operatorname{cl} I_\alpha] = p^{t+s}[\operatorname{cl} I_\alpha].$$

Hence $p^t \circ p^s = p^{t+s}$ on $\operatorname{cl} I_\alpha$ for $t, s \in \operatorname{Sem}(f,g)$ and $\alpha \in A$ and, in a consequence, by the continuity of p^t, $\{p^t\colon t \in \operatorname{Sem}(f,g)\}$ is a T-iteration semigroup. The uniqueness is obvious.

Since $p^t[\operatorname{cl} I_\alpha] = F^t[\operatorname{cl} I_\alpha]$, by Theorem 5 and Corollary 8, we get

$$f_-^t(x) \leqslant p^t(x) \leqslant f_+^t(x), \quad t \in \operatorname{Sem}(f,g),\ x \in I.$$

Let φ be a continuous solution of (1). Since φ is decreasing, by Lemma 8, $\varphi(x) + t = \varphi(f_+^t(x)) \leqslant \varphi(p^t(x)) \leqslant \varphi(f_-^t(x)) = \varphi(x) + t$, so

$$\varphi\big(p^t(x)\big) = \varphi(x) + t, \quad t \in \operatorname{Sem}(f,g),\ x \in I. \tag{15}$$

Thus $p^t \in \operatorname{Realm}(f,g)$ and $\operatorname{ind} p^t = t$ for $t \in \operatorname{Sem}(f,g)$. $\square$

Remind that $G_f^+ = \{h^t\colon t \in V^+\}$, where $V^+ = \{n - sm \geqslant 0\colon n, m \in \mathcal{N}_+\}$ and

$$h^t := g^{-m} \circ f^n \quad \text{for } t = n - sm, \tag{16}$$

is a semigroup in which every element commutes with f and g.

Lemma 14. *If $b \in L_{f,g}$ and $t \in V^+$ then for every $\alpha \in A$, $h^t[\operatorname{cl} I_\alpha] = \operatorname{cl} I_\beta$ for a $\beta \in A$ and for every $I_\beta \subset h^t[I]$ there exists an $\alpha \in A$ such that $\operatorname{cl} I_\beta = h^t[\operatorname{cl} I_\alpha]$.*

Proof. Let $t \in V^+$. In a view of Theorem 1 h^t commutes with f and g thus, by Lemma 10, $h^t[L_{f,g}] = L_{f,g} \cap h^t[I]$, so $h^t[I \setminus L_{f,g}] = h^t[I] \setminus L_{f,g} = (0, h^t(b)] \setminus L_{f,g}$ and $h^t(b) \in L_{f,g}$. Hence we get the equality $\bigcup_{\alpha \in A} h^t[I_\alpha] = \bigcup_{\alpha \in A'} I_\alpha$, where $A' = \{\alpha\colon\ I_\alpha \subset (0, h^t(b))\}$, which gives our assertion since h^t are continuous injections. □

Lemma 15. *If $b \in L_{f,g}$ then for every $t \in V^+$ and $\alpha \in A$, $h^t[\operatorname{cl} I_\alpha] = F^t[\operatorname{cl} I_\alpha]$ and $V^+ \subset \operatorname{Sem}(f,g)$.*

Proof. Let $t \in V^+$. Then $h^t \in G_{f,g}$ and, by Theorem 6, $f_-^t \leqslant h^t \leqslant f_+^t$, what together with Theorem 5 and Lemma 6 implies that $h^t[\operatorname{cl} I_\alpha] \subset F^t[\operatorname{cl} I_\alpha]$. We know by Lemma 14 that $h^t[\operatorname{cl} I_\alpha] = \operatorname{cl} I_\beta$ for a $\beta \in A$. On the other hand, by Lemmas 7 and 6, we see that $F^t[\operatorname{cl} I_\alpha]$ is either one of intervals I_γ or a singleton. Hence $h^t[\operatorname{cl} I_\alpha] = F^t[\operatorname{cl} I_\alpha]$. This relation together with Theorem 11 and Lemma 14 implies that $t \in \operatorname{Sem}(f,g)$. □

The above properties let us to prove the main result of the paper: $\operatorname{Sem}(f,g)$ is the maximal support of refinement iteration semigroups as well as the maximal semigroup of indices of Abelian semigroups $\mathcal{A} \subset C^*(I,I)$ containing f and g.

Theorem 12. *There exist the disjoint refinement iteration semigroups $\{f^t\colon\ t \in T\}$ such that $f^1 = f$, $f^{s(f,g)} = g$ and $T = \operatorname{Sem}(f,g)$.*

Proof. In a view of Proposition 6 we may restrict our considerations to the case where $b \in L_{f,g}$. Firstly we shall show that there exists a homeomorphic solution γ of the system of equations

$$\begin{cases} \gamma(p^1(x)) = f(\gamma(x)) \\ \gamma(p^s(x)) = g(\gamma(x)) \end{cases}, \quad x \in I, \tag{17}$$

where p^1, p^s are determine in Lemma 13. In the decomposition (13) we can assume that $I_\alpha < I_\beta$ if and only if $\alpha < \beta$.

Define on the set of indices A the following equivalence relation $\alpha \sim \beta$ if and only if there exist $u, v \in V^+$ such that $F^u[I_\alpha] = F^v[I_\beta]$.

Note that if $\alpha \sim \beta$ then there exists a $t \in V^+$ such that either $F^t[\operatorname{cl} I_\alpha] = \operatorname{cl} I_\beta$ or $F^t[\operatorname{cl} I_\beta] = \operatorname{cl} I_\alpha$. Denote by S a selector of the relation $\sim$ and define $\bar{\alpha} := [\alpha] \cap S$, where $[\alpha]$ is the equivalent class of α. Now we can define the mapping $A \ni \alpha \to t_\alpha \in V^+$ by the following way.

(i) If $\alpha \leqslant \bar{\alpha}$ then there exists a unique $t_\alpha \in V^+$ such that

$$F^{t_\alpha}[\operatorname{cl} I_{\bar{\alpha}}] = \operatorname{cl} I_\alpha$$

and (ii) if $\alpha > \bar{\alpha}$ there exists a unique $t_\alpha \in V^+$ such that

$$F^{t_\alpha}[\operatorname{cl} I_\alpha] = \operatorname{cl} I_{\bar{\alpha}}.$$

For every $\alpha \in A$ define α' and α'' by

$$F^1[\operatorname{cl} I_\alpha] = \operatorname{cl} I_{\alpha'} \quad \text{and} \quad F^s[\operatorname{cl} I_\alpha] = \operatorname{cl} I_{\alpha''}.$$

Note that $\alpha' < \alpha$ and $\alpha'' < \alpha$. They have also the following properties:

(P1) $t_{\alpha'} = t_\alpha + 1$ for $\alpha \leqslant \bar{\alpha}$,
(P2) $t_{\alpha'} = t_\alpha - 1$ for $\alpha > \bar{\alpha}$ and $\alpha' > \bar{\alpha}$,
(P3) $t_{\alpha'} + t_\alpha = 1$ for $\alpha > \bar{\alpha}$ and $\alpha' \leqslant \bar{\alpha}$,
(P4) $t_{\alpha''} = t_\alpha + s$ for $\alpha \leqslant \bar{\alpha}$,
(P5) $t_{\alpha''} = t_\alpha - s$ for $\alpha > \bar{\alpha}$ and $\alpha'' > \bar{\alpha}$,
(P6) $t_{\alpha''} + t_\alpha = s$ for $\alpha > \bar{\alpha}$ and $\alpha'' \leqslant \bar{\alpha}$.

In fact, note that $\overline{\alpha''} = \overline{\alpha'} = \bar{\alpha}$ and $\alpha' < \alpha$, $\alpha'' < \alpha$.

Suppose $\alpha \leqslant \bar{\alpha}$, then $\alpha' < \bar{\alpha}$, $\alpha'' < \bar{\alpha}$ and by (i)

$$F^{t_\alpha+1}[\operatorname{cl} I_{\overline{\alpha'}}] = F^{t_\alpha+1}[\operatorname{cl} I_{\bar{\alpha}}] = F^1\bigl[F^{t_\alpha}[\operatorname{cl} I_{\bar{\alpha}}]\bigr] = F^1[\operatorname{cl} I_\alpha] = \operatorname{cl} I_{\alpha'} = F^{t_{\alpha'}}[\operatorname{cl} I_{\overline{\alpha'}}].$$

Hence $t_\alpha + 1 = t_{\alpha'}$, i.e. (P1) holds. Similarly

$$F^{t_\alpha+s}[\operatorname{cl} I_{\overline{\alpha''}}] = F^{t_\alpha+s}[\operatorname{cl} I_{\bar{\alpha}}] = F^s\bigl[F^{t_\alpha}[\operatorname{cl} I_{\bar{\alpha}}]\bigr] = F^s[\operatorname{cl} I_\alpha] = \operatorname{cl} I_{\alpha''} = F^{t_{\alpha''}}[\operatorname{cl} I_{\overline{\alpha''}}].$$

Thus $t_\alpha + s = t_{\alpha''}$, i.e. (P4) holds. Suppose $\bar{\alpha} < \alpha'$. Then $\alpha > \alpha'$ and

$$F^{t_{\alpha'}+1}[\operatorname{cl} I_\alpha] = F^{t_{\alpha'}}\bigl[F^1[\operatorname{cl} I_\alpha]\bigr] = F^{t_{\alpha'}}[\operatorname{cl} I_{\alpha'}] = \operatorname{cl} I_{\overline{\alpha'}} = \operatorname{cl} I_{\bar{\alpha}} = F^{t_\alpha}[\operatorname{cl} I_\alpha],$$

so $t_{\alpha'} + 1 = t_\alpha$, i.e. (P2) holds.

Now, let $\alpha' \leqslant \bar{\alpha} < \alpha$. Then by (ii) and (i)

$$F^{t_\alpha+t_{\alpha'}}[\operatorname{cl} I_\alpha] = F^{t_{\alpha'}}\bigl[F^{t_\alpha}[\operatorname{cl} I_\alpha]\bigr] = F^{t_{\alpha'}}[\operatorname{cl} I_{\bar{\alpha}}] = F^{t_{\alpha'}}[\operatorname{cl} I_{\overline{\alpha'}}] = \operatorname{cl} I_{\alpha'} = F^1[\operatorname{cl} I_\alpha],$$

so $t_\alpha + t_{\alpha'} = 1$ and (P3) is proved. The proof for the cases (P5) and (P6) is the same as for (P2) and (P3).

Now we deal with the functions h^t given by (16) and p^t defined in Lemma 13. By Lemma 15 they are defined simultaneously for $t \in V^+$. Put $h^{-t} := (h^t)^{-1}$ and $p^{-t} := (p^t)^{-1}$ for $t \in V^+$. It is easy to verify that the relations

$$h^1 \circ h^{t-1} = h^t, \qquad p^{-t} \circ p^1 = p^{-t+1}, \qquad h^1 \circ h^{-(t+1)} = h^{-t}, \quad t \in V^+ \tag{18}$$

and

$$h^s \circ h^{t-s} = h^t, \qquad p^{-t} \circ p^s = p^{-t+s}, \qquad h^s \circ h^{-(t+s)} = h^{-t}, \quad t \in V^+ \tag{19}$$

hold in this intervals I_α, where both sides of the equalities are correctly defined.

Let $\omega_\alpha : \operatorname{cl} I_\alpha \to \operatorname{cl} I_\alpha$ for $\alpha \in S$ be an increasing bijection. Define the homeomorphisms $\gamma_\alpha : \operatorname{cl} I_\alpha \to \operatorname{cl} I_\alpha$ by the following way

$$\gamma_\alpha = h^{t_\alpha} \circ \omega_{\bar{\alpha}} \circ p^{-t_\alpha}, \quad \text{if } \alpha \leqslant \bar{\alpha} \tag{20}$$

and

$$\gamma_\alpha = h^{-t_\alpha} \circ \omega_{\bar{\alpha}} \circ p^{t_\alpha}, \quad \text{if } \alpha > \bar{\alpha}. \tag{21}$$

Note that for $\alpha \leqslant \bar{\alpha}$, p^{-t_α} is defined on $\operatorname{cl} I_\alpha$, because $\operatorname{cl} I_\alpha = F^{t_\alpha}[\operatorname{cl} I_{\bar{\alpha}}] \subset F^{t_\alpha}[I] = p^{t_\alpha}[I]$. Moreover, $p^{-t_\alpha}[\operatorname{cl} I_\alpha] = \operatorname{cl} I_{\bar{\alpha}}$ and $h^{t_\alpha}[\operatorname{cl} I_{\bar{\alpha}}] = F^{t_\alpha}[\operatorname{cl} I_{\bar{\alpha}}] = \operatorname{cl} I_{t_\alpha}$, hence γ_α is an increasing bijection. Similarly is for $\alpha > \bar{\alpha}$, since $h^{t_\alpha}[\operatorname{cl} I_\alpha] = F^{t_\alpha}[\operatorname{cl} I_\alpha] = \operatorname{cl} I_{\bar{\alpha}}$ and h^{-t_α} maps $\operatorname{cl} I_{\bar{\alpha}}$ onto $\operatorname{cl} I_\alpha$.

Hence we get that the mapping

$$\gamma(x) := \begin{cases} \gamma_\alpha(x) & x \in \operatorname{cl} I_\alpha \\ x & x \in L^{**}_{f,g} \end{cases}$$

is a homeomorphism. We show that γ satisfies system (17).

Let $x \in \operatorname{cl} I_\alpha$. Then $p^1(x) \in \operatorname{cl} I_{\alpha'}$. If $\alpha \leqslant \bar{\alpha}$, then $\alpha' < \bar{\alpha}$ and by (P1), (18), (20)

$$\begin{aligned} \gamma \circ p^1(x) &= \gamma_{\alpha'} \circ p^1(x) = h^{t_{\alpha'}} \circ \omega_{\bar{\alpha}} \circ p^{-t_{\alpha'}} \circ p^1(x) = h^1 \circ h^{(t_{\alpha'}-1)} \circ \omega_{\bar{\alpha}} \circ p^{(1-t_{\alpha'})}(x) \\ &= h^1 \circ h^{t_\alpha} \circ \omega_{\bar{\alpha}} \circ p^{-t_\alpha}(x) = f \circ \gamma_\alpha(x), \end{aligned}$$

since $h^1 = f$. Let now $\alpha > \bar{\alpha}$ and $\alpha' > \bar{\alpha}$. Then by (21), (P2) and (18) we have

$$\begin{aligned} \gamma \circ p^1(x) &= \gamma_{\alpha'} \circ p^1(x) = h^{-t_{\alpha'}} \circ \omega_{\bar{\alpha}} \circ p^{t_{\alpha'}} \circ p^1(x) \\ &= h^1 \circ h^{-(t_{\alpha'}+1)} \circ \omega_{\bar{\alpha}} \circ p^{(1+t_{\alpha'})}(x) = h^1 \circ h^{-t_\alpha} \circ \omega_{\bar{\alpha}} \circ p^{t_\alpha}(x) = f \circ \gamma_\alpha(x). \end{aligned}$$

Finally assume $\alpha > \bar{\alpha}$ and $\alpha' \leqslant \bar{\alpha}$. Then by (20), (P3), (18) and (21)

$$\begin{aligned} \gamma \circ p^1(x) &= \gamma_{\alpha'} \circ p^1(x) = h^{t_{\alpha'}} \circ \omega_{\bar{\alpha}} \circ p^{-t_{\alpha'}} \circ p^1(x) \\ &= h^1 \circ h^{(t_{\alpha'}-1)} \circ \omega_{\bar{\alpha}} \circ p^{(1-t_{\alpha'})}(x) = h^1 \circ h^{-t_\alpha} \circ \omega_{\bar{\alpha}} \circ p^{t_\alpha}(x) = f \circ \gamma_\alpha(x). \end{aligned}$$

Similarly, using (19), we verify that $\gamma \circ p^s(x) = g \circ \gamma(x)$ for $x \in \operatorname{cl} I_\alpha$. Since $p^1(x) = f(x) \in L^{**}_{f,g}$ and $p^s(x) = g(x) \in L^{**}_{f,g}$ for $x \in L^{**}_{f,g}$ we infer that γ satisfies system (17). Let us note that the homeomorphic solution of (17) depends on an arbitrary function.

Now we can define our T-iteration semigroup. Put

$$f^t := \gamma \circ p^t \circ \gamma^{-1}, \quad t \in \operatorname{Sem}(f,g).$$

It is easy to see that $\{f^t\colon\ t \in \operatorname{Sem}(f,g)\}$ is a T-iteration semigroup such that $f^1 = f$ and $f^s = g$.

Let φ be a continuous solution of (1) and put $\psi := \varphi \circ \gamma^{-1}$. Since $\{p^t\colon\ t \in \operatorname{Sem}(f,g)\}$ satisfies (15) we have $\psi \circ f^t = (\varphi \circ \gamma^{-1}) \circ (\gamma \circ p^t \circ \gamma^{-1}) = \varphi \circ p^t \circ \gamma^{-1} = \varphi \circ \gamma^{-1} + t = \psi + t$, for $t \in \operatorname{Sem}(f,g)$. Putting $t = 1$ and $t = s(f,g)$ we see that ψ is a continuous solution of (1). Hence, by the uniqueness, we get $\psi = \varphi + c$ for a $c \in \mathbb{R}$, so $\varphi(\gamma^{-1}(x)) = \varphi(x) + c$. Since $\gamma^{-1}(x) = x$ for $x \in L^{**}_{f,g}$ we get $c = 0$. Thus $\varphi = \psi$ which gives that $\varphi \circ f^t = \varphi + t$. Hence $\operatorname{ind} f^t = t$ and $T = \operatorname{Sem}(f,g)$. □

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