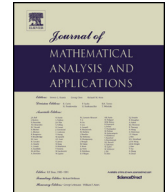




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Positive ground state solutions and multiple nontrivial solutions for coupled critical elliptic systems

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ABSTRACT

In this study, we consider the following coupled elliptic system with a Sobolev critical exponent:

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \nu_1 u_1^{p_1-1} + \mu_1 u_1^{2^*-1} + \beta u_1^{\frac{2^*}{2}-1} u_2^{\frac{2^*}{2}}, & x \in \Omega, \\ -\Delta u_2 + \lambda_2 u_2 = \nu_2 u_2^{p_2-1} + \mu_2 u_2^{2^*-1} + \beta u_1^{\frac{2^*}{2}} u_2^{\frac{2^*}{2}-1}, & x \in \Omega, \\ u_1, u_2 \geq 0 \text{ in } \Omega, \quad u_1 = u_2 = 0 \text{ on } \partial\Omega, \end{cases} \quad (\mathcal{P})$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $N \geq 5$, $2 < p_1, p_2 < 2^*$, $\lambda_j \in (-\lambda_1(\Omega), 0)$, $\mu_j > 0$ for $j = 1, 2$, and $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$ with the Dirichlet boundary condition. We demonstrate the existence of a positive ground state solution for problem $(\mathcal{P})$ when the coupling parameter $\beta \geq -\sqrt{\mu_1 \mu_2}$. Under some other conditions, we show the nonexistence of positive solutions for $(\mathcal{P})$ when $N \geq 3$. We also construct multiple nontrivial solutions and sign-changing solutions for the following system:

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \mu_1 u_1^3 + \beta u_2^2 u_1, & x \in \Omega, \\ -\Delta u_2 + \lambda_2 u_2 = \mu_2 u_2^3 + \beta u_1^2 u_2, & x \in \Omega, \\ u_1 = u_2 = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and $N \leq 4$.

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1. Introduction

In recent years, there have been extensive mathematical investigations of the following coupled elliptic system:

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$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \mu_1 u_1^3 + \beta u_2^2 u_1, & x \in \Omega, \\ -\Delta u_2 + \lambda_2 u_2 = \mu_2 u_2^3 + \beta u_1^2 u_2, & x \in \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain or $\Omega = \mathbb{R}^N$. System (1.1) arises when we consider the standing wave solutions to the following time-dependent Schrödinger system, which comprises two coupled Gross–Pitaevskii equations:

$$\begin{cases} -i \frac{\partial}{\partial t} \Phi_1 = \Delta \Phi_1 + \mu_1 |\Phi_1|^2 \Phi_1 + \beta |\Phi_2|^2 \Phi_1, & x \in \Omega, \ t > 0, \\ -i \frac{\partial}{\partial t} \Phi_2 = \Delta \Phi_2 + \mu_2 |\Phi_2|^2 \Phi_2 + \beta |\Phi_1|^2 \Phi_2, & x \in \Omega, \ t > 0, \\ \Phi_j = \Phi_j(x, t) \in \mathbb{C}, \ j = 1, 2, \\ \Phi_j(x, t) = 0, \ x \in \partial\Omega, \ t > 0, \ j = 1, 2, \end{cases} \quad (1.2)$$

which has many applications in physics and nonlinear optics (see [27]). Physically, the solution Φ_j denotes the j th component of the beam in Kerr-like photorefractive media (see [1]), μ_j represents self-focusing in the j th component, and the coupling constant β is the interaction between the two components of the beam. System (1.2) also arises in the Hartree–Fock theory for a double condensate, which is a binary mixture of Bose–Einstein condensates in two different hyperfine states $|1\rangle$ and $|2\rangle$ (see [18] and the references therein). In order to obtain solitary wave solutions of system (1.2), we set $\Phi_j(x, t) = e^{i\lambda_j t} u_j(x)$ for $j = 1, 2$. Then, it is reduced to system (1.1). The existence of least energy and other finite energy solutions were studied by [2,6,7,13,15–17,20,25–27,30,31] and the references therein. The existence and multiplicity of positive and sign-changing solutions were studied by [2,5–7,10,11,13,20–24,26,29] and the references therein.

However, we note that most of the studies mentioned above considered the subcritical case, i.e., $N \leq 3$. When $N = 4$, which is the critical case ($2^* := \frac{2N}{N-2} = 4$), the authors of [12] demonstrated the existence of a positive least energy solution for all negative β , positive small β , and positive large β . Later, in [14], they considered the following critical case for higher dimensional $N \geq 5$:

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \mu_1 u_1^{2^*-1} + \beta u_1^{\frac{2^*}{2}-1} u_2^{\frac{2^*}{2}}, & x \in \Omega, \\ -\Delta u_2 + \lambda_2 u_2 = \mu_2 u_2^{2^*-1} + \beta u_1^{\frac{2^*}{2}} u_2^{\frac{2^*}{2}-1}, & x \in \Omega, \\ u_1, u_2 \geq 0 \text{ in } \Omega, \ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.3)$$

In [19], the authors also considered system (1.3) and showed that when $N \geq 3$, it has a positive solution for $\beta \rightarrow 0$ or $+\infty$. When $N = 4$, system (1.3) returns to (1.1). Interestingly, the authors of [14] obtained different results from the case when $N = 4$. Indeed, they showed that system (1.3) has a positive least energy solution for any $\beta \neq 0$. When $\beta = 0$, system (1.3) is reduced to the following critical exponent equation, which was first studied by Brezis and Nirenberg in [8]:

$$\begin{cases} -\Delta u_j + \lambda_j u_j = \mu_j u_j^{2^*-1}, & x \in \Omega, \\ u_j \geq 0 \text{ in } \Omega, \ u_j = 0 & \text{on } \partial\Omega, \ j = 1, 2. \end{cases} \quad (1.4)$$

In [8], they also included the following more general case:

$$\begin{cases} -\Delta u + \lambda_j u = \nu_j g_j(x, u) + \mu_j u^{2^*-1}, & x \in \Omega, \\ u \geq 0 \text{ in } \Omega, \ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.5)$$

By taking $g_j(x, u) = u^{p_j-1}$, where $2 < p_j < 2^*$ for $j = 1, 2$, $N \geq 4$, $\nu_j, \mu_j > 0$, $\lambda_j \in (-\lambda_1(\Omega), 0)$, problem (1.5) has positive ground state solutions $v_1, v_2 \in C^2(\Omega) \cap C(\bar{\Omega})$ for $j = 1, 2$ respectively, with the energy

$$0 < B_j := \frac{1}{2} \int_{\Omega} (|\nabla v_j|^2 + \lambda_j v_j^2) - \frac{1}{p_j} \int_{\Omega} \nu_j v_j^{p_j} - \frac{1}{2^*} \int_{\Omega} \mu_j v_j^{2^*} < \frac{1}{N} \mu_j^{-\frac{N-2}{2}} S^{\frac{N}{2}}, \quad (1.6)$$

where S is the sharp embedding constant from $D^{1,2}(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$.

It is natural to ask whether the more general system that corresponds to system (1.3) has a similar result related to the existence of positive ground state solutions, in the same way as the results related to Eqs. (1.4) and (1.5). In particular, we consider the following more general critical elliptic system:

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \nu_1 u_1^{p_1-1} + \mu_1 u_1^{2^*-1} + \beta u_1^{\frac{2^*}{2}-1} u_2^{\frac{2^*}{2}}, & x \in \Omega, \\ -\Delta u_2 + \lambda_2 u_2 = \nu_2 u_2^{p_2-1} + \mu_2 u_2^{2^*-1} + \beta u_1^{\frac{2^*}{2}} u_2^{\frac{2^*}{2}-1}, & x \in \Omega, \\ u_1, u_2 \geq 0 \text{ in } \Omega, \quad u_1 = u_2 = 0 \text{ on } \partial\Omega. \end{cases} \quad (1.7)$$

We obtain the following results.

Theorem 1.1. *If we assume that $N \geq 5$, $2 < p_j < 2^*$, $\lambda_j \in (-\lambda_1(\Omega), 0)$, $\nu_j, \mu_j > 0$, for $j = 1, 2$, then system (1.7) has a positive ground state solution for any $\beta \geq -\sqrt{\mu_1 \mu_2}$.*

We can then obtain the result related to the nonexistence of solutions to (1.7).

Theorem 1.2. *If we assume that $N \geq 3$, $2 < p_1, p_2 < 2^*$, then system (1.7) has no positive solutions if one of the following conditions holds:*

- (1) $\lambda_j \leq -\lambda_1(\Omega)$, $\nu_j, \mu_j > 0$ for $j = 1, 2$, and $\beta > 0$;
- (2) $\mu_j, \beta \in \mathbb{R}$, $\lambda_j \geq 0$, $\nu_j \leq 0$ for $j = 1, 2$, and Ω is star-shaped.

Consider the case where $\lambda_1 = \lambda_2 = \lambda$, $p_1 = p_2 = p$. Let v be a positive solution of

$$\Delta u + \lambda u = \nu u^{p-1} + u^{2^*-1}, \quad u \in H_0^1(\Omega),$$

where $\nu > 0$. By Lemma 2.1 in [14], the following system has a solution:

$$\begin{cases} \mu_1 a_1^{2^*-2} + \beta a_1^{\frac{2^*}{2}-2} a_2^{\frac{2^*}{2}} = 1, \\ \mu_2 a_2^{2^*-2} + \beta a_1^{\frac{2^*}{2}} a_2^{\frac{2^*}{2}-2} = 1, \\ a_1, a_2 > 0. \end{cases} \quad (1.8)$$

Then we can construct solutions of (1.7) using v , as described in [12,14,27]. In particular, we have the following result.

Theorem 1.3. *Let (a_1, a_2) be the solution of (1.8). Assume that $N \geq 5$, $-\lambda_1(\Omega) < \lambda_1 = \lambda_2 < 0$, $\mu_1, \mu_2, \beta > 0$ and $\nu_1 = \nu/a_1^{p-2}$, $\nu_2 = \nu/a_2^{p-2}$. Then, $(a_1 v, a_2 v)$ is a positive solution of system (1.7).*

We recall that in [3,9], when $N \geq 4$, $\lambda < 0$, the authors obtained a nontrivial solution for the following equation:

$$\begin{cases} -\Delta u + \lambda u = u|u|^{2^*-2}, & x \in \Omega \\ u = 0 \text{ on } \partial\Omega. \end{cases} \quad (1.9)$$

Denote the nontrivial solution by w , then by a similar method to Theorem 1.3, we can also obtain the following result for system (1.1).

Theorem 1.4. *Assume that $N = 4$, $\lambda_1 = \lambda_2 := \lambda < 0$, and one of the following conditions holds:*

- (1) $\mu_1, \mu_2 > 0$, $\beta \in (-\sqrt{\mu_1 \mu_2}, \min\{\mu_1, \mu_2\}) \cup (\max\{\mu_1, \mu_2\}, \infty)$;

- (2) $\mu_1, \mu_2 < 0, \beta > \sqrt{\mu_1 \mu_2}$;
 (3) $\mu_1 \mu_2 < 0, \beta > \max\{\mu_1, \mu_2\}$.

Then, system (1.1) has four nontrivial solutions of the form $(\pm a_1 w, \pm a_2 w)$, where $a_1, a_2 > 0$ are defined by

$$\begin{cases} a_1^2 := \frac{\mu_2 - \beta}{\mu_1 \mu_2 - \beta^2}, \\ a_2^2 := \frac{\mu_1 - \beta}{\mu_1 \mu_2 - \beta^2}, \end{cases} \quad (1.10)$$

which are well defined by the assumption. Moreover, when $\lambda \leq -\lambda_1(\Omega)$, then the four solutions obtained above are sign-changing solutions, whereas they are signed solutions when $-\lambda_1(\Omega) < \lambda < 0$.

For the subcritical case, we have the following stronger conclusion.

Theorem 1.5. *When the dimension $N \leq 3$, under the same conditions based on μ_1, μ_2, β , as in Theorem 1.4, then system (1.1) has infinitely many sign-changing solutions for any $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$ and four signed solutions for $\lambda > -\lambda_1(\Omega)$.*

Next, we give the proof of Theorem 1.1 for $\beta < 0$ and $\beta > 0$ in Sections 2 and 3, respectively. In Section 4, we prove Theorem 1.2. We present Theorems 1.3–1.5 in Section 5.

2. Proof of Theorem 1.1 for $-\sqrt{\mu_1 \mu_2} \leq \beta < 0$

In this section, we consider the case where $-\sqrt{\mu_1 \mu_2} \leq \beta < 0$. We assume that all the following integrations are taken over Ω unless specified otherwise. Let $\mathbf{H} := H_0^1(\Omega) \times H_0^1(\Omega)$ be the product space and we write $\mathbf{u} = (u_1, u_2)$. Since $\lambda_1, \lambda_2 > -\lambda_1(\Omega)$, we define the equivalent inner product in $H_0^1(\Omega)$ by

$$(u, v)_j = \int \langle \nabla u, \nabla v \rangle + \lambda_j uv, \quad j = 1, 2,$$

and the induced norm $\|\cdot\|_j$. Then, the product space $\mathbf{H}$ is endowed with the inner product, as follows:

$$(\mathbf{u}, \mathbf{v}) = \sum_{j=1}^2 (u_j, v_j)_j, \quad \mathbf{u} = (u_1, u_2), \quad \mathbf{v} = (v_1, v_2),$$

which gives rise to a norm on $\mathbf{H}$ denoted by $\|\cdot\|$. Thus, the energy functional of system (1.7) is

$$I(\mathbf{u}) = \frac{1}{2} \|\mathbf{u}\|^2 - \int \sum_{j=1}^2 \frac{\nu_j}{p_j} |u_j|^{p_j} - \frac{1}{2^*} \int \left(\sum_{j=1}^2 \mu_j |u_j|^{2^*} + 2\beta |u_1|^{\frac{2^*}{2}} |u_2|^{\frac{2^*}{2}} \right).$$

We refer to a solution $\mathbf{u}$ of system (1.7) as a ground state solution if $\mathbf{u}$ is nontrivial and $I(\mathbf{u}) \leq I(\mathbf{U})$ for any other nontrivial solution $\mathbf{U}$ of (1.7). Define the manifold

$$\mathcal{M} := \{(u_1, u_2) \in \mathbf{H} : u_1, u_2 \neq 0, I'(u_1, u_2)(u_1, 0) = I'(u_1, u_2)(0, u_2) = 0\}. \quad (2.1)$$

Then, any nontrivial solution of (1.7) must belong to $\mathcal{M}$ and the constraint in (2.1) means that

$$\begin{aligned} \|u_1\|_1^2 &= \int (\nu_1 |u_1|^{p_1} + \mu_1 |u_1|^{2^*} + \beta |u_1|^{\frac{2^*}{2}} |u_2|^{\frac{2^*}{2}}), \\ \|u_2\|_2^2 &= \int (\nu_2 |u_2|^{p_2} + \mu_2 |u_1|^{2^*} + \beta |u_1|^{\frac{2^*}{2}} |u_2|^{\frac{2^*}{2}}). \end{aligned}$$

By taking $U_1, U_2 \in C_0^\infty(\Omega)$ with $U_1, U_2 \not\equiv 0$ and $\text{supp } U_1 \cap \text{supp } U_2 = \emptyset$, it is easy to see that $t_1, t_2 > 0$ exist such that $(t_1 U_1, t_2 U_2) \in \mathcal{M}$, i.e., $\mathcal{M} \neq \emptyset$. Define

$$B := \inf_{\mathbf{u} \in \mathcal{M}} I(\mathbf{u}).$$

It is known that the existence of nontrivial solutions for critical system (1.7) depends greatly on the existence of ground state solutions of the limit problem, as follows:

$$\begin{cases} -\Delta u_1 = \mu_1 |u_1|^{2^*-2} u_1 + \beta |u_1|^{\frac{2^*}{2}-2} u_1 |u_2|^{\frac{2^*}{2}}, & x \in \mathbb{R}^N, \\ -\Delta u_2 = \mu_2 |u_2|^{2^*-2} u_2 + \beta |u_1|^{\frac{2^*}{2}} |u_2|^{\frac{2^*}{2}-2} u_2, & x \in \mathbb{R}^N, \\ u_1, u_2 \in D^{1,2}(\mathbb{R}^N). \end{cases} \quad (2.2)$$

Let $\mathbf{D} := D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ and the energy functional

$$\bar{I}(\mathbf{u}) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u_1|^2 + |\nabla u_2|^2) - \frac{1}{2^*} \int_{\mathbb{R}^N} (\mu_1 |u_1|^{2^*} + \mu_2 |u_2|^{2^*} + 2\beta |u_1|^{\frac{2^*}{2}} |u_2|^{\frac{2^*}{2}}).$$

Similar to the manifold $\mathcal{M}$, set

$$\bar{\mathcal{M}} := \{(u_1, u_2) \in \mathbf{D} : u_1, u_2 \not\equiv 0, \bar{I}'(u_1, u_2)(u_1, 0) = \bar{I}'(u_1, u_2)(0, u_2) = 0\}.$$

Then, $\bar{\mathcal{M}} \neq \emptyset$ and we define

$$\bar{B} := \inf_{\mathbf{u} \in \bar{\mathcal{M}}} \bar{I}(\mathbf{u}).$$

First, we have a lower bound estimate for B .

Lemma 2.1. *If we assume that $\beta \in [-\sqrt{\mu_1 \mu_2}, 0)$, then I is coercive on $\mathcal{M}$ and $B > 0$.*

Proof. We may assume that $p_2 \geq p_1$ without any loss of generality. For $\mathbf{u} \in \mathcal{M}$, we have

$$\begin{aligned} I(\mathbf{u}) &= I(\mathbf{u}) - \frac{1}{p_1} (I'(\mathbf{u})(u_1, 0) + I'(\mathbf{u})(0, u_2)) \\ &= \left(\frac{1}{2} - \frac{1}{p_1}\right) \|\mathbf{u}\|^2 + \left(\frac{1}{p_1} - \frac{1}{p_2}\right) \int \nu_2 |u_2|^{p_2} \\ &\quad + \left(\frac{1}{p_1} - \frac{1}{2^*}\right) \int \left(\sum_j \mu_j |u_j|^{2^*} + 2\beta |u_1|^{\frac{2^*}{2}} |u_2|^{\frac{2^*}{2}}\right) \\ &\geq \left(\frac{1}{2} - \frac{1}{p_1}\right) \|\mathbf{u}\|^2. \end{aligned} \quad (2.3)$$

Hence,

$$\|\mathbf{u}\|^2 \leq \int \sum_j (\nu_j |u_j|^{p_j} + (\mu_j + |\beta|) |u_j|^{2^*}) \leq C_1 \|\mathbf{u}\|^{\tilde{p}} + C_2 \|\mathbf{u}\|^{2^*}, \quad (2.4)$$

where $C_1, C_2 > 0$ are constants that are independent of $\mathbf{u}$ and $\tilde{p} = p_1$ or p_2 . It follows that a constant $C_0 > 0$ exists such that $\|\mathbf{u}\| \geq C_0$. Thus, $B > 0$. $\square$

We also have an upper bound for B . This idea is similar to that in [14], but it is more complex and some new tricks are used in the current study. We give the details of the proof.

Lemma 2.2. For $\beta < 0$, we have

$$B < \min\{B_1 + \frac{1}{N}\mu_2^{-\frac{N-2}{2}}S^{\frac{N}{2}}, B_2 + \frac{1}{N}\mu_1^{-\frac{N-2}{2}}S^{\frac{N}{2}}, \bar{B}\}.$$

Proof. First, we show that $B < B_1 + \frac{1}{N}\mu_2^{-\frac{N-2}{2}}S^{\frac{N}{2}}$. Recall that v_1 is a solution of (1.5) since $v_1 \in C(\bar{\Omega})$ and $v_1 = 0$ on $\partial\Omega$, so $B(x_0, 2r) := \{x : |x - x_0| \leq 2r\} \subset \Omega$ exists such that

$$\varrho := \max_{B(x_0, 2r)} v_1 < \min\left\{\left(\frac{\int \mu_1 v_1^{2^*}}{2|\beta| \int v_1^2}\right)^{\frac{2}{2^*-2}}, \left(\frac{\mu_2}{2|\beta|}\right)^{\frac{2}{2^*-2}}, \left(\frac{\mu_1 \mu_2 \int v_1^{2^*}}{\beta^2 \int v_1^2}\right)^{\frac{1}{2^*-2}}\right\}.$$

Take a cutoff function $\phi \in C_0^1(B(x_0, 2r))$ with $0 \leq \phi \leq 1$ and $\phi(x) \equiv 1$ for $|x - x_0| \leq r$. Denote

$$U_{\varepsilon, x_0}(x) := \frac{[N(N-2)\varepsilon^2]^{(N-2)/4}}{[\varepsilon^2 + |x - x_0|^2]^{(N-2)/2}}, \quad x \in \mathbb{R}^N.$$

Then (see [4,28]), U_{ε, x_0} is the solution of the equation

$$-\Delta u = |u|^{2^*-2}u, \quad u \in D^{1,2}(\mathbb{R}^N).$$

Let $u_\varepsilon := \phi U_{\varepsilon, x_0}$, then (see [32])

$$\begin{aligned} \int |\nabla u_\varepsilon|^2 &= S^{\frac{N}{2}} + O(\varepsilon^{N-2}), \quad \int |u_\varepsilon|^{2^*} = S^{\frac{N}{2}} + O(\varepsilon^N), \\ \int |u_\varepsilon|^2 &\geq C\varepsilon^2 + O(\varepsilon^{N-2}), \quad \int |u_\varepsilon|^{p_j} \geq C\varepsilon^{N - \frac{(N-2)p_j}{2}} + O(\varepsilon^{\frac{(N-2)p_j}{2}}), \end{aligned} \quad (2.5)$$

where $C > 0$ is a constant and we also have

$$\int |u_\varepsilon|^{\frac{2^*}{2}} \leq C \int_{B(0, 2r)} \left(\frac{\varepsilon}{\varepsilon^2 + |x|^2}\right)^{\frac{N}{2}} = o(\varepsilon^2). \quad (2.6)$$

For any $t, s > 0$, by selecting ϱ and the Hölder inequality, we have

$$\begin{aligned} 2|\beta|t^{\frac{2^*}{2}}s^{\frac{2^*}{2}} \int v_1^{\frac{2^*}{2}} u_\varepsilon^{\frac{2^*}{2}} &\leq \varrho^{\frac{2^*}{2}-1} \cdot 2|\beta|t^{\frac{2^*}{2}}s^{\frac{2^*}{2}} \int v_1 u_\varepsilon^{\frac{2^*}{2}} \\ &\leq \varrho^{\frac{2^*}{2}-1} |\beta|t^{2^*} \int v_1^2 + \varrho^{\frac{2^*}{2}-1} |\beta|s^{2^*} \int u_\varepsilon^{2^*} \\ &\leq \frac{1}{2}t^{2^*} \int \mu_1 v_1^{2^*} + \frac{1}{2}s^{2^*} \int \mu_2 u_\varepsilon^{2^*}. \end{aligned}$$

Then, we have

$$\begin{aligned} I(tv_1, su_\varepsilon) &= \frac{1}{2}t^2\|v_1\|_1^2 - \frac{t^{p_1}}{p_1} \int \nu_1 v_1^{p_1} - \frac{t^{2^*}}{2^*} \int \mu_1 v_1^{2^*} \\ &\quad + \frac{1}{2}s^2\|u_\varepsilon\|_2^2 - \frac{s^{p_2}}{p_2} \int \nu_2 u_\varepsilon^{p_2} - \frac{s^{2^*}}{2^*} \int \mu_2 u_\varepsilon^{2^*} \\ &\quad - \frac{t^{\frac{2^*}{2}}s^{\frac{2^*}{2}}}{2^*} \int 2\beta v_1^{\frac{2^*}{2}} u_\varepsilon^{\frac{2^*}{2}} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2}t^2\|v_1\|_1^2 - \frac{t^{p_1}}{p_1} \int \nu_1 v_1^{p_1} - \frac{t^{2^*}}{2 \cdot 2^*} \int \mu_1 v_1^{2^*} \\
&\quad + \frac{1}{2}s^2\|u_\varepsilon\|_2^2 - \frac{s^{p_2}}{p_2} \int \nu_2 u_\varepsilon^{p_2} - \frac{s^{2^*}}{2 \cdot 2^*} \int \mu_2 u_\varepsilon^{2^*} \\
&:= f_1(t) + f_{2\varepsilon}(s).
\end{aligned}$$

It is known (see [32]) that

$$\max_{s>0} f_{2\varepsilon}(s) < \frac{1}{N} \left(\frac{\mu_2}{2}\right)^{-\frac{N-2}{2}} S^{\frac{N}{2}}.$$

Take a sufficiently large $t_0 > 0$ such that for any $t > t_0$, it holds that

$$f_1(t) + \frac{1}{N} \left(\frac{\mu_2}{2}\right)^{-\frac{N-2}{2}} S^{\frac{N}{2}} < 0.$$

Then,

$$\max_{t,s>0} I(tv_1, su_\varepsilon) = \max_{0<t\leq t_0, s>0} I(tv_1, su_\varepsilon).$$

Let

$$g_{2\varepsilon}(s) := \frac{1}{2}s^2\|u_\varepsilon\|_2^2 - \frac{s^{p_2}}{p_2} \int \nu_2 u_\varepsilon^{p_2} - \frac{s^{2^*}}{2^*} \int \mu_2 u_\varepsilon^{2^*},$$

and thus it is easy to see that a unique maximum point $0 < \tilde{s} < +\infty$ of $g_{2\varepsilon}(s)$ exists such that $g'_{2\varepsilon}(\tilde{s}) = 0$, i.e.,

$$\begin{aligned}
\|u_\varepsilon\|_2^2 &= \tilde{s}^{p_2-2} \int \nu_2 u_\varepsilon^{p_2} + \tilde{s}^{2^*-2} \int \mu_2 u_\varepsilon^{2^*} \\
&\leq (\xi + C_\xi \tilde{s}^{2^*-2}) \|u_\varepsilon\|^{p_2} + C_1 \tilde{s}^{2^*-2} \|u_\varepsilon\|^{2^*},
\end{aligned} \tag{2.7}$$

where $\xi > 0$, $C_\xi > 0$ is a constant that is only dependent on ξ . Therefore, (2.7) implies that

$$\tilde{s}^{2^*-2} \geq \frac{C}{\|u_\varepsilon\|^{p_2-2} + \|u_\varepsilon\|^{2^*-2}}.$$

Thus, by (2.5), some $0 < s_0 < \tilde{s}$ exists, so we have

$$\max_{t,s>0} I(tv_1, su_\varepsilon) = \max_{0<t\leq t_0, s\geq s_0} I(tv_1, su_\varepsilon).$$

For $t \in (0, t_0]$, $s \geq s_0$, by (2.6) and $1 < 2^*/2 < 2$, we have

$$2|\beta|t^{\frac{2^*}{2}}s^{\frac{2^*}{2}} \int v_1^{\frac{2^*}{2}} u_\varepsilon^{\frac{2^*}{2}} \leq 2|\beta|t_0^{\frac{2^*}{2}}s_0^{\frac{2^*}{2}-2}s^2 \varrho^{\frac{2^*}{2}} \int u_\varepsilon^{\frac{2^*}{2}} \leq s^2 \cdot o(\varepsilon^2).$$

Thus,

$$\begin{aligned}
I(tv_1, su_\varepsilon) &\leq \frac{1}{2}t^2\|v_1\|_1^2 - \frac{t^{p_1}}{p_1} \int \nu_1 v_1^{p_1} - \frac{t^{2^*}}{2^*} \int \mu_1 v_1^{2^*} \\
&\quad + \frac{1}{2}s^2(\|u_\varepsilon\|_2^2 + o(\varepsilon^2)) - \frac{s^{p_2}}{p_2} \int \nu_2 u_\varepsilon^{p_2} - \frac{s^{2^*}}{2^*} \int \mu_2 u_\varepsilon^{2^*} \\
&:= h_1(t) + h_{2\varepsilon}(s).
\end{aligned}$$

It is easy to confirm that

$$\max_{t>0} h_1(t) = h_1(1) = B_1, \quad \max_{s>0} h_{2\varepsilon}(s) < \frac{1}{N}(\mu_2)^{-\frac{N-2}{2}} S^{\frac{N}{2}}, \quad \varepsilon < 1.$$

Now, we only need to show that $t_{1\varepsilon}, s_{1\varepsilon} > 0$ exist such that $(t_{1\varepsilon}v_1, s_{1\varepsilon}u_\varepsilon) \in \mathcal{M}$. For convenience, we denote

$$\begin{aligned} A_1 &= \int \nu_1 v_1^{p_1}, \quad A_2 = \int \nu_2 u_\varepsilon^{p_2}, \\ D_1 &= \int \mu_1 v_1^{2^*}, \quad D_2 = \int \mu_2 u_\varepsilon^{2^*}, \\ D &= \int \beta v_1^{\frac{2^*}{2}} u_\varepsilon^{\frac{2^*}{2}}, \quad E = \|u_\varepsilon\|_2^2. \end{aligned} \quad (2.8)$$

Then, $D < 0$, $\|v_1\|_1^2 = A_1 + D_1$. By selecting ϱ and the Hölder inequality, we have

$$\begin{aligned} \left(\int \beta v_1^{\frac{2^*}{2}} u_\varepsilon^{\frac{2^*}{2}} \right)^2 &\leq \beta^2 \varrho^{2^*-2} \left(\int v_1 u_\varepsilon^{\frac{2^*}{2}} \right)^2 \\ &\leq \left(\frac{\beta^2}{\mu_1 \mu_2} \varrho^{2^*-2} \right) \int \mu_1 v_1^2 \int \mu_2 u_\varepsilon^{2^*} \\ &< \int \mu_1 v_1^{2^*} \int \mu_2 u_\varepsilon^{2^*}, \end{aligned}$$

i.e., $D_1 D_2 - D^2 > 0$. Note that $(tv_1, su_\varepsilon) \in \mathcal{M}$ for some $t, s > 0$ is equivalent to

$$t^2(A_1 + D_1) = t^{p_1} A_1 + t^{2^*} D_1 + t^{\frac{2^*}{2}} s^{\frac{2^*}{2}} D; \quad (2.9)$$

$$s^2 E = s^{p_2} A_2 + s^{2^*} D_2 + t^{\frac{2^*}{2}} s^{\frac{2^*}{2}} D, \quad t, s > 0. \quad (2.10)$$

In this case, (2.9) implies that

$$s^{\frac{2^*}{2}} = \frac{t^{2-\frac{2^*}{2}}(A_1 + D_1) - t^{p_1-\frac{2^*}{2}} A_1 - t^{\frac{2^*}{2}} D_1}{D} := \frac{G(t)}{D} := \frac{t^{2-\frac{2^*}{2}} \tilde{G}(t)}{D} > 0, \quad (2.11)$$

i.e., $\tilde{G}(t) < 0$. Since $\tilde{G}(1) = 0$ and $\tilde{G}'(t) < 0$, we have $t > 1$. By combining (2.10) and (2.11), we simply need to show that

$$F(t) := \left(\frac{G(t)}{D} \right)^{\frac{4}{2^*}-1} t^{-\frac{2^*}{2}} E - \left(\frac{G(t)}{D} \right)^{\frac{2p_2}{2^*}-1} t^{-\frac{2^*}{2}} A_2 - \frac{G(t)}{D} t^{-\frac{2^*}{2}} D_2 - D$$

has a solution $t > 1$. Note that $2 - \frac{2^*}{2} < p_1 - \frac{2^*}{2} < \frac{2^*}{2}$ and $\frac{4}{2^*} - 1 < \frac{2p_2}{2^*} - 1 < 1$, and it is easy to confirm that

$$\lim_{t \rightarrow +\infty} F(t) = \frac{D_1 D_2 - D^2}{D} < 0, \quad F(1) = -D > 0.$$

Thus, we confirmed it for this case. Similarly, we can show that $B < B_2 + \frac{1}{N} \mu_1^{-\frac{N-2}{2}} S^{\frac{N}{2}}$. In [14], it was proved that for $\beta < 0$, $\bar{B} = \frac{1}{N} \mu_1^{-\frac{N-2}{2}} S^{\frac{N}{2}} + \frac{1}{N} \mu_2^{-\frac{N-2}{2}} S^{\frac{N}{2}}$. Since $B_j < \frac{1}{N} \mu_j^{-\frac{N-2}{2}} S^{\frac{N}{2}}$ for $j = 1, 2$, we find that $B < \bar{B}$. $\square$

Lemma 2.3. *If we assume that $\beta < 0$, $\mathbf{u} = (u_1, u_2) \in \mathcal{M}$, $I(\mathbf{u})$ is bounded, then the constants $C_1, C_2 > 0$ exist such that*

$$0 < C_1 \leq \int |u_1|^{2^*}, \quad \int |u_2|^{2^*} \leq C_2.$$

Proof. By Lemma 2.1 and the Sobolev inequality, it is easy to show that $C_2 > 0$ exists such that $\int |u_1|^{2^*}, \int |u_2|^{2^*} \leq C_2$. In addition, because $\beta < 0$, we have

$$C_3 \left(\int |u_j|^{2^*} \right)^{\frac{2}{2^*}} \leq \|u_j\|_j^2 \leq \nu_j \int |u_j|^{p_j} + \mu_j \int |u_j|^{2^*} \leq C_4 \left[\left(\int |u_j|^{2^*} \right)^{\frac{p_j}{2^*}} + \int |u_j|^{2^*} \right],$$

where $C_3, C_4 > 0$ are constants that are independent of u_j , for $j = 1, 2$. This implies that $\int |u_1|^{2^*}, \int |u_2|^{2^*} \geq C_1$ for some $C_1 > 0$. $\square$

We include the following lemma, which was proved in [14].

Lemma 2.4. (See [14].) *If we assume that $(u_{1n}, u_{2n}) \rightharpoonup (u_1, u_2)$ weakly in $\mathbf{H}$ as $n \rightarrow \infty$, then by passing to a subsequence, it holds that*

$$\lim_{n \rightarrow \infty} \int (|u_{1n}|^{\frac{2^*}{2}} |u_{2n}|^{\frac{2^*}{2}} - |u_1|^{\frac{2^*}{2}} |u_2|^{\frac{2^*}{2}} - |u_{1n} - u_1|^{\frac{2^*}{2}} |u_{2n} - u_2|^{\frac{2^*}{2}}) = 0.$$

Lemma 2.5. *If we assume that $-\sqrt{\mu_1 \mu_2} \leq \beta < 0$ and $(u_1, u_2) \in \mathcal{M}$ with $I(u_1, u_2) = B$, then $I'(u_1, u_2) = 0$.*

Proof. Set

$$J_1(u_1, u_2) := I'(u_1, u_2)(u_1, 0), \quad J_2(u_1, u_2) := I'(u_1, u_2)(0, u_2).$$

Then, two Lagrange multipliers $L_1, L_2 \in \mathbb{R}$ exist such that

$$J(u_1, u_2) := I'(u_1, u_2) + L_1 J'_1(u_1, u_2) + L_2 J'_2(u_1, u_2) = 0. \quad (2.12)$$

Thus, $(J(u_1, u_2), (u_1, 0)) = 0$ and $(J(u_1, u_2), (0, u_2)) = 0$, i.e.,

$$\begin{aligned} L_1 [(p_1 - 2) \int \nu_1 |u_1|^{p_1} + (2^* - 2) \int \mu_1 |u_1|^{2^*} + (2 - \frac{2^*}{2}) \int |\beta| |u_1|^{\frac{2^*}{2}} |u_2|^{\frac{2^*}{2}}] \\ = L_2 \frac{2^*}{2} \int |\beta| |u_1|^{\frac{2^*}{2}} |u_2|^{\frac{2^*}{2}}, \end{aligned} \quad (2.13)$$

$$\begin{aligned} L_2 [(p_2 - 2) \int \nu_2 |u_2|^{p_2} + (2^* - 2) \int \mu_2 |u_2|^{2^*} + (2 - \frac{2^*}{2}) \int |\beta| |u_1|^{\frac{2^*}{2}} |u_2|^{\frac{2^*}{2}}] \\ = L_1 \frac{2^*}{2} \int |\beta| |u_1|^{\frac{2^*}{2}} |u_2|^{\frac{2^*}{2}}. \end{aligned} \quad (2.14)$$

Similar to (2.8), we denote

$$\begin{aligned} A_1 &= \int \nu_1 |u_1|^{p_1}, \quad A_2 = \int \nu_2 |u_2|^{p_2}, \\ D_1 &= \int \mu_1 |u_1|^{2^*}, \quad D_2 = \int \mu_2 |u_2|^{2^*}, \quad D = \int |\beta| |u_1|^{\frac{2^*}{2}} |u_2|^{\frac{2^*}{2}}. \end{aligned}$$

Therefore, they are all nonnegative. Note that $(u_1, u_2) \in \mathcal{M}$ and we have

$$\|u_1\|_1^2 = A_1 + D_1 - D, \quad \|u_2\|_2^2 = A_2 + D_2 - D. \quad (2.15)$$

Consequently, by Lemma 2.1, we have $A_1 + D_1 > D$, $A_2 + D_2 > D$. Set

$$\begin{aligned} S_1 &:= (p_1 - 2)(p_2 - 2)A_1A_2 + (p_1 - 2)(2^* - 2)A_1D_2 + (p_1 - 2)\left(2 - \frac{2^*}{2}\right)A_1D \\ &\quad + (p_2 - 2)(2^* - 2)A_2D_1 + (p_2 - 2)\left(2 - \frac{2^*}{2}\right)A_2D, \\ S_2 &:= (2^* - 2)^2D_1D_2 + (2^* - 2)\left(2 - \frac{2^*}{2}\right)D_1D + (2^* - 2)\left(2 - \frac{2^*}{2}\right)D_2D - 2(2^* - 2)D^2. \end{aligned}$$

Then, by (2.13) and (2.14), we have

$$0 = L_1L_2(S_1 + S_2) := L_1L_2S. \quad (2.16)$$

Note that $D \leq \frac{1}{2}(D_1 + D_2)$, $D^2 \leq D_1D_2$, and we have

$$S_2 \geq (2^* - 2)^2(D_1D_2 - D^2) \geq 0.$$

Thus, $S > 0$. By (2.16), we have $L_1L_2 = 0$. Without any loss of generality, we may assume that $L_1 = 0$, and thus by (2.14) and Lemma 2.3, we have $L_2 = 0$. Then, (2.12) implies that $I'(u_1, u_2) = 0$. $\square$

Proof of Theorem 1.1 for the case where $-\sqrt{\mu_1\mu_2} \leq \beta < 0$. By Lemma 2.1 and the Ekeland variational principle, a minimizing sequence $\{\mathbf{u}_n\} = \{(u_{1n}, u_{2n})\} \subset \mathcal{M}$ exists such that

$$I(\mathbf{u}_n) \leq \min\left\{B + \frac{1}{n}, \bar{B}\right\}, \quad I(\mathbf{u}_n) - \frac{1}{n}\|\mathbf{u}_n - \mathbf{u}\| \leq I(\mathbf{u}), \quad \forall \mathbf{u} \in \mathcal{M}, \quad (2.17)$$

and $\{\mathbf{u}_n\}$ is bounded in $\mathbf{H}$. Therefore, an $\mathbf{u}$ exists such that $\mathbf{u}_n \rightharpoonup \mathbf{u}$ weakly in $\mathbf{H}$ up to a subsequence. Now, we show that $\{\mathbf{u}_n\}$ is a (PS) sequence of I . Take $(\phi, \psi) \in \mathbf{H}$ with $\|(\phi, \psi)\| \leq 1$. Define two functions $\rho_{1n}, \rho_{2n} : \mathbb{R}^3 \rightarrow \mathbb{R}$ by

$$\begin{aligned} \rho_{1n}(t, k, l) &:= \|u_{1n} + t\phi + ku_{1n}\|_1^2 - \nu_1 \int |u_{1n} + t\phi + ku_{1n}|^{p_1} \\ &\quad - \mu_1 \int |u_{1n} + t\phi + ku_{1n}|^{2^*} - \beta \int |u_{1n} + t\phi + ku_{1n}|^{\frac{2^*}{2}} |u_{2n} + t\psi + lu_{2n}|^{\frac{2^*}{2}}, \\ \rho_{2n}(t, k, l) &:= \|u_{2n} + t\psi + lu_{2n}\|_1^2 - \nu_2 \int |u_{2n} + t\psi + lu_{2n}|^{p_2} \\ &\quad - \mu_2 \int |u_{2n} + t\psi + lu_{2n}|^{2^*} - \beta \int |u_{1n} + t\phi + ku_{1n}|^{\frac{2^*}{2}} |u_{2n} + t\psi + lu_{2n}|^{\frac{2^*}{2}}. \end{aligned}$$

Then, $\rho_{1n}(0, 0, 0) = \rho_{2n}(0, 0, 0) = 0$. We denote

$$\begin{aligned} A_{1n} &= \int \nu_1 |u_{1n}|^{p_1}, \quad A_{2n} = \int \nu_2 |u_{2n}|^{p_2}, \\ D_{1n} &= \int \mu_1 |u_{1n}|^{2^*}, \quad D_{2n} = \int \mu_2 |u_{2n}|^{2^*}, \quad D_n = \int |\beta| |u_{1n}|^{\frac{2^*}{2}} |u_{2n}|^{\frac{2^*}{2}}. \end{aligned}$$

Thus, they are all nonnegative. By direct calculations, we obtain

$$\begin{aligned}
\frac{\partial \rho_{1n}}{\partial k}(0,0,0) &= 2\|u_{1n}\|_1^2 - p_1 A_{1n} - 2^* D_{1n} + \frac{2^*}{2} D_n \\
&= (2 - p_1) A_{1n} + (2 - 2^*) D_{1n} + \left(\frac{2^*}{2} - 2\right) D_n, \\
\frac{\partial \rho_{2n}}{\partial l}(0,0,0) &= (2 - p_2) A_{2n} + (2 - 2^*) D_{2n} + \left(\frac{2^*}{2} - 2\right) D_n \\
\frac{\partial \rho_{1n}}{\partial l}(0,0,0) &= \frac{\partial \rho_{2n}}{\partial k}(0,0,0) = \frac{2^*}{2} D_n.
\end{aligned}$$

Set

$$H_n := \begin{pmatrix} \frac{\partial \rho_{1n}}{\partial k}(0,0,0) & \frac{\partial \rho_{1n}}{\partial l}(0,0,0) \\ \frac{\partial \rho_{2n}}{\partial k}(0,0,0) & \frac{\partial \rho_{2n}}{\partial l}(0,0,0) \end{pmatrix}.$$

Similar to the proof of Lemma 2.5, we have $\det(H_n) > 0$. Thus, by the implicit function theorem, $k_n(t), l_n(t) \in C^1$ are well defined on $(-\delta_n, \delta_n)$ for some $\delta_n > 0$ and

$$\begin{aligned}
k_n(0) &= l_n(0) = 0, \\
\rho_{1n}(t, k_n(t), l_n(t)) &\equiv 0, \quad \rho_{2n}(t, k_n(t), l_n(t)) \equiv 0, \quad t \in (-\delta_n, \delta_n).
\end{aligned} \tag{2.18}$$

Then,

$$\begin{aligned}
k'_n(0) &= \frac{1}{\det(H_n)} \left(\frac{\rho_{2n}}{\partial t} \frac{\rho_{1n}}{\partial l} - \frac{\rho_{2n}}{\partial l} \frac{\rho_{1n}}{\partial t} \right) \Big|_{(0,0,0)}, \\
l'_n(0) &= \frac{1}{\det(H_n)} \left(\frac{\rho_{2n}}{\partial t} \frac{\rho_{1n}}{\partial k} - \frac{\rho_{2n}}{\partial k} \frac{\rho_{1n}}{\partial t} \right) \Big|_{(0,0,0)}.
\end{aligned}$$

By Lemmas 2.1 and 2.3, a constant $C > 0$ exists that is independent of n such that

$$|k'_n(0)|, |l'_n(0)| \leq C. \tag{2.19}$$

Let

$$\phi_{nt} := u_{1n} + t\phi + k_n(t)u_{1n}, \quad \psi_{nt} := u_{2n} + t\psi + l_n(t)u_{2n}.$$

Then, $(\phi_{nt}, \psi_{nt}) \in \mathcal{M}$ for $t \in (-\delta_n, \delta_n)$. Since $(u_{1n}, u_{2n}) \in \mathcal{M}$, by the Taylor expansion, we have

$$\begin{aligned}
I(\phi_{nt}, \psi_{nt}) - I(u_{1n}, u_{2n}) &= I'(u_{1n}, u_{2n})(t\phi + k_n(t)u_{1n}, t\psi + l_n(t)u_{2n}) + r(n, t) \\
&= tI'(u_{1n}, u_{2n})(\phi, \psi) + o(\|t\phi + k_n(t)u_{1n}, t\psi + l_n(t)u_{2n}\|), \quad t \rightarrow 0.
\end{aligned} \tag{2.20}$$

By (2.18) and (2.19), it holds that

$$\limsup_{t \rightarrow 0} \left\| \phi + \frac{k_n(t)}{t} u_{1n}, \psi + \frac{l_n(t)}{t} u_{2n} \right\| \leq C \tag{2.21}$$

for some $C > 0$ that is independent of n . Furthermore, (2.17) implies that

$$I(\phi_{nt}, \psi_{nt}) - I(u_{1n}, u_{2n}) \geq -\frac{1}{n} \|t\phi + k_n(t)u_{1n}, t\psi + l_n(t)u_{2n}\|. \tag{2.22}$$

Thus, by combining (2.20), (2.21) and (2.22), and if we let $t \rightarrow 0$, we have

$$|I'(u_{1n}, u_{2n})(\phi, \psi)| \leq \frac{C}{n}$$

for some $C > 0$ that is independent of n . Therefore, $I'(u_{1n}, u_{2n}) \rightarrow 0$ as $n \rightarrow \infty$. It follows that $I'(u_1, u_2) = 0$. Now, we show that $u_1, u_2 \not\equiv 0$. Let $w_{1n} = u_{1n} - u_1$, $w_{2n} = u_{2n} - u_2$, then by the Brezis–Lieb Lemma, we have

$$\int |u_{jn}|^{2^*} = \int |u_j|^{2^*} + \int |w_{jn}|^{2^*} + o(1), \quad j = 1, 2.$$

By the fact that $(u_{1n}, u_{2n}) \in \mathcal{M}$, $I'(u_1, u_2) = 0$ and Lemma 2.4, we have

$$\int |\nabla w_{1n}|^2 = \int (\mu_1 |w_{1n}|^{2^*} + \beta |w_{1n}|^{\frac{2^*}{2}} |w_{2n}|^{\frac{2^*}{2}}) + o(1), \quad (2.23)$$

$$\int |\nabla w_{2n}|^2 = \int (\mu_2 |w_{2n}|^{2^*} + \beta |w_{1n}|^{\frac{2^*}{2}} |w_{2n}|^{\frac{2^*}{2}}) + o(1), \quad (2.24)$$

$$I(u_{1n}, u_{2n}) = I(u_1, u_2) + \bar{I}(w_{1n}, w_{2n}) + o(1), \quad n \rightarrow \infty. \quad (2.25)$$

We may assume that $\int |\nabla w_{jn}|^2 \rightarrow b_j$ as $n \rightarrow \infty$, via a subsequence if necessary. Then, by (2.23) and (2.24), we have

$$\bar{I}(w_{1n}, w_{2n}) = \frac{1}{N}(b_1 + b_2) + o(1), \quad n \rightarrow \infty.$$

By combining this with (2.25), we obtain

$$B = \lim_{n \rightarrow \infty} I(u_{1n}, u_{2n}) = I(u_1, u_2) + \frac{1}{N}(b_1 + b_2) \geq I(u_1, u_2) \geq 0. \quad (2.26)$$

Case 1: Assume that $u_1, u_2 \equiv 0$. Then, $0 < b_1, b_2 < \infty$, $w_{1n}, w_{2n} \not\equiv 0$ for a sufficiently large n . Similar to the proof of Lemma 2.2, it holds that $t_{1n}, t_{2n} > 0$ exist such that $(t_{1n}w_{1n}, t_{2n}w_{2n}) \in \bar{\mathcal{M}}$. We claim that

$$\lim_{n \rightarrow +\infty} t_{1n} = \lim_{n \rightarrow +\infty} t_{2n} = 1 \quad (2.27)$$

by a subsequence if necessary. We may assume that

$$\begin{aligned} E_{1n} &= \int |\nabla w_{1n}|^2 \rightarrow b_1, \quad E_{2n} = \int |\nabla w_{2n}|^2 \rightarrow b_2; \\ D_{1n} &= \int \mu_1 |w_{1n}|^{2^*} \rightarrow d_1, \quad D_{2n} = \int \mu_2 |w_{2n}|^{2^*} \rightarrow d_2, \\ D_n &= |\beta| \int |w_{1n}|^{\frac{2^*}{2}} |w_{2n}|^{\frac{2^*}{2}} \rightarrow d, \quad n \rightarrow +\infty. \end{aligned}$$

Then, $d_1, d_2, d < +\infty$, and by (2.23) and (2.24), we have

$$d_1 = b_1 + d \geq b_1 > 0, \quad d_2 = b_2 + d \geq b_2 > 0. \quad (2.28)$$

Since $(t_{1n}w_{1n}, t_{2n}w_{2n}) \in \bar{\mathcal{M}}$, we have

$$t_{1n}^{2^*} D_{1n} = t_{1n}^2 E_{1n} + t_{1n}^{\frac{2^*}{2}} t_{2n}^{\frac{2^*}{2}} D_n \geq t_{1n}^2 E_{1n}, \quad (2.29)$$

$$t_{2n}^{2^*} D_{2n} = t_{2n}^2 E_{2n} + t_{1n}^{\frac{2^*}{2}} t_{2n}^{\frac{2^*}{2}} D_n \geq t_{2n}^2 E_{2n}. \quad (2.30)$$

Thus,

$$t_{1n}^{2^*-2} \geq \frac{E_{1n}}{D_{1n}} \rightarrow \frac{b_1}{d_1} > 0, \quad t_{2n}^{2^*-2} \geq \frac{E_{2n}}{D_{2n}} \rightarrow \frac{b_2}{d_2} > 0. \quad (2.31)$$

Assume that $t_{1n} \rightarrow +\infty$ up to a subsequence, then by (2.29) and (2.30), we have $t_{2n} \rightarrow +\infty$ and thus

$$\begin{aligned} d^2 &= \lim_{n \rightarrow +\infty} D_n^2 \\ &= \lim_{n \rightarrow +\infty} \frac{t_{1n}^{\frac{2^*}{2}} D_{1n} - t_{1n}^{2-\frac{2^*}{2}} E_{1n}}{t_{2n}^{\frac{2^*}{2}}} \cdot \frac{t_{2n}^{\frac{2^*}{2}} D_{2n} - t_{2n}^{2-\frac{2^*}{2}} E_{2n}}{t_{1n}^{\frac{2^*}{2}}} \\ &= \lim_{n \rightarrow +\infty} (D_{1n} - t_{1n}^{2-2^*} E_{1n})(D_{2n} - t_{2n}^{2-2^*} E_{2n}) \\ &= d_1 d_2 > d^2, \end{aligned} \quad (2.32)$$

which is a contradiction. Thus, t_{1n}, t_{2n} are uniformly bounded, and then by (2.31), we may assume that $t_{1n} \rightarrow t_1 > 0$, $t_{2n} \rightarrow t_2 > 0$. Let $n \rightarrow +\infty$ in (2.29) and (2.30), we have

$$t_1^{\frac{2^*}{2}} d_1 = t_1^{2-\frac{2^*}{2}} b_1 + t_2^{\frac{2^*}{2}} d, \quad t_2^{\frac{2^*}{2}} d_2 = t_2^{2-\frac{2^*}{2}} b_2 + t_1^{\frac{2^*}{2}} d. \quad (2.33)$$

If $d = 0$, by combining with (2.28), it holds that $t_1 = t_2 = 1$. Now, we assume that $d > 0$. Let

$$h(t) := t^{\frac{2^*}{2}} d_1 - t^{2-\frac{2^*}{2}} b_1, \quad t > 0.$$

Then, $h(t)$ is increasing when $t > t_0$, where

$$t_0^{2^*-2} = \frac{(2 - \frac{2^*}{2})b_1}{\frac{2^*}{2}d_1} < \frac{b_1}{d_1}.$$

By (2.31), $t_1 > t_0$. If $t_1 < 1$, then from (2.33), we have $t_2^{\frac{2^*}{2}} d < h(1) = d_1 - b_1 = d$, i.e., $t_2 < 1$. Then, by (2.28) and (2.32), we have

$$d^2 = (d_1 - t_1^{2-2^*} b_1)(d_2 - t_2^{2-2^*} b_2) < d^2,$$

which is a contradiction. Similarly, if $t_1 > 1$, we have $t_2 > 1$ and thus

$$d^2 = (d_1 - t_1^{2-2^*} b_1)(d_2 - t_2^{2-2^*} b_2) > d^2,$$

which is also a contradiction. Thus, we prove the claim (2.27). Then, by (2.26) and the definition of $\bar{B}$, we have

$$\bar{B} \leq \lim_{n \rightarrow +\infty} \bar{I}(t_{1n} w_{1n}, t_{2n} w_{2n}) = \lim_{n \rightarrow +\infty} \bar{I}(w_{1n}, w_{2n}) = \frac{1}{N}(b_1 + b_2) \leq B,$$

which contradicts Lemma 2.2. Therefore, Case 1 is impossible.

Case 2: Assume that $u_1 \not\equiv 0$, $u_2 \equiv 0$ or $u_1 \equiv 0$, $u_2 \not\equiv 0$. We may simply consider the former case, and thus $b_2 > 0$. By the proof of Case 1, we may assume that $b_1 = 0$. By the Hölder inequality and using the same denotations employed in Case 1, we have $d = 0$. Then, $E_{2n} = D_{2n} + o(1) \leq \mu_2 S^{-\frac{2^*}{2}} E_{2n}^{\frac{2^*}{2}}$. Thus, $b_2 \geq \mu_2^{-\frac{N-2}{2}} S^{\frac{N}{2}}$. Since $u_2 \equiv 0$, we find that u_1 is a nontrivial solution of

$$-\Delta u + \lambda_1 u = \nu_1 |u|^{p-2} u + \mu_1 |u|^{2^*-2} u.$$

Then, by (1.6) and (2.26), we have

$$B = I(u_1, 0) + \frac{1}{N} b_2 \geq B_1 + \frac{1}{N} \mu_2^{-\frac{N-2}{2}} S^{\frac{N}{2}},$$

which also contradicts Lemma 2.2. Therefore, Case 2 is also impossible.

Thus, we have shown $u_1, u_2 \not\equiv 0$ and then $(u_1, u_2) \in \mathcal{M}$. By the definition of B and (2.26), we have $I(u_1, u_2) = B$. Then, $(|u_1|, |u_2|) \in \mathcal{M}$ and $I(|u_1|, |u_2|) = B$. By Lemma 2.5, we have $I'(|u_1|, |u_2|) = 0$. By the maximum principle, we have $|u_1|, |u_2| > 0$ in Ω . Consequently, $(|u_1|, |u_2|)$ is a positive ground state solution of system (1.7). Next, we complete the proof. $\square$

Remark 2.1. The condition $\beta \geq -\sqrt{\mu_1 \mu_2}$ ensures that

$$\int (\mu_1 |u_1|^{2^*} + \mu_2 |u_2|^{2^*}) + \int 2\beta |u_1|^{\frac{2^*}{2}} |u_2|^{\frac{2^*}{2}} \geq 0,$$

which is very important in many estimations, such as the boundedness of the minimizing sequence. However, it might not be the sharp interval of β for the existence of a positive ground state solution to system (1.7). This remains an open problem, as mentioned in the introduction.

3. Proof of Theorem 1.1 for $\beta > 0$

In this section, we always assume that $\beta > 0$. Define the minimax value

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$

where

$$\Gamma := \{\gamma \in C([0,1], \mathbf{H}) : \gamma(0) = (0,0), I(\gamma(1)) < 0\}.$$

It is easy to confirm that $\Gamma \neq \emptyset$. For any given $\mathbf{u} = (u_1, u_2) \in \mathbf{H} \setminus (0,0)$, we have

$$\max_{t>0} I(tu_1, tu_2) = I(t_{\mathbf{u}}u_1, t_{\mathbf{u}}u_2) \tag{3.1}$$

for a unique $t_{\mathbf{u}} > 0$ that satisfies

$$\|\mathbf{u}\|^2 = \int \sum_j t_{\mathbf{u}}^{p_j-2} \nu_j |u_j|^{p_j} + t_{\mathbf{u}}^{2^*-2} \int (\sum_j \mu_j |u_j|^{2^*} + 2\beta |u_1|^{\frac{2^*}{2}} |u_2|^{\frac{2^*}{2}}). \tag{3.2}$$

Set

$$\mathcal{M}' := \{(u_1, u_2) \in \mathbf{H} \setminus \{0,0\} : I'(u_1, u_2)(u_1, u_2) = 0\}. \tag{3.3}$$

The condition in (3.3) is

$$K(\mathbf{u}) := \|\mathbf{u}\|^2 - \int \sum_j \nu_j |u_j|^{p_j} - \int (\sum_j \mu_j |u_j|^{2^*} + 2\beta |u_1|^{\frac{2^*}{2}} |u_2|^{\frac{2^*}{2}}) = 0.$$

Then, (3.2) implies that $(t_{\mathbf{u}}u_1, t_{\mathbf{u}}u_2) \in \mathcal{M}'$. It is well known that (see [32])

$$c = \inf_{\mathbf{u} \in \mathbf{H} \setminus (0,0)} \max_{t > 0} I(tu_1, tu_2) = \inf_{\mathbf{u} \in \mathcal{M}'} I(\mathbf{u}).$$

By the Sobolev inequality and since $\mathcal{M} \subset \mathcal{M}'$, we find that

$$0 < c \leq B. \quad (3.4)$$

Similar to [Lemma 2.2](#), we also have an upper bound for c . First, we need the conclusion proved in [\[14\]](#). Define

$$I_0(\mathbf{u}) := \frac{1}{2} \|\mathbf{u}\|^2 - \frac{1}{2^*} \int_{\mathbb{R}^N} (\mu_1 |u_1|^{2^*} + \mu_2 |u_2|^{2^*} + 2\beta u_1^{\frac{2^*}{2}} u_2^{\frac{2^*}{2}}),$$

and

$$c_0 := \inf_{\gamma \in \Gamma_0} \max_{t \in [0,1]} I_0(\gamma(t)),$$

where

$$\Gamma_0 := \{\gamma \in C([0,1], \mathbf{H}) : \gamma(0) = (0,0), I_0(\gamma(1)) < 0\}.$$

Lemma 3.1. *If we assume that $\beta > 0$, then $c_0 < \bar{B}$.*

Next, we have the following lemma.

Lemma 3.2. *If we assume that $\beta > 0$, then*

$$c < \min\{B_1, B_2, \bar{B}\}.$$

Proof. First, since for any $\mathbf{u} \in \mathbf{H}$, it holds that $I(\mathbf{u}) \leq I_0(\mathbf{u})$, so we find that $\Gamma_0 \subset \Gamma$. Then, by [Lemma 3.1](#), we obtain

$$c \leq \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_0(\gamma(t)) \leq c_0 < \bar{B}.$$

Now, we show that $c < B_1$. Recall that v_1, v_2 are defined in [\(1.6\)](#) and let $\mathbf{v}(l) := (v_1, lv_2)$, $t(l) := t_{\mathbf{v}(l)}$, as defined in [\(3.2\)](#), i.e.,

$$\begin{aligned} & \int (\nu_1 v_1^{p_1} + \mu_1 v_1^{2^*}) + \int (\nu_2 |l|^{p_2} v_2^{p_2} + \mu_2 |l|^{2^*} v_2^{2^*}) \\ &= \int (t(l)^{p_1-2} \nu_1 v_1^{p_1} + t(l)^{p_2-2} \nu_2 |l|^{p_2} v_2^{p_2}) \\ &+ t(l)^{2^*-2} \int (\mu_1 v_1^{2^*} + \mu_2 |l|^{2^*} v_2^{2^*} + 2\beta |l|^{\frac{2^*}{2}} v_1^{\frac{2^*}{2}} v_2^{\frac{2^*}{2}}). \end{aligned}$$

Thus, $t(0) = 1$. By direct calculations, we obtain

$$\lim_{l \rightarrow 0} \frac{t'(l)}{|l|^{\frac{2^*}{2}-2} l} = - \frac{2^* \beta \int v_1^{\frac{2^*}{2}} v_2^{\frac{2^*}{2}}}{(p_1 - 2) \int \nu_1 v_1^{p_1} + (2^* - 2) \int \mu_1 v_1^{2^*}} := -2^* V.$$

Therefore,

$$t'(l) = -2^*V|l|^{\frac{2^*}{2}-2}l(1+o(1)), \quad l \rightarrow 0,$$

and thus

$$t(l) = 1 - 2V|l|^{\frac{2^*}{2}}(1+o(1)), \quad l \rightarrow 0.$$

Consequently, for $j = 1, 2$,

$$t(l)^{p_j} = 1 - 2p_jV|l|^{\frac{2^*}{2}}(1+o(1)), \quad t(l)^{2^*} = 1 - 2 \cdot 2^*V|l|^{\frac{2^*}{2}}(1+o(1)), \quad l \rightarrow 0.$$

Therefore, a constant $C > 0$ exists such that

$$\begin{aligned} c &\leq I(t(l)v_1, t(l)lv_2) \\ &= \int \left[\left(\frac{1}{2} - \frac{1}{p_1} \right) t(l)^{p_1} \nu_1 v_1^{p_1} + \left(\frac{1}{2} - \frac{1}{p_2} \right) t(l)^{p_2} \nu_2 |l|^{p_2} v_2^{p_2} \right] \\ &\quad + \left(\frac{1}{2} - \frac{1}{2^*} \right) t(l)^{2^*} \int (\mu_1 v_1^{2^*} + \mu_2 |l|^{2^*} v_2^{2^*} + 2\beta |l|^{\frac{2^*}{2}} v_1^{\frac{2^*}{2}} v_2^{\frac{2^*}{2}}) \\ &\leq \left(\frac{1}{2} - \frac{1}{p_1} \right) \int \nu_1 v_1^{p_1} + \left(\frac{1}{2} - \frac{1}{2^*} \right) \int \mu_1 v_1^{2^*} - C|l|^{2^*} + o(|l|^{2^*}) \\ &< B_1, \quad \text{for } l \text{ that is sufficiently small.} \end{aligned}$$

Similarly, we can check that $c < B_2$. This completes the proof. $\square$

We also need to consider the corresponding minimax value of the problem (2.2). Define

$$\bar{B}' := \inf_{\mathbf{u} \in \bar{\mathcal{M}}'} \bar{I}(\mathbf{u}),$$

where

$$\bar{\mathcal{M}}' := \{(u_1, u_2) \in \mathbf{H} \setminus (0, 0) : \bar{I}'(u_1, u_2)(u_1, u_2) = 0\}.$$

In [14], the authors proved the following conclusion.

Lemma 3.3. *If we assume that $\beta > 0$, then $\bar{B}' = \bar{B}$.*

Proof of Theorem 1.1 for $\beta > 0$. Since $\beta > 0$, it is easy to see that I satisfies the mountain pass geometry. Thus, a sequence $\{\mathbf{u}_n\} \subset \mathbf{H}$ exists such that

$$I(\mathbf{u}_n) \rightarrow c, \quad I'(\mathbf{u}_n) \rightarrow 0, \quad \text{as } n \rightarrow +\infty.$$

Similar to the proof of (2.3), we can show that $\{\mathbf{u}_n\}$ is bounded. We may assume that $\mathbf{u}_n \rightharpoonup \mathbf{u}$ weakly in $\mathbf{H}$ for some $\mathbf{u} := (u_1, u_2) \in \mathbf{H}$ and $I'(\mathbf{u}) = 0$. Now, we show that $u_1, u_2 \not\equiv 0$. We still employ the same notations used in Section 2, i.e., let $w_{1n} = u_{1n} - u_1$, $w_{2n} = u_{2n} - u_2$. As in (2.26), we have

$$c = \lim_{n \rightarrow \infty} I(u_{1n}, u_{2n}) = I(u_1, u_2) + \frac{1}{N}(b_1 + b_2) \geq I(u_1, u_2) \geq 0. \quad (3.5)$$

Case 1: Assume that $u_1, u_2 \equiv 0$. Then, by (3.5), we have $b_1 + b_2 > 0$ and thus $(w_{1n}, w_{2n}) \neq (0, 0)$ for a sufficiently large n . Similar to (3.1), $t_n > 0$ exists such that $(t_n w_{1n}, t_n w_{2n}) \in \bar{\mathcal{M}}'$. Recall that w_{1n}, w_{2n}

satisfy (2.23) and (2.24), and thus we find that $t_n \rightarrow 1$. Then, by (3.5), the definition of $\bar{B}'$ and Lemma 3.3, we obtain

$$c = \frac{1}{N}(b_1 + b_2) = \lim_{n \rightarrow +\infty} \bar{I}(w_{1n}, w_{2n}) = \lim_{n \rightarrow +\infty} \bar{I}(t_n w_{1n}, t_n w_{2n}) \geq \bar{B}' = \bar{B},$$

which is a contradiction with Lemma 3.2. Therefore, Case 1 is impossible.

Case 2: Assume that $u_1 \equiv 0$, $u_2 \not\equiv 0$ or $u_1 \not\equiv 0$, $u_2 \equiv 0$. We may simply consider the latter case. Then, u_1 is a nontrivial solution of the equation

$$-\Delta u + \lambda_1 u = \nu_1 |u|^{p_1-2} u_1 + \mu_1 |u|^{2^*-2} u.$$

By (1.6) and (3.5), we have

$$c \geq I(u_1, 0) \geq B_1,$$

which is also a contradiction of Lemma 3.2. Therefore, Case 2 is also impossible.

Thus, we have shown that $u_1, u_2 \not\equiv 0$ and thus $(u_1, u_2) \in \mathcal{M}$. By (3.4) and (3.5), we have $B \leq I(u_1, u_2) \leq c \leq B$, i.e., $I(u_1, u_2) = B = c$. Then, $(|u_1|, |u_2|) \in \mathcal{M} \subset \mathcal{M}'$ and $I(|u_1|, |u_2|) = c$. Similar to the proof of Lemma 2.5, we have $I'(|u_1|, |u_2|) = 0$. Indeed, we find that a Lagrange multiplier $L \in \mathbb{R}$ exists such that

$$I'(|u_1|, |u_2|) - LK'(|u_1|, |u_2|) = 0.$$

Then,

$$\begin{aligned} 0 &= I'(|u_1|, |u_2|)(|u_1|, |u_2|) = LK'(|u_1|, |u_2|)(|u_1|, |u_2|) \\ &= L[2\|\mathbf{u}\|^2 - \int \sum_j p_j \nu_j |u_j|^{p_j} - 2^* \int (\sum_j \mu_j |u_j|^{2^*} + 2\beta |u_1|^{\frac{2^*}{2}} |u_2|^{\frac{2^*}{2}})] \\ &= L[\int \sum_j (2 - p_j) \nu_j |u_j|^{p_j} + (2 - 2^*) \int (\sum_j \mu_j |u_j|^{2^*} + 2\beta |u_1|^{\frac{2^*}{2}} |u_2|^{\frac{2^*}{2}})], \end{aligned}$$

which implies that $L = 0$, and thus $I'(|\mathbf{u}|) = 0$. By the maximum principle, we have $|u_1|, |u_2| > 0$ in Ω . Consequently, $(|u_1|, |u_2|)$ is a positive ground state solution of system (1.7). This completes the proof. $\square$

4. Proof of Theorem 1.2

Proof of Theorem 1.2. First, we present (1) in Theorem 1.2. By multiplying the equations in system (1.7) with the first eigenfunction ϕ_1 and integrating over Ω , we obtain

$$\begin{aligned} (\lambda_1(\Omega) + \lambda_1) \int_{\Omega} u_1 \phi_1 &= \int_{\Omega} (\nu_1 u_1^{p_1-1} \phi_1 + \mu_1 u_1^{2^*-1} \phi_1 + \beta u_1^{\frac{2^*-1}{2}} u_2^{\frac{2^*}{2}} \phi_1), \\ (\lambda_1(\Omega) + \lambda_2) \int_{\Omega} u_2 \phi_1 &= \int_{\Omega} (\nu_2 u_2^{p_2-1} \phi_1 + \mu_2 u_2^{2^*-1} \phi_1 + \beta u_1^{\frac{2^*}{2}} u_2^{\frac{2^*-1}{2}} \phi_1). \end{aligned}$$

Then, it is easy to see that when $\lambda_j \leq -\lambda_1(\Omega)$, $\nu_j, \mu_j, \beta > 0$ for $j = 1, 2$, system (1.7) does not have any positive solutions.

If Ω is star-shaped with respect to some point $y_0 \in \mathbb{R}^N$ and (u_1, u_2) is a solution of (1.7), and then by the Pohozaev Identity, we have

$$\begin{aligned} & \frac{1}{2^*} \int \sum_j |\nabla u_j|^2 + \frac{1}{2N} \int_{\partial\Omega} (|\frac{\partial u_1}{\partial \mathbf{n}}|^2 + |\frac{\partial u_2}{\partial \mathbf{n}}|^2)(x - y_0) \cdot \mathbf{n} dS \\ &= \int \sum_j \frac{\nu_j}{p_j} |u_j|^{p_j} + \frac{1}{2^*} \int (\sum_j \mu_j |u_j|^{2^*} + 2\beta |u_1|^{\frac{2^*}{2}} |u_2|^{\frac{2^*}{2}}) - \frac{1}{2} \int \sum_j \lambda_j u_j^2. \end{aligned}$$

Since (u_1, u_2) also satisfies

$$\int \sum_j |\nabla u_j|^2 = \int \sum_j \nu_j |u_j|^{p_j} + \int (\sum_j \mu_j |u_j|^{2^*} + 2\beta |u_1|^{\frac{2^*}{2}} |u_2|^{\frac{2^*}{2}}) - \int \sum_j \lambda_j u_j^2,$$

if $\lambda_j \geq 0$ and $\nu_j \leq 0$ for $j = 1, 2$, then we have

$$\begin{aligned} 0 &\leq \frac{1}{2N} \int_{\partial\Omega} (|\frac{\partial u_1}{\partial \mathbf{n}}|^2 + |\frac{\partial u_2}{\partial \mathbf{n}}|^2)(x - y_0) \cdot \mathbf{n} dS \\ &= (\frac{1}{2^*} - \frac{1}{2}) \int \sum_j \lambda_j u_j^2 + \int \sum_j (\frac{1}{p_j} - \frac{1}{2^*}) \nu_j |u_j|^{p_j} \leq 0, \end{aligned}$$

where $\mathbf{n}$ is the exterior unit normal. Therefore, (2) follows in [Theorem 1.2](#). $\square$

5. Proof of [Theorems 1.3–1.5](#)

Proof of Theorem 1.3. The proof depends on direct calculation. Since (a_1, a_2) satisfies [\(1.8\)](#), for any $\varphi \in C_0^\infty(\Omega)$, then we have

$$\begin{aligned} & \nu_1 \int |a_1 v|^{p-2} a_1 v \varphi + \mu_1 \int |a_1 v|^{2^*-2} a_1 v \varphi + \beta \int |a_1 v|^{\frac{2^*}{2}-2} a_1 v |a_2 v|^{\frac{2^*}{2}} \varphi \\ &= a_1 (\nu \int |v|^{p-2} v \varphi + \int |v|^{2^*-2} v \varphi) = (a_1 v, \varphi)_1. \end{aligned}$$

Therefore, $(a_1 v, a_2 v)$ satisfies the first equation in system [\(1.7\)](#). Thus, [Theorem 1.2](#) follows easily. $\square$

Proof of Theorems 1.4 and 1.5. When $-\lambda_1(\Omega) < \lambda < 0$, the solution w of [\(1.9\)](#) that we use is the positive one obtained in [\[8\]](#), whereas when $\lambda \leq -\lambda_1(\Omega)$, as shown in [Theorem 1.2\(1\)](#), any solution of system [\(1.1\)](#) must be sign-changing, including w . Since a_1, a_2 in [\(1.10\)](#) satisfy

$$\mu_1 a_1^2 + \beta a_2^2 = 1, \quad \mu_2 a_2^2 + \beta a_1^2 = 1.$$

Similarly, we can also construct solutions as $(a_1 w, a_2 w)$. We can verify that [Theorem 1.4](#) holds.

For [Theorem 1.5](#), we recall a result about the equation:

$$\begin{cases} -\Delta u + \lambda u = u^3, & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (5.1)$$

where $\Omega \subset \mathbb{R}^N$ ($N \leq 3$). For any $\lambda \in \mathbb{R}$, problem [\(5.1\)](#) has infinitely many nontrivial solutions (see [\[32, Corollary 3.9\]](#)). When $\lambda > -\lambda_1(\Omega)$, [\(5.1\)](#) has a positive solution (see [\[32, Theorem 1.19\]](#)) as well as infinitely many sign-changing solutions (see [\[33, Theorem 5.7\]](#)). Therefore, for any $\lambda \in \mathbb{R}$, [\(5.1\)](#) has infinitely many sign-changing solutions, which can be used to construct solutions of system [\(1.1\)](#) when $\lambda_1 = \lambda_2 = \lambda$. Then, the last theorem follows, as shown in [Theorem 1.4](#). $\square$

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