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Optimality conditions for the buckling of a clamped plate

Kathrin Stollenwerk, Alfred Wagner *

Institut für Mathematik, RWTH Aachen, Templergraben 55, D-52062 Aachen, Germany

A R T I C L E I N F O

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A B S T R A C T

We prove the following uniqueness result for the buckling plate: Assume there exists a smooth domain that minimizes the first buckling eigenvalue for a plate among all smooth domains of given volume and connected boundary. Then the domain must be a ball. The proof uses the second domain variation and an inequality by L.E. Payne to establish this result.

1. Introduction

We consider the following variational problem. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let

$$\mathcal{R}(u,\Omega) := \frac{\int\limits_\Omega |\Delta u|^2\,dx}{\int\limits_\Omega |\nabla u|^2\,dx}$$

for $u \in H_0^{2,2}(\Omega)$. We set $\mathcal{R}(u,\Omega) = \infty$ if the denominator vanishes. We define

$$\Lambda(\Omega) := \inf\left\{\mathcal{R}(u,\Omega)\ :\ u \in H_0^{2,2}(\Omega)\right\}. \tag{1.1}$$

The infimum is attained by the first eigenfunction u, which solves the Euler Lagrange equation

$$\Delta^2 u + \Lambda(\Omega)\Delta u = 0 \quad \text{in } \Omega \tag{1.2}$$

$$u = \partial_\nu u = 0 \quad \text{in } \partial\Omega. \tag{1.3}$$

If we normalize u by $\|\nabla u\|_{L^2(\Omega)} = 1$, the first eigenfunction is uniquely determined. Otherwise any multiple of u is an eigenfunction as well. The sign of the first eigenfunction may change depending on Ω.

* Corresponding author.
E-mail address: wagner@instmath.rwth-aachen.de (A. Wagner).

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The quantity $\Lambda(\Omega)$ is called the buckling eigenvalue of Ω. It is well known that there is a discrete spectrum of positive eigenvalues of finite multiplicity and their only accumulation point is ∞. The corresponding eigenfunctions form an orthonormal basis of $H_0^{2,2}(\Omega)$.

In the sequel, we will assume that u is normalized. If we multiply (1.2) with $x \cdot \nabla u$ and integrate by parts, we obtain

$$\Lambda(\Omega) = \frac{1}{2} \int_{\partial\Omega} |\Delta u|^2 \, x \cdot \nu \, dS. \tag{1.4}$$

In 1951, G. Polya and G. Szegö formulated the following conjecture (see [8]).

Among all domains Ω of given volume, the ball minimizes $\Lambda(\Omega)$.

This conjecture is still open. However, partial results are known. In [9] Szegö proved the conjecture for all smooth plane domains under the additional assumption that $u > 0$ in Ω. M.S. Ashbaugh and D. Bucur proved that among simply connected plane domains of prescribed volume there exists an optimal domain [1]. In [10] H. Weinberger and B. Willms proved the following uniqueness result for $n = 2$. If an optimal simply connected bounded plane domain Ω exists and if $\partial\Omega$ is smooth (at least $C^{2,\alpha}$), then Ω is a disc.

There also exist bounds for $\Lambda(\Omega)$. We mention only Payne's inequality (see [7]), which states that

$$\Lambda(\Omega) \geq \lambda_2(\Omega), \tag{1.5}$$

where λ_2 denotes the second Dirichlet eigenvalue for the Laplacian. Equality holds if and only if Ω is a ball. This result holds for any dimension.

In this paper, we assume that there exists an optimal domain $\Omega \subset \mathbb{R}^n$, which is smooth and such that the boundary $\partial\Omega$ is connected. For planar domains this is the assumption of simply connectedness of Ω. We then prove that Ω must be a ball. Thus we generalize the result of H. Weinberger and B. Willms in [10] to higher dimensions.

Considering the second domain variation for $\Lambda(\Omega)$ is motivated by the work of E. Mohr in [6]. He was interested in the clamped plate eigenvalue, where

$$\mathcal{R}(u, \Omega) = \frac{\int_\Omega |\Delta u|^2 \, dx}{\int_\Omega u^2 \, dx}$$

and Ω is a smoothly bounded domain in $\mathbb{R}^2$. For the corresponding eigenvalue he computed the second domain variation. The explicit computation of the kernel of the second domain variation then implies that the disc is a unique minimizer among smooth domains of equal volume.

Our strategy will be as follows. In Section 2 we introduce a smooth family $(\Omega_t)_t$ of perturbations of Ω of equal volume. We denote by $\Lambda(t) := \Lambda(\Omega_t)$ the corresponding first buckling eigenvalue of Ω_t. As a consequence of the optimality of Ω, the eigenfunction u satisfies the overdetermined boundary value problem

$$\Delta^2 u + \Lambda(\Omega)\Delta u = 0 \text{ in } \Omega \tag{1.6}$$

$$u = \partial_\nu u = 0 \text{ in } \partial\Omega \tag{1.7}$$

$$\Delta u = c_0 \text{ in } \partial\Omega, \text{ where } c_0 = \frac{2\Lambda(\Omega)}{|\Omega|} \text{ by (1.4)}. \tag{1.8}$$

This follows from the fact that the first domain variation of $\Lambda(\Omega)$ – computed in Section 3 – for an optimal domain necessarily vanishes. The equations (1.6)–(1.8) are the starting point for the proof of Weinberger and Willms. In a first step they substitute $v(x,y) = \Delta u(x,y)$ and $w(x,y) = x\partial_y v - y\partial_x v$ (for $n = 2$). For w they derive the equation

$$\Delta w + \Lambda(\Omega)w = 0 \quad \text{in } \Omega \qquad\qquad w = 0 \quad \text{in } \partial\Omega.$$

This implies that $\Lambda(\Omega)$ is equal to an eigenvalue $\lambda_k(\Omega)$ of the Dirichlet Laplace operator for some $k \in \mathbb{N}$. Clearly $w \equiv 0$ implies radial symmetry of u and the conjecture is proved. To exclude $w \not\equiv 0$ they apply more involved arguments. In particular the authors show that in the case $w \not\equiv 0$ the ball of equal volume as Ω has a strictly smaller first buckling eigenvalue. This contradicts the assumed minimality of Ω. A good reference is Chapter 11.3.4 in [3].

It is not immediate how to deduce $\Lambda(\Omega) = \lambda_k(\Omega)$ in higher dimensions. Even if one could show this, the discussion of the case $w \not\equiv 0$ would need completely new ideas.

This is why we use (1.6)–(1.8) to get more information about the nonnegative second domain variation of $\Lambda(\Omega)$. The main result of this paper is the derivation of the inequality $\Lambda(\Omega) \leq \lambda_2(\Omega)$. Payne's inequality (1.5) gives us the reverse inequality. Thus we are in the case of equality in (1.5). This implies Ω is a ball.

We now describe how we get the inequality $\Lambda(\Omega) \leq \lambda_2(\Omega)$. In Section 4 we compute the second domain variation of $\Lambda(\Omega)$. It turns out that

$$\ddot{\Lambda}(0) = \frac{d^2}{dt^2}\Lambda(t)\bigg|_{t=0} = 2\int_\Omega |\Delta u'|^2 - 2\Lambda(\Omega)\int_\Omega |\nabla u'|^2\,dx, \tag{1.9}$$

where u' is the so-called shape derivative of u. It solves

$$\Delta^2 u' + \Lambda(\Omega)\Delta u' = 0 \text{ in } \Omega \tag{1.10}$$

$$u' = 0 \text{ in } \partial\Omega \tag{1.11}$$

$$\partial_\nu u' = -c_0 v.\nu \text{ in } \partial\Omega \tag{1.12}$$

and

$$\int_\Omega \nabla u.\nabla u'\,dx = 0. \tag{1.13}$$

The vector field v is the first order approximation of Ω_t in the sense that for $y \in \Omega_t$ there exists an $x \in \Omega$ such that

$$y = x + tv(x) + o(t).$$

Thus, $\ddot{\Lambda}(0)$ is equal to a quadratic functional in the shape derivative u' which we denote by $\mathcal{E}(u')$ and $\mathcal{E}(u')$ is given by the right hand side of (1.9). Since we assumed the optimality of Ω, we have $\mathcal{E}(u') \geq 0$. It turns out that the kernel of $\mathcal{E}(u')$ contains the directional derivatives $\partial_1 u, \ldots, \partial_n u$ of u. Each directional derivative is a shape derivative, which corresponds to a domain perturbation given by translations.

The key idea is to enlarge the class of shape derivatives on which $\mathcal{E}$ is defined. This new class will be denoted by $\mathcal{Z}$ and contains the shape derivatives as a true subset. Nevertheless we can show that $\mathcal{E}$ is still bounded from below and even nonnegative on $\mathcal{Z}$. Moreover $\min_{\mathcal{Z}} \mathcal{E} = 0$ since the directional derivatives of

u are in $\mathcal{Z}$. This is done in Section 5. In Section 6 we construct a function $\psi \in \mathcal{Z}$ for which we will show

$$0 \le \mathcal{E}(\psi) \le (\lambda_2(\Omega) - \Lambda(\Omega))\, \lambda_2(\Omega).$$

This implies $\Lambda(\Omega) \le \lambda_2(\Omega)$.

Some of these results were obtained in the Diploma thesis of the first author [5].

2. Domain variations

Let Ω be a bounded smooth (at least $C^{2,\alpha}$) domain in $\mathbb{R}^n$ for which $\partial\Omega$ is connected. We denote by ν the unit normal vector field on $\partial\Omega$. Let δ be the distance function to the boundary, i.e. for $x \in \overline{\Omega}$ we have

$$\delta(x) := \inf\{|x - z| \ : \ z \in \partial\Omega\}.$$

Then, for smooth $\partial\Omega$, $\nu := \nabla\delta$ defines a smooth extension of ν into a sufficiently small tubular neighborhood of $\partial\Omega$. With this the following identities hold.

$$\nu \cdot \nu = 1, \qquad \nu \cdot D\nu = 0 \qquad \text{and} \qquad D\nu \cdot \nu = 0 \tag{2.1}$$

on $\partial\Omega$. See e.g. Proposition 5.4.14 in [4] for a proof.

Moreover, the mean curvature of $\partial\Omega$ is bounded since Ω is smooth, i.e. for each $x \in \partial\Omega$ there holds

$$|H_{\partial\Omega}(x)| \le \max_{\partial\Omega} |H_{\partial\Omega}| < \infty. \tag{2.2}$$

We will frequently use integration by parts on $\partial\Omega$. Let $f \in C^1(\partial\Omega)$ and $v \in C^{0,1}(\partial\Omega, \mathbb{R}^n)$. The next formula is often called the *Gauss theorem on surfaces*.

$$\oint_{\partial\Omega} f \,\mathrm{div}_{\partial\Omega}\, v \, dS = -\oint_{\partial\Omega} v \cdot \nabla^\tau f \, dS + (n-1) \oint_{\partial\Omega} f (v \cdot \nu)\, H_{\partial\Omega}\, dS, \tag{2.3}$$

where

$$\nabla^\tau f = \nabla f - (\nabla f \cdot \nu)\nu \tag{2.4}$$

denotes the tangential gradient of f.

In this section, we describe the class of admissible variations for the domain functional $\Lambda(\Omega)$. For given $t_0 > 0$ and $t \in (-t_0, t_0)$ let $(\Omega_t)_t$ be a family of perturbations of the domain $\Omega \subset \mathbb{R}^n$ of the form

$$\Omega_t = \Phi_t(\Omega)$$

where

$$\Phi_t \ : \ \overline{\Omega} \to \mathbb{R}^n$$

is a diffeomorphism which is smooth in t and x. Thus we may write

$$\Omega_t := \{y = x + tv(x) + \frac{t^2}{2} w(x) + o(t^2) : x \in \Omega,\ t \text{ small}\},$$

where

$$v = (v_1(x), v_2(x), \dots, v_n(x)) = \partial_t \Phi_t(x)|_{t=0}$$

and

$$w = (w_1(x), w_2(x), \dots, w_n(x)) = \partial_t^2 \Phi_t(x)|_{t=0}$$

are smooth vector fields and where $o(t^2)$ collects terms such that $\frac{o(t^2)}{t^2} \to 0$ as $t \to 0$. For small t_0 the sets Ω_t and Ω are diffeomorphic. We will frequently use the notation $y := \Phi_t(x)$. Consider the functional

$$\Lambda(\Omega_t) := \inf \left\{ \mathcal{R}(u, \Omega_t) \; : \; u \in H_0^{2,2}(\Omega_t) \right\},$$

which only depends on Ω_t. Let $u(t, y) \in H_0^{2,2}(\Omega_t)$ be the minimizer. For short we will write

$$\tilde{u}(t) := u(t, y). \tag{2.5}$$

Then $\tilde{u}(t)$ solves

$$\Delta^2 \tilde{u}(t) + \Lambda(\Omega_t)\Delta\tilde{u}(t) = 0 \quad \text{in } \Omega_t \tag{2.6}$$

$$\tilde{u}(t) = |\nabla \tilde{u}(t)| = 0 \quad \text{in } \partial\Omega_t \tag{2.7}$$

for each $t \in (-t_0, t_0)$. With this notation we define

$$\Lambda(t) := \mathcal{R}(\tilde{u}(t), \Omega_t).$$

Since we assume smoothness of Ω and Φ_t, the eigenfunction $\tilde{u}$ is also smooth in t and x. This has several consequences, which we list as remarks.

Remark 1. Since $\partial\Omega_t$ is smooth and since $\tilde{u}(t) = 0$ on $\partial\Omega_t$, then necessarily

$$\Delta\tilde{u} = \partial_\nu^2 \tilde{u} + (n-1)\partial_\nu \tilde{u}\, H_{\partial\Omega_t} \quad \text{in } \partial\Omega_t, \tag{2.8}$$

where $H_{\partial\Omega_t}$ denotes the mean curvature of $\partial\Omega_t$. Clearly, if $\tilde{u} = |\nabla\tilde{u}| = 0$ on $\partial\Omega_t$, then necessarily

$$\Delta\tilde{u} = \partial_\nu^2 \tilde{u} \quad \text{in } \partial\Omega_t. \tag{2.9}$$

Remark 2. Since (2.7) holds for all $t \in (-t_0, t_0)$, we also have

$$\dot{\tilde{u}}(t) = |\nabla \dot{\tilde{u}}(t)| = 0 \quad \text{in } \partial\Omega_t \tag{2.10}$$

for all $t \in (-t_0, t_0)$.

Remark 3. Straightforward computations yield

$$\dot{\tilde{u}}(t) = \frac{d}{dt} u(t, y) = \partial_t u(t, \Phi_t(\Phi_t^{-1}(y)) + \partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot \nabla u(t, y)$$

for all $t \in (-t_0, t_0)$. Let $y \in \partial\Omega_t$. Then (2.10) and (2.7) imply

$$0 = \dot{\tilde{u}}(t) = \partial_t u(t, y) \quad \text{for } y \text{ in } \partial\Omega_t \tag{2.11}$$

for all $t \in (-t_0, t_0)$.

In particular for $t = 0$ we compute $\tilde{u}(0) = u(x)$ and

$$\begin{aligned}\dot{\tilde{u}}(0) &= \partial_t u(0,x) + v(x)\cdot Du(0,x)\\ \ddot{\tilde{u}}(0) &= \partial_t^2 u(0,x) + 2v(x)\cdot D\partial_t u(0,x) + w(x)\cdot Du(0,x) + v(x)\cdot D\left(v(x)\cdot Du(0,x)\right).\end{aligned}$$

We will use the notation

$$u'(x) := \partial_t u(0,x) \qquad \text{and} \qquad u''(x) := \partial_t^2 u(0,x).$$

Hence,

$$\dot{\tilde{u}}(0) = u'(x) + v(x)\cdot Du(x) \tag{2.12}$$

$$\ddot{\tilde{u}}(0) = u''(x) + 2v(x)\cdot Du'(x) + w(x)\cdot Du(x) + v(x)\cdot D\left(v(x)\cdot Du(x)\right). \tag{2.13}$$

Note that all these quantities are defined for $x \in \overline{\Omega}$. For $x \in \partial\Omega$ we thus get

$$0 = \dot{\tilde{u}}(0) = u'(x) \qquad \text{and} \qquad 0 = \nabla\dot{\tilde{u}}(0) = \nabla u'(x) + v(x)\cdot D^2u(x),$$

where $(v(x)\cdot D^2u(x))_j = \sum_{i=1}^n v_i(x)\partial_i\partial_j u(x)$ for $j = 1,\dots,n$. Thus, we get the following boundary conditions for u'.

$$u'(x) = 0 \qquad \text{and} \qquad \partial_\nu u'(x) = -v(x)\cdot D^2u(x)\cdot\nu(x) \quad \text{for} \quad x \in \partial\Omega. \tag{2.14}$$

Here we used the notation $v(x)\cdot D^2u(x)\cdot\nu(x) = \sum_{i,j=1}^n v_i(x)\partial_i\partial_j u(x)\nu_j(x)$.

Let $\nu_t(y)$ be the unit normal vector in $y \in \partial\Omega_t$. We also write this as

$$\nu_t(y) = \nu(t,\Phi_t(x)) \qquad \forall\, t \in (-t_0,t_0) \qquad x \in \partial\Omega. \tag{2.15}$$

Then we have

$$\nu' = -\nabla^\tau(v\cdot\nu), \qquad \nu\cdot\nu' = 0. \tag{2.16}$$

This follows from direct calculations (see e.g. (5.64) in [4]).

Lemma 1. *With the notation from above the following equality holds.*

$$\nu_t\cdot\nabla(\partial_t u(t,y)) = -\Delta u(t,y)\,\nu_t\cdot\partial_t\Phi_t(\Phi_t^{-1}(y)) \quad \textit{for } y \textit{ in } \partial\Omega_t \tag{2.17}$$

for all $t \in (-t_0,t_0)$. Alternatively, we write this for all $t \in (-t_0,t_0)$ and $x \in \partial\Omega$ as

$$\nu(t,\Phi_t(x))\cdot\nabla\{\partial_t u(t,\Phi_t(x))\} = -\Delta u(t,\Phi_t(x))\,\nu(t,\Phi_t(x))\cdot\partial_t\Phi_t(x). \tag{2.18}$$

Proof. Since $\nabla u(t,\Phi_t(x)) = 0$ for all $|t| < t_0$ and all $x \in \partial\Omega$, we have

$$0 = \frac{d}{dt}\,\nabla u(t,\Phi_t(x)) = \nabla\partial_t u(t,\Phi_t(x)) + D^2u(t,\Phi_t(x))\cdot\partial_t\Phi_t(x).$$

This implies

$$0 = \nu_t\cdot\nabla(\partial_t u(t,y)) + \nu_t\cdot D^2u(t,y)\cdot\partial_t\Phi_t(\Phi_t^{-1}(y)) \quad \text{for } y \text{ in } \partial\Omega_t$$

for all $t \in (-t_0, t_0)$. Here we used the notation

$$\nu_t \cdot D^2 u(t,y) \cdot \partial_t \Phi_t(\Phi_t^{-1}(y)) = \sum_{ij=1}^{n} \nu_{t,i} \cdot \partial_i \partial_j u(t,y) \cdot \partial_t \Phi_t(\Phi_t^{-1}(y))_j .$$

Since $\nabla \tilde{u}(t) = 0$ in $\partial\Omega_t$, we get

$$\nu_t \cdot D^2 u(t,y) \cdot \partial_t \Phi_t(\Phi_t^{-1}(y)) = \nu_t \cdot D^2 u(t,y) \cdot \nu_t \, \nu_t \cdot \partial_t \Phi_t(\Phi_t^{-1}(y)).$$

Thus,

$$\nu_t \cdot \nabla(\partial_t u(t,y)) = -\nu_t \cdot D^2 u(t,y) \cdot \nu_t \, \nu_t \cdot \partial_t \Phi_t(\Phi_t^{-1}(y)) \quad \text{for } y \text{ in } \partial\Omega_t.$$

Formula (2.9) simplifies to

$$\nu_t \cdot \nabla(\partial_t u(t,y)) = -\Delta u(t,y) \, \nu_t \cdot \partial_t \Phi_t(\Phi_t^{-1}(y)) \quad \text{for } y \text{ in } \partial\Omega_t.$$

This proves the lemma. □

The first derivative of $\Lambda(t)$ with respect to the parameter t is called *the first domain variation* and the second derivative is called *the second domain variation*.

Our domain variations will be chosen within the class of volume preserving perturbations up to order 2. Hence, they are chosen such that

$$\mathcal{L}^n(\Omega_t) = \mathcal{L}^n(\Omega) + o(t^2) \tag{2.19}$$

holds. This puts constraints on the vector fields v and w. They were discussed e.g. in [2], formula (2.13) and Lemma 1.

Lemma 2. *Let $v, w \in C^{0,1}(\Omega, \mathbb{R}^n)$ be such that (2.19) holds. Then*

$$\int_\Omega \operatorname{div} v \, dx = 0 \tag{2.20}$$

and

$$\int_\Omega \left((\operatorname{div} v)^2 - Dv : Dv + \operatorname{div} w \right) dx = 0,$$

where $Dv : Dv = \sum_{i,j=1}^n \partial_i v_j \, \partial_j v_i$. The second equality is equivalent to

$$\int_{\partial\Omega} (v \cdot \nu) \operatorname{div} v \, dS - \int_{\partial\Omega} v \cdot Dv \cdot \nu \, dS + \int_{\partial\Omega} (w \cdot \nu) \, dS = 0. \tag{2.21}$$

Note that rotations do not satisfy these conditions (see e.g. Remark 1 in [2]).

3. The first domain variation

We will use the following formula for the computations of the first domain variation of Λ. It is well known as Reynold's transport theorem and is analyzed in detail in Chapter 5.2.3 in [4].

Theorem 1. *Let $t \in (-t_0, t_0)$ for some $t_0 > 0$. Let $\Phi_t \in C^{0,1}(\mathbb{R}^n)$ be differentiable in t and let $t \to f(t) \in L^1(\mathbb{R}^n)$ be a function which is differentiable in t. Moreover, let $f(t) \in W^{1,1}(\mathbb{R}^n)$. Then $t \to I(t) := \int_{\Omega_t} f(t)\, dy$ is differentiable in t and we have the formula*

$$\dot{I}(t) = \int_{\Omega_t} \partial_t f(t) + \operatorname{div}\left(f(t)\, \partial_t \Phi_t(\Phi_t^{-1}(y))\right)\, dy.$$

If $\partial\Omega$ is sufficiently smooth (at least Lipschitz continuous), this is equivalent to

$$\dot{I}(t) = \int_{\Omega_t} \partial_t f(t)\, dy + \int_{\partial\Omega_t} f(t)\, \partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot \nu(y)\, dS(y).$$

In particular, for $t = 0$ we get

$$\dot{I}(0) = \int_{\Omega} \partial_t f(t)|_{t=0} + \operatorname{div}\left(f(0)\, v(x)\right)\, dx.$$

Again, if $\partial\Omega$ is sufficiently smooth, this is equivalent to

$$\dot{I}(0) = \int_{\Omega} \partial_t f(t)|_{t=0}\, dx + \int_{\partial\Omega} f(0)\, v(x) \cdot \nu(x)\, dS(x).$$

We apply this formula to $\Lambda(t) = \frac{D(t)}{N(t)}$ where

$$D(t) := \int_{\Omega_t} |\Delta\tilde{u}(t)|^2\, dy \qquad \text{and} \qquad N(t) := \int_{\Omega_t} |\nabla\tilde{u}(t)|^2\, dy$$

and we assume the normalization

$$N(t) = \int_{\Omega_t} |\nabla\tilde{u}(t)|^2\, dy = 1 \quad \forall\, t \in (-t_0, t_0). \tag{3.1}$$

We then obtain

$$\begin{aligned}\dot{\Lambda}(t) = 2 \int_{\Omega_t} \Delta\tilde{u}(t)\, \Delta\partial_t\tilde{u}(t)\, dy - 2\,\Lambda(t) \int_{\Omega_t} \nabla\tilde{u}(t) \cdot \nabla\partial_t\tilde{u}(t)\, dy \\ + \int_{\partial\Omega_t} |\Delta\tilde{u}(t)|^2\, \partial_t\Phi_t(\Phi_t^{-1}(y)) \cdot \nu_t(y)\, dS(y),\end{aligned} \tag{3.2}$$

where $\nu_t(y)$ denotes the unit normal vector in $y \in \partial\Omega_t$. We integrate by parts and use (2.10). Then

$$\dot{\Lambda}(t) = 2 \int_{\Omega_t} \left\{ \Delta^2 \tilde{u}(t) + \Lambda(t)\, \Delta \tilde{u}(t) \right\} \partial_t \tilde{u}(t)\, dy + 2 \int_{\partial\Omega_t} \Delta \tilde{u}(t)\, \partial_{\nu_t} \partial_t \tilde{u}(t)\, dS(y)$$
$$- 2 \int_{\partial\Omega_t} \partial_{\nu_t} \Delta \tilde{u}(t) \partial_t \tilde{u}(t)\, dS(y) + \int_{\partial\Omega_t} |\Delta \tilde{u}(t)|^2\, \partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot \nu_t(y)\, dS(y).$$

The first integral vanishes since $\tilde{u}(t)$ solves (2.6). The third integral vanishes since (2.11) holds. Finally we use (2.17). This proves the following lemma.

Lemma 3. *Let $\tilde{u}(t)$ be an eigenfunction (i.e. a solution of (2.6)–(2.7)) and assume (3.1) holds. Let*

$$\Lambda(t) = \int_{\Omega_t} |\Delta \tilde{u}(t)|^2\, dy.$$

Then

$$\dot{\Lambda}(t) = - \int_{\partial\Omega_t} |\Delta \tilde{u}(t)|^2\, \partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot \nu_t(y)\, dS(y). \tag{3.3}$$

Remark 4. Note that if $\partial_t \Phi_t(\Phi_t^{-1}(y)) \cdot \nu_t(y) > 0$, this implies $\mathcal{L}^n(\Omega_t) > \mathcal{L}^n(\Omega)$ for small t. Thus, $\dot{\Lambda}(t)$ is negative in this case. We conclude, that the first buckling eigenvalue is decreasing under set inclusion.

From Lemma 3 we get in particular

$$\dot{\Lambda}(0) = - \int_{\partial\Omega} |\Delta u|^2\, v(x) \cdot \nu(x)\, dS(x).$$

From Lemma 2 and (2.20) we deduce $|\Delta u| = \mathit{const.}$ if Ω is a critical point of $\Lambda(t)$. Due to formula (1.4), this constant is equal to

$$c_0 := \frac{2\Lambda(0)}{|\Omega|}. \tag{3.4}$$

We summarize this result as a theorem.

Theorem 2. *Let Ω_t be a family of volume preserving perturbations of Ω as described in Section 2. Then Ω is a critical point of the energy $\Lambda(t)$, i.e. $\dot{\Lambda}(0) = 0$, if and only if*

$$\Delta u = c_0 \qquad \text{on} \quad \partial\Omega. \tag{3.5}$$

In particular, u is a solution of the overdetermined boundary value problem

$$\Delta^2 u + \Lambda(\Omega) \Delta u = 0 \quad \text{in } \Omega \tag{3.6}$$
$$u = \partial_\nu \nabla u = 0 \quad \text{in } \partial\Omega \tag{3.7}$$
$$\Delta u = c_0 > 0 \quad \text{in } \partial\Omega. \tag{3.8}$$

Note that if we set $U := \Delta u + \Lambda(\Omega) u$ (3.6)–(3.8) imply

$$\Delta U = 0 \text{ in } \Omega \;\text{ and }\; U = c_0 \text{ in } \partial\Omega.$$

Hence,

$$U = \Delta u + \Lambda(\Omega)u = c_0 \qquad \text{in } \overline{\Omega}. \tag{3.9}$$

From [10] we know that for $n = 2$ this implies that Ω is a ball. In particular,

$$\partial_\nu \Delta u = 0 \qquad \text{in } \partial\Omega. \tag{3.10}$$

These considerations are only valid if we assume that $\partial\Omega$ consists of one connected component only.

4. The second domain variation

Throughout this section we assume that Ω is an optimal domain, i.e. $\dot{\Lambda}(0) = 0$ and $\ddot{\Lambda}(0) \geq 0$. This implies that u solves (3.6)–(3.8) and (3.9). As a consequence (2.14) reads as

$$u'(x) = 0 \qquad \text{and} \qquad \partial_\nu u'(x) = -c_0\, v(x) \cdot \nu(x) \quad \text{for} \quad x \in \partial\Omega. \tag{4.1}$$

Note that if we differentiate (2.6)–(2.7) in $t = 0$ and use the fact that $\dot{\Lambda}(0) = 0$, we obtain an equation for u':

$$\Delta^2 u'(x) + \Lambda(\Omega)\Delta u'(x) = 0 \qquad \text{in } \Omega. \tag{4.2}$$

The boundary conditions for u' are given by (4.1). Furthermore, the normalization (3.1) implies

$$\int_\Omega \nabla u \cdot \nabla u' \, dx = 0. \tag{4.3}$$

We recall formula (3.3). Before we differentiate with respect to t again we state the following consequence of Reynold's theorem (see e.g. Chapter 5.4.2 in [4]).

Theorem 3. *Let Ω be a bounded smooth domain of class C^3. Let $t \in (-t_0, t_0)$ and let $\Phi_t \in C^{0,1}(\mathbb{R}^n)$ be differentiable in t. Let $t \to g(t) \in L^1(\mathbb{R}^n)$ be a function which is differentiable in t. Moreover, let $g(t) \in W^{1,1}(\mathbb{R}^n)$. Then $t \to J(t) := \int_{\partial\Omega_t} g(t)\, dS(y)$ is differentiable in t. For $t = 0$ we have the formula*

$$\dot{J}(0) = \int_{\partial\Omega} \partial_t g(0) + (v(x) \cdot \nu(x)) \left\{\partial_\nu g(0) + (n-1) g(0)\, H_{\partial\Omega}(x)\right\} dS(x),$$

where $H_{\partial\Omega}$ denotes the mean curvature of $\partial\Omega$ in x.

We apply this theorem to (3.3). It is convenient to apply (2.17) and to rewrite (3.3) as

$$\dot{\Lambda}(t) = \int_{\partial\Omega_t} \Delta\tilde{u}(t)\, \nu_t \cdot \nabla(\partial_t u(t, y))\, dS(y).$$

Let

$$g(t) := \Delta\tilde{u}(t)\, \nu_t \cdot \nabla(\partial_t u(t, y)).$$

An application of Theorem 3 yields

$$\ddot{\Lambda}(0) = \int_{\partial\Omega} \Delta u' \, \partial_\nu u' \, dS + \int_{\partial\Omega} \Delta u \, \nu' \cdot \nabla u' \, dS + \int_{\partial\Omega} \Delta u \, \partial_\nu u'' \, dS$$
$$+ \int_{\partial\Omega} (v \cdot \nu) \, \partial_\nu (\Delta u \, \partial_\nu u') \, dS + (n-1) \int_{\partial\Omega} (v \cdot \nu) \, \Delta u \, \partial_\nu u' \, H_{\partial\Omega} \, dS. \tag{4.4}$$

Note that

$$\nu_t \cdot \nu_t = 1 \ \text{ in } \ \partial\Omega_t \quad \Longrightarrow \quad \nu \cdot \nu' = 0 \ \text{ in } \ \partial\Omega,$$

where

$$\nu'(x) = \partial_t \nu(t, \Phi_t(x))|_{t=0} \quad \text{for} \quad x \in \partial\Omega.$$

Since (4.1) implies $\nabla u' = \partial_\nu u' \, \nu$, this implies

$$\int_{\partial\Omega} \Delta u \, \nu' \cdot \nabla u' \, dS = 0.$$

For the fourth integral we apply (3.5) and (3.10).

$$\int_{\partial\Omega} (v \cdot \nu) \, \partial_\nu (\Delta u \, \partial_\nu u') \, dS = \int_{\partial\Omega} (v \cdot \nu) \, \partial_\nu \Delta u \, \partial_\nu u' \, dS + \int_{\partial\Omega} (v \cdot \nu) \, \Delta u \, \partial_\nu^2 u' \, dS$$
$$= 0 + c_0 \int_{\partial\Omega} (v \cdot \nu) \, \partial_\nu^2 u' \, dS.$$

With the help of (4.1) and (2.8) we write

$$\partial_\nu^2 u' = \Delta u' - (n-1) \partial_\nu u' \, H_{\partial\Omega}.$$

Hence,

$$\int_{\partial\Omega} (v \cdot \nu) \, \partial_\nu (\Delta u \, \partial_\nu u') \, dS = c_0 \int_{\partial\Omega} (v \cdot \nu) \, \Delta u' \, dS - c_0 (n-1) \int_{\partial\Omega} (v \cdot \nu) \, \partial_\nu u' \, H_{\partial\Omega} \, dS.$$

Our computations yield a first simplification of (4.4):

$$\ddot{\Lambda}(0) = \int_{\partial\Omega} \Delta u' \, \partial_\nu u' \, dS + \int_{\partial\Omega} \Delta u \, \partial_\nu u'' \, dS + c_0 \int_{\partial\Omega} (v \cdot \nu) \, \Delta u' \, dS.$$

In the first integral on the right hand side we use (4.1) again. Thus, we get

$$\ddot{\Lambda}(0) = c_0 \int_{\partial\Omega} \partial_\nu u'' \, dS \tag{4.5}$$

In order to find a lower bound for $\ddot{\Lambda}(0)$, we analyze the integral in (4.5). Recall (2.18). We differentiate this equation with respect to t in $t = 0$. Then (3.10) and (3.5) yield

$$\begin{aligned}&\nu' \cdot \nabla u' + v \cdot D\nu \cdot \nabla u' + \partial_\nu u'' + \nu \cdot D^2 u' \cdot v \\ &\quad = -\Delta u' \, (v \cdot \nu) - c_0 \, (v \cdot \nu') - c_0 v \cdot D\nu \cdot v - c_0 \, (w \cdot \nu).\end{aligned}$$

As before, $\nu' \cdot \nabla u' = 0$ on $\partial\Omega$. Moreover, by (4.1)

$$v \cdot D\nu \cdot \nabla u' = -c_0 v \cdot D\nu \cdot \nu \, (v \cdot \nu) = 0,$$

where the last equality follows from (2.1). Thus,

$$\begin{aligned}\ddot{\Lambda}(0) = &-c_0 \int_{\partial\Omega} (v \cdot \nu) \, \Delta u' \, dS - c_0 \int_{\partial\Omega} \nu \cdot D^2 u' \cdot v \, dS \\ &- c_0^2 \int_{\partial\Omega} (v \cdot \nu') \, dS - c_0^2 \int_{\partial\Omega} v \cdot D\nu \cdot v \, dS - c_0^2 \int_{\partial\Omega} (w \cdot \nu) \, dS.\end{aligned} \tag{4.6}$$

For the first integral we use (4.1) and we observe that Gauß theorem, partial integration and equation (4.2) for u' give

$$-c_0 \int_{\partial\Omega} (v \cdot \nu) \, \Delta u' \, dS = \int_{\partial\Omega} \Delta u' \, \partial_\nu u' \, dS = \int_{\Omega} |\Delta u'|^2 \, dx - \Lambda(\Omega) \int_{\Omega} |\nabla u'|^2 \, dx. \tag{4.7}$$

The second integral is slightly more involved. We set $v^\tau = v - (v \cdot \nu)\nu$. Since $\nabla u' = (\partial_\nu u')\nu$ and since (2.8) can be applied to u', we get

$$\begin{aligned}-c_0 \int_{\partial\Omega} v \cdot D^2 u' \cdot \nu \, dS &= -c_0 \int_{\partial\Omega} v^\tau \cdot D^2 u' \cdot \nu \, dS - c_0 \int_{\partial\Omega} (v \cdot \nu) \left(\Delta u' - (n-1)\partial_\nu u' \, H_{\partial\Omega}\right) dS \\ &= -c_0 \int_{\partial\Omega} v^\tau \cdot D \left(\partial_\nu u' \, \nu\right) \cdot \nu \, dS - c_0 \int_{\partial\Omega} (v \cdot \nu) \, \Delta u' \, dS \\ &\quad - c_0^2 \, (n-1) \int_{\partial\Omega} (v \cdot \nu)^2 \, H_{\partial\Omega} \, dS.\end{aligned}$$

For the last equality we also used

$$v^\tau \cdot D\nu \cdot \nu = v^\tau \cdot D^\tau \nu \cdot \nu = 0 \quad \text{in } \partial\Omega.$$

Next we note that with (4.1) we have

$$\begin{aligned}-c_0 \int_{\partial\Omega} v^\tau \cdot D \left(\partial_\nu u' \, \nu\right) \cdot \nu \, dS &= -c_0 \int_{\partial\Omega} v^\tau \cdot D^\tau \left(\partial_\nu u' \, \nu\right) \cdot \nu \, dS = c_0^2 \int_{\partial\Omega} v^\tau \cdot D^\tau \left((v \cdot \nu) \, \nu\right) \cdot \nu \, dS \\ &= c_0^2 \int_{\partial\Omega} v^\tau \cdot \nabla^\tau (v \cdot \nu) \, dS,\end{aligned}$$

where the last equality uses (2.1).

For the third integral in (4.6) we apply formula (2.16):

$$-c_0^2 \int_{\partial\Omega} (v \cdot \nu') \, dS = c_0^2 \int_{\partial\Omega} v \cdot \nabla^\tau (v \cdot \nu) \, dS = c_0^2 \int_{\partial\Omega} v^\tau \cdot \nabla^\tau (v \cdot \nu) \, dS.$$

These computations simplify (4.6) and we obtain

$$\ddot{\Lambda}(0) = 2\int_{\partial\Omega} \partial_\nu u'\, \Delta u'\, dS + 2c_0^2 \int_{\partial\Omega} v^\tau \cdot \nabla^\tau (v\cdot\nu)\, dS - c_0^2\,(n-1)\int_{\partial\Omega} (v\cdot\nu)^2\, H_{\partial\Omega}\, dS$$
$$- c_0^2 \int_{\partial\Omega} v \cdot D\nu \cdot v\, dS - c_0^2 \int_{\partial\Omega} (w\cdot\nu)\, dS. \tag{4.8}$$

Next we use the volume constraint (2.21).

$$-c_0^2 \int_{\partial\Omega} (w\cdot\nu)\, dS = c_0^2 \int_{\partial\Omega} (v\cdot\nu)\,\mathrm{div}\, v\, dS - c_0^2 \int_{\partial\Omega} v\cdot Dv\cdot\nu\, dS$$
$$= c_0^2 \int_{\partial\Omega} (v\cdot\nu)\,\mathrm{div}_{\partial\Omega}\, v\, dS - c_0^2 \int_{\partial\Omega} v^\tau\cdot D^\tau v\cdot\nu\, dS.$$

We integrate by parts in the first integral (see formula (2.3) and (2.4)).

$$-c_0^2 \int_{\partial\Omega} (w\cdot\nu)\, dS = -c_0^2 \int_{\partial\Omega} v^\tau\cdot D^\tau (v\cdot\nu)\, dS + c_0^2(n-1)\int_{\partial\Omega} (v\cdot\nu)^2\, H_{\partial\Omega}\, dS$$
$$- c_0^2 \int_{\partial\Omega} v^\tau\cdot D^\tau v\cdot\nu\, dS.$$

Thus, (4.8) becomes

$$\ddot{\Lambda}(0) = 2\int_{\partial\Omega} \partial_\nu u'\, \Delta u'\, dS + c_0^2 \int_{\partial\Omega} v^\tau\cdot\nabla^\tau(v\cdot\nu)\, dS - c_0^2 \int_{\partial\Omega} v^\tau\cdot D^\tau v\cdot\nu\, dS$$
$$- c_0^2 \int_{\partial\Omega} v\cdot D\nu\cdot v\, dS.$$

An application of (2.1) and (2.16) yields

$$v^\tau\cdot\nabla^\tau(v\cdot\nu) - v^\tau\cdot D^\tau v\cdot\nu - v\cdot D\nu\cdot v = v^\tau\cdot D^\tau\nu\cdot v - v\cdot D\nu\cdot v$$
$$= -(v\cdot\nu)\nu\cdot D\nu\cdot v = 0.$$

Thus, with (4.8) we proved the following lemma.

Lemma 4. *Let u' be the shape derivative of u resulting from a volume preserving perturbation of Ω. Then there holds*

$$\ddot{\Lambda}(0) = 2\mathcal{E}(u'),$$

where

$$\mathcal{E}(u') = \int_\Omega |\Delta u'|^2\, dx - \Lambda(\Omega)\int_\Omega |\nabla u'|^2\, dx.$$

5. Minimization of the second domain variation

In this section we consider the quadratic functional

$$\mathcal{E}(\varphi) := \int_\Omega |\Delta\varphi|^2\,dx - \Lambda(\Omega)\int_\Omega |\nabla\varphi|^2\,dx \tag{5.1}$$

for $\varphi \in H_0^{1,2} \cap H^{2,2}(\Omega)$. It will be convenient to work with an alternative representation of $\mathcal{E}$. For $\varphi \in H_0^{1,2} \cap H^{2,2}(\Omega)$ there holds

$$\mathcal{E}(\varphi) = \int_\Omega |D^2\varphi|^2 - \Lambda(\Omega)|\nabla\varphi|^2\,dx + \int_{\partial\Omega} \Delta\varphi\partial_\nu\varphi - \varphi\cdot D^2\varphi\cdot\nu\,dS.$$

We apply (2.8) and (2.1).

$$\begin{aligned}\Delta\varphi\partial_\nu\varphi - \varphi\cdot D^2\varphi\cdot\nu &= \partial_\nu^2\varphi\,\partial_\nu\varphi + (n-1)(\partial_\nu\varphi)^2\,H_{\partial\Omega} - \varphi\cdot D^2\varphi\cdot\nu\\ &= \nu\cdot D^2\varphi\cdot\nu\,(\nu\cdot\nabla\varphi) + (n-1)(\partial_\nu\varphi)^2\,H_{\partial\Omega} - \varphi\cdot D^2\varphi\cdot\nu\\ &= (n-1)(\partial_\nu\varphi)^2\,H_{\partial\Omega}.\end{aligned}$$

Consequently, we get

$$\mathcal{E}(\varphi) = \int_\Omega |D^2\varphi|^2\,dx - \Lambda(\Omega)\int_\Omega |\nabla\varphi|^2\,dx + (n-1)\int_{\partial\Omega}(\partial_\nu\varphi)^2 H_{\partial\Omega}\,dS. \tag{5.2}$$

Remark 5. The functional $\mathcal{E}$ is lower semicontinuous with respect to weak convergence in $H_0^{1,2} \cap H^{2,2}(\Omega)$.

Since Ω is optimal, we know from Lemma 4 that

$$\mathcal{E}(\varphi) \geq 0$$

for all φ which are shape derivatives of u. Recall that φ is a shape derivative, if it solves (1.10)–(1.13) for some vector field v in the class described in Section 2 (Lemma 2).

The following remark shows a property of shape derivatives we have not yet mentioned.

Remark 6. Let φ be a shape derivative and assume that $\partial_\nu\varphi \equiv 0$ in $\partial\Omega$. Then $\varphi \in H_0^{2,2}(\Omega)$ and, since φ satisfies equation (4.2), φ is a buckling eigenfunction in Ω. Thus by uniqueness of u we get $\varphi = \alpha u$ for any $\alpha \in \mathbb{R}$. Then formula (1.4) yields

$$\Lambda(\Omega) = \int_{\partial\Omega} |\Delta\varphi|^2\,x\cdot\nu\,dS = \alpha^2 c_o^2\int_{\partial\Omega} x\cdot\nu\,dS = \alpha^2\int_{\partial\Omega}|\Delta u|^2\,x\cdot\nu\,dS = \alpha^2\Lambda(\Omega).$$

Thus, $\alpha^2 = 1$ and there holds

$$\left|\int_\Omega \nabla u\cdot\nabla\varphi\,dx\right| = 1.$$

This is contradictory to (4.3) and thus $\partial_\nu\varphi$ cannot vanish identically on $\partial\Omega$.

This motivates the following definition.

$$\mathcal{Z} := \left\{ \varphi \in H_0^{1,2} \cap H^{2,2}(\Omega) : \int_{\partial\Omega} \partial_\nu \varphi \, dS = 0, \int_{\partial\Omega} (\partial_\nu \varphi)^2 \, dS > 0, \int_{\Omega} \nabla u \cdot \nabla \varphi \, dx = 0 \right\}.$$

Note that $\mathcal{Z}$ contains elements which are not shape derivatives. Nevertheless we will show that

$$\mathcal{E}\big|_{\mathcal{Z}} \geq 0.$$

The next lemma ensures that $\mathcal{Z}$ is not empty and that at least for a specific shape derivative $\mathcal{E}$ is equal to zero.

Lemma 5. *For each $1 \leq k \leq n$ the directional derivative $\partial_k u$ satisfies $\partial_k u \in \mathcal{Z}$. Furthermore, $\mathcal{E}(\partial_k u) = 0$.*

Proof. Let $1 \leq k \leq n$. Due to (1.2) and (1.3) $\partial_k u$ satisfies

$$\begin{aligned} \Delta^2 \partial_k u + \Lambda(\Omega) \Delta \partial_k u = 0 \quad & \text{in } \Omega \\ \partial_k u = 0 \quad & \text{in } \partial\Omega. \end{aligned} \tag{5.3}$$

According to (2.9) there holds $\partial_\nu \partial_k u = c_0 \nu_k$ on $\partial\Omega$. Hence,

$$\int_{\partial\Omega} \partial_\nu \partial_k u \, dS = c_0 \int_{\partial\Omega} \nu_k \, dS = 0.$$

In addition, we find that

$$\int_{\Omega} \nabla u \cdot \nabla \partial_k u \, dx = \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 \nu_k \, dS = 0.$$

Following the idea of Remark 6, we obtain that $\partial_\nu \partial_k u$ does not vanish identically on $\partial\Omega$. Thus, $\partial_k u \in \mathcal{Z}$. Moreover, (3.10) and (5.3) imply

$$\mathcal{E}(\partial_k u) = \int_{\Omega} (\Delta^2 \partial_k u + \Lambda(\Omega) \Delta \partial_k u) \partial_k u \, dx + \int_{\partial\Omega} \partial_k \Delta u \, \partial_\nu \partial_k u \, dS = 0.$$

This proves the lemma. $\square$

Note that each directional derivative of u is a shape derivative resulting from translations of Ω.
We consider the functional

$$\tilde{\mathcal{E}}(\varphi) := \frac{\mathcal{E}(\varphi)}{\int_{\partial\Omega} (\partial_\nu \varphi)^2 \, dS}, \tag{5.4}$$

where $\varphi \in \mathcal{Z}$ and we set $\tilde{\mathcal{E}} = \infty$ if $\int_{\partial\Omega} (\partial_\nu \varphi)^2 \, dS = 0$. By scaling we may assume

$$\int_{\partial\Omega} (\partial_\nu \varphi)^2 \, dS = 1$$

With this normalization we prove the following statement.

Theorem 4. *The infimum of the functional $\tilde{\mathcal{E}}$ in $\mathcal{Z}$ is finite.*

Proof. We argue by contradiction. Let us assume that $\inf_{\mathcal{Z}} \tilde{\mathcal{E}} = -\infty$ and consider a sequence $(\hat{w}_k)_k \subset \mathcal{Z}$ such that

$$\int_{\partial\Omega} (\partial_\nu \hat{w}_k)^2 \, dS = 1$$

and

$$\lim_{k\to\infty} \tilde{\mathcal{E}}(\hat{w}_k) = -\infty.$$

Assumption (2.2) gives

$$\left| \int_{\partial\Omega} H_{\partial\Omega} \, (\partial_\nu \hat{w}_k)^2 \, dS \right| \leq \max_{\partial\Omega} |H_{\partial\Omega}| < \infty.$$

We use (5.2) and obtain

$$\tilde{\mathcal{E}}(\hat{w}_k) \geq -\Lambda(0) \int_{\Omega} |\nabla \hat{w}_k| \, dx - (n-1) \max_{\partial\Omega} |H_{\partial\Omega}|. \tag{5.5}$$

The assumption $\lim_{k\to\infty} \mathcal{E}(\hat{w}_k) = -\infty$ implies

$$\int_{\Omega} |\nabla \hat{w}_k|^2 \, dx \overset{k\to\infty}{\longrightarrow} \infty.$$

We define

$$w_k := \frac{1}{\|\nabla \hat{w}_k\|_{L^2(\Omega)}} \hat{w}_k.$$

Then there holds

$$\|\nabla w_k\|_{L^2(\Omega)} = 1 \quad \text{and} \quad \int_{\partial\Omega} (\partial_\nu w_k)^2 \, dS \overset{k\to\infty}{\longrightarrow} 0. \tag{5.6}$$

Moreover, for each $k \in \mathbb{N}$ estimate (5.5) implies

$$\tilde{\mathcal{E}}(w_k) \geq -\Lambda(0) - C$$

and the infimum of $\tilde{\mathcal{E}}$ in $M := \{w_k : k \in \mathbb{N}\}$ is finite. Therefore, we can choose a subsequence of $(w_k)_k$, denote by $(w_k)_k$ as well, such that

$$\lim_{k\to\infty} \tilde{\mathcal{E}}(w_k) = \inf_M \mathcal{E}.$$

Now Poincaré's inequality and the previous estimates imply

$$\begin{aligned}\|w_k\|^2_{H^{2,2}(\Omega)} &= \int\limits_\Omega |D^2 w_k|^2 + |\nabla w_k|^2 + w_k^2\,dx \\ &\le \tilde{\mathcal{E}}(w_k) + C\int\limits_\Omega |\nabla w_k|^2\,dx + (n-1)\int\limits_{\partial\Omega} |H_{\partial\Omega}|(\partial_\nu w_k)^2\,dS \\ &\le C.\end{aligned}$$

Thus, the sequence $(w_k)_k$ is uniformly bounded in $H^{2,2}(\Omega)$ and there exists a $w \in H^{2,2}(\Omega)$ such that $(w_k)_k$ weakly converges to w. In view of (5.6), the limit function w satisfies $\|\nabla w\|_{L^2(\Omega)} = 1$ and $\partial_\nu w = 0$ on $\partial\Omega$. Since $w_k = 0$ in $\partial\Omega$ for each $k \in \mathbb{N}$, we conclude that $w \in H^{2,2}_0(\Omega)$.

Now let us recall that $\tilde{\mathcal{E}}(\hat{w}_k)$ converges to $-\infty$. Thus there exists a $k_0 \in \mathbb{N}$ such that

$$\tilde{\mathcal{E}}(w_k) = \frac{1}{\|\nabla \hat{w}_k\|_{L^2(\Omega)}}\,\mathcal{E}(\hat{w}_k) < 0$$

for all $k \ge k_0$. Since the functional $\tilde{\mathcal{E}}$ is lower semicontinuous with respect to weak convergence in $H^{2,2}(\Omega)$, we find that $\tilde{\mathcal{E}}(w) < 0$. According to the definition of $\mathcal{E}$ in (5.1), this immediately leads to

$$\frac{\int\limits_\Omega |\Delta w|^2\,dx}{\int\limits_\Omega |\nabla w|^2\,dx} < \Lambda(\Omega).$$

Since $w \in H^{2,2}_0(\Omega)$ this is contradictory to the minimum property of $\Lambda(\Omega)$. □

We now consider a minimizing sequence $(\varphi_k)_k \subset \mathcal{Z}$ which satisfies

$$\int\limits_{\partial\Omega} (\partial_\nu \varphi_k)^2\,dS = 1 \tag{5.7}$$

for all $k \in \mathbb{N}$. As before we obtain the inequality

$$\|\varphi_k\|^2_{H^{2,2}(\Omega)} \le \tilde{\mathcal{E}}(\varphi_k) + C\int\limits_\Omega |\nabla \varphi_k|^2\,dx.$$

Thus, $(\varphi_k)_k$ is uniformly bounded in $H^{2,2}(\Omega)$ and φ_k converges weakly to a $\varphi^* \in H^{2,2}(\Omega)$. We find that $\varphi^* \in \mathcal{Z}$ and $\tilde{\mathcal{E}}(\varphi^*) = \inf_{\mathcal{Z}} \tilde{\mathcal{E}}$. In addition, there holds

$$\int\limits_{\partial\Omega} (\partial_\nu \varphi^*)^2\,dS = 1.$$

Hence, φ^* minimizes $\tilde{\mathcal{E}}$ in $\mathcal{Z}$. Suppose $\theta \in \mathcal{Z}$, then the minimality of φ^* implies

$$\frac{d}{dt}\left.\frac{\mathcal{E}(\varphi^* + t\theta)}{\int\limits_{\partial\Omega} (\partial_\nu(\varphi^* + t\theta))^2\,dS)}\right|_{t=0} = 0$$

and we obtain

$$\int\limits_\Omega [\Delta^2 \varphi^* + \Lambda(\Omega)\Delta\varphi]\,\theta\,dx - \int\limits_{\partial\Omega} [\Delta\varphi^* + \rho\,\partial_\nu\varphi^*]\,\partial_\nu\theta\,dS = 0.$$

Since $\theta \in \mathcal{Z}$ was chosen arbitrary, φ^* satisfies the Euler–Lagrange equalities

$$\Delta^2\varphi^* + \Lambda(\Omega)\Delta\varphi^* = 0 \quad \text{in} \quad \Omega$$
$$\Delta\varphi^* + \rho\partial_\nu\varphi^* = const. \quad \text{in} \quad \partial\Omega,$$

where $\rho := \min_{\mathcal{Z}} \tilde{\mathcal{E}}$. The following theorem collects the previous results.

Theorem 5. *There exists a function $\varphi^* \in \mathcal{Z}$ such that $\tilde{\mathcal{E}}(\varphi^*) = \min_{\mathcal{Z}} \tilde{\mathcal{E}}$. Furthermore, any minimizer $\varphi^* \in \mathcal{Z}$ satisfies*

$$\Delta^2\varphi^* + \Lambda(\Omega)\Delta\varphi^* = 0 \quad \text{in} \quad \Omega \tag{5.8}$$
$$\Delta\varphi^* + \rho\,\partial_\nu\varphi^* = const. \quad \text{in} \quad \partial\Omega \tag{5.9}$$
$$\varphi^* = 0 \quad \text{in} \quad \partial\Omega,$$

where $\rho := \min_{\mathcal{Z}} \tilde{\mathcal{E}}$.

The next theorem shows that in fact $\rho = 0$.

Theorem 6. *Suppose $\varphi^* \in \mathcal{Z}$ is a minimizer of $\tilde{\mathcal{E}}$. Then there holds $\tilde{\mathcal{E}}(\varphi^*) = 0$. In particular, $\mathcal{E} \geq 0$ in $\mathcal{Z}$.*

Proof. Let $\varphi^* \in \mathcal{Z}$ be a minimizer of $\tilde{\mathcal{E}}$. Since φ^* satisfies equation (5.8) and $\partial\Omega$ is smooth, φ^* is a smooth function on $\overline{\Omega}$. Hence, we may define a volume preserving perturbation Φ_t of Ω such that

$$\partial_\nu u'(x) = \partial_\nu\varphi^*(x) \quad \text{for} \quad x \in \partial\Omega.$$

Note that this can be achieved by setting $v = c_0^{-1}\nabla\varphi^*$ in $\partial\Omega$. In this way, each minimizer φ^* implies the existence of vector fields v and w in the sense of Section 2. We define $\psi := u' - \varphi^*$, then $\psi \in H_0^{2,2}(\Omega)$ and

$$\Delta^2\psi + \Lambda(\Omega)\Delta\psi = 0 \quad \text{in} \quad \Omega.$$

The uniqueness of u implies $\psi = \alpha\, u$ for an $\alpha \in \mathbb{R}$. Since $\varphi^* \in \mathcal{Z}$, equation (4.3) yields

$$0 = \int_\Omega \nabla u \cdot \nabla u'\, dx - \int_\Omega \nabla u \cdot \nabla\varphi^*\, dx = \int_\Omega \nabla u \cdot \nabla\psi\, dx = \alpha.$$

Consequently, $u' \equiv \varphi^*$. Thus φ^* is a shape derivative. Since Ω is optimal $\tilde{\mathcal{E}}(\varphi^*) \geq 0$. Finally we apply Lemma 5. This gives

$$0 \leq \tilde{\mathcal{E}}(\varphi^*) = \min_{\mathcal{Z}} \tilde{\mathcal{E}} \leq \tilde{\mathcal{E}}(\partial_k u) = 0. \qquad \square$$

6. The optimal domain is a ball

We will use an inequality due to L.E. Payne to show that the optimal domain Ω is a ball. Payne's inequality (see [7]) states that for each domain G there holds

$$\lambda_2(G) \leq \Lambda(G)$$

and equality holds if and only if G is a ball. Thereby λ_2 denotes the second Dirichlet eigenfunction of the Laplacian. In the sequel, we construct a suitable function $\psi \in \mathcal{Z}$ such that the condition $\mathcal{E}(\psi) \geq 0$ (due to

Theorem 6) will imply that the optimal domain Ω is a ball. For this purpose, we denote by u_1 and u_2 the first and the second Dirichlet eigenfunction for the Laplacian in Ω. Thus, for $k = 1, 2$ there holds

$$\begin{aligned} \Delta u_k + \lambda_k(\Omega) u_k &= 0 \quad \text{in } \Omega \\ u_k &= 0 \quad \text{in } \partial\Omega, \end{aligned}$$

where $\lambda_k(\Omega)$ is the k-th Dirichlet eigenvalue for the Laplacian in Ω. Note that $0 < \lambda_1(\Omega) < \lambda_2(\Omega)$. For the sake of brevity, we will write λ_k instead of $\lambda_k(\Omega)$ and Λ instead of $\Lambda(\Omega)$. In addition, we assume $\|u_k\|_{L^2(\Omega)} = 1$ and

$$\int_\Omega u_1 u_2 \, dx = 0.$$

Without loss of generality, we may assume that

$$\int_\Omega u_1 \, dx > 0 \quad \text{and} \quad \int_\Omega u_2 \, dx \leq 0.$$

Consequently, there exists a $t \in (0, 1]$ such that

$$\int_\Omega (1-t)\, \lambda_1\, u_1 + t\, \lambda_2\, u_2 \, dx = 0. \tag{6.1}$$

This fixes t. Next we define

$$\psi(x) := (1-t)\, u_1(x) + t\, u_2(x) + c\, u(x) \qquad \text{for } x \in \overline{\Omega},$$

where u is the first buckling eigenfunction in Ω. The constant c is given by

$$c := -\frac{1}{\Lambda} \int_\Omega (1-t)\lambda_1 \nabla u . \nabla u_1 + t\lambda_2 \nabla u . \nabla u_2 \, dx.$$

In a first step we show that $\psi \in \mathcal{Z}$. Note that $\psi \in H_0^{1,2} \cap H^{2,2}(\Omega)$. Moreover the definition of ψ, the fact that $\partial_\nu u = 0$ on $\partial\Omega$, the equations for u_1 and u_2, and (6.1) imply

$$\int_{\partial\Omega} \partial_\nu \psi \, dS = \int_\Omega (1-t)\, \Delta u_1 + t\, \Delta u_2 \, dx = -\int_\Omega (1-t)\lambda_1 u_1 + t\lambda_2 u_2 \, dx = 0.$$

By the unique continuation principle $\partial_\nu \psi$ does not vanish identically in $\partial\Omega$. Thus, to show that $\psi \in \mathcal{Z}$, it remains to prove that

$$\int_\Omega \nabla u . \nabla \psi \, dx = 0. \tag{6.2}$$

We recall that $\Delta u = c_0$ in $\partial\Omega$. Hence

$$\begin{aligned} 0 &= \int_\Omega (\Delta^2 u + \Lambda \Delta u)\psi \, dx = \int_\Omega \Delta u \Delta \psi \, dx - \Lambda \int_\Omega \nabla u . \nabla \psi \, dx \\ &= -\int_\Omega [(1-t)\lambda_1 u_1 + t\lambda_2 u_2] \Delta u \, dx + c \int_\Omega |\Delta u|^2 \, dx - \Lambda \int_\Omega \nabla u . \nabla \psi \, dx. \end{aligned}$$

Since $\|\nabla u\|_{L^2(\Omega)} = 1$, the second integral is equal to Λ. Thus, the definition of c implies (6.2). Note that ψ is not a shape derivative since it fails to satisfy (4.2) – unless $t = 1$ and Ω equals a ball. However, $\psi \in \mathcal{Z}$ and, according to Theorem 6, there holds $\tilde{\mathcal{E}}(\psi) \geq 0$. Consequently, $\mathcal{E}(\psi) \geq 0$. Thus

$$
\begin{aligned}
\mathcal{E}(\psi) &= \int_\Omega |\Delta\psi|^2 - \Lambda|\nabla\psi|^2\,dx \\
&= (1-t)^2\lambda_1(\lambda_1 - \Lambda) + t^2\lambda_2(\lambda_2 - \Lambda) + 2\,c\,c_0 \int_\Omega (1-t)\lambda_1 u_1 + t\lambda_2 u_2\,dx \\
&\overset{(6.1)}{=} (1-t)^2\lambda_1(\lambda_1 - \Lambda) + t^2\lambda_2(\lambda_2 - \Lambda) \geq 0.
\end{aligned}
$$

Since $\lambda_1 - \Lambda < 0$ and $\lambda_2 - \Lambda \leq 0$, both summands in $\mathcal{E}(\psi)$ have to vanish. Consequently $t = 1$ and $\lambda_2(\Omega) = \Lambda(\Omega)$. Payne's inequality implies that Ω is a ball. This proves the main theorem of the paper.

Theorem 7. *Let Ω be within the class of bounded, smooth domains in $\mathbb{R}^n$ for which the boundary $\partial\Omega$ is connected. Assume Ω minimizes the first buckling eigenvalue among all domains in $\mathbb{R}^n$ in this class with given measure. Then Ω is a ball.*

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