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Nontrivial periodic motions for resonant type asymptotically linear lattice dynamical systems ☆

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A B S T R A C T

In this paper, we consider the following one dimensional lattices consisting of infinitely many particles with nearest neighbor interaction

$$\ddot{q}_i(t) = \Phi'_{i-1}\big(t, q_{i-1}(t) - q_i(t)\big) - \Phi'_i\big(t, q_i(t) - q_{i+1}(t)\big), \quad i \in \mathbb{Z},$$

where $\Phi_i(t,x) = -(\alpha_i/2)|x|^2 + V_i(t,x)$ is T-periodic in t for $T > 0$ and satisfies $\Phi_{i+N} = \Phi_i$ for some $N \in \mathbb{N}$, $q_i(t)$ is the state of the i-th particle. Assume that $\alpha_i = 0$ for some $i \in \mathbb{Z}$ and $V'_i(t,x)$ denoting the derivative of V_i respect to x is asymptotically linear with x both at origin and at infinity. We would like to point out that this system is resonant both at origin and at infinity and not studied up to now. Based on some new results concerning the precise computations of the critical groups, for a given $m \in \mathbb{Z}$, we obtain the existence of nontrivial periodic solutions satisfying $q_{i+mN}(t+T) = q_i(t)$ for all $t \in \mathbb{R}$ and $i \in \mathbb{Z}$ under some additional conditions.

1. Introduction and main results

In this paper, we consider one dimensional lattices consisting of infinitely many particles with nearest neighbor interaction. We represent the state of the non-autonomous dynamical system at time t by a sequence of functions $q(t) = \{q_i(t)\}$, $i \in \mathbb{Z}$, where $q_i(t)$ is the state of the i-th particle. Let $\Phi_i(t,\cdot)$ denote the potential of the interaction between the i-th and the $(i+1)$-th particle (whose displacement is $q_i(t) - q_{i+1}(t)$), then the equation governing the state of $q_i(t)$ reads

$$\ddot{q}_i(t) = \Phi'_{i-1}\big(t, q_{i-1}(t) - q_i(t)\big) - \Phi'_i\big(t, q_i(t) - q_{i+1}(t)\big), \quad i \in \mathbb{Z}, \tag{1.1}$$

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where $t \in \mathbb{R}$. Here and in the sequel $\Phi_i'(t,x)$ denotes the derivative of Φ_i respect to x. We define the potential $\Phi : \mathbb{R} \times \mathbb{R}^\infty \to \mathbb{R}$ by

$$\Phi\big(t, q(t)\big) = \sum_{i=-\infty}^{+\infty} \Phi_i\big(t, q_i(t) - q_{i+1}(t)\big), \quad t \in \mathbb{R}.$$

Then infinitely many equations (1.1) can be written in a vectorial form

$$\ddot{q}(t) = -\Phi'\big(t, q(t)\big), \quad t \in \mathbb{R}. \tag{1.2}$$

We first recall some historical comments on related work. After the pioneering numerical experiment of Fermi, Pasta and Ulam [9] on finite lattices, an autonomous dynamical system with finitely or infinitely many degrees of freedom whose dynamics is described by the equations

$$\ddot{q}_i = \Phi_{i-1}'(q_{i-1} - q_i) - \Phi_i'(q_i - q_{i+1}), \quad i \in \mathbb{Z}, \tag{1.3}$$

has been widely studied under different kinds of potentials [19]. Let us now briefly recall the main results obtained before. Due to the implementation of variational methods, a number of rigorous results was obtained in the case of the general equations (1.3) in the 1990s. In [10,16,18], travelling waves, i.e. solutions of the form $q_i(t) = u(i - ct)$ were studied by using a constrained minimization approach, Nehari manifold approach and mountain pass theorem, respectively. We would like to point out the paper [17] considering the existence of infinitely many travelling waves of multibump type for the non-autonomous case.

The existence and multiplicity of periodic motions for system (1.3) have been studied, restricting the system to periodic potentials, that is $\Phi_i = \Phi_{i+N}$ for some integer N. In [2] a nontrivial solution is obtained as a mountain pass point for the corresponding Lagrangian functional, under the assumption that $\Phi_i(x) := -\alpha_i x^2 + V_i(x)$, satisfying $\alpha_i > 0$ and $V_i(x) \geqslant 0$ for all $i \in \mathbb{Z}$, is quadratically repulsive for small displacements and superquadratically attractive for large displacements; note that $\alpha_i > 0$ implies the minimum of the spectrum of the quadratic part of the Lagrangian functional is strictly positive. In [1], Arioli and Gazzola extended the result to the purely attractive potentials ($\alpha_i = 0$ for all $i \in \mathbb{Z}$) which are strictly superquadratic at both the origin and the infinity; in this case the minimum of the spectrum is 0. In [4] the existence of infinitely many periodic non-constant solutions of multibump type has been proved, with the same assumptions taken in [2]. In [3], a nonzero periodic solutions of finite energy has been obtain with to the potentials quadratically attractive and the coefficients α_i take both signs.

We assume that the potentials Φ_i are given by

$$\Phi_i(t,x) = -\frac{\alpha_i}{2}|x|^2 + V_i(t,x),$$

where $V_i(t,x)$ is C^2 in x and T-periodic in t for some $T > 0$. Similar to [3] (also [1]), if $\alpha_i = 0$ for some i, it is easy to check that 0 lies in the spectrum of the quadratic part of the Lagrangian functional corresponding to system (1.2) (see Remark 2.1 below for more details). Thus a nature question is whether system (1.2) has nonzero periodic solutions when $V_i'(t,x)$ is asymptotically linear respect to x both at origin and at infinity; in other words, this problem is resonant both at origin and at infinity. To the best of our knowledge, resonant type asymptotically linear lattice dynamical systems have not been studied up to now. Furthermore $\alpha_i = 0$ for some i is "necessary" if one wants to study resonant type problem (1.2) in some sense, since the properties of the spectrum except 0 are not very clear. In this paper, we give a positive answer to this question. It is worth mentioning that the autonomous case can be treated similarly.

Before we state our main results, we give some assumptions on the potentials Φ_i.

(Φ) There exists $N \in \mathbb{N}$ such that $\Phi_{i+N} = \Phi_i$;

(Φ^0) There exists constant $\beta \in [1, +\infty)$ such that $\lim_{|x|\to 0} \frac{|V_i'(t,x)|}{|x|^\beta} = 0$ uniformly for $t \in [0,T]$;

($\Phi_{\pm}^{0}$) For every i, there exists function $h_i^0 \in C([0,T],\mathbb{R})$ satisfying $\int_0^T h_i^0(t)\,dt < 0$, such that $\limsup_{|x|\to 0} \pm\frac{V_i'(t,x)x}{|x|^{2\beta}} \leqslant h_i^0(t) \leqslant 0$ uniformly for $t \in [0,T]$;

(Φ^{∞}) There exists constant $\alpha \in (0,1]$ such that $\lim_{|x|\to\infty} \frac{|V_i'(t,x)|}{|x|^{\alpha}} = 0$ uniformly for $t \in [0,T]$;

($\Phi_{\pm}^{\infty}$) For every i, there exists function $h_i^\infty \in C([0,T],\mathbb{R})$ satisfying $\int_0^T h_i^\infty(t)\,dt < 0$, such that $\limsup_{|x|\to\infty} \pm\frac{V_i'(t,x)x}{|x|^{2\alpha}} \leqslant h_i^\infty(t) \leqslant 0$ uniformly for $t \in [0,T]$.

We shall prove the following theorem.

Theorem 1.1. *Assume that* (Φ), (Φ^0), (Φ^∞) *and* $\alpha_i = 0$ *for some* i. *If one of the following conditions holds:*

(a) (Φ_{+}^{0}) *and* (Φ_{-}^{∞});
(b) (Φ_{-}^{0}) *and* (Φ_{+}^{∞}),

then for any positive integer m, *system* (1.2) *admits a nontrivial* T*-periodic solution* q^* *satisfying*

$$q_{i+mN}^*(t+T) = q_i^*(t) \quad \text{for all } t \in \mathbb{R} \text{ and } i \in \mathbb{Z}.$$

Moreover, if the nullity of $q^*|_{[-n,n-1]}$ *(in the sense,* $q^*|_{[-n,n-1]} \in H$, *see Section* 2 *for* H*) is* ν^* *with* $\nu^* \leqslant \nu$ *(see Section* 2 *for the definition of* ν*), where* $2n = mN$, *then system* (1.2) *has another nontrivial* T*-periodic solution* $q \neq q^*$.

Remark 1.1. Here we give an example to illustrate our main result. Let $\rho(t) \in C^1(\mathbb{R},\mathbb{R})$ satisfy $\rho(t) \geqslant 0$, $\rho(t) \not\equiv 0$ and $\rho(t) = \rho(t+T)$ for any $t \in \mathbb{R}$. Consider

$$V_i'(t,x) = \begin{cases} x^{2\beta-1}\rho(t), & \text{for } |x| < 1, \\ -x^{2\alpha-1}\rho(t)(2+\sin\frac{2\pi t}{T}), & \text{for } |x| > M \gg 2, \\ \text{smooth}, & \text{for } 1 \leqslant |x| \leqslant M, \end{cases}$$

where $\alpha \in (0,1)$ and $\beta > 1$. Then

$$\lim_{|x|\to 0} \frac{|V_i'(t,x)|}{|x|^{\beta}} = 0 = \lim_{|x|\to\infty} \frac{|V_i'(t,x)|}{|x|^{\alpha}}, \quad \text{uniformly for } t \in [0,T],$$

and

$$\lim_{|x|\to 0} \frac{V_i'(t,x)x}{|x|^{2\beta}} = \rho(t), \qquad \limsup_{|x|\to\infty} \frac{V_i'(t,x)x}{|x|^{2\alpha}} = -\rho(t), \quad \text{uniformly for } t \in [0,T].$$

Hence the corresponding system has at least one nontrivial solution by Theorem 1.1 (b) with $h_i^0(t) = h_i^\infty(t) = \rho(t)$. Case (a) can be treated similarly.

Corollary 1. *The conclusion in Theorem 1.1 holds true if* (Φ^0), (Φ^∞), ($\Phi_{\pm}^{0}$) *and* ($\Phi_{\pm}^{\infty}$) *are replaced respectively by the following corresponding conditions:*

($A\Phi^0$) *There exist* $\beta \in (1,+\infty)$ *and* $M_\beta > 0$ *such that* $\lim_{|x|\to 0} \frac{|V_i'(t,x)|}{|x|^{\beta}} \leqslant M_\beta$ *uniformly for* $t \in [0,T]$;

($C\Phi_{\pm}^{0}$) *For every* i, *there exist* $\xi \in [1+\beta, 2\beta)$ *and function* $h_i^0 \in C([0,T],\mathbb{R})$ *satisfying* $\int_0^T h_i^0(t)\,dt < 0$, *such that* $\limsup_{|x|\to 0} \pm\frac{V_i'(t,x)x}{|x|^{\xi}} \leqslant h_i^0(t) \leqslant 0$ *uniformly for* $t \in [0,T]$;

($A\Phi^\infty$) *There exist* $\alpha \in (0,1)$ *and* $M_\alpha > 0$ *such that* $\lim_{|x|\to\infty} \frac{|V_i'(t,x)|}{|x|^{\alpha}} \leqslant M_\alpha$ *uniformly for* $t \in [0,T]$;

($C\Phi_{\pm}^{\infty}$) *For every i, there exist $\eta \in (2\alpha, 1+\alpha]$ and function $h_i^{\infty} \in C([0,T],\mathbb{R})$ satisfying $\int_0^T h_i^{\infty}(t)\,dt < 0$, such that $\limsup_{|x|\to\infty} \pm\frac{V_i'(t,x)x}{|x|^{\eta}} \leqslant h_i^{\infty}(t) \leqslant 0$ uniformly for $t \in [0,T]$.*

Remark 1.2. We point out that the above corollary is a consequence of Theorem 1.1. Indeed, we set $\alpha' = \eta/2$ and $\beta' = \xi/2$, then $\alpha' > \alpha$, $\beta' < \beta$ and it is easy to check that all the conditions of Theorem 1.1 hold true with α and β replaced by α' and β', respectively.

Moreover, we can obtain the following result.

Theorem 1.2. *The conclusion of Theorem 1.1 remains valid if the assumptions (Φ^0), (Φ^{∞}), ($\Phi_{\pm}^0$) and ($\Phi_{\mp}^{\infty}$) are replaced by ($A\Phi^0$), ($A\Phi^{\infty}$) and the following corresponding conditions, respectively.*

($A\Phi_{\pm}^0$) *For every i, $\limsup_{|x|\to 0} \pm\frac{V_i'(t,x)x}{|x|^{2\beta}} \leqslant M_{\beta}$ uniformly for $t \in [0,T]$, and $\lim_{|x|\to 0} \pm\frac{V_i'(t,x)x}{|x|^{2\beta}} = -\infty$ on some positive measure subset $E_{\beta}^{\pm} \subset [0,T]$;*

($A\Phi_{\pm}^{\infty}$) *For every i, $\limsup_{|x|\to\infty} \pm\frac{V_i'(t,x)x}{|x|^{2\alpha}} \leqslant M_{\alpha}$ uniformly for $t \in [0,T]$, and $\lim_{|x|\to\infty} \pm\frac{V_i'(t,x)x}{|x|^{2\alpha}} = -\infty$ on some positive measure subset $E_{\alpha}^{\pm} \subset [0,T]$.*

This paper is organized as follows. In Section 2, we study the variational structure of system (1.2) and recall some abstract results concerning the precise computations of the critical groups. In Section 3, we compute the critical groups of the functional corresponding to (1.2) at origin and at infinity. We prove our main results in Section 4.

2. Variational setting and preliminary lemmas

We look for solution q of (1.2) satisfying $q_{i+mN}(t+T) = q_i(t)$ for all $t \in \mathbb{R}$ and $i \in \mathbb{Z}$. To this end, we reduce the non-autonomous dynamical system with infinitely many degrees of freedom to the finite system, whose motion is described by

$$\ddot{q}_i(t) = \Phi_{i-1}'\big(t, q_{i-1}(t) - q_i(t)\big) - \Phi_i'\big(t, q_i(t) - q_{i+1}(t)\big), \quad t \in [0,T], \tag{2.1}$$

where $i \in \{-n, \dots, n-1\}$. Here, without loss of generality, we set $2n := mN$. Obviously, any solution of the finite system is periodic, that is $q_{-n} = q_n$. We can also define the potential $\Psi : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$ by

$$\Psi\big(t, q(t)\big) = \sum_{i=-n}^{n-1} \Phi_i\big(t, q_i(t) - q_{i+1}(t)\big), \quad t \in [0,T].$$

Then finitely many equations (2.1) can be written in a vectorial form

$$\ddot{q}(t) = -\Psi'\big(t, q(t)\big), \quad t \in [0,T]. \tag{2.2}$$

In order to obtain the periodic motions of the finite system, we consider the following space

$$H = \left\{ q \in H^1\big(S^1, \mathbb{R}\big)^{2n} \colon \int_0^T q_0(t)\,dt = 0 \right\},$$

which is a Hilbert space when endowed with the scalar product

$$(q,p) := \sum_{i=-n}^{n-1} \int_0^T \left[\dot{q}_i(t)\dot{p}_i(t) + \big(q_i(t) - q_{i+1}(t)\big)\big(p_i(t) - p_{i+1}(t)\big)\right] dt. \tag{2.3}$$

Note that $q_n = q_{-n}$. We need to set $\int_0^T q_0(t)\,dt = 0$ in the definition of H in order to have (2.3) defining a scalar product; this causes no loss of generality as the following observation: if $q = \{q_i\}$ is a solution of the system (2.2), then so is $\hat{q} = \{q_i + \sigma\}$ for any constant σ. Here $H^1(S^1, \mathbb{R})$ is the usual Hilbert space equipped with the norm $\|q_i\|_{H^1} = (\int_0^T |\dot{q}_i(t)|^2 + |q_i(t)|^2)^{1/2}$. We denote by $\|\cdot\|$ the norm induced by (2.3) and $L_n^p := L^p(S^1, \mathbb{R})^{2n}$ with $1 \leqslant p \leqslant +\infty$. It is easy to see that H is compactly embedded in L_n^p with $1 \leqslant p \leqslant +\infty$. We define a self-adjoint linear operator $L : H \to H$ by

$$(Lq,p) = \sum_{i=-n}^{n-1} \int_0^T \left[\dot{q}_i(t)\dot{p}_i(t) + \alpha_i\big(q_i(t) - q_{i+1}(t)\big)\big(p_i(t) - p_{i+1}(t)\big)\right] dt$$

and a functional $\psi : H \to \mathbb{R}$ by

$$\psi(q) = \sum_{i=-n}^{n-1} \int_0^T V_i\big(t, q_i(t) - q_{i+1}(t)\big)\,dt.$$

It is easy to check that the functional $\varphi : H \to \mathbb{R}$ defined by

$$\varphi(q) = \frac{1}{2}(Lq,q) - \psi(q) \tag{2.4}$$

is C^2 on H. By a standard procedure one shows that critical points of φ are periodic motions of the periodic lattice.

Before we state our main result, we firstly study the spectrum of the linear operator L (we denote it by $\sigma(L)$). We split the space H into $H = H^- \oplus H^0 \oplus H^+$, where H^-, H^0 and H^+ denote the negative, the kernel and the positive subspaces of L, respectively. Note that the negative subspace H^- is empty if $\alpha_i \geqslant 0$ for all i. If there exists $\alpha_j < 0$ for some integer j, let $\varrho := -\min_{i \in \mathbb{Z}}\{\alpha_i\} > 0$; otherwise we set $\varrho := 0$. The following lemma, which is a slight modification of Lemma 6.1 in [3], describes some properties of the spectrum.

Lemma 2.1. *If $0 < T < \pi/\sqrt{\varrho}$, then there exists $\lambda > 0$, such that $(Lq,q) \leqslant -\lambda\|q\|^2$ for all $q \in H^-$ and $(Lq,q) \geqslant \lambda\|q\|^2$ for all $q \in H^+$. Furthermore $\dim H^0 = \sharp\{\alpha_i\colon\ \alpha_i = 0\} - 1$ and $\dim H^- = \sharp\{\alpha_i\colon\ \alpha_i < 0\}$.*

Remark 2.1. The kernel H^0 is never empty if we assume that $\alpha_i = 0$ for some integer i. In fact, without loss of generality, we assume $\alpha_i = 0$ for some integer $i \geqslant 0$, and we define the nonzero vector $q \in H$ by $q_k = 0$ if $k \leqslant i$; and $q_k = 2$ if $k > i$. Then it is easy to check that $Lq = 0$ which implies $q \in H^0$ and $0 \in \sigma(L)$. In the following, we set $\nu = \dim H^0$ and $\mu = \dim H^-$, then by Lemma 2.1, ν is positive and finite; $\mu \geqslant 0$ is finite. As a T/l ($l \in \mathbb{N}$) periodic solution is T periodic as well, we only consider the existence of T-periodic solution of system (2.2) for $T < \pi/\sqrt{\varrho}$.

Now we recall some new results concerning the precise computations of the critical groups [12]. Let H be a real Hilbert space and $I \in C^1(H, \mathbb{R})$. Denote $K = \{u \in H \mid I'(u) = 0\}$ and $I^c := \{u \in H \mid I(u) \leqslant c\}$. Suppose u_0 is an isolated critical point of I with $I(u_0) = c$, then the critical groups of I at u_0 are defined as $C_k(I, u_0) := H_k(I^c, I^c \setminus \{u_0\}, G)$, $k \in \mathbb{Z}$, where $H_*(-, G)$ denotes the singular homology with coefficients in a field G. Moreover, if u_0 is nondegenerate with Morse index μ_0, then from the Morse lemma (see [8,15]), $C_k(I,u_0) \cong \delta_{k,\mu_0} G = G$ if $k = \mu_0$, or $= 0$, if $k \neq \mu_0$. To compute the critical groups at a degenerate critical point, we have the following result (see Gromoll and Meyer [11]).

Proposition 2.1. *Suppose*

(A_0) *I has an isolated critical point u_0 and is of class C^2 near u_0. In addition, 0 is isolated in the spectrum of $A_0 := I''(u_0)$ and $\nu_0 := \dim \ker A_0 < \infty$,*

holds, then $C_k(I,u_0) \cong 0$ for $k \notin [\mu_0, \mu_0 + \nu_0]$, where μ_0 is the Morse index of u_0. If $\nu_0 = 0$ and $\mu_0 < \infty$, then $C_k(I,u_0) \cong \delta_{k,\mu_0} G$, $k \in \mathbb{Z}$.

Assume (A_0) holds. Set $V_0 = \ker A_0$ and $W_0 = (V_0)^\perp$ and split $W_0 = W_0^+ \oplus W_0^-$ such that these subspaces are invariant under A_0, $A_0|_{W_0^+}$ is positive definite and $A_0|_{W_0^-}$ is negative definite. The following proposition plays an important role in the computations of $C_k(\varphi, 0)$ (see (2.4) for φ).

Proposition 2.2. *(See [12, Theorem 2.1].) Let I satisfy (A_0) and $V_0 = V_{10} \oplus V_{20}$. If for some $\beta \geqslant 1$, $\|I'(u_0+u) - A_0 u\| = o(\|u\|^\beta)$ as $\|u\| \to 0$, and there exist $\eta > 0$, $\varpi < 1$ and $\theta \in (0,1)$ such that $\langle I'(u_0+u), v_1 - v_2\rangle + \varpi \|I'(u_0+u)\| \cdot \|v\| \geqslant 0$, for any $u = v + w \in H = V_{10} \oplus V_{20} \oplus W_0^+ \oplus W_0^-$, where $v = v_1 + v_2$, $v_1 \in V_{10}$, $v_2 \in V_{20}$, $w \in W_0^+ \oplus W_0^-$, $\|u\| \leqslant \eta$ and $\|w\| \leqslant \theta \|u\|^\beta$, then $C_k(I,u_0) \cong \delta_{k,\mu_0 + \dim V_{20}} G$ for $k \in \mathbb{Z}$, where $\mu_0 = \dim W_0^-$.*

Recall that I satisfies $(C)_c$ condition if any sequence $\{u_j\} \subset H$ with $I(u_j) \to c$ and $(1+\|u_j\|)\|I'(u_j)\| \to 0$ as $j \to \infty$ have a convergent subsequence [7]. Suppose that $I(K)$ is bounded from below by $a \in \mathbb{R}$ and I satisfies the compactness condition $(C)_c$ for all $c \leqslant a$, then denote $C_k(I,\infty) := H_k(H, I^a)$, $k \in \mathbb{Z}$ the k-th critical groups of I at infinity [6]. To state the abstract result about the computations of the critical groups at infinity, we need the following assumption.

(A_∞) *$I(u) = \frac{1}{2}\langle A_\infty u, u\rangle + J(u)$, where $A_\infty : H \to H$ is a self-adjoint linear operator such that 0 is isolated in the spectrum of A_∞. The map $J \in C^1(H,\mathbb{R})$ is of class C^2 in a neighborhood of infinity and satisfies $\|J'(u)\|/\|u\| \to 0$ as $\|u\| \to \infty$. Moreover, J and J' map bounded sets to bounded sets. Finally, the critical values of I are bounded from below and I satisfies $(C)_c$ for $c \ll 0$.*

Denote $V := \ker A_\infty$ and $W := V^\perp$ with $W = W^+ \oplus W^-$ where $W^\pm$ is invariant under A_∞ and $A_\infty|_{W^+}$ is positive definite, $A_\infty|_{W^-}$ is negative definite. Let $\mu_\infty := \dim W^-$ be the Morse index of I at infinity and $\nu_\infty := \dim V$ the nullity of I at infinity. The following proposition will play an important part in the computations of $C_k(\varphi, \infty)$.

Proposition 2.3. *(See [12, Theorem 2.4].) Suppose that (A_∞) holds. If $\|J'(u)\| = o(\|u\|^\alpha)$ as $\|u\| \to \infty$, for some $\alpha \in (0,1]$, then:*

(a) *$C_k(I,\infty) \cong \delta_{k,\mu_\infty} G$, $k \in \mathbb{Z}$, if there exist $R > 0$ and $\theta \in (0,1)$ such that $\langle I'(u), v\rangle \geqslant 0$ for any $u = v + w \in C_\infty(R,\theta,\alpha)$ with $v \in V$, $w \in W$, and $\langle u, J'(u)\rangle \geqslant 0$ for any $u \in C_\infty(R,\theta,\frac{\alpha+1}{2}) \backslash C_\infty(R,\theta,\alpha)$, where for $R > 0, \vartheta > 0$ and $\rho > 0$, $C_\infty(R,\vartheta,\rho) := \{u \in H \mid u = v + w \in V \oplus W,\ v \in V,\ w \in W,\ \|u\| \geqslant R, \|w\| \leqslant \vartheta \|u\|^\rho\}$.*
(b) *$C_k(I,\infty) \cong \delta_{k,\mu_\infty + \nu_\infty} G$, $k \in \mathbb{Z}$, if there exist $R > 0$ and $\theta \in (0,1)$ such that $\langle I'(u), v\rangle \leqslant 0$ for any $u = v + w \in C_\infty(R,\theta,\alpha)$ with $v \in V$, $w \in W$, and $\langle u, J'(u)\rangle \leqslant 0$ for any $u \in C_\infty(R,\theta,\frac{\alpha+1}{2}) \backslash C_\infty(R,\theta,\alpha)$.*

Suppose that K is finite and all critical points have finite Morse index and finite nullity. Let $\beta_k(I,u) := \dim C_k(I,u)$ for $u \in K \cup \{\infty\}$ be the Betti numbers of I at $u \in K$ respectively at infinity. If I satisfies $(C)_c$ condition for all $c \in \mathbb{R}$, then we have the following Morse inequalities (see [6] or [15]).

Proposition 2.4. *Let $P(I,u) := \sum_{k=0}^{\infty} \beta_k(I,u)t^k$ be the Morse polynomial for I at $u \in K \cup \{\infty\}$. Then there exists a polynomial $Q(t)$ with nonnegative integer coefficients such that*

$$P(I,\infty) + (1+t)Q(t) = \sum_{u \in K} P(I,u).$$

3. The computations of the critical groups

In this section, we will prove that the functional φ satisfies $(C)_c$ condition for every $c \in \mathbb{R}$ and compute precisely the critical groups of φ at origin and at infinity. Throughout this section, we denote $d_i := q_i - q_{i+1}$ for $q \in H$. In the sequel, for simplicity, "if $(\Phi^i_\pm)$ then $\pm A$" means (Φ^i_+) ((Φ^i_-) resp.) implies $+A$ ($-A$ resp.).

Lemma 3.1. *Assume (Φ) and $(\Phi^0_\pm)$. Let $H_0 := \{q \in H\colon\ \|q^+ + q^-\| = o(\|q\|)$ as $\|q\| \to 0\}$, then*

$$\limsup_{q \in H_0,\, \|q\| \to 0} \pm \sum_{i=-n}^{n-1} \int_0^T \frac{V_i'(t, d_i(t)) d_i(t)}{\|q\|^{2\beta}}\, dt < 0.$$

Proof. The proof is standard (see [12,13]). For the convenience of the reader, we sketch the proof here briefly. We only consider the case that (Φ^0_+) holds. The other case can be treated similarly. Note that

$$\sum_{i=-n}^{n-1} \|d_i\|_2^2 \leqslant \|q\|^2, \quad \text{and} \quad \sup_{-n \leqslant i \leqslant n-1} \|d_i\|_\infty \leqslant \|q\|. \tag{3.1}$$

For any small $\varepsilon > 0$, by using an argument as used in the proof of Lemma 3.2 in [5] and (Φ), it is not hard to show that there exist small $\gamma(\varepsilon) \in (0,1)$ and large $\Gamma(\varepsilon) > 1$, such that for all $q^0 \in H^0 \setminus \{0\}$,

$$\sup_{-n \leqslant i \leqslant n-1} \operatorname{meas}\bigl\{t \in [0,T] \bigm| \bigl|d_i^0(t)\bigr| < \gamma(\varepsilon)\bigl\|q^0\bigr\|\bigr\} < \varepsilon,$$

and for all $q^+ + q^- \in H^+ + H^-$,

$$\sup_{-n \leqslant i \leqslant n-1} \operatorname{meas}\bigl\{t \in [0,T] \bigm| \bigl|d_i^+(t) + d_i^-(t)\bigr| > \Gamma(\varepsilon)\bigl\|q^+ + q^-\bigr\|\bigr\} < \gamma^{2\beta}(\varepsilon)\varepsilon \leqslant \varepsilon.$$

Set $E_1^i(q,\varepsilon) = \{t \in [0,T] \mid |d_i^0(t)| \geqslant \gamma(\varepsilon)\|q^0\|\}$, and $E_2^i(q,\varepsilon) = \{t \in [0,T] \mid |d_i^+(t) + d_i^-(t)| \leqslant \Gamma(\varepsilon)\|q^+ + q^-\|\}$. Obviously,

$$\operatorname{meas}\bigl([0,T] \setminus E_1^i(q,\varepsilon)\bigr) < \varepsilon, \quad \text{and} \quad \operatorname{meas}\bigl([0,T] \setminus E_2^i(q,\varepsilon)\bigr) < \gamma^{2\beta}(\varepsilon)\varepsilon \leqslant \varepsilon, \tag{3.2}$$

uniformly for $i \in \{-n, \ldots, n-1\}$. Then, for all $q \in H$, one has

$$\operatorname{meas}\bigl([0,T]\bigr) \geqslant \operatorname{meas}\bigl(E_1^i(q,\varepsilon)\bigr) - \operatorname{meas}\bigl([0,T] \setminus E_2^i(q,\varepsilon)\bigr) \geqslant T - 2\varepsilon.$$

For any $q \in H_0$ and any $t \in E_1^i(q,\varepsilon) \cap E_2^i(q,\varepsilon)$, we have

$$\frac{|d_i(t)|}{\|q\|} \geqslant \frac{|d_i^0(t)|}{\|q\|_n} - \frac{|d_i^+(t) + d_i^-(t)|}{\|q\|} \geqslant \gamma(\varepsilon)\frac{\|q^0\|}{\|q\|} - \Gamma(\varepsilon)\frac{\|q^+ + q^-\|}{\|q\|} \to \gamma(\varepsilon), \tag{3.3}$$

as $\|q\| \to 0$. For any $q \in H_0$ and $t \in E_2^i(q,\varepsilon) \setminus E_1^i(q,\varepsilon)$, we have

$$\frac{|d_i(t)|}{\|q\|} \leqslant \gamma(\varepsilon)\frac{\|q^0\|}{\|q\|} + \Gamma(\varepsilon)\frac{\|q^+ + q^-\|}{\|q\|} \to \gamma(\varepsilon), \quad \text{as } \|q\| \to 0. \tag{3.4}$$

By virtue of (Φ) and (Φ_+^0), there exists small $\eta_1 > 0$ such that for $i \in \{-n,\dots,n-1\}$

$$\frac{V_i'(t,x)x}{|x|^{2\beta}} \leqslant h_i^0(t) + \gamma^{2\beta}(\varepsilon)\varepsilon, \quad \text{uniformly for } t \in [0,T] \text{ and } |x| \leqslant \eta_1. \tag{3.5}$$

It follows from (3.1) and (3.3) that for $q \in H_n$ with $\|q\|_n$ small enough, $E_1^i(q,\varepsilon) \cap E_2^i(q,\varepsilon) \subset E_3^i(q) := \{t \in [0,T] \mid |d_i(t)| \leqslant \eta_1\}$, and $|d_i(t)|/\|q\| \geqslant \gamma(\varepsilon)/2$ for all $t \in E_1(q,\varepsilon) \cap E_2^i(q,\varepsilon)$ and all i. Hence, it follows from (3.1), (3.5) and the fact that $h_i^0(t) \leqslant 0$ that

$$\sum_{i=-n}^{n-1} \int\limits_{E_1^i(q,\varepsilon)\cap E_2^i(q,\varepsilon)} \frac{V_i'(t,d_i(t))d_i(t)}{\|q\|^{2\beta}}\,dt \leqslant c(n)\gamma^{2\beta}(\varepsilon)\varepsilon + \frac{\gamma^{2\beta}(\varepsilon)}{4^\beta}\sum_{i=-n}^{n-1} \int\limits_{E_1^i(q,\varepsilon)\cap E_2^i(q,\varepsilon)} h_i^0(t)\,dt, \tag{3.6}$$

here and in what follows, $c(n)$ denotes a positive constant depending on n. Again, by (Φ) and (Φ_+^0), we can find a large number $\eta' > 0$ such that for every $i \in \{-n,\dots,n-1\}$

$$V_i'(t,x)x \leqslant |x|^{2\beta}, \quad \text{uniformly for } t \in [0,T] \text{ and } |x| \leqslant \eta'. \tag{3.7}$$

Set $E_4^i(q) = \{t \in [0,T] \mid |d_i(t)| \leqslant \eta'\}$, then for $q \in H_0$ with $\|q\|$ small enough, $E_4^i(q) = [0,T]$ for all i and hence, by (3.2), (3.4) and (3.7), it follows that

$$\sum_{i=-n}^{n-1} \int\limits_{E_2^i(q,\varepsilon)\setminus E_1^i(q,\varepsilon)} \frac{V_i'(t,d_i(t))d_i(t)}{\|q\|^{2\beta}}\,dt \leqslant \sum_{i=-n}^{n-1} \int\limits_{E_2^i(q,\varepsilon)\setminus E_1^i(q,\varepsilon)} \frac{|d_i(t)|^{2\beta}}{\|q\|^{2\beta}}\,dt \leqslant c\gamma^{2\beta}(\varepsilon)\varepsilon, \tag{3.8}$$

and

$$\sum_{i=-n}^{n-1} \int\limits_{[0,T]\setminus E_2^i(q,\varepsilon)} \frac{V_i'(t,d_i(t))d_i(t)}{\|q\|^{2\beta}}\,dt \leqslant \sum_{i=-n}^{n-1} \int\limits_{[0,T]\setminus E_2^i(q,\varepsilon)} \frac{|d_i(t)|^{2\beta}}{\|q\|^{2\beta}}\,dt \leqslant c(n)\gamma^{2\beta}(\varepsilon)\varepsilon. \tag{3.9}$$

Therefore, for any $q \in H_0$ with $\|q\|$ sufficiently small, it follows from (3.6), (3.8) and (3.9) that

$$\sum_{i=-n}^{n-1} \int_0^T \frac{V_i'(t,d_i(t))d_i(t)}{\|q\|^{2\beta}}\,dt \leqslant c(n)\gamma^{2\beta}(\varepsilon)\varepsilon + \frac{\gamma^{2\beta}(\varepsilon)}{4^\beta}\sum_{i=-n}^{n-1} \int\limits_{E_1^i(q,\varepsilon)\cap E_2^i(q,\varepsilon)} h_i^0(t)\,dt,$$

which together with the fact that $\lim_{\varepsilon\to 0}\int_{E_1^i(q,\varepsilon)\cap E_2^i(q,\varepsilon)} h_i^0(t)\,dt = \int_0^T h_i^0(t)\,dt < 0$ yields

$$\limsup_{q\in H_0,\, \|q\|\to 0} \sum_{i=-n}^{n-1} \int_0^T \frac{V_i'(t,d_i(t))d_i(t)}{\|q\|^{2\beta}}\,dt < 0,$$

ending the proof. □

Lemma 3.2. *Assume* (Φ) *and* $(\Phi_{\pm}^{\infty})$. *Let* $H_\infty := \{q \in H\colon\ \|q^+ + q^-\| = o(\|q\|)$ *as* $\|q\| \to \infty\}$, *then*

$$\limsup_{q\in H_\infty,\, \|q\|\to\infty} \pm \sum_{i=-n}^{n-1} \int_0^T \frac{V_i'(t, d_i(t))d_i(t)}{\|q\|^{2\alpha}}\, dt < 0.$$

Proof. We only consider the case that (Φ_+^∞) holds. The other case can be treated similarly. As in the proof of Lemma 3.1, using the same notation, one obtains

$$\operatorname{meas}\bigl([0,T] \setminus E_1^i(q,\varepsilon)\bigr) < \varepsilon, \quad \text{and} \quad \operatorname{meas}\bigl([0,T] \setminus E_2^i(q,\varepsilon)\bigr) < \gamma^{2\alpha}(\varepsilon)\varepsilon, \tag{3.10}$$

and for any $q \in H_\infty$, $t \in E_1^i(q,\varepsilon) \cap E_2^i(q,\varepsilon)$,

$$\frac{|d_i(t)|}{\|q\|} \geqslant \gamma(\varepsilon)\frac{\|q^0\|}{\|q\|} - \Gamma(\varepsilon)\frac{\|q^+ + q^-\|}{\|q\|} \to \gamma(\varepsilon), \quad \text{as } \|q\| \to \infty. \tag{3.11}$$

Also for any $q \in H_\infty$ and $t \in E_2^i(q,\varepsilon) \setminus E_1^i(q,\varepsilon)$, we can show that

$$\frac{|d_i(t)|}{\|q\|} \leqslant \gamma(\varepsilon)\frac{\|q^0\|}{\|q\|} + \Gamma(\varepsilon)\frac{\|q^+ + q^-\|}{\|q\|} \to \gamma(\varepsilon), \quad \text{as } \|q\| \to \infty. \tag{3.12}$$

It follows from (Φ) and (Φ_+^∞) that, there exists small $R_1 > 0$ such that for every $i \in \{-n, \ldots, n-1\}$

$$\frac{V_i'(t,x)x}{|x|^{2\alpha}} \leqslant h_i^\infty(t) + \gamma^{2\alpha}(\varepsilon)\varepsilon, \quad \text{uniformly for } t \in [0,T] \text{ and } |x| \geqslant R_1. \tag{3.13}$$

It follows from (3.11) that for $q \in H_\infty$ with $\|q\|$ large enough, $E_1^i(q,\varepsilon) \cap E_2^i(q,\varepsilon) \subset E_5^i(q) := \{t \in [0,T] \mid |d_i(t)| \geqslant R_1\}$, and $|d_i(t)|/\|q\| \geqslant \gamma(\varepsilon)/2$ for all $t \in E_1^i(q,\varepsilon) \cap E_2^i(q,\varepsilon)$ and all i. Then, it follows from (3.1), (3.13) and the fact that $h_i^\infty(t) \leqslant 0$ that

$$\sum_{i=-n}^{n-1} \int\limits_{E_1^i(q,\varepsilon)\cap E_2^i(q,\varepsilon)} \frac{V_i'(t,d_i(t))d_i(t)}{\|q\|^{2\alpha}} \leqslant c(n)\gamma^{2\alpha}(\varepsilon)\varepsilon + \frac{\gamma^{2\alpha}(\varepsilon)}{4^\alpha} \sum_{i=-n}^{n-1} \int\limits_{E_1^i(q,\varepsilon)\cap E_2^i(q,\varepsilon)} h_i^\infty(t)\, dt. \tag{3.14}$$

Again, by (Φ) and (Φ_+^∞), we can find a large number $R' > 0$ such that $V_i'(t,x)x \leqslant |x|^{2\alpha}$, uniformly for every $i \in \{-n, \ldots, n-1\}$, $t \in [0,T]$ and $|x| \geqslant R'$. Set $E_6^i(q) = \{t \in [0,T] \mid |d_i(t)| \geqslant R'\}$ and $C = \max_{-n\leqslant i\leqslant n-1}\{|V_i'(t,x)x\colon\ t \in [0,T],\ |x| \leqslant R'\}$. It follows (3.10) and (3.12) that

$$\begin{aligned}
&\sum_{i=-n}^{n-1} \int\limits_{E_2^i(q,\varepsilon)\setminus E_1^i(q,\varepsilon)} \frac{V_i'(t,d_i(t))d_i(t)}{\|q\|^{2\alpha}}\, dt \\
&\quad\leqslant \sum_{i=-n}^{n-1} \int\limits_{(E_2^i(q,\varepsilon)\setminus E_1^i(q,\varepsilon))\cap E_6^i(q)} \frac{|d_i(t)|^{2\alpha}}{\|q\|^{2\alpha}}\, dt + \sum_{i=-n}^{n-1} \int\limits_{(E_2^i(q,\varepsilon)\setminus E_1^i(q,\varepsilon))\setminus E_6^i(q)} \frac{C}{\|q\|^{2\alpha}}\, dt \leqslant c(n)\gamma^{2\alpha}(\varepsilon)\varepsilon,
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i=-n}^{n-1} \int\limits_{[0,T]\setminus E_2^i(q,\varepsilon)} \frac{V_i'(t,d_i(t))d_i(t)}{\|q\|^{2\alpha}}\, dt \\
&\quad\leqslant \sum_{i=-n}^{n-1} \int\limits_{([0,T]\setminus E_2^i(q,\varepsilon))\cap E_6^i(q)} \frac{|d_i(t)|^{2\alpha}}{\|q\|^{2\alpha}}\, dt + \sum_{i=-n}^{n-1} \int\limits_{([0,T]\setminus E_2^i(q,\varepsilon))\setminus E_6^i(q)} \frac{C}{\|q\|^{2\alpha}}\, dt \leqslant c(n)\gamma^{2\alpha}(\varepsilon)\varepsilon,
\end{aligned}$$

which, jointly with (3.14), imply that, for any $q \in H_\infty$ with $\|q\|$ sufficiently large,

$$\sum_{i=-n}^{n-1} \int_0^T \frac{V_i'(t, d_i(t))d_i(t)}{\|q\|^{2\alpha}}\, dt \leqslant c(n)\gamma^{2\alpha}(\varepsilon)\varepsilon + \frac{\gamma^{2\alpha}(\varepsilon)}{4^\alpha} \sum_{i=-n}^{n-1} \int_{E_1^i(q,\varepsilon)\cap E_2^i(q,\varepsilon)} h_i^\infty(t)\, dt.$$

Noting that $\lim_{\varepsilon\to 0} \int_{E_1^i(q,\varepsilon)\cap E_2^i(q,\varepsilon)} h_i^\infty(t)\, dt = \int_0^T h_i^\infty(t)\, dt < 0$, the last inequality and the arbitrariness of ε yield that

$$\limsup_{q\in H_\infty,\, \|q\|\to\infty} \sum_{i=-n}^{n-1} \int_0^T \frac{V_i'(t, d_i(t))d_i(t)}{\|q\|^{2\alpha}}\, dt < 0.$$

The proof is complete. □

Lemma 3.3. *Assume* (Φ), (Φ^0) *and* (Φ^0_-). *Then* $C_k(\varphi, 0) \cong \delta_{k,\mu+\nu}G$, $k \in \mathbb{Z}$.

Proof. For any $\varepsilon > 0$, it follows from (Φ), (Φ^0), (3.1) and Hölder inequality that for $\|q\|$ sufficiently small,

$$\big|\big(\varphi'(q) - Lq, p\big)\big| \leqslant \sum_{i=-n}^{n-1} \int_0^T \varepsilon\big|d_i(t)\big|^\beta \big|p_i(t) - p_{i+1}(t)\big|\, dt \leqslant \varepsilon\|q\|^\beta\|p\|$$

for any $p \in H$, hence $\|\varphi'(q) - Lq\| = o(\|q\|^\beta)$ as $\|q\| \to 0$. By virtue of Proposition 2.2, it suffices to show that there exist $\eta > 0$ and $\theta \in (0, 1)$ such that $(\varphi'(q), q^0) \leqslant 0$, for all $q \in C_0(\eta, \theta, \beta) := \{q \in H \mid \|q\| \leqslant \eta, \|q^+ + q^-\| \leqslant \theta\|q\|^\beta\}$. Arguing indirectly, assume by contradiction that for any $\eta = \theta = 1/m$, there exists $q_m \in H$ such that $\|q_m\| \leqslant 1/m$ and $\|q_m^+ + q_m^-\| \leqslant \frac{1}{m}\|q_m\|^\beta$, but $(\varphi'(q_m), q_m^0) > 0$. Clearly, $\|q_m\| \to 0$, $\|q_m^+ + q_m^-\|/\|q_m\|^\beta \to 0$, and $(\varphi'(q_m), q_m^0) = -\sum_{i=-n}^{n-1} \int_0^T V_i'(t, d_{m,i}(t))d_{m,i}^0(t)\, dt > 0$, which implies

$$\limsup_{m\to\infty} \sum_{i=-n}^{n-1} \int_0^T \frac{V_i'(t, d_{m,i}(t))d_{m,i}^0(t)}{\|q_m\|^{2\beta}} \leqslant 0, \tag{3.15}$$

where $d_{m,i} := q_{m,i} - q_{m,i+1}$. On the other hand, for any $\varepsilon > 0$, by (Φ), (Φ^0) and (3.1), for $\|q_m\|$ small enough, we have

$$\left|\sum_{i=-n}^{n-1} \int_0^T \frac{V_i'(t, d_{m,i}(t))(d_{m,i}^+(t) + d_{m,i}^-(t))}{\|q_m\|^{2\beta}}\, dt\right| \leqslant \sum_{i=-n}^{n-1} \int_0^T \frac{\varepsilon|d_{m,i}|^\beta|d_{m,i}^+ + d_{m,i}^-|}{\|q_m\|^{2\beta}} \leqslant \varepsilon\frac{\|q_m^+ + q_m^-\|}{\|q_m\|^\beta} \to 0.$$

Therefore, by Lemma 3.2, we get

$$\liminf_{m\to\infty} \sum_{i=-n}^{n-1} \int_0^T \frac{V_i'(t, d_{m,i}(t))d_{m,i}^0(t)}{\|q_m\|^{2\beta}}\, dt = \liminf_{m\to\infty} \sum_{i=-n}^{n-1} \int_0^T \frac{V_i'(t, d_{m,i}(t))d_{m,i}(t)}{\|q_m\|^{2\beta}}\, dt > 0,$$

which contradicts with (3.15). Therefore, applying Proposition 2.2, we conclude that $C_k(\varphi, 0) \cong \delta_{k,\mu+\nu}G$, $k \in \mathbb{Z}$. □

Lemma 3.4. *Assume* (Φ), (Φ^0) *and* (Φ^0_+). *Then* $C_k(\varphi, 0) \cong \delta_{k,\mu}G$, $k \in \mathbb{Z}$.

Proof. Since the proof is similar to that of Lemma 3.3, we omit the details. □

Now we are in a position to show the computations of the critical groups at infinity. The following lemma plays an important part in proving the compactness conditions.

Lemma 3.5. *Assume (Φ), (Φ^∞) and $(\Phi^\infty_\pm)$. Then there exist $R>0$, $\delta>0$ and $\theta\in(0,1)$ such that*

$$\pm\big(\varphi'(q),q^0\big)\geqslant\delta,$$

for all $q\in C_\infty(R,\theta,\alpha):=\{q\in H\mid \|q\|\geqslant R,\ \|q^++q^-\|\leqslant\theta\|q\|^\alpha\}$.

Proof. We assume that (Φ^∞_+) holds. The other case can be treated similarly.

Suppose for the contrary that, for all $m>0$ and $\delta=\theta=1/m$, there exists $q_m\in H$ with $\|q_m\|\geqslant m$ and $\|q_m^++q_m^-\|\leqslant\theta\|q_m\|^\alpha$, but $(\varphi'(q_m),q_m^0)<1/m$. It then follows that $\|q_m\|\to\infty$, $\|q_m^++q_m^-\|/\|q_m\|^\alpha\to0$, and $(\varphi'(q_m),q_m^0)=-\sum_{i=-n}^{n-1}\int_0^T V_i'(t,d_{m,i}(t))d^0_{m,i}(t)\,dt<\frac{1}{m}$, which implies

$$\liminf_{m\to\infty}\sum_{i=-n}^{n-1}\int_0^T\frac{V_i'(t,d_{m,i}(t))d^0_{m,i}(t)}{\|q_m\|^{2\alpha}}\,dt\geqslant0,\tag{3.16}$$

where $d_{m,i}:=q_{m,i}-q_{m,i+1}$. On the other hand, by (Φ) and (Φ^∞), for any $\varepsilon>0$, there exists a positive constant $D:=D(\varepsilon)$ such that $|V_i'(t,x)|\leqslant\varepsilon|x|^\alpha+D$ for all $x\in\mathbb{R}$, $t\in[0,T]$ and $i\in\{-n,\dots,n-1\}$. Then, we have

$$\begin{aligned}\left|\sum_{i=-n}^{n-1}\int_0^T\frac{V_i'(t,d_{m,i}(t))(d^+_{m,i}(t)+d^-_{m,i}(t))}{\|q_m\|^{2\alpha}}\,dt\right|&\leqslant\sum_{i=-n}^{n-1}\int_0^T\frac{(\varepsilon|d_{m,i}|^\alpha+D)|d^+_{m,i}+d^-_{m,i}|}{\|q_m\|^{2\alpha}}\\&\leqslant c(n)\varepsilon\frac{\|q_m^++q_m^-\|}{\|q_m\|^\alpha}+c(n)\frac{\|q_m^++q_m^-\|}{\|q_m\|^{2\alpha}}\to0.\end{aligned}$$

Noting that $\|q_m^++q_m^-\|=o(\|q_m\|)$ as $m\to\infty$, by Lemma 3.3, we get

$$\limsup_{m\to\infty}\sum_{i=-n}^{n-1}\int_0^T\frac{V_i'(t,d_{m,i}(t))d^0_{m,i}(t)}{\|q_m\|^{2\alpha}}\,dt=\limsup_{m\to\infty}\sum_{i=-n}^{n-1}\int_0^T\frac{V_i'(t,d_{m,i}(t))d_{m,i}(t)}{\|q_m\|^{2\alpha}}\,dt<0,$$

which contradicts with (3.16). The lemma is proved. □

Lemma 3.6. *Assume (Φ), (Φ^∞) and either (Φ^∞_+) or (Φ^∞_-) hold. Then φ satisfies the compactness condition $(C)_c$ for all $c\in\mathbb{R}$.*

Proof. We assume that (Φ^∞_+) holds. The other case can be treated similarly.

Let $\{q_m\}\subset H$ be such that $\varphi(q_m)\to c$ and $\|\varphi'(q_m)\|(1+\|q_m\|)\to0$ as $m\to\infty$. Firstly we show that $\{q_m\}$ is bounded. Suppose for the contrary that $\|q_m\|\to\infty$ as $m\to\infty$. Since there exists a positive constant $c_0>0$ such that $\pm(Lq^\pm,q^\pm)\geqslant c_0\|q^\pm\|^2$, for all $q^\pm\in H^\pm$, then for any $\varepsilon>0$, by (Φ^∞), we get

$$\begin{aligned}c_0\|q_m^+\|^2&\leqslant\big(\varphi'(q_m),q_m^+\big)+\sum_{i=-n}^{n-1}\int_0^T V_i'\big(t,d_{m,i}(t)\big)d^+_{m,i}(t)\,dt\\&\leqslant\varepsilon\|q_m^+\|+\sum_{i=-n}^{n-1}\int_0^T\big(\varepsilon|d_{m,i}(t)|^\alpha+D\big)\big|d^+_{m,i}(t)\big|\,dt\leqslant c(n)\|q_m^+\|+\varepsilon\|q_m\|^\alpha\|q_m^+\|,\end{aligned}$$

which implies $\lim_{m\to\infty} \|q_m^+\|/\|q_m\|^\alpha = 0$. Similarly, we have $\lim_{m\to\infty} \|q_m^-\|/\|q_m\|^\alpha = 0$. Therefore, we get $\lim_{m\to\infty} \frac{\|q_m^+ + q_m^-\|}{\|q_m\|^\alpha} = 0$. Therefore, noting that $\|q_m\| \to \infty$ as $m \to \infty$, it follows from Lemma 3.5 that there exist $R > 0$, $\delta > 0$ and $\theta \in (0,1)$ such that $q_m \in C_\infty(R,\theta,\alpha)$ for m large enough and $(\varphi'(q_m), q_m^0) \geqslant \delta$, which contradicts to the fact that $|(\varphi'(q_m), q_m^0)| \leqslant \|\varphi'(q_m)\| \cdot (1 + \|q_m\|) \to 0$ as $m \to \infty$. Thus $\{q_m\}$ is bounded.

By the boundedness of $\{q_m\}$ and the compact embedding $H \hookrightarrow L_n^\infty$ we infer that there exists $q \in H$ such that, up to a subsequence, $q_m \rightharpoonup q$ in H and $q_m \to q$ in L_n^∞. Then it follows from $(\varphi'(q_m) - \varphi'(q), q_m - q) \to 0$ that $\dot{q}_m \to \dot{q}$ in L_n^2 which implies $q_m \to q$ in H. The proof is complete. □

Lemma 3.7. *Assume* (Φ) *and* (Φ^∞). *Then*

$$C_k(\varphi,\infty) \cong \begin{cases} \delta_{k,\mu}G, \ k \in \mathbb{Z}, & \textit{if } (\Phi_+^\infty) \textit{ holds}, \\ \delta_{k,\mu+\nu}G, \ k \in \mathbb{Z}, & \textit{if } (\Phi_-^\infty) \textit{ holds}. \end{cases}$$

Proof. We assume (Φ_+^∞) holds. The other case can be treated similarly.

Recall that for any $\varepsilon > 0$, there exists a positive constant $D > 0$ such that $|V'(t,x)| \leqslant \varepsilon|x|^\alpha + D$ for all $t \in [0,T]$, $x \in \mathbb{R}$ and $i \in \{-n,\dots,n-1\}$, then

$$\big|\big(\psi'(q),p\big)\big| = \left|\sum_{i=-n}^{n-1} \int_0^T V_i'\big(t, q_i(t) - q_{i+1}(t)\big)\big(p_i(t) - p_{i+1}(t)\big)\,dt\right| \leqslant c(n)\big(\varepsilon\|q\|^\alpha + 1\big)\|p\|,$$

for any $q,p \in H$, and hence $\|\psi'(q)\| = o(\|q\|^\alpha)$ as $\|q\| \to \infty$. By Lemma 3.5, there exist $R > 0$, $\delta > 0$ and $\theta \in (0,1)$ such that if $q \in H$ satisfies $\|q\| \geqslant R$ and $\|q^+ + q^-\| \leqslant \theta\|q\|^\alpha$, there holds $(\varphi'(q), q^0) \geqslant \delta > 0$.

If $\alpha \in (0,1)$, $q \in H$ satisfies $\|q\| \geqslant R$ and $\theta\|q\|^\alpha < \|q^+ + q^-\| \leqslant \theta\|q\|^{\frac{\alpha+1}{2}}$, then $\|q^+ + q^-\| = o(\|q\|)$, as $\|q\| \to \infty$. Therefore, by Lemma 3.2, there exists $R_1 \geqslant R$ such that

$$\big(\psi'(q),q\big) = -\sum_{i=-n}^{n-1} \int_0^T V_i'\big(t, q_i(t) - q_{i+1}(t)\big)\big(q_i(t) - q_{i+1}(t)\big)\,dt > 0,$$

for $q \in C_\infty(R_1,\theta,\frac{\alpha+1}{2}) \setminus C_\infty(R_1,\theta,\alpha)$. By Lemma 3.6, the functional φ satisfies the compactness condition $(C)_c$ for all $c \in \mathbb{R}$. It is also easy to check that other conditions of (A_∞) in Section 2 are satisfied. So by virtue of Proposition 2.3, we conclude that $C_k(\varphi,\infty) \cong \delta_{k,\mu}G$, $k \in \mathbb{Z}$. The proof is complete. □

4. Proofs of main results

Proof of Theorem 1.1. As mentioned in Section 2, we only consider the existence and multiplicity of T-periodic solutions for system (2.2).

Existence. Assume (Φ) and (Φ^0), then by Lemmas 3.3 and 3.4, we conclude that

$$C_k(\varphi,0) \cong \begin{cases} \delta_{k,\mu}G, \ k \in \mathbb{Z}, & \text{if } (\Phi_+^0) \text{ holds}, \\ \delta_{k,\mu+\nu}G, \ k \in \mathbb{Z}, & \text{if } (\Phi_-^0) \text{ holds}. \end{cases}$$

Thus the existence of a nontrivial critical point of the functional φ follows from Lemma 3.7 and Proposition 2.1.

Multiplicity. Here we adapt a technique in the proof of Theorem 1 in [13] (see also in [20]). We only consider the case that (a) holds, the another one is similar. As in the proof of *Existence* and by the $(\mu+\nu)$-th Morse inequality, φ has at least one nontrivial critical point q^* satisfying $C_{\mu+\nu}(\varphi,q^*) \ncong 0$.

If the nullity of q^* satisfies $\nu^* = 0$, then $C_k(\varphi, q^*) \cong \delta_{k,\mu+\nu}G$, $k \in \mathbb{Z}$. If φ does not have other critical points, by Proposition 2.4, $(-1)^{\mu+\nu} = (-1)^{\mu} + (-1)^{\mu+\nu}$, it is impossible.

Now, we suppose that $0 < \nu^* \leqslant \nu$. Let μ^* be the Morse index of q^*, by Proposition 2.1, one concludes that $C_k(\varphi, q^*) \cong 0$ for all $k \notin [\mu^*, \mu^* + \nu^*]$, which implies $\mu^* \leqslant \mu + \nu \leqslant \mu^* + \nu^*$.

In order to prove the existence of the second nontrivial critical point of φ, arguing indirectly, we suppose φ only has critical point 0 and q^*. We distinguish the following two cases.

Case 1: $\mu + \nu = \mu^*$ or $\mu + \nu = \mu^* + \nu^*$. In this case, by the splitting theorem and the characterization of critical groups at the local minimum and the local maximum (see [8,15]), $C_k(\varphi, q^*) \cong \delta_{k,\mu+\nu}G$, $k \in \mathbb{Z}$. Thus, from Proposition 2.4, we have $(-1)^{\mu+\nu} = (-1)^{\mu} + (-1)^{\mu+\nu}$. It is impossible.

Case 2: $\mu^* < \mu + \nu < \mu^* + \nu^*$. Using again the splitting theorem and the critical group characterization of the local minimum and local maximum, one has $C_{\mu^*}(\varphi, q^*) \cong C_{\mu^*+\nu^*}(\varphi, q^*) \cong 0$. We claim that $\mu^* \leqslant \mu$. Otherwise, we have $\mu < \mu^* < \mu+\nu$, and the μ^*-th and (μ^*-1)-th Morse inequalities read as $(-1)^{\mu^*}(-1)^{\mu} \geqslant 0$ and $(-1)^{\mu^*-1}(-1)^{\mu} \geqslant 0$. As a consequence, $(-1)^{\mu} = 0$, which leads to a contradiction. Therefore, we obtain $\mu^* \leqslant \mu < \mu + \nu < \mu^* + \nu^*$, which jointly with the assumption $\nu^* \leqslant \nu$ yields $\mu + \nu < \mu^* + \nu^* \leqslant \mu + \nu$, a contradiction. The proof is complete. □

Proof of Theorem 1.2. Similar to the proof of Lemmas 4.4 and 4.5 in [14], we can prove Theorem 1.2 by using the similar argument as used in proof of Theorem 1.1. We omitted the details. □

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