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# Ground states for the nonlinear Kirchhoff type problems ☆

Hui Zhang [a,*], Fubao Zhang [b]

[a] *Department of Mathematics, Jinling Institute of Technology, Nanjing 211169, China*
[b] *Department of Mathematics, Southeast University, Nanjing 210096, China*

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ABSTRACT

In this paper we study the existence of ground states for the asymptotically periodic Kirchhoff type problems with critical growth and three times growth. The proof is based on the method of Nehari manifold and concentration compactness principle. In particular, we improve the method of Nehari manifold for Kirchhoff type equations with three times growth.



## 1. Introduction and statement of the main result

The following Kirchhoff type problem

$$-\left(a+b\int|\nabla u|^2\right)\Delta u+V(x)u=l(x,u),\quad u\in H^1(\mathbb{R}^3),\tag{1.1}$$

has been widely investigated, where $a, b$ are positive constants, $V:\mathbb{R}^3\mapsto\mathbb{R}$ and $l:\mathbb{R}^3\times\mathbb{R}\mapsto\mathbb{R}$. Problem (1.1) is related to the stationary analogue of the equation

$$\rho\frac{\partial^2u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2dx\right)\frac{\partial^2u}{\partial x^2}=0,\tag{1.2}$$

which was presented by Kirchhoff [7], where $L$ is the length of the string, $h$ is the area of cross-section, $E$ is the Young modulus of the material, $\rho$ is the mass density and $P_0$ is the initial tension. For some early

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* Corresponding author.
*E-mail address:* huihz0517@126.com (H. Zhang).

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researches of Eq. (1.2) we refer to [2,16]. In [11], J.L. Lions introduced an abstract variational framework for Eq. (1.2). After that, the problem (1.2) received much attention, see [1,3,12].

There are many results about the existence of nontrivial solutions, sign-changing solutions, ground states, multiplicity of solutions and concentration of solutions. See [4–10,13–15,18–21,23,25,26]. Some of them are based on the bounded domain $\Omega$ of $\mathbb{R}^3$. Then Eq. (1.1) turns out to be

$$\begin{cases} -\left(a+b\displaystyle\int_{\Omega}|\nabla u|^2\right)\Delta u+V(x)u=l(x,u), & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

In [15], Perera and Zhang showed that the problem (1.3) possesses a nontrivial solution by using the Yang index and critical groups. Later, using Nehari manifold and fibering map, in [4] the authors considered the existence of multiple positive solutions of Eq. (1.3) involving sign-changing weight functions. Recently, most papers are researched on the whole space $\mathbb{R}^3$. He and Zou [6] studied the existence, multiplicity and concentration behavior of positive solutions for the problem (1.1) where the nonlinearity $l$ is differential and of subcritical growth. Later, Wang et al. [20] treated with Eq. (1.1) when $l$ is merely continuous and of critical growth, namely

$$-\left(\epsilon^2a+b\epsilon\int|\nabla u|^2\right)\Delta u+V(x)u=|u|^4u+\lambda f(u),\quad u\in H^1\left(\mathbb{R}^3\right). \tag{1.4}$$

Using the method of Nehari manifold and minimax methods, they obtained the multiplicity and concentration of positive solutions and the existence of ground states of the problem (1.4) when $\epsilon$ is small enough and $\lambda$ is sufficiently large. More recently, Li and Ye [8] considered Eq. (1.1) with the nonlinearity $l(x,u)=|u|^{p-1}u$, where $p\in(2,5)$. Assuming that $V$ is continuous and weakly differentiable and satisfies a certain conditions, they proved that the problem (1.1) has a positive ground state by using a monotonicity trick and a new version of global compactness lemma. However, the nonlinearity $l$ of (1.1) in [6,8,20] is autonomous. Concerning the non-autonomous nonlinearity $l$, with the use of some parameters, the authors in [9] and [14] studied that the existence and multiplicity of nontrivial solutions for Eq. (1.1); Sun and Wu [18] considered the existence and the nonexistence of nontrivial solutions, the existence of ground states of Eq. (1.1) with indefinite nonlinearity and steep potential well.

Motivated by the above works, without use of any parameter, we want to look for ground states of Eq. (1.1) when the nonlinearity $l$ is non-autonomous. Furthermore, inspired by [17], where the authors considered the existence of nontrivial solutions of asymptotically periodic quasilinear Schrödinger equations, we shall study the case where $V$ and $l$ in Eq. (1.1) are asymptotically periodic. In addition, as we know, the results of the problem (1.1) with three times growth are few since there is no higher-order term in the nonlinearity. Then there is no Mountain-Pass structure and the standard variational methods cannot be used. So some techniques or a new variational framework are needed. In [8], by the virtue of the Pohožaev type identity, Li and Ye obtained ground states for Eq. (1.1) with the pure nonlinearity $u^3$. We shall discuss the existence of ground states for two cases where the nonlinearity $l$ is with critical growth and three times growth respectively.

Firstly, we consider Eq. (1.1) with $l(x,u)=K(x)|u|^4u+f(x,u)$, namely:

$$-\left(a+b\int|\nabla u|^2\right)\Delta u+V(x)u=K(x)|u|^4u+f(x,u),\quad u\in H^1\left(\mathbb{R}^3\right). \tag{KH}$$

Let $\mathcal{F}$ be the class of functions $h\in L^\infty(\mathbb{R}^3)$ such that, for every $\epsilon>0$ the set $\{x\in\mathbb{R}^3:|h(x)|\geq\epsilon\}$ has finite Lebesgue measure. Suppose that:

($V_1$) $V \in L^\infty(\mathbb{R}^3)$, and there exists a function $V_p \in L^\infty(\mathbb{R}^3)$, 1-periodic in $x_i$, $1 \leq i \leq 3$, such that $V - V_p \in \mathcal{F}$.

($V_2$) There exists a constant $a_0 > 0$ such that $a_0 < V(x) \leq V_p(x)$, $x \in \mathbb{R}^3$, where $V_p$ is given in ($V_1$).

(K) $K \in L^\infty(\mathbb{R}^3)$, and there exist a constant $b_0 > 0$, a function $K_p \in L^\infty(\mathbb{R}^3)$, 1-periodic in $x_j$, $1 \leq j \leq 3$, and a point $x_0 \in \mathbb{R}^3$ such that $K - K_p \in \mathcal{F}$ and:
  (i) $K(x) \geq K_p(x) > b_0$ for all $x \in \mathbb{R}^3$,
  (ii) $K(x) = |K|_\infty + O(|x - x_0|)$, as $x \to x_0$.

We also suppose that $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ satisfies the following conditions:

($H_1$) $|f(x,u)| \leq a_1(1 + |u|^{q-1})$ for some $a_1 > 0$ and $2 < q < 6$,
($H_2$) $f(x,u) = o(u^3)$ uniformly in $x$ as $u \to 0$,
($H_3$) $u \mapsto \frac{f(x,u)}{|u|^3}$ is nondecreasing on $(-\infty, 0)$ and $(0, \infty)$,
($H_4$) there exists a function $f_p \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, 1-periodic in $x_j$, $1 \leq j \leq 3$, such that:
  (i) $|f(x,u)| \geq |f_p(x,u)|$, $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$,
  (ii) $|f(x,u) - f_p(x,u)| \leq |h(x)|(|u| + |u|^{q-1})$, $(x,u) \in \mathbb{R}^3 \times \mathbb{R}$, $h \in \mathcal{F}$, $q$ is given by ($H_1$),
  (iii) $u \mapsto \frac{f_p(x,u)}{|u|^3}$ is nondecreasing on $(-\infty, 0)$ and $(0, \infty)$.
($H_5$) $\frac{F(x,u)}{|u|^4} \to \infty$ uniformly in $x$ as $|u| \to \infty$, where $F(x,u) = \int_0^u f(x,s)ds$.

In order to obtain positive ground states of (KH), we still assume that:

($H_6$) $F(x,u) \leq F(x,|u|)$, $u \in \mathbb{R}$.

**Theorem 1.1.** *If ($V_1$), ($V_2$), (K) and ($H_1$)–($H_6$) are satisfied, then the problem (KH) has a positive ground state.*

**Remark 1.1.** (1) A typical example of the assumption ($H_6$) with $F(x,u) = F(u)$ is the condition that: $f(t) > 0$ for $t > 0$ and $f(t) = 0$ for $t \leq 0$, which was considered in [20].

(2) In order to restrict the functional level in a certain interval and then overcome the difficulties brought about by the critical growth term, Wang et al. [20] made use of the sufficiently large $\lambda$ in (1.4). Here we take advantage of the condition ($H_5$) that $F$ is 4-superlinear at infinity.

We also consider that Eq. (1.1) with $l(x,u) = Q(x)u^3$, that is:

$$-\left(a + b\int |\nabla u|^2\right)\Delta u + V(x)u = Q(x)u^3, \quad u \in H^1(\mathbb{R}^3). \tag{QH}$$

Assume that:

($Q_1$) $Q \in L^\infty(\mathbb{R}^3)$, and there exists a function $Q_p \in L^\infty(\mathbb{R}^3)$, 1-periodic in $x_i$, $1 \leq i \leq 3$, such that $Q - Q_p \in \mathcal{F}$.

($Q_2$) There exists a constant $q_0 > 0$ such that $q_0 < Q_p(x) \leq Q(x)$, where $Q_p$ is given in ($Q_1$).

**Theorem 1.2.** *Let ($V_1$), ($V_2$), ($Q_1$) and ($Q_2$) hold. Then Eq. (QH) has a positive ground state.*

*The outline for the proof*. We shall use the method of Nehari manifold and concentration compactness principle to prove the main results.

For Theorem 1.1, as in [20], we reduce the problem of looking for a ground state into that of finding a minimizer on the Nehari manifold. Then we use the concentration compactness lemma to solve the minimizing problem. Comparing with the existence result of ground states in [20], the main difficulties are as follows: firstly, the nonlinearity is non-autonomous and not including any parameter, which causes the estimation of the least energy is difficult. We shall give a more accurate estimation to restrict the least energy into a wider interval. Secondly, in the process of looking for the minimizer, since our problem is generalized asymptotically periodic, the weak limit of the minimizing sequence of the functional on Nehari manifold may be trivial, we will make use of the periodicity of the limit equation of (KH) and the relation of the functionals and derivatives of (KH) and its limit equation to find the minimizer.

For Theorem 1.2, the absence of higher-order term of the nonlinearity and the competing effect of the nonlocal term $-(\int |\nabla u|^2)\Delta u$ with the nonlinearity $Q(x)u^3$ of our problem (QH) prevent us from using the standard method of Nehari manifold [19]. Partially inspired by [5], where the authors considered the existence of infinitely many nontrivial solutions of quasilinear Schrödinger equations with three times growth, we find that, although the Nehari manifold is not homeomorphic to the unit sphere, it is homeomorphic to an open set of the unit sphere. So we can still reduce the problem of looking for a ground state into that of finding a minimizer of the functional on Nehari manifold. Then we use concentration compactness principle to deal with the minimizing problem.

The paper is organized as follows. In Section 2 we give some preliminaries. In Section 3 we study Eq. (KH) and prove Theorem 1.1. In Section 4 we study the problem (QH) and prove Theorem 1.2.

## 2. Preliminaries

In this paper we use the following notation. For $1 \le p \le \infty$, the norm in $L^p(\mathbb{R}^3)$ is denoted by $|\cdot|_p$. For any $r > 0$ and $x \in \mathbb{R}^3$, $B_r(x)$ denotes the ball centered at $x$ with the radius $r$. $\int_{\mathbb{R}^3} f(x)dx$ is represented by $\int f(x)$. Let $E$ be a Banach space and $\Phi : E \to \mathbb{R}$ be a functional of class $C^1$, the Fréchet derivative of $\Phi$ at $u$, $\Phi'(u)$, is an element of the dual space $E^*$ and we denote $\Phi'(u)$ evaluated at $v \in E$ by $\langle \Phi'(u), v\rangle$. A solution $\tilde{u} \in H^1(\mathbb{R}^3)$ of the equation $\Phi'(u) = 0$ is called a ground state if

$$\Phi(\tilde{u}) = \min\{\Phi(u) : u \in H^1(\mathbb{R}^3)\backslash\{0\},\ \Phi'(u) = 0\}.$$

The best constant for the Sobolev embedding $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ is given by

$$S = \inf_{u \in D^{1,2}(\mathbb{R}^3)\backslash\{0\}} \frac{|\nabla u|_2^2}{|u|_6^2}.$$

We consider the Sobolev space $H^1(\mathbb{R}^3)$ endowed with one of the following norms

$$\|u\|^2 = \int \left(a|\nabla u|^2 + V(x)u^2\right), \qquad \|u\|_p^2 = \int \left(a|\nabla u|^2 + V_p(x)u^2\right).$$

In view of ($V_2$), the norms $\|\cdot\|$ and $\|\cdot\|_p$ are equivalent to the standard norm in $H^1(\mathbb{R}^3)$. $S_1 = \{u \in H^1(\mathbb{R}^3) : \|u\|^2 = 1\}$.

The functional corresponding to the problem (KH) is

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int |\nabla u|^2\right)^2 - \frac{1}{6}\int K(x)u^6 - \int F(x,u), \quad u \in H^1(\mathbb{R}^3).$$

By our assumptions, $I$ is differentiable and its critical points are solutions of (KH).

In the process of looking for ground states of (KH), its corresponding periodic equation is very important and defined by

$$-\left(a+b\int|\nabla u|^2\right)\Delta u+V_p(x)u=K_p(x)|u|^4u+f_p(x,u),\quad u\in H^1(\mathbb{R}^3),\tag{KH$_p$}$$

and the functional is

$$I_p(u)=\frac{1}{2}\|u\|_p^2+\frac{b}{4}\left(\int|\nabla u|^2\right)^2-\frac{1}{6}\int K_p(x)u^6-\int F_p(x,u),\quad u\in H^1(\mathbb{R}^3).$$

Set

$$\begin{aligned} g(x,u)&=K(x)|u|^4u+f(x,u), &\qquad g_p(x,u)&=K_p(x)|u|^4u+f_p(x,u),\\ G(x,u)&:=\int_0^u g(x,s)ds, &\qquad G_p(x,u)&:=\int_0^u g_p(x,s)ds. \end{aligned}\tag{2.1}$$

Below we give some properties of $f$, $f_p$, $F$, $F_p$ and above functions.

**Lemma 2.1.** *If ($H_1$) and ($H_2$) are satisfied, then for all $\epsilon>0$ there exist $a_\epsilon>0$ such that*

$$\left|f(x,u)\right|\le\epsilon|u|+a_\epsilon|u|^{q-1},\quad u\in\mathbb{R}.\tag{2.2}$$

*If ($H_2$) and ($H_3$) are satisfied, then*

$$0\le 4G(x,u)\le g(x,u)u,\quad u\in\mathbb{R},\tag{2.3}$$

$$s\mapsto\frac{1}{4}g(x,su)su-G(x,su)\ \textit{is nondecreasing in}\ (0,\infty).\tag{2.4}$$

*If ($H_2$), ($H_4$)-(i) and (iii) are satisfied, then*

$$0\le 4G_p(x,u)\le g_p(x,u)u,\quad u\in\mathbb{R},\tag{2.5}$$

$$s\mapsto\frac{1}{4}g_p(x,su)su-G_p(x,su)\ \textit{is nondecreasing in}\ (0,\infty).\tag{2.6}$$

*Moreover, if ($H_3$) is also satisfied, then*

$$F(x,u)\ge F_p(x,u),\quad u\in\mathbb{R}.\tag{2.7}$$

**Proof.** We only prove (2.4) and (2.6), the others we can see [24, Lemma 2.1]. By ($H_2$) and ($H_3$) one easily has that $g(x,u)u\ge 0$ for all $u$, and

$$s\mapsto\frac{g(x,su)u}{s^3}\ \text{is nondecreasing in}\ (0,\infty).$$

Then for $0<s_1<s_2$, we get

$$\begin{aligned} G(x,s_2u)-G(x,s_1u)&=\int_{s_1}^{s_2}s^3\frac{g(x,su)u}{s^3}ds\\ &\le\frac{g(x,s_2u)u}{4s_2^3}\left(s_2^4-s_1^4\right)\le\frac{1}{4}g(x,s_2u)s_2u-\frac{1}{4}g(x,s_1u)s_1u. \end{aligned}$$

Therefore, (2.4) yields. Similarly, (2.6) yields. □

In addition, one easily has that the functional

$$J(u)=\frac{1}{2}\|u\|^2+\frac{b}{4}\left(\int|\nabla u|^2\right)^2-\frac{1}{4}\int Q(x)u^4$$

is of class $C^1$ and its critical points are solutions of (QH). The corresponding periodic equation is defined by

$$-\left(a+b\int|\nabla u|^2\right)\Delta u+V_p(x)u=Q_p(x)u^3,\quad u\in H^1(\mathbb{R}^3). \tag{(QH)$_p$}$$

Moreover, the functional of (QH)$_p$ is given by

$$J_p(u)=\frac{1}{2}\|u\|_p^2+\frac{b}{4}\left(\int|\nabla u|^2\right)^2-\frac{1}{4}\int Q_p(x)u^4.$$

## 3. Eq. (KH)

This section is devoted to describing the variational framework for the study of ground states of (KH) and giving the proof of Theorem 1.1.

### 3.1. The method of Nehari manifold

The Nehari manifold corresponding to Eq. (KH) is

$$M=\left\{u\in H^1\left(\mathbb{R}^3\right)\backslash\{0\}:\left\langle I'(u),u\right\rangle=0\right\},$$

where

$$\left\langle I'(u),u\right\rangle=\|u\|^2+b\left(\int|\nabla u|^2\right)^2-\int K(x)u^6-\int f(x,u)u, \tag{3.1}$$

and the least energy on $M$ is defined by $c:=\inf_M I$.

Similar to [20, Lemma 2.2], we have the following two lemmas:

**Lemma 3.1.** *Let $V,K\in L^\infty(\mathbb{R}^3)$ be such that $\inf_{\mathbb{R}^3}V>0$ and $\inf_{\mathbb{R}^3}K>0$. Under the conditions ($H_1$)–($H_4$), then:*

- (i) *For all $u\in S_1$, there exists a unique $t_u>0$ such that $t_uu\in M$ and $I(t_uu)=\max_{t>0}I(tu)$.*
- (ii) *$M$ is bounded away from 0. Furthermore, $M$ is closed in $H^1(\mathbb{R}^3)$.*
- (iii) *There is $\alpha>0$ such that $t_u\ge\alpha$ for each $u\in S_1$; and for each compact subset $W\subset S_1$, there exists $C_W>0$ such that $t_u\le C_W$, for all $u\in W$.*

**Lemma 3.2.** *Under the assumptions of Lemma 3.1, then there exists $\rho>0$ such that $\inf_{S_\rho}I>0$ and then $c=\inf_M I\ge\inf_{S_\rho}I>0$.*

Define the mapping $m:S_1\to M$ by setting $m(w):=t_ww$, where $t_w$ is as in Lemma 3.1 (i) and as [19, Proposition 3.1] we have:

**Lemma 3.3.** *The mapping $m$ is a homeomorphism between $S_1$ and $M$.*

Considering the functional $\Psi : S_1 \to \mathbb{R}$ given by $\Psi(w) := I(m(w))$, and similar to [19, Corollary 3.3] and [24, Lemma 3.2], we have:

**Lemma 3.4.** *Under the assumptions of Lemma 3.1, then:*

(1) *If $\{w_n\}$ is a PS sequence for $\Psi$, then $\{m(w_n)\}$ is a PS sequence for $I$. If $\{u_n\} \subset M$ is a bounded PS sequence for $I$, then $\{m^{-1}(u_n)\}$ is a PS sequence for $\Psi$.*
(2) *$w$ is a critical point of $\Psi$ if and only if $m(w)$ is a nontrivial critical point of $I$. Moreover, $\inf_M I = inf_{S_1} \Psi$.*
(3) *A minimizer of $I|_M$ is a ground state of Eq. (KH).*

From Lemma 3.4 (3), we know that the problem of seeking for a ground state for (KH) can be transformed into that of finding a minimizer of $I|_M$. Comparing with [20], we will overcome the difficulties which brought by the asymptotically periodicity of (KH) and give a wider interval. First we deal with the difficulty brought by the asymptotically periodicity of (KH).

By (2.7), one easily has the following lemma:

**Lemma 3.5.** *Let $(V_1)$, $(V_2)$, $(K)$ and $(H_1)$–$(H_4)$ hold. Then $I(u) \le I_p(u)$, for all $u \in H^1(\mathbb{R}^3)$.*

By Lemma 3.1 (i), we easily obtain that the infimum of $I$ on $M$ has the following minimax characterization:

$$c = \inf_M I = \inf_{w \in S_1} \max_{t>0} I(tw).$$

Define $M_p$ and $c_p$ as follows:

$$M_p := \{u \in H^1(\mathbb{R}^3) \backslash \{0\} : \langle I_p'(u), u \rangle = 0\}, \qquad c_p := \inf_{M_p} I_p. \tag{3.2}$$

Similarly, the infimum of $I_p$ on $M_p$ has the following minimax characterization:

$$c_p = \inf_{w \in S_1} \max_{t>0} I_p(tw).$$

Combining with Lemma 3.5, we have that:

**Lemma 3.6.** *Let $(V_1)$, $(V_2)$, $(K)$ and $(H_1)$–$(H_4)$ hold. Then $c \le c_p$.*

As in the proof of [10, Lemma 5.1] and [24, Lemma 3.6, Remark 3.1], we have the following lemmas respectively:

**Lemma 3.7.** *Let $(K)$ hold. Assume that $\{u_n\} \subset H^1(\mathbb{R}^3)$ is bounded and $\varphi_n(x) = \varphi(x - x_n)$, where $\varphi \in H^1(\mathbb{R}^3)$ and $x_n \in \mathbb{R}^3$. If $|x_n| \to \infty$, then*

$$\int (K(x) - K_p(x))|u_n|^4 u_n \varphi_n \to 0.$$

**Lemma 3.8.** *Let $(V_1)$, $(H_1)$ and $(H_4)$-(ii) hold. Assume that $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$ and $\{\varphi_n\} \subset H^1(\mathbb{R}^3)$ is bounded. Then*

$$\int \big[V(x) - V_p(x)\big] u_n \varphi_n \to 0,$$

$$\int \big[f(x,u_n) - f_p(x,u_n)\big] \varphi_n \to 0,$$

$$\int \big[F(x,u_n) - F_p(x,u_n)\big] \to 0.$$

Define

$$c_0 := \frac{ab|K|_\infty^{-1} S^3}{4} + \frac{|K|_\infty^{-2} S^6}{24}\big[\big(b^2 + 4a|K|_\infty S^{-3}\big)^{\frac{3}{2}} + b^3\big], \tag{3.3}$$

where $S$ is the best constant for the Sobolev embedding $D^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$.

**Lemma 3.9.** *Let $(V_1)$, $(V_2)$, $(K)$ and $(H_1)$–$(H_4)$ hold. Then the minimizing sequence of $c$ is bounded. Moreover, if $c \in (0, c_0)$, then the minimizing sequence of $c$ is non-vanishing.*

**Proof.** Let $\{u_n\}$ be a minimizing sequence of $I$ on $M$. Namely,

$$I(u_n) \to c, \qquad \big\langle I'(u_n), u_n\big\rangle = 0. \tag{3.4}$$

We firstly prove that $\{u_n\}$ is bounded. Indeed, noting that

$$\begin{aligned} c + o_n(1) &= I(u_n) - \frac{1}{4}\big\langle I'(u_n), u_n\big\rangle \\ &= \frac{1}{4}\|u_n\|^2 + \int \left[\frac{1}{4} g(x,u_n)u_n - G(x,u_n)\right] \\ &\geq \frac{1}{4}\|u_n\|^2, \end{aligned}$$

where we used (2.3). Then $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$.

Now we show that $\{u_n\}$ is non-vanishing. We argue by contradiction. Suppose $\{u_n\}$ is vanishing. Namely

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} u_n^2(x)\,dx = 0.$$

Then P.L. Lions Compactness Lemma implies that $u_n \to 0$ in $L^q(\mathbb{R}^3)$. By (2.2), we easily have $\int F(x,u_n) \to 0$ and $\int f(x,u_n)u_n \to 0$. With the use of (3.4), we get

$$\begin{aligned} c &= \frac{1}{2}\int \big(a|\nabla u_n|^2 + V(x)u_n^2\big) + \frac{b}{4}\Big(\int |\nabla u_n|^2\Big)^2 - \frac{1}{6}\int K(x)|u_n|^6 + o_n(1), \\ &\int \big(a|\nabla u_n|^2 + V(x)u_n^2\big) + b\Big(\int |\nabla u_n|^2\Big)^2 = \int K(x)|u_n|^6 + o_n(1). \end{aligned} \tag{3.5}$$

If $\int |\nabla u_n|^2 \to 0$, then

$$\int K(x)|u_n|^6 \leq |K|_\infty \Big(\int |\nabla u_n|^2\Big)^3 S^{-3} \to 0.$$

Then it follows from (3.5) that $c = 0$. This contradicts with the condition $c > 0$. Then $\int |\nabla u_n|^2 \nrightarrow 0$. By (3.5) we have

$$a + b\int |\nabla u_n|^2 \le |K|_\infty S^{-3}\left(\int |\nabla u_n|^2\right)^2 + o_n(1). \tag{3.6}$$

So

$$\int |\nabla u_n|^2 \ge \frac{|K|_\infty^{-1} S^3}{2}\left(b + \sqrt{b^2 + 4a|K|_\infty S^{-3}}\right) + o_n(1). \tag{3.7}$$

Then by (3.6) we infer

$$\left(\int |\nabla u_n|^2\right)^2 \ge a|K|_\infty^{-1} S^3 + \frac{b|K|_\infty^{-2} S^6}{2}\left(b + \sqrt{b^2 + 4a|K|_\infty S^{-3}}\right) + o_n(1). \tag{3.8}$$

From (3.5) we deduce

$$c \ge \frac{a}{3}\int |\nabla u_n|^2 + \frac{b}{12}\left(\int |\nabla u_n|^2\right)^2 + o_n(1).$$

Then combining with (3.7) and (3.8) we easily conclude that $c \ge c_0$, given in (3.3), contradicting with the assumption $c < c_0$. Hence $\{u_n\}$ is non-vanishing. The proof is completed. $\square$

**Remark 3.1.** In [20, Lemma 3.7], the authors obtained that when

$$c \in \left(0, \frac{1}{3}(aS)^{\frac{3}{2}} + \frac{1}{12}b^3 S^6\right), \tag{3.9}$$

the minimizing sequence of $c$ is non-vanishing. Here we give a wider interval for $c$, since if we assume that $|K|_\infty = 1$, then $c_0$ in (3.3) turns out to be

$$\frac{abS^3}{4} + \frac{S^6}{24}\left[\left(4aS^{-3} + b^2\right)^{\frac{3}{2}} + b^3\right],$$

which is larger than $\frac{1}{3}(aS)^{\frac{3}{2}} + \frac{1}{12}b^3 S^6$.

### *3.2. Estimates*

Now we devote to estimating the least energy $c$ and proving that $c \in (0, c_0)$. By Lemma 3.2, it suffices to show that $c < c_0$. We shall choose a function $u \in M$ and show that $I(u) < c_0$. The construct of $u$ is based on a test function in $H^1(\mathbb{R}^3)$. The test function is standard, see [22].

Without loss of generality, in the condition (K), we assume that $x_0 = 0$. For $\epsilon > 0$, the function $w_\epsilon : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$w_\epsilon(x) = 3^{\frac{1}{4}} \frac{\epsilon^{\frac{1}{4}}}{(\epsilon + |x|^2)^{\frac{1}{2}}},$$

is a family of functions on which $S$ is attained. Let $\phi \in C_0^\infty(\mathbb{R}^3, [0, 1])$, $\phi \equiv 1$ in $B_{\frac{\rho}{2}}(0)$, $\phi \equiv 0$ in $\mathbb{R}^3 \backslash B_\rho(0)$. Then define the test function by

$$v_\epsilon = \frac{u_\epsilon}{(\int K(x)u_\epsilon^6)^{\frac{1}{6}}},$$

where $u_\epsilon = \phi w_\epsilon$.

We give some properties for $v_\epsilon$ and $u_\epsilon$, which is given in [10].

**Lemma 3.10.** *If (K) is satisfied, then:*

$$\int |\nabla v_\epsilon|^2 \le |K|_\infty^{-\frac{1}{3}} S + O\big(\epsilon^{\frac{1}{2}}\big), \quad \text{as } \epsilon \to 0^+,$$
$$|v_\epsilon|_2^2 = O\big(\epsilon^{\frac{1}{2}}\big), \quad \text{as } \epsilon \to 0^+. \tag{3.10}$$

*Moreover, there exist positive constants $k_1$, $k_2$ and $\epsilon_0$ such that*

$$k_1 < \int K(x)u_\epsilon^6 < k_2, \quad \text{for all } 0 < \epsilon < \epsilon_0. \tag{3.11}$$

**Lemma 3.11.** *Let $V \in L^\infty(\mathbb{R}^3)$ be such that $\inf_{\mathbb{R}^3} V > 0$. Suppose (K) and ($H_1$)–($H_5$) are satisfied. Then $c < c_0$.*

**Proof.** By the definition of $c$, we just need to verify that there exists $v \in M$ such that

$$I(v) < c_0. \tag{3.12}$$

We first claim that for $\epsilon > 0$ small enough, there exist constants $t_\epsilon$, $A_1$ and $A_2$ independent of $\epsilon$ such that

$$I(t_\epsilon v_\epsilon) = \max_{t>0} I(tv_\epsilon)$$

and

$$0 < A_1 < t_\epsilon < A_2 < \infty. \tag{3.13}$$

In fact, by Lemma 3.1 (i), there exists $t_\epsilon > 0$ such that

$$t_\epsilon v_\epsilon \in M, \quad I(t_\epsilon v_\epsilon) = \max_{t\ge 0} I(tv_\epsilon). \tag{3.14}$$

Then $I(t_\epsilon v_\epsilon) \ge c > 0$. By (2.3), we get

$$\frac{t_\epsilon^2}{2}\|v_\epsilon\|^2 + \frac{b}{4}t_\epsilon^4\left(\int |\nabla v_\epsilon|^2\right)^2 \ge c.$$

Then $t_\epsilon^2\|v_\epsilon\|^2 \ge C_1$. Moreover, since $\|v_\epsilon\|$ is bounded for $\epsilon$ small enough by (3.10), then there exists $A_1 > 0$ such that $t_\epsilon \ge A_1$, for every $\epsilon > 0$ sufficiently small. On the other hand, since $t_\epsilon v_\epsilon \in M$, we get $\langle I'(t_\epsilon v_\epsilon), t_\epsilon v_\epsilon\rangle = 0$. Noting that $\int K(x)v_\epsilon^6 = 1$, we have

$$t_\epsilon^2\|v_\epsilon\|^2 + bt_\epsilon^4\|v_\epsilon\|^4 \ge t_\epsilon^6 + \int f(x, t_\epsilon v_\epsilon)t_\epsilon v_\epsilon.$$

By (2.2) we find that

$$\begin{aligned} t_\epsilon^6 &\le t_\epsilon^2\|v_\epsilon\|^2 + bt_\epsilon^4\|v_\epsilon\|^4 + \epsilon t_\epsilon^2|v_\epsilon|_2^2 + C_\epsilon t_\epsilon^q|v_\epsilon|_q^q \\ &\le (1+\epsilon)t_\epsilon^2\|v_\epsilon\|^2 + CC_\epsilon t_\epsilon^q\|v_\epsilon\|^q + bt_\epsilon^4\|v_\epsilon\|^4. \end{aligned}$$

Noting that $q < 6$, then there exists $A_2 > 0$ such that $t_\epsilon \le A_2$ since $\|v_\epsilon\|$ is bounded for small $\epsilon$.

Now we estimate $I(t_\epsilon v_\epsilon)$. Note that

$$\begin{aligned} I(t_\epsilon v_\epsilon) &\le \left(\frac{at_\epsilon^2}{2}B_\epsilon + \frac{t_\epsilon^2}{2}|V|_\infty |v_\epsilon|_2^2\right) + \frac{bt_\epsilon^4}{4}B_\epsilon^2 - \frac{t_\epsilon^6}{6} - \int F(x, t_\epsilon v_\epsilon) \\ &= \left(\frac{at_\epsilon^2}{2}B_\epsilon + \frac{bt_\epsilon^4}{4}B_\epsilon^2 - \frac{t_\epsilon^6}{6}\right) + \left(\frac{t_\epsilon^2}{2}|V|_\infty |v_\epsilon|_2^2 - \int F(x, t_\epsilon v_\epsilon)\right) \\ &:= I_1 + I_2, \end{aligned} \tag{3.15}$$

where $B_\epsilon := \int |\nabla v_\epsilon|^2$. For $I_1$, considering the function $\theta : [0, \infty) \to \mathbb{R}$, $\theta(t) = \frac{a}{2}B_\epsilon t^2 + \frac{b}{4}B_\epsilon^2 t^4 - \frac{1}{6}t^6$, we have that $t_0 = (\frac{bB_\epsilon^2 + \sqrt{b^2B_\epsilon^4 + 4aB_\epsilon}}{2})^{\frac{1}{2}}$ is a maximum point of $\theta$ and

$$\theta(t_0) = \left(\frac{aB_\epsilon}{3} + \frac{b^2B_\epsilon^4}{12}\right)\frac{bB_\epsilon^2 + \sqrt{b^2B_\epsilon^4 + 4aB_\epsilon}}{2} + \frac{ab}{12}B_\epsilon^3.$$

Then

$$I_1 \le \left(\frac{aB_\epsilon}{3} + \frac{b^2B_\epsilon^4}{12}\right)\frac{bB_\epsilon^2 + \sqrt{b^2B_\epsilon^4 + 4aB_\epsilon}}{2} + \frac{ab}{12}B_\epsilon^3.$$

Combining with (3.10) we have

$$I_1 \le \frac{ab|K|_\infty^{-1}S^3}{4} + \frac{|K|_\infty^{-2}S^6}{24}\left[\left(b^2 + 4a|K|_\infty S^{-3}\right)^{\frac{3}{2}} + b^3\right] + O\left(\epsilon^{\frac{1}{2}}\right), \tag{3.16}$$

where we applying the inequality

$$(a_1 + a_2)^\zeta \le a_1^\zeta + \zeta(a_1 + a_2)^{\zeta-1}a_2, \quad a_1, a_2 \ge 0,\ \zeta \ge 1.$$

It suffices to estimate $I_2$. For $|x| < \epsilon^{\frac{1}{2}} < \frac{\varrho}{2}$, noting that $\phi \equiv 1$ in $B_{\frac{\varrho}{2}}(0)$, by the definition of $v_\epsilon$ and (3.11), we find a constant $\alpha > 0$ such that

$$t_\epsilon v_\epsilon(x) \ge \frac{A_1}{(k_2)^{\frac{1}{6}}}u_\epsilon(x) \ge \frac{A_1}{(k_2)^{\frac{1}{6}}}w_\epsilon(x) = \frac{A_1 3^{\frac{1}{4}}}{(k_2)^{\frac{1}{6}}}\frac{\epsilon^{\frac{1}{4}}}{(\epsilon + |x|^2)^{\frac{1}{2}}} \ge \alpha\epsilon^{-\frac{1}{4}}, \tag{3.17}$$

here $A_1$ is given by (3.13). Given $A_0 > 0$, we invoke $(\mathrm{H}_5)$ to obtain $R = R(A_0) > 0$ such that, for $x \in \mathbb{R}^3$, $s \ge R$,

$$F(x, s) \ge A_0 s^4.$$

Then we may choose $\epsilon_1 > 0$ such that

$$t_\epsilon v_\epsilon(x) \ge \alpha\epsilon^{-\frac{1}{4}} \ge R,$$

for $|x| < \epsilon^{\frac{1}{2}}$, $0 < \epsilon < \epsilon_1$. So

$$F\left(x, t_\epsilon v_\epsilon(x)\right) \ge A_0 t_\epsilon^4 v_\epsilon^4,$$

for $|x| < \epsilon^{\frac{1}{2}}$, $0 < \epsilon < \epsilon_1$. Then for any $0 < \epsilon < \epsilon_1$, by (3.17) we infer

$$\int_{B_{\epsilon^{\frac{1}{2}}}(0)} F(x, t_\epsilon v_\epsilon)dx \geq A_0 \int_{B_{\epsilon^{\frac{1}{2}}}(0)} t_\epsilon^4 v_\epsilon^4 dx \geq A_0\alpha^4 \int_{B_{\epsilon^{\frac{1}{2}}}(0)} \epsilon^{-1} dx$$
$$\geq A_0\alpha^4\epsilon^{-1}\omega_3 \int_0^{\epsilon^{\frac{1}{2}}} r^2 dr = A_0\alpha^4 \frac{\omega_3}{3}\epsilon^{\frac{1}{2}}, \tag{3.18}$$

where $\omega_3$ is the surface area of the unit sphere in $\mathbb{R}^3$.

For $|x| > \epsilon^{\frac{1}{2}}$, by $(H_2)$ and $(H_3)$, we get

$$F(x, s) \geq 0, \quad s \in \mathbb{R}.$$

Combining with (3.10) and (3.18), we have

$$I_2 \leq C\epsilon^{\frac{1}{2}} - A_0\alpha^4 \frac{\omega_3}{3}\epsilon^{\frac{1}{2}}.$$

Inserting the above inequality and (3.16) into (3.15), we find

$$I(t_\epsilon v_\epsilon) \leq \frac{ab|K|_\infty^{-1}S^3}{4} + \frac{|K|_\infty^{-2}S^6}{24}\left[\left(b^2 + 4a|K|_\infty S^{-3}\right)^{\frac{3}{2}} + b^3\right] + \left(C - A_0\alpha^4\frac{\omega_3}{3}\right)\epsilon^{\frac{1}{2}},$$

where $C$ may denote different constant. Since $A_0 > 0$ is arbitrary, we choose large enough $A_0$ such that $C - A_0\alpha^4\frac{\omega_3}{3} < 0$. Then for small $\epsilon > 0$ we have

$$I(t_\epsilon v_\epsilon) < \frac{ab|K|_\infty^{-1}S^3}{4} + \frac{|K|_\infty^{-2}S^6}{24}\left[\left(b^2 + 4a|K|_\infty S^{-3}\right)^{\frac{3}{2}} + b^3\right].$$

Namely $I(t_\epsilon v_\epsilon) < c_0$, where $c_0$ is given in (3.3). Noting that $t_\epsilon v_\epsilon \in M$ by (3.14), then (3.12) establishes. This ends the proof. □

**Remark 3.2.** In [20, Lemma 3.5], for Eq. (1.4), with the use of the sufficient large $\lambda$, it is easy to restrict the least energy $c$ in the interval (3.9). Here we do not use any parameter, and then the estimation of $c$ turns out to be difficult. We make use of the condition $(H_5)$ that $F$ is 4-superlinear at infinity to restrict $c$ into $(0, c_0)$.

### *3.3. Proof of Theorem 1.1*

We are now in a position to give the proof of Theorem 1.1. By Lemma 3.4 (3), it suffices to show that the infimum $c$ is attained. By differentiability of $S_1$, we easily obtain a minimizing sequence of $c$. If the weak limit of the minimizing sequence is nontrivial, then we show the weak limit is the desired ground state. Noting that the nonlocal term $-(\int |\nabla u|^2)\Delta u$ causes that $I'$ may not be weakly sequentially continuous, then we cannot prove that the weak limit is a ground state as the previous papers dealing with other variational problems, for example, see [19]. Partially inspired by [21, Lemma 4.1], we use some techniques to show the weak limit is in fact a ground state. Otherwise, if the weak limit is trivial, by concentration compactness principle and the periodicity of $(KH)_p$, we can still find a minimizer. Moreover, in this process, we still do not know whether $I'$ and $I'_p$ are weakly sequentially continuous. Since our problem is asymptotically periodic, using only the idea in [21] is not enough to find the new minimizer. With some new skills we realize this process.

**Proof of Theorem 1.1.** Assume that $\{w_n\} \subset S_1$ is a minimizing sequence satisfying $\Psi(w_n) \to \inf_{S_1} \Psi$. By the Ekeland variational principle, we suppose $\Psi'(w_n) \to 0$. Then, from Lemma 3.4 (1) it follows that $I'(u_n) \to 0$, where $u_n = m(w_n) \in M$. Moreover, by Lemma 3.4 (2), we have $I(u_n) = \Psi(w_n) \to c$. Applying Lemma 3.9, we get that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Up to a subsequence, we assume that $u_n \rightharpoonup \tilde{u}$ in $H^1(\mathbb{R}^3)$, $u_n \to \tilde{u}$ in $L^2_{loc}(\mathbb{R}^3)$ and $u_n \to \tilde{u}$ a.e. on $\mathbb{R}^3$. Below we distinguish two cases where $\tilde{u} \neq 0$ and $\tilde{u} = 0$.

*Case* 1: $\tilde{u} \neq 0$.

We first claim that $I'(\tilde{u}) = 0$. Since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$, passing to a subsequence, we may assume that there exists $l \geq 0$ such that $\int |\nabla u_n|^2 \to l^2$. Note that $I'(u_n) \to 0$, then $\tilde{u}$ is a solution of the following equation

$$-\left(a + bl^2\right)\triangle u + V(x)u = K(x)u^5 + f(x, u), \quad u \in H^1\left(\mathbb{R}^3\right).$$

It suffices to show that $l^2 = \int |\nabla \tilde{u}|^2$. From the weakly lower semi-continuous of the norm it follows that $l^2 \geq \int |\nabla \tilde{u}|^2$. Then

$$\begin{aligned} &\left(a + b\int |\nabla \tilde{u}|^2\right)\int |\nabla \tilde{u}|^2 + \int V(x)\tilde{u}^2 \\ &\quad \leq \left(a + bl^2\right)\int |\nabla \tilde{u}|^2 + \int V(x)\tilde{u}^2 = \int f(x, \tilde{u})\tilde{u} + \int K(x)\tilde{u}^6. \end{aligned} \tag{3.19}$$

So $\langle I'(\tilde{u}), \tilde{u}\rangle \leq 0$. By Lemma 3.1 (i), we get that there exists $t_1 > 0$ such that $t_1\tilde{u} \in M$. Then we claim that $t_1 \leq 1$. Otherwise, $t_1 > 1$. Noting that $\langle I'(t_1\tilde{u}), t_1\tilde{u}\rangle = 0$, then we infer that

$$\begin{aligned} \int \left(a|\nabla \tilde{u}|^2 + V(x)\tilde{u}^2\right) + b\left(\int |\nabla \tilde{u}|^2\right)^2 &> \frac{1}{t_1^2}\int \left(a|\nabla \tilde{u}|^2 + V(x)\tilde{u}^2\right) + b\left(\int |\nabla \tilde{u}|^2\right)^2 \\ &= \int \frac{f(x, t_1\tilde{u})t_1\tilde{u}}{t_1^4} + t_1^2\int K(x)\tilde{u}^6 \\ &> \int f(x, \tilde{u})\tilde{u} + \int K(x)\tilde{u}^6, \end{aligned}$$

where the last inequality follows from the condition $(\mathrm{H}_3)$. This contradicts with (3.19). So $t_1 \leq 1$. Let

$$\tilde{G}(x, u) = \frac{1}{4}\left(a|\nabla u|^2 + V(x)u^2\right) + \frac{1}{4}g(x, u)u - G(x, u). \tag{3.20}$$

By (2.4) we have

$$\int \tilde{G}(x, t_1\tilde{u}) \leq \int \tilde{G}(x, \tilde{u}).$$

Then

$$\begin{aligned} c &\leq I(t_1\tilde{u}) - \frac{1}{4}\left\langle I'(t_1\tilde{u}), t_1\tilde{u}\right\rangle = \int \tilde{G}(x, t_1\tilde{u}) \leq \int \tilde{G}(x, \tilde{u}) \\ &\leq \int \tilde{G}(x, u_n) + o_n(1) = I(u_n) - \frac{1}{4}\left\langle I'(u_n), u_n\right\rangle + o_n(1) = c + o_n(1), \end{aligned} \tag{3.21}$$

where we used (2.3) and Fatou Lemma. So $t_1 = 1$, $\int \tilde{G}(x, u_n) \to \int \tilde{G}(x, \tilde{u})$, and then $\int |\nabla u_n|^2 \to \int |\nabla \tilde{u}|^2$ by (2.3). Therefore, $l^2 = \int |\nabla \tilde{u}|^2$. Then, $I'(\tilde{u}) = 0$.

Note that

$$I(\tilde{u}) - \frac{1}{4}\langle I'(\tilde{u}), \tilde{u}\rangle = \int \tilde{G}(x, \tilde{u}) = c,$$

by (3.21). Moreover, $I'(\tilde{u}) = 0$. Therefore, $I(\tilde{u}) = c$.

*Case* 2: $\tilde{u} = 0$.

This case is more complicated. We discuss that $\{u_n\}$ is vanishing or non-vanishing. It is easy to see that the case of vanishing does not happen since the energy $c \in (0, c_0)$.

Suppose $\{u_n\}$ is vanishing. By Lemmas 3.2, 3.11 and 3.9, then we infer that $\{u_n\}$ is non-vanishing. Then there exist $x_n \in \mathbb{R}^3$ and $\delta_0 > 0$ such that

$$\int\limits_{B_1(x_n)} u_n^2(x)dx > \delta_0. \tag{3.22}$$

Without loss of generality, we assume that $x_n \in \mathbb{Z}^3$. Since $u_n \to \tilde{u}$ in $L^2_{loc}(\mathbb{R}^3)$ and $\tilde{u} = 0$, we may suppose that $|x_n| \to \infty$ up to a subsequence. Denote $\bar{u}_n$ by $\bar{u}_n(\cdot) = u_n(\cdot + x_n)$. Similarly, passing to a subsequence, we assume that $\bar{u}_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^3)$, $\bar{u}_n \to \bar{u}$ in $L^2_{loc}(\mathbb{R}^3)$, and $\bar{u}_n \to \bar{u}$ a.e. on $\mathbb{R}^3$. By (3.22) one easily has that $\bar{u} \neq 0$.

For all $\psi \in H^1(\mathbb{R}^3)$, set $\psi_n(\cdot) := \psi(\cdot + x_n)$. From Lemmas 3.7 and 3.8, replacing $\varphi_n$ by $\psi_n$ it follows that

$$\int \big(V(x) - V_p(x)\big)u_n\psi_n \to 0,$$

$$\int \big(K(x) - K_p(x)\big)|u_n|^4 u_n\psi_n \to 0,$$

$$\int \big[f(x, u_n) - f_p(x, u_n)\big]\psi_n \to 0.$$

Consequently

$$\langle I'(u_n), \psi_n\rangle - \langle I'_p(u_n), \psi_n\rangle \to 0.$$

Noting that $I'(u_n) \to 0$ and $\|\psi_n\|_{H^1} = \|\psi\|_{H^1}$, we have $\langle I'(u_n), \psi_n\rangle \to 0$. So

$$\langle I'_p(u_n), \psi_n\rangle \to 0.$$

Moreover, by the periodicity of $V_p$, $K_p$ and $f_p$ in the variable $x$ and $x_n \in \mathbb{Z}^3$, we get

$$\langle I'_p(\bar{u}_n), \psi\rangle = \langle I'_p(u_n), \psi_n\rangle.$$

Then $\langle I'_p(\bar{u}_n), \psi\rangle \to 0$. By the arbitrary of $\psi$, $I'_p(\bar{u}_n) \rightharpoonup 0$ in $(H^1(\mathbb{R}^3))^*$. Since $\{\bar{u}_n\}$ is bounded in $H^1(\mathbb{R}^3)$, passing to a subsequence, we may assume that there exists $l_0 \geq 0$ such that $\int |\nabla \bar{u}_n|^2 \to l_0^2$. Note that $I'_p(\bar{u}_n) \rightharpoonup 0$, then $\bar{u}$ is a solution of the following equation

$$-\big(a + bl_0^2\big)\triangle u + V_p(x)u = K_p(x)u^5 + f_p(x, u), \quad u \in H^1\big(\mathbb{R}^3\big).$$

From the weakly lower semi-continuous of the norm it follows that $l_0^2 \geq \int |\nabla \bar{u}|^2$. Then

$$\begin{aligned}\left(a + b\int |\nabla \bar{u}|^2\right)\int |\nabla \bar{u}|^2 + \int V_p(x)\bar{u}^2 &\leq \big(a + bl_0^2\big)\int |\nabla \bar{u}|^2 + \int V_p(x)\bar{u}^2 \\ &= \int f_p(x, \bar{u})\bar{u} + K_p(x)\bar{u}^6. \end{aligned} \tag{3.23}$$

Now we prove that

$$I_p(\bar{u}) - \frac{1}{4}\langle I_p'(\bar{u}), \bar{u}\rangle \le c. \tag{3.24}$$

Replacing $\varphi_n$ by $u_n$, Lemma 3.8 yields

$$\int \big(V(x) - V_p(x)\big)u_n^2 \to 0, \int \big[F(x,u_n) - F_p(x,u_n)\big] \to 0,$$
$$\int \big[f(x,u_n)u_n - f_p(x,u_n)u_n\big] \to 0. \tag{3.25}$$

Similar to (3.20), set

$$\tilde{G}_p(x,u) = \frac{1}{4}\big(a|\nabla u|^2 + V_p(x)u^2\big) + \frac{1}{4}g_p(x,u)u - G_p(x,u).$$

By the condition $K \ge K_p$ in (K) and (3.25), we get

$$\begin{aligned}\int \tilde{G}_p(x,u_n) &= \frac{1}{4}\left(a|\nabla u_n|_2^2 + \int V_p(x)u_n^2\right) + \frac{1}{12}\int K_p(x)|u_n|^6 + \int\left(\frac{1}{4}f_p(x,u_n)u_n - F_p(x,u_n)\right)\\ &\le \frac{1}{4}\left(a|\nabla u_n|_2^2 + \int V(x)u_n^2\right) + \frac{1}{12}\int K(x)|u_n|^6 + \int\left(\frac{1}{4}f(x,u_n)u_n - F(x,u_n)\right) + o_n(1)\\ &= \int \tilde{G}(x,u_n) + o_n(1).\end{aligned}$$

Noting that $\tilde{G}_p$ is 1-periodic in $x$, we have

$$\int \tilde{G}_p(x,\bar{u}_n) = \int \tilde{G}_p(x,u_n).$$

Therefore,

$$\int \tilde{G}_p(x,\bar{u}_n) \le \int \tilde{G}(x,u_n) + o_n(1).$$

Note that $\bar{u}_n \to \bar{u}$ a.e. on $\mathbb{R}^3$. Then from (2.5) and Fatou Lemma it follows that

$$\int \tilde{G}_p(x,\bar{u}) + o_n(1) \le \int \tilde{G}_p(x,\bar{u}_n).$$

So

$$\int \tilde{G}_p(x,\bar{u}) \le \int \tilde{G}(x,u_n) + o_n(1).$$

Note that

$$\int \tilde{G}(x,u) = I(u) - \frac{1}{4}\langle I'(u), u\rangle, \qquad \int \tilde{G}_p(x,u) = I_p(u) - \frac{1}{4}\langle I_p'(u), u\rangle,$$

we infer that

$$\begin{aligned}c + o_n(1) = \int \tilde{G}(x,u_n) &\ge \int \tilde{G}_p(x,\bar{u}) + o_n(1)\\ &= I_p(\bar{u}) - \frac{1}{4}\langle I_p'(\bar{u}), \bar{u}\rangle + o_n(1).\end{aligned}$$

Then we have $I_p(\bar{u}) - \frac{1}{4}\langle I_p'(\bar{u}), \bar{u}\rangle \le c$.

Similar to Lemma 3.1 (i), one easily has that there exists $t_2 > 0$ such that $t_2\bar{u} \in M_p$ and $I_p(t_2\bar{u}) = \max_{t>0} I_p(t\bar{u})$, where $M_p$ is defined in (3.2). Moreover, we claim that $t_2 \le 1$. Otherwise, $t_2 > 1$. Noting that $\langle I_p'(t_2\bar{u}), t_2\bar{u}\rangle = 0$, then we infer that

$$\begin{aligned}\int \left(a|\nabla \bar{u}|^2 + V_p(x)\bar{u}^2\right) + b\left(\int |\nabla \bar{u}|^2\right)^2 &> \frac{1}{t_2^2}\int \left(a|\nabla \bar{u}|^2 + V_p(x)\bar{u}^2\right) + b\left(\int |\nabla \bar{u}|^2\right)^2 \\ &= \int \frac{f_p(x, t_2\bar{u})t_2\bar{u}}{t_2^4} + t_2^2 \int K_p(x)\bar{u}^6 \\ &> \int f_p(x, \bar{u})\bar{u} + \int K_p(x)\bar{u}^6,\end{aligned}$$

where we used the condition (H$_4$)-(iii). This contradicts with (3.23). So $t_2 \le 1$. Note that

$$\int \tilde{G}_p(x, t_2\bar{u}) \le \int \tilde{G}_p(x, \bar{u}).$$

Then

$$\begin{aligned} c_p \le I_p(t_2\bar{u}) - \frac{1}{4}\langle I_p'(t_2\bar{u}), t_2\bar{u}\rangle &= \int \tilde{G}_p(x, t_2\bar{u}) \\ \le \int \tilde{G}_p(x, \bar{u}) = I_p(\bar{u}) - \frac{1}{4}\langle I_p'(\bar{u}), \bar{u}\rangle &\le c, \end{aligned} \tag{3.26}$$

since (3.24). With the use of Lemma 3.6, we get that $c \le c_p$. Consequently, $t_2 = 1$ and then

$$\langle I_p'(\bar{u}), \bar{u}\rangle = 0, \qquad c_p = I_p(\bar{u}) = c. \tag{3.27}$$

Using Lemma 3.1 (i), there exists $\tilde{t} > 0$ such that $\tilde{t}\bar{u} \in M$. Then by Lemma 3.5 and (3.27) we infer

$$c \le I(\tilde{t}\bar{u}) \le I_p(\tilde{t}\bar{u}) \le I_p(\bar{u}) = c.$$

Then $I(\tilde{t}\bar{u}) = c$.

In a word, we deduce that $c$ is attained, and then the corresponding minimizer is a ground state of (KH). Below we shall look for a positive ground state for (KH). Assume that the ground state we found is $u_0$. Then $u_0 \in M$ and $I(u_0) = c$. By Lemma 3.1 (i) there exists $t_0 > 0$ such that $t_0|u_0| \in M$. Then $I(t_0|u_0|) \ge c$. Noting that $I(t_0|u_0|) \le I(t_0u_0)$ by (H$_6$) and $I(t_0u_0) \le I(u_0)$, we get $I(t_0|u_0|) \le c$. So $I(t_0|u_0|) = c$. Then $t_0|u_0|$ is also a ground state of (KH). Applying the maximum principle to (KH), we easily infer that $t_0|u_0| > 0$ by (2.3). Namely, we find a positive ground state for (KH). This ends the proof. □

## 4. Eq. (QH)

In this section we describe the variational framework for the problem (QH), and prove Theorem 1.2.

### *4.1. The new method of Nehari manifold*

In order to find ground states, we usually use the method of Nehari manifold [19]. A very important condition using the method of Nehari manifold is that the functional has a unique maximum point along the direction of nontrivial $u$. However, the lack of the higher-order term in the nonlinearity and the competing effect of the nonlocal term $-b(\int |\nabla u|^2)\Delta u$ with the nonlinearity $Q(x)u^3$ cause that $J(tu)$ $(t \ge 0)$ may not have the maximum, and therefore the standard method of Nehari manifold cannot be used. Partially inspired by [5], we find that if we restrict the functional $J(tu)$ in a set, then the functional has a unique maximum. Then we can use the one-to-one correspondence of the functionals on the manifold and the intersection of the above set and the unit sphere to improve the method of Nehari manifold in [19], and therefore find ground states.

Below we firstly give the Nehari manifold: the Nehari manifold $N$ corresponding to $J$ is

$$N = \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : \langle J'(u), u\rangle = 0\},$$

where

$$\langle J'(u), u\rangle = \|u\|^2 + b\left(\int |\nabla u|^2\right)^2 - \int Q(x)u^4,$$

and the least energy on $N$ is defined by $d := \inf_N J$.

**Lemma 4.1.** *Let $V, Q \in L^\infty(\mathbb{R}^3)$ be such that $\inf_{\mathbb{R}^3} V > 0$ and $\inf_{\mathbb{R}^3} Q > 0$. Then $J$ is coercive on $N$.*

**Proof.** For all $u \in N$, we have

$$J(u) = J(u) - \frac{1}{4}\langle J'(u), u\rangle = \frac{1}{4}\|u\|^2. \tag{4.1}$$

Then $J|_N$ is coercive. □

By the above statement, we need a new set to construct the variational framework. We define

$$\varTheta := \left\{u \in H^1(\mathbb{R}^3) : b\left(\int |\nabla u|^2\right)^2 < \int Q(x)u^4\right\}.$$

It is easy to see that $\varTheta \neq \varnothing$ since $b > 0$ and $Q > q_0 > 0$ by ($Q_2$).

Set

$$h(t) := J(tu) = \frac{t^2}{2}\|u\|^2 + \frac{t^4}{4}\left[b\left(\int |\nabla u|^2\right)^2 - \int Q(x)u^4\right].$$

**Lemma 4.2.** *Under the assumptions of Lemma 4.1, we have that:*

(i) *For all $u \in \varTheta$, there exists a unique $t_u > 0$ such that $h'(t) > 0$ for $0 < t < t_u$, and $h'(t) < 0$ for $t > t_u$. Moreover, $t_u u \in N$ and $J(t_u u) = \max_{t>0} J(tu)$.*

(ii) *For each compact subset $W$ of $\varTheta \cap S_1$, there exists $C_W > 0$ such that $t_w \le C_W$ for all $w \in W$.*

**Proof.** (i) For each $u \in \varTheta$, one easily has that $h(t) > 0$ when $t$ is sufficiently small, and $h(t) < 0$ when $t$ is large enough. Then $h$ has a positive maximum point in $(0, \infty)$. Moreover, the maximum point $t$ satisfies that

$$\|u\|^2 = t^2\left[\int Q(x)u^4 - b\left(\int |\nabla u|^2\right)^2\right].$$

Then the maximum point is unique, and denoted by $t_u$. Therefore the conclusion (i) yields.

(ii) Suppose that there exist a compact subset $W \subset \varTheta \cap S_1$ and a sequence $w_n \in W$ such that $t_{w_n} \to \infty$. Assume $w \in W$ satisfies $w_n \to w$ in $H^1(\mathbb{R}^3)$. Without loss of generality, one easily has that

$$b\left(\int |\nabla w_n|^2\right)^2 - \int Q(x)w_n^4 \to b\left(\int |\nabla w|^2\right)^2 - \int Q(x)w^4 < 0.$$

So

$$\frac{J(t_{w_n}w_n)}{t_{w_n}^2} = \frac{1}{2} + \frac{t_{w_n}^2}{4}\left[b\left(\int |\nabla w_n|^2\right)^2 - \int Q(x)w_n^4\right] \to -\infty.$$

However, by (4.1), we know that $J(t_{w_n}w_n) \geq 0$. This is a contradiction. This ends the proof. □

**Lemma 4.3.** *Under the assumptions of Lemma 4.1,*

(1) *There exists $\rho > 0$ such that $\inf_{S_\rho} J > 0$ and then $d = \inf_N J \geq \inf_{S_\rho} J > 0$, where $S_\rho = \{u \in H^1(\mathbb{R}^3) : \|u\|^2 = \rho\}$.*
(2) *$\|u\|^2 \geq 4d$ for all $u \in N$.*

**Proof.** One easily has that there exists $\rho > 0$ such that $\inf_{S_\rho} J > 0$. For any $u \in N$, there is $t > 0$ such that $tu \in S_\rho$. As a consequence of Lemma 4.2 (i), $J(u) \geq J(tu)$, then $\inf_{S_\rho} J \leq \inf_N J = d$. Hence $d > 0$. Then the conclusion (1) yields. By (4.1), the conclusion (2) easily yields. □

From Lemma 4.3 (1), we define the mapping $\hat{m} : \Theta \to N$ by $\hat{m}(u) = t_u u$. In addition, $\forall v \in \mathbb{R}^+ u$ we have $\hat{m}(v) = \hat{m}(u)$. Let $U := \Theta \cap S_1$, we easily infer that $U$ is an open subset of $S_1$. Define $m_0 := \hat{m}|_U$. Then $m_0$ is a bijection from $U$ to $N$. Moreover, by Lemmas 4.2 and 4.3, as in the proof of [19, Proposition 3.1], we have

**Lemma 4.4.** *Under the assumptions of Lemma 4.1, the mapping $m_0$ is a homeomorphism between $U$ and $N$.*

Considering the functional $\Phi : U \to \mathbb{R}$ given by $\Phi(w) := J(m_0(w))$, and we easily deduce that:

**Lemma 4.5.** *Under the assumptions of Lemma 4.1, the following results hold:*

(1) *If $\{w_n\}$ is a PS sequence for $\Phi$, then $\{m_0(w_n)\}$ is a PS sequence for $J$. If $\{u_n\} \subset N$ is a bounded PS sequence for $J$, then $\{m_0^{-1}(u_n)\}$ is a PS sequence for $\Phi$.*
(2) *$w$ is a critical point of $\Phi$ if and only if $m_0(w)$ is a nontrivial critical point of $J$. Moreover, $\inf_N J = \inf_U \Phi$.*
(3) *A minimizer of $J$ on $N$ is a ground state of (QH).*

From Lemma 4.5 (3), we know that the problem of seeking for a ground state of (QH) can be reduced into that of finding a minimizer of $J|_N$. Since (QH) is non-periodic, similar to the proof of Theorem 1.1, we shall solve the above minimization problem by means of the periodicity of $(\mathrm{QH})_p$, and the relationship of the functionals and derivatives of (QH) and $(\mathrm{QH})_p$.

One easily has that:

**Lemma 4.6.** *Let $(V_1)$, $(V_2)$, $(Q_1)$ and $(Q_2)$ hold. Then $J(u) \leq J_p(u)$, for all $u \in H^1(\mathbb{R}^3)$.*

### *4.2. Proof of Theorem 1.2*

We now give the proof of Theorem 1.2.

**Proof of Theorem 1.2.** Assume that $w_n \in U$ satisfies $\Phi(w_n) \to \inf_U \Phi$. By the Ekeland variational principle, we may suppose that $\Phi'(w_n) \to 0$. Then from Lemma 4.5 (1) it follows that $J'(u_n) \to 0$, where $u_n = m_0(w_n) \in N$. By Lemma 4.5 (2), we have $J(u_n) = \Phi(w_n) \to d$. By Lemma 4.1, we get that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Up to a subsequence, we assume that $u_n \rightharpoonup \tilde{u}$ in $H^1(\mathbb{R}^3)$, $u_n \to \tilde{u}$ in $L^2_{loc}(\mathbb{R}^3)$ and $u_n \to \tilde{u}$ a.e. on $\mathbb{R}^3$. We discuss for two cases where $\tilde{u} \neq 0$ and $\tilde{u} = 0$.

*Case* 1: $\tilde{u} \neq 0$.

We first claim that $J'(\tilde{u}) = 0$. Since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$, passing to a subsequence, we may assume that there exists $l \geq 0$ such that $\int |\nabla u_n|^2 \to l^2$. Note that $J'(u_n) \to 0$, then $\tilde{u}$ is a solution of the following equation

$$-\big(a + bl^2\big)\triangle u + V(x)u = Q(x)u^3, \quad u \in H^1\big(\mathbb{R}^3\big).$$

It suffices to show that $l^2 = \int |\nabla \tilde{u}|^2$. From the weakly lower semi-continuous of the norm it follows that $l^2 \geq \int |\nabla \tilde{u}|^2$. Then

$$\left(a + b\int |\nabla \tilde{u}|^2\right) \int |\nabla \tilde{u}|^2 + \int V(x)\tilde{u}^2 \leq \big(a + bl^2\big) \int |\nabla \tilde{u}|^2 + \int V(x)\tilde{u}^2 = \int Q(x)\tilde{u}^4. \tag{4.2}$$

So $\langle J'(\tilde{u}), \tilde{u}\rangle \leq 0$. By Lemma 4.2 (i), we get that there exists $t_1 > 0$ such that $t_1\tilde{u} \in N$. Then we claim that $t_1 \leq 1$. Otherwise, $t_1 > 1$. Noting that $\langle J'(t_1\tilde{u}), t_1\tilde{u}\rangle = 0$, then we infer that

$$\int \big(a|\nabla \tilde{u}|^2 + V(x)\tilde{u}^2\big) + b\bigg(\int |\nabla \tilde{u}|^2\bigg)^2 > \frac{1}{t_1^2} \int \big(a|\nabla \tilde{u}|^2 + V(x)\tilde{u}^2\big) + b\bigg(\int |\nabla \tilde{u}|^2\bigg)^2 = \int Q(x)\tilde{u}^4.$$

This contradicts with (4.2). So $t_1 \leq 1$. Then

$$\begin{aligned} d &\leq J(t_1\tilde{u}) - \frac{1}{4}\big\langle J'(t_1\tilde{u}), t_1\tilde{u}\big\rangle = \frac{t_1^2}{4}\|\tilde{u}\|^2 \\ &\leq \frac{1}{4}\|\tilde{u}\|^2 \leq \frac{1}{4}\|u_n\|^2 + o_n(1) \\ &= J(u_n) - \frac{1}{4}\big\langle J'(u_n), u_n\big\rangle + o_n(1) = d + o_n(1), \end{aligned} \tag{4.3}$$

where we used Fatou Lemma. So $t_1 = 1$ and $\int |\nabla u_n|^2 \to \int |\nabla \tilde{u}|^2$. Then, $l^2 = \int |\nabla \tilde{u}|^2$. Therefore, $J'(\tilde{u}) = 0$.

By (4.3), we get

$$d = J(\tilde{u}) - \frac{1}{4}\big\langle J'(\tilde{u}), \tilde{u}\big\rangle.$$

Note that $J'(\tilde{u}) = 0$, then $J(\tilde{u}) = d$.

*Case* 2: $\tilde{u} = 0$.

By $d > 0$ in Lemma 4.3, it is easy to see that $\{u_n\}$ is non-vanishing. Then there exists $x_n \in \mathbb{R}^3$ and $\delta_0 > 0$ such that

$$\int\limits_{B_1(x_n)} u_n^2(x)dx > \delta_0. \tag{4.4}$$

Without loss of generality, we assume that $x_n \in \mathbb{Z}^3$. Since $u_n \to \tilde{u}$ in $L^2_{loc}(\mathbb{R}^3)$ and $\tilde{u} = 0$, we may suppose that $|x_n| \to \infty$ up to a subsequence. Denote $\bar{u}_n$ by $\bar{u}_n(\cdot) = u_n(\cdot + x_n)$. Similarly, passing to a subsequence, we assume that $\bar{u}_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^3)$, $\bar{u}_n \to \bar{u}$ in $L^2_{loc}(\mathbb{R}^3)$, and $\bar{u}_n \to \bar{u}$ a.e. on $\mathbb{R}^3$. By (4.4) we have $\bar{u} \neq 0$.

For all $\psi \in H^1(\mathbb{R}^3)$, set $\psi_n(\cdot) := \psi(\cdot + x_n)$. With the use of $|Q(x)u^3| \leq C(1 + |u|^3)$, from Lemma 3.8, replacing $\varphi_n$ by $\psi_n$ it follows that

$$\int \big(V(x) - V_p(x)\big) u_n \psi_n \to 0,$$
$$\int \big(Q(x) - Q_p(x)\big) u_n^3 \psi_n \to 0.$$

Consequently

$$\big\langle J'(u_n), \psi_n \big\rangle - \big\langle J_p'(u_n), \psi_n \big\rangle \to 0.$$

Noting that $J'(u_n) \to 0$ and $\|\psi_n\| = \|\psi\|$, we have $\langle J'(u_n), \psi_n\rangle \to 0$. So

$$\big\langle J_p'(u_n), \psi_n \big\rangle \to 0.$$

Moreover, by the periodicity of $V_p$ and $Q_p$ in the variable $x$ and $x_n \in \mathbb{Z}^3$, we get

$$\big\langle J_p'(\bar{u}_n), \psi \big\rangle = \big\langle J_p'(u_n), \psi_n \big\rangle.$$

Then $\langle J_p'(\bar{u}_n), \psi\rangle \to 0$. By the arbitrary of $\psi$, $J_p'(\bar{u}_n) \rightharpoonup 0$ in $(H^1(\mathbb{R}^3))^*$.

Since $\{\bar{u}_n\}$ is bounded in $H^1(\mathbb{R}^3)$, passing to a subsequence, we may assume that there exists $l_0 \geq 0$ such that $\int |\nabla \bar{u}_n|^2 \to l_0^2$. Note that $J_p'(\bar{u}_n) \rightharpoonup 0$, then $\bar{u}$ is a solution of the following equation

$$-\big(a + bl_0^2\big) \triangle u + V_p(x) u = Q_p(x) u^3, \quad u \in H^1\big(\mathbb{R}^3\big).$$

From the weakly lower semi-continuous of the norm it follows that $l_0^2 \geq \int |\nabla \bar{u}|^2$. Then

$$\left(a + b\int |\nabla \bar{u}|^2\right) \int |\nabla \bar{u}|^2 + \int V_p(x) \bar{u}^2 \leq \big(a + bl_0^2\big) \int |\nabla \bar{u}|^2 + \int V_p(x)\bar{u}^2 = \int Q_p(x) \bar{u}^4. \tag{4.5}$$

Now we prove that

$$J_p(\bar{u}) - \frac{1}{4}\big\langle J_p'(\bar{u}), \bar{u}\big\rangle \leq d. \tag{4.6}$$

Replacing $\varphi_n$ by $u_n$, Lemma 3.8 yields

$$\int \big(V(x) - V_p(x)\big) u_n^2 \to 0.$$

Then

$$\begin{aligned} J_p(\bar{u}) - \frac{1}{4}\big\langle J_p'(\bar{u}), \bar{u}\big\rangle &= \frac{1}{4}\int \big(a|\nabla \bar{u}|^2 + V_p(x)\bar{u}^2\big) \leq \frac{1}{4}\int \big(a|\nabla \bar{u}_n|^2 + V_p(x)\bar{u}_n^2\big) + o_n(1) \\ &= \frac{1}{4}\int \big(a|\nabla u_n|^2 + V_p(x) u_n^2\big) + o_n(1) = \frac{1}{4}\int \big(a|\nabla u_n|^2 + V(x) u_n^2\big) + o_n(1) \\ &= J(u_n) - \frac{1}{4}\big\langle J'(u_n), u_n\big\rangle + o_n(1) = d + o_n(1). \end{aligned}$$

Hence $J_p(\bar{u}) - \frac{1}{4}\langle J_p'(\bar{u}), \bar{u}\rangle \leq d$.

Letting

$$N_p := \big\{u \in H^1\big(\mathbb{R}^3\big)\backslash\{0\} : \big\langle J_p'(u), u\big\rangle = 0\big\},$$
$$\Theta_p := \left\{u \in H^1\big(\mathbb{R}^3\big) : b\left(\int |\nabla u|^2\right)^2 < \int Q_p(x) u^4\right\}, \qquad d_p := \inf_{N_p} J_p.$$

One easily has that $\bar{u} \in \Theta_p$ by (4.5). Similar to Lemma 4.2 (i), there exists $t_2 > 0$ such that $t_2\bar{u} \in N_p$, and $J_p(t_2\bar{u}) = \max_{t>0} J_p(t\bar{u})$. Moreover, we claim that $t_2 \leq 1$. Otherwise, $t_2 > 1$. Noting that $\langle J_p'(t_2\bar{u}), t_2\bar{u}\rangle = 0$, then we infer that

$$\int \left(a|\nabla \bar{u}|^2 + V_p(x)\bar{u}^2\right) + b\left(\int |\nabla \bar{u}|^2\right)^2 > \frac{1}{t_2^2}\int \left(a|\nabla \bar{u}|^2 + V_p(x)\bar{u}^2\right) + b\left(\int |\nabla \bar{u}|^2\right)^2 = \int Q_p(x)\bar{u}^4.$$

This contradicts with (4.5). So $t_2 \leq 1$. Then by (4.6) we conclude that

$$\begin{aligned} d_p &\leq J_p(t_2\bar{u}) - \frac{1}{4}\left\langle J_p'(t_2\bar{u}), t_2\bar{u}\right\rangle \\ &= \frac{t_2^2}{4}\|\bar{u}\|_p^2 \leq \frac{1}{4}\|\bar{u}\|_p^2 = J_p(\bar{u}) - \frac{1}{4}\left\langle J_p'(\bar{u}), \bar{u}\right\rangle \leq d. \end{aligned} \tag{4.7}$$

With the use of Lemma 4.6, as Lemma 3.6, we infer that

$$d = \inf_{w\in U}\max_{t>0} J(tw) \leq \inf_{w\in \Theta_p\cap S_1}\max_{t>0} J_p(tw) = d_p.$$

Then from (4.7) it follows that $t_2 = 1$ and

$$d_p = J_p(\bar{u}) = d. \tag{4.8}$$

By (4.5) and $Q \geq Q_p$, we have $\bar{u} \in \Theta$. Using Lemma 4.2 (i), there exists $\tilde{t} > 0$ such that $\tilde{t}\bar{u} \in N$. Then by Lemma 4.6 and (4.8), we infer

$$d \leq J(\tilde{t}\bar{u}) \leq J_p(\tilde{t}\bar{u}) \leq J_p(\bar{u}) = d.$$

Then $J(\tilde{t}\bar{u}) = d$.

In a word, we deduce that $d$ is attained, and then the corresponding minimizer is a ground state of (QH). Below we shall look for a positive ground state for (QH). Assume that the ground state we found is $u_0$. Then $u_0 \in N$ and $J(u_0) = d$. We easily have that $u_0 \in \Theta$. Then $|u_0| \in \Theta$. By Lemma 4.2 (i) there exists $t_0 > 0$ such that $t_0|u_0| \in N$. Then $J(t_0|u_0|) \geq d$. Noting that $J(t_0|u_0|) \leq J(t_0u_0)$ and $J(t_0u_0) \leq J(u_0)$, we get $J(t_0|u_0|) \leq d$. So $J(t_0|u_0|) = d$. Then $t_0|u_0|$ is also a ground state of (QH). Applying the maximum principle to (QH), we easily infer that $t_0|u_0| > 0$. Namely, we find a positive ground state for (QH). This ends the proof. □

## References


[1] A. Arosio, S. Panizzi, On the well-posedness of the Kirchhoff string, Trans. Amer. Math. Soc. 348 (1996) 305–330.
[2] S. Bernstein, Sur une classe d'équations fonctionelles aux dérivées partielles, Bull. Acad. Sci. URSS Sér. Math. 4 (1940) 17–26.
[3] M.M. Cavalcanti, V.N. Domingos Cavalcanti, J.A. Soriano, Global existence and uniform decay rates for the Kirchhoff–Carrier equation with nonlinear dissipation, Adv. Differential Equations 6 (2001) 701–730.
[4] C.Y. Chen, Y.C. Kuo, T.F. Wu, The Nehari manifold for a Kirchhoff type problem involving sign-changing weight functions, J. Differential Equations 250 (2011) 1876–1908.
[5] X.D. Fang, A. Szulkin, Multiple solutions for a quasilinear Schrödinger equation, J. Differential Equations 254 (2013) 2015–2032.
[6] X.M. He, W.M. Zou, Existence and concentration behavior of positive solutions for a Kirchhoff equation in $\mathbb{R}^3$, J. Differential Equations 2 (2012) 1813–1834.
[7] G. Kirchhoff, Mechanik, Teubner, Leipzig, 1883.
[8] G.B. Li, H.Y. Ye, Existence of positive ground state solutions for the nonlinear Kirchhoff type equations in $\mathbb{R}^3$, J. Differential Equations 257 (2014) 566–600.
[9] S.H. Liang, J.H. Zhang, Existence of solutions for Kirchhoff type problems with critical nonlinearity in $\mathbb{R}^3$, Nonlinear Anal. 17 (2014) 126–136.

[10] Haendel F. Lins, Elves A.B. Silva, Quasilinear asymptotically periodic elliptic equations with critical growth, Nonlinear Anal. 71 (2009) 2890–2905.

[11] J.L. Lions, On some questions in boundary value problems of mathematical physics, in: Contemporary Development in Continuum Mechanics and Partial Differential Equations, in: North-Holland Math. Stud., vol. 30, North-Holland, Amsterdam, New York, 1978, pp. 284–346.

[12] T.F. Ma, J.E. Munoz Rivera, Positive solutions for a nonlinear nonlocal elliptic transmission problem, Appl. Math. Lett. 16 (2003) 243–248.

[13] A. Mao, Z.T. Zhang, Sign-changing and multiple solutions of Kirchhoff type problems without the P.S. condition, Nonlinear Anal. 70 (2009) 1275–1287.

[14] J.J. Nie, Existence and multiplicity of nontrivial solutions for a class of Schrödinger–Kirchhoff-type equations, J. Math. Anal. Appl. 417 (2014) 65–79.

[15] K. Perera, Z.T. Zhang, Nontrivial solutions of Kirchhoff-type problems via the Yang index, J. Differential Equations 221 (2006) 246–255.

[16] S.I. Pohožaev, A certain class of quasilinear hyperbolic equations, Mat. Sb. (N.S.) 96 (1975) 152–166 (in Russian).

[17] E.A.B. Silva, G.F. Vieira, Quasilinear asymptotically periodic Schrödinger equations with critical growth, Calc. Var. Partial Differential Equations 39 (2010) 1–33.

[18] J.T. Sun, T.F. Wu, Ground state solutions for an indefinite Kirchhoff type problem with steep potential well, J. Differential Equations 256 (2014) 1771–1792.

[19] A. Szulkin, T. Weth, The method of Nehari manifold, in: Handbook of Nonconvex Analysis and Applications, Int. Press, Somerville, MA, 2010.

[20] J. Wang, L.X. Tian, J.X. Xu, F.B. Zhang, Multiplicity and concentration of positive solutions for a Kirchhoff type problem with critical growth, J. Differential Equations 253 (2012) 2314–2351.

[21] J. Wang, F.B. Zhang, J.X. Xu, On the concentration of positive solutions for a Kirchhoff type problem with competing potentials, preprint.

[22] M. Willem, Minimax Theorems, Progr. Nonlinear Differential Equations Appl., vol. 24, Birkhäuser, Basel, 1996.

[23] X. Wu, Existence of nontrivial solutions and high energy solutions for Schrödinger–Kirchhoff-type equations in $\mathbb{R}^N$, Nonlinear Anal. 12 (2011) 1278–1287.

[24] H. Zhang, J.X. Xu, F.B. Zhang, Ground state solutions for asymptotically periodic Schrödinger equations with critical growth, Electron. J. Differential Equations 2013 (227) (2013), 16 pp.

[25] J. Zhang, X.H. Tang, W. Zhang, Existence of multiple solutions of Kirchhoff type equation with sign-changing potential, Appl. Math. Comput. 242 (2014) 491–499.

[26] Z.T. Zhang, K. Perera, Sign changing solutions of Kirchhoff type problems via invariant sets of descent flow, J. Math. Anal. Appl. 317 (2006) 456–463.