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Value distribution for the derivatives of the logarithm of L-functions from the Selberg class in the half-plane of absolute convergence

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In the present paper, we show that, for every $\delta > 0$, the function $(\log \mathcal{L}(s))^{(m)}$, where $m \in \mathbb{N} \cup \{0\}$ and $\mathcal{L}(s) := \sum_{n=1}^{\infty} a(n)n^{-s}$ is an element of the Selberg class $\mathcal{S}$, takes any value infinitely often in the strip $1 < \mathrm{Re}(s) < 1 + \delta$, provided $\sum_{p \le x} |a(p)|^2 \sim \kappa\pi(x)$ for some $\kappa > 0$. In particular, $\mathcal{L}(s)$ takes any non-zero value infinitely often in the strip $1 < \mathrm{Re}(s) < 1 + \delta$, and the first derivative of $\mathcal{L}(s)$ has infinitely many zeros in the half-plane $\mathrm{Re}(s) > 1$.

1. Introduction and statement of main results

Let $\mathcal{S}_A$ consist of functions defined, for $\sigma := \mathrm{Re}(s) > 1$, by

$$\mathcal{L}(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}}\right), \tag{1.1}$$

where $a(n) \ll n^{\varepsilon}$ for any $\varepsilon > 0$ and $b(p^k) \ll p^{k\theta}$ for some $\theta < 1/2$. Then it is well known that both the Dirichlet series and the Euler product converge absolutely when $\sigma > 1$, and $a(p) = b(p)$ for every prime p (e.g. [26, p. 112]). Moreover, the set $\mathcal{S}_A$ includes the Selberg class $\mathcal{S}$ (for the definition we refer to [13] or [26, Section 6]), which contains a lot of L-functions in number theory. As mentioned in [13, Section 2.1], the Riemann zeta function $\zeta(s)$, Dirichlet L-functions $L(s + i\theta, \chi)$ with $\theta \in \mathbb{R}$ and a primitive

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character χ, L-functions associated with holomorphic newforms of a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ (after some normalization) are elements of the Selberg class. However, it should be noted that $\mathcal{S} \subsetneq \mathcal{S}_A$, since for example $\zeta(s)/\zeta(2s) \in \mathcal{S}_A$ but $\zeta(s)/\zeta(2s) \notin \mathcal{S}$ by the fact that $\zeta(s)/\zeta(2s)$ has poles on the line $\mathrm{Re}(s) = 1/4$.

Many mathematicians have been studying the distribution of the logarithmic derivative of the Riemann zeta function (see e.g. [10]). For instance it is known that there are some relationships between mean value of products of logarithmic derivatives of $\zeta(s)$ near the critical line, correlations of the zeros of $\zeta(s)$ and the distribution of integers representable as a product of a fixed number of prime powers (see [9] and [10]). Moreover, it is known that the second derivative of the logarithm of the Riemann zeta function appears in the pair correlation for the zeros of $\zeta(s)$ (see for example [2]). We refer also to [27], where Stopple investigated zeros of $(\log \zeta(s))''$.

In the present paper, we show the following result on value distribution of the m-th derivative of the logarithm of L-function from $\mathcal{S}_A$.

Theorem 1.1. *Let $m \in \mathbb{N} \cup \{0\}$, $z \in \mathbb{C}$ and $\mathcal{L}(s) := \sum_{n=1}^{\infty} a(n)n^{-s} \in \mathcal{S}_A$ satisfy*

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\le x} |a(p)|^2 = \kappa \tag{1.2}$$

for some $\kappa > 0$. Then, for any $\delta > 0$, we have

$$\#\big\{s : 1 < \mathrm{Re}(s) < 1 + \delta,\ \ \mathrm{Im}(s) \in [0, T] \ \ \textit{and} \ \ (\log \mathcal{L}(s))^{(m)} = z\big\} \gg T \tag{1.3}$$

for sufficiently large T.

Remark 1.2. The condition (1.2) is closely related to the widely believed Selberg conjecture

$$\sum_{p\le x} \frac{|a(p)|^2}{p} = \kappa \log\log x + O(1), \qquad (\kappa > 0). \tag{1.4}$$

However, (1.4) is weaker than (1.2), since in order to deduce (1.2) we need to assume that the error term in (1.4) is $C_1 + C_2/\log x + O\left((\log x)^{-2}\right)$ for $C_1, C_2 \ge 0$.

Remark 1.3. It is known (see for example [21, Theorem 3.6 (vi)]) that the assumption (1.2) implies that the abscissa of absolute convergence of $\mathcal{L}(s) \not\equiv 1$ is equal to 1, which is also a necessary condition for (1.3). The main reason, why the assumption that the abscissa of absolute convergence is 1 is not enough in our case, is the fact that we need to estimate the number of primes p for which $a(p)$ is not too close to 0. Thus, if $|a(p)| > c$ for every prime p and some constant $c > 0$, then (1.3) is equivalent to the fact that the abscissa of absolute convergence is 1.

As an immediate consequence of Theorem 1.1, we obtain the following result.

Corollary 1.4. *Let $z \in \mathbb{C} \setminus \{0\}$ and $\mathcal{L}(s) \in \mathcal{S}_A$ satisfy (1.2). Then, for any $\delta > 0$, we have*

$$\#\big\{s : 1 < \mathrm{Re}(s) < 1 + \delta,\ \ \mathrm{Im}(s) \in [0, T] \ \ \textit{and} \ \ \mathcal{L}(s) = z\big\} \gg T \tag{1.5}$$

for sufficiently large T.

The truth of (1.5) with $\mathcal{L}(s) = \zeta(s)$ was already proved by Bohr in [3] (see also Remark 2.5). Moreover, it was conjectured in [26, p. 188, l. 12–13] that it holds even for all L-functions with the following polynomial

Euler product

$$L(s)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}=\prod_p\prod_{j=1}^{m}\left(1-\frac{\alpha_j(p)}{p^s}\right)^{-1},\qquad \sigma>1,$$

where the $\alpha_j(p)$'s are complex numbers with $|\alpha_j(p)|\le 1$ (see [26, Section 2.2]). One can easily show (see [26, Lemma 2.2]) that the coefficients $a(n)$ satisfy $a(n)\ll n^{\varepsilon}$ for every $\varepsilon>0$, and hence $\mathcal{S}_A$ contains all functions of such kind. Therefore, Corollary 1.4 shows that we have (1.5) not only for $L(s)$ as above, but also for $\mathcal{L}(s)\in\mathcal{S}_A$.

The above result on c-values is related to the following uniqueness theorem proved by Li [16, Theorem 1]. If two L-functions $\mathfrak{L}_1$ and $\mathfrak{L}_2$ (Euler product is not necessary) satisfy the same functional equation, $a_1(1)=1=a_2(1)$, and $\mathfrak{L}_1^{-1}(c_j)=\mathfrak{L}_2^{-1}(c_j)$ for two distinct complex numbers c_1 and c_2, then $\mathfrak{L}_1=\mathfrak{L}_2$. It turns out (see Ki [14, Theorem 1]) that in the case of two functions $\mathfrak{L}_1$ and $\mathfrak{L}_2$ from the extended Selberg class $\mathcal{S}^{\#}$ (for definition we refer to [13, p. 160] or [26, p. 217]), in order to prove that $\mathfrak{L}_1=\mathfrak{L}_2$, it is sufficient to check that they have the same functional equation with positive degree, $a_1(1)=1=a_2(1)$ and $\mathfrak{L}_1^{-1}(c)=\mathfrak{L}_2^{-1}(c)$ for some nonzero complex number c. Recently, Gonek, Haan and Ki [11] improved Ki's result by showing that the assumption that they satisfy the same functional equation is superfluous. Now let $\mathcal{L}(s)\in\mathcal{S}_A$ satisfy all assumptions of Corollary 1.4. Then we see that for any $c\in\mathbb{C}\setminus\{0\}$ and sufficiently large T, we have

$$\#\mathcal{L}^{-1}(c)\ge\#\bigl\{s\in\mathbb{C}:\mathcal{L}(s)=c,\ \ \mathrm{Re}(s)>1,\ \ \mathrm{Im}(s)\in[0,T]\bigr\}\gg T,$$

which means that it is not trivial to check the condition $\mathfrak{L}_1^{-1}(c)=\mathfrak{L}_2^{-1}(c)$ if $\mathfrak{L}_1,\mathfrak{L}_2\in\mathcal{S}_A$.

Next, since $\mathcal{L}(s)$ has no zeros in the half-plane of absolute convergence and $(\log\mathcal{L}(s))'=\mathcal{L}'(s)/\mathcal{L}(s)$, we obtain immediately the following result by using Theorem 1.1 for $m=1$ and $z=0$.

Corollary 1.5. *Let $\mathcal{L}(s)\in\mathcal{S}_A$ satisfy (1.2). Then for any $\delta>0$, one has*

$$\#\bigl\{s:1<\mathrm{Re}(s)<1+\delta,\ \ \mathrm{Im}(s)\in[0,T]\ \ \textit{and}\ \ \mathcal{L}'(s)=0\bigr\}\gg T \tag{1.6}$$

for sufficiently large T.

The above corollary generalizes the well-known result that the first derivative of the Riemann zeta function has infinitely many zeros in the region of absolute convergence $\sigma>1$ (see [28, Theorem 11.5 (B)]). Moreover, it should be mentioned that although there are a lot of papers on zeros of the derivatives of the Riemann zeta function (see for instance [1,15,25] and articles which cite them), there are few papers treating zeros of the derivatives of other zeta and L-functions. On result concerning this matter is the following fact due to Yildirim [29, Theorem 2]. Let χ be a Dirichlet character to the modulus q and m be the smallest prime that does not divide q. Then the k-th derivatives of the Dirichlet L-function $L^{(k)}(s,\chi)$ does not vanish in the half-plane

$$\sigma>1+\frac{m}{2}\left(1+\sqrt{1+\frac{4k^2}{m\log m}}\right),\qquad k\in\mathbb{N}.$$

As an application of our method we show the following result concerning zeros of combinations of L-functions.

Corollary 1.6. *Let $c_1,c_2\in\mathbb{C}\setminus\{0\}$ and $\mathcal{L}_j(s):=\sum_{n=1}^{\infty}a_j(n)n^{-s}\in\mathcal{S}_A$ for $j=1,2$. Assume*

$$\lim_{x\to\infty}\frac{1}{\pi(x)}\sum_{p\le x}|a_1(p)-a_2(p)|^2=\kappa,\qquad(\kappa>0). \tag{1.7}$$

Then for any $\delta > 0$, it holds that

$$\#\{s : 1 < \mathrm{Re}(s) < 1 + \delta,\ \ \mathrm{Im}(s) \in [0, T] \ \text{ and } c_1\mathcal{L}_1(s) + c_2\mathcal{L}_2(s) = 0\} \gg T$$

for sufficiently large T.

Now we mention earlier works related to zeros of zeta functions in the half plane $\sigma > 1$. Davenport and Heilbronn [8] showed that the Hurwitz zeta function $\zeta(s, \alpha) = \sum_{n=0}^{\infty}(n + \alpha)^{-s}$ has infinitely many zeros in the region $\mathrm{Re}(s) > 1$, provided $0 < \alpha \neq 1/2, 1$ is rational or transcendental. Later, Cassels [6] extended their result to algebraic irrational parameter α. Recently, Saias and Weingartner [24] showed that a Dirichlet series with periodic coefficients and non-vanishing in the half-plane $\sigma > 1$ equals $F(s) = P(s)L(s, \chi)$, where $P(s)$ is a Dirichlet polynomial that does not vanish in $\sigma > 1$. Afterwards, Booker and Thorne [5], and very recently Righetti [23] generalized the work of Saias and Weingartner to general L-functions with bounded coefficients at primes.

Nevertheless Corollary 1.6 gives new examples, which cannot be treated by Saias–Weingartner approach and its known generalizations. For example Corollary 1.6 implies that the Euler–Zagier double zeta function $\zeta_2(s, s) = (\zeta^2(s) - \zeta(2s))/2$ has zeros for $\sigma > 1$. Moreover, we can prove that the zeta functions associated to symmetric matrices treated by Ibukiyama and Saito in [12, Theorem 1.2] vanish infinitely often in the region of absolute convergence. In addition, it follows that some Epstein zeta functions, for example,

$$\zeta(s; I_6) = -4\big(\zeta(s)L(s-2, \chi_{-4}) - 4\zeta(s-2)L(s, \chi_{-4})\big),$$
$$\zeta(s; \mathfrak{L}_{24}) = \frac{65520}{691}\big(\zeta(s)\zeta(s-11) - L(s; \Delta)\big),$$

have infinitely many zeros for $\sigma > 3$ and $\sigma > 12$, respectively, since $|\tau(p)| < 2p^{11/2}$ and

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p \le x} \big|\big(1 + p^{-2}\big)\big(1 - \chi_{-4}(p)\big)\big|^2 = \frac{0}{\varphi(4)} + \frac{2^2}{\varphi(4)} = 2,$$
$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p \le x} \big|p^{-11} + 1 - \tau(p)p^{-11}\big|^2 = 1.$$

It should be noted that it is already known that $\zeta_2(s, s)$ and $\zeta(s; \mathfrak{L}_{24})$ vanish in the half-plane $\sigma > 1$ and $\sigma > 12$ from the numerical computations [18, Figure 1] and [22, Fig. 1]. Furthermore, we have to remark that such zeta functions have infinitely many zeros outside of the region of absolute convergence (see [19, Main Theorem 1] and [20, Theorem 3.1]).

In Section 2, we prove Theorem 1.1 and its corollaries. Some topics related to almost periodicity are discussed in Section 3. More precisely, we prove that for any $\mathrm{Re}(\eta) > 0$, the function $\zeta(s) \pm \zeta(s + \eta)$ has zeros when $\sigma > 1$ (see Corollary 3.1) but for any $\delta > 0$, there exists $\theta \in \mathbb{R} \setminus \{0\}$ such that the function $\zeta(s) + \zeta(s + i\theta)$ does not vanish in the region $\sigma \ge 1 + \delta$ (see Proposition 3.2).

2. Proofs of Theorem 1.1 and its corollaries

Lemma 2.1. *Let $r_1, \ldots, r_n \in \mathbb{C}$ be such that $0 < |r_1| \le |r_2| \le \cdots \le |r_n|$ and $R_0 = 0$, $R_j = |r_1| + \cdots + |r_j|$. Then*

$$\left\{\sum_{j=1}^{n} c_j r_j : |c_j| = 1,\ c_j \in \mathbb{C}\right\} = \{z \in \mathbb{C} : T_n \le z \le R_n\},$$

where

$$T_n = \begin{cases} |r_n| - R_{n-1} & \text{if } R_{n-1} \le |r_n|, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. From [7, Proposition 3.3] every complex number z with $T_n \le |z| \le R_n$ can be written as

$$z = \sum_{j=1}^{n} c'_j |r_j|, \qquad |c'_j| = 1.$$

Hence, taking $c_j = c'_j |r_j|/r_j$ completes the proof. □

Lemma 2.2. *Let $b(p)$ be a sequence of complex numbers indexed by primes. Assume that $b(p) \ll p^\varepsilon$ for every $\varepsilon > 0$ and*

$$\lim_{x\to\infty} \frac{1}{(\log x)^m \pi(x)} \sum_{p\le x} |b(p)|^2 = \kappa$$

for some $\kappa > 0$ and a non-negative integer m. Then for any $c > 1$, $\eta > 0$ and $\varepsilon > 0$ we have

$$\sum_{\substack{x<p\le cx \\ |b(p)|>p^{-\eta}}} 1 \gg x^{1-\varepsilon}.$$

Proof. One can easily get that

$$\sum_{x<p\le cx} |b(p)|^2 \ll x^\varepsilon \sum_{\substack{x<p\le cx \\ |b(p)|>p^{-\eta}}} 1 + x^{-2\eta} \sum_{\substack{x<p\le cx \\ |b(p)|\le p^{-\eta}}} 1 \ll x^\varepsilon \sum_{\substack{x<p\le cx \\ |b(p)|>p^{-\eta}}} 1 + \frac{x^{1-2\eta}}{\log x}.$$

On the other hand, we have

$$\sum_{x<p\le cx} |b(p)|^2 \gg x(\log x)^{l-1}.$$

Hence the proof is complete. □

Lemma 2.3. *Let $L(s) = \sum_p \sum_{k\ge1} b(p^k)p^{-ks}$ for $\sigma > 1$ be such that $b(p^k) \ll p^{k\theta}$ for some $\theta < 1/2$, $b(p) \ll p^\varepsilon$ for every $\varepsilon > 0$ and*

$$\lim_{x\to\infty} \frac{1}{(\log x)^m \pi(x)} \sum_{p\le x} |b(p)|^2 = \kappa \tag{2.1}$$

for some $\kappa > 0$ and a non-negative integer m. Then, for every complex z and $\delta > 0$ there exist $1 < \sigma < 1+\delta$ and a sequence $\chi(p)$ of complex number indexed by primes such that $|\chi(p)| = 1$ and

$$\sum_p \sum_{k\ge1} \frac{\chi(p)^k b(p^k)}{p^{k\sigma}} = z.$$

Proof. We follow the idea introduced by Cassels in [6].

Assume that N_1 is a positive integer, $\varepsilon > 0$ and $c_0 > 0$; we determine these parameters later on. Put $M_j = [c_0 N_j]$ and $N_{j+1} = N_j + M_j$. We shall show that there exist $\sigma \in (1, 1+\delta)$ and a sequence $\chi(p)$ with $|\chi(p)| = 1$ such that

$$\left| \sum^*_{(p,k):\, p^k \le N_j} \frac{\chi(p)^k b(p^k)}{p^{k\sigma}} - z + \sum_{(p,k):\, |b(p)| \le p^{-\varepsilon}} \frac{b(p^k)}{p^{k\sigma}} \right| \le 10^{-2} \sum^*_{(p,k):\, p^k > N_j} \frac{|b(p^k)|}{p^{k\sigma}}, \tag{2.2}$$

where $\sum^*$ denotes the double sum over (p, k) satisfying $|b(p)| > p^{-\varepsilon}$, p is prime and $k \in \mathbb{N}$. Let us note that for every $\sigma \in (1, 1+\delta)$ we have

$$\sum_{(p,k):\, |b(p)| \le p^{-\varepsilon}} \frac{|b(p^k)|}{p^{k\sigma}} \le \sum_{p:\, |b(p)| \le p^{-\varepsilon}} \frac{1}{p^{1+\varepsilon}} + \sum_p \sum_{k \ge 2} \frac{|b(p^k)|}{p^k} =: S_0 < \infty.$$

From (2.1) and [21, Theorem 3.6 (vi)], the abscissa of convergence of the series $\sum_p \sum_{k\ge 1} |b(p^k)| p^{-k\sigma}$ is 1, then by Landau's theorem, this series has a pole at $\sigma = 1$, which implies that

$$\sum^*_{(p,k)} \frac{|b(p^k)|}{p^{k\sigma}} \to \infty \qquad \text{as} \quad \sigma \to 1^+.$$

Therefore, we can find $\sigma \in (1, 1+\delta)$ such that

$$\sum^*_{(p,k):\, p^k \le N_1} \frac{|b(p^k)|}{p^{k\sigma}} + |z| + S_0 \le 10^{-2} \sum^*_{(p,k):\, p^k > N_1} \frac{|b(p^k)|}{p^{k\sigma}},$$

and hence (2.2) holds for $j = 1$ and arbitrary $\chi(p)$'s with $p \le N_1$.

Now, let us assume that complex numbers $\chi(p)$ are chosen for all $p \le N_j$. We shall find $\chi(p)$ with $N_j < p \le N_{j+1}$ and $|b(p)| > p^{-\varepsilon}$ such that (2.2) holds with $j+1$ instead of j.

Let $\mathfrak{A}$ denote the set of pairs $(p, 1)$ satisfying $p \in (N_j, N_{j+1}]$ is a prime number and $|b(p)| > p^{-\varepsilon}$. Moreover, define

$$\mathfrak{B} = \{(p,k) : p^k \in (N_j, N_{j+1}],\ p \text{ is prime},\ k \ge 2,\ |b(p)| > p^{-\varepsilon}\}.$$

Note that the $\chi(p)^k$'s are already defined for $(p, k) \in \mathfrak{B}$, since for suitable N_1 and c_0 we have $p \le \sqrt{N_{j+1}} < N_j$ if $(p, k) \in \mathfrak{B}$.

Using Lemma 2.2 gives that

$$|\mathfrak{A}| \gg N_j^{1-\varepsilon}$$

and since $k \ge 2$ for every $(p, k) \in \mathfrak{B}$ we have

$$|\mathfrak{B}| \ll N_j^{\frac{1}{2}}.$$

Moreover, note that for every p_1, p_2 satisfying $(p_1, 1), (p_2, 1) \in \mathfrak{A}$, by the Ramanujan conjecture, we have

$$\left| \frac{b(p_1)}{b(p_2)} \right| \ll N_j^{2\varepsilon} \qquad \text{and} \qquad \left(\frac{p_2}{p_1} \right)^{\sigma} \le \left(\frac{N_{j+1}}{N_j} \right)^{\sigma} \le (c_0 + 1)^{1+\delta},$$

so

$$\frac{|b(p_2)|}{p_2^\sigma} \gg N_j^{-2\varepsilon} \frac{|b(p_1)|}{p_1^\sigma}.$$

Hence, using Lemma 2.1 with the sequence $b(p)p^{-\sigma}$, where $(p,1) \in \mathfrak{A}$, we obtain that

$$\sideset{}{^*}\sum_{(p,1)\in\mathfrak{A}} \frac{b(p)\chi(p)}{p^\sigma}, \qquad |\chi(p)| = 1,$$

takes all values z_0 with $|z_0| \le \sideset{}{^*}\sum_{(p,1)\in\mathfrak{A}} |b(p)|p^{-\sigma} =: S_3$, since for sufficiently large N_1 and arbitrary p_0 satisfying $(p_0,1) \in \mathfrak{A}$, we have

$$\sideset{}{^*}\sum_{(p_0,1)\neq(p,1)\in\mathfrak{A}} \frac{|b(p)|}{p^\sigma} \gg N_j^{1-3\varepsilon} \frac{|b(p_0)|}{p_0^\sigma} > \frac{|b(p_0)|}{p_0^\sigma}.$$

Hence the inner radius $T_{|\mathfrak{A}|}$ in Lemma 2.1 is 0.

Write

$$\Lambda := \sideset{}{^*}\sum_{(p,k):\, p^k \le N_j} \frac{\chi(p)^k b(p^k)}{p^{k\sigma}} - z + \sum_{(p,k):\, |b(p)| \le p^{-\varepsilon}} \frac{b(p^k)}{p^{k\sigma}} + \sideset{}{^*}\sum_{(p,k)\in\mathfrak{B}} \frac{\chi(p)^k b(p^k)}{p^{k\sigma}}$$

and put

$$z_0 = \begin{cases} -\Lambda & \text{if } 0 < |\Lambda| \le S_3, \\ -S_3\Lambda/|\Lambda| & \text{if } |\Lambda| > S_3, \\ 0 & \text{if } \Lambda = 0. \end{cases}$$

Then, from Lemma 2.1 we can choose $\chi(p)$ for $(p,1) \in \mathfrak{A}$ such that

$$\begin{aligned} &\left| \sideset{}{^*}\sum_{(p,k):\, p^k \le N_j + M_j} \frac{\chi(p)^k b(p^k)}{p^{k\sigma}} - z + \sum_{(p,k):\, |b(p)| \le p^{-\varepsilon}} \frac{b(p^k)}{p^{k\sigma}} \right| \\ &\quad = \left| \Lambda + \sideset{}{^*}\sum_{(p,1)\in\mathfrak{A}} \frac{b(p)\chi(p)}{p^\sigma} \right| \le \max(0, S_1 + S_2 - S_3), \end{aligned}$$

where

$$S_1 := \left| \sideset{}{^*}\sum_{(p,k):\, p^k \le N_j} \frac{\chi(p)^k b(p^k)}{p^{k\sigma}} - z + \sum_{(p,k):\, |b(p)| \le p^{-\varepsilon}} \frac{b(p^k)}{p^{k\sigma}} \right|$$

and

$$S_2 := \sideset{}{^*}\sum_{(p,k)\in\mathfrak{B}} \frac{|b(p^k)|}{p^{k\sigma}},$$

so $|\Lambda| \le S_1 + S_2$.

Now, let us notice that

$$\frac{S_3}{S_2} \geq \frac{N_j^{\sigma-\theta}}{N_{j+1}^{\sigma+\varepsilon}} \frac{|\mathfrak{A}|}{|\mathfrak{B}|} \gg N_j^{1/2-\theta-2\varepsilon} \geq \frac{101}{99}$$

for sufficiently small $\varepsilon > 0$ and sufficiently large N_1. Hence

$$S_2 - S_3 \leq -10^{-2}(S_2 + S_3).$$

Moreover, from (2.2) we have

$$S_1 \leq 10^{-2}(S_2 + S_3 + S_4),$$

where

$$S_4 := \sum_{(p,k):\, p^k > N_{j+1}}^{*} \frac{|b(p^k)|}{p^{k\sigma}}.$$

Thus $S_1 + S_2 - S_3 < 10^{-2} S_4$ and, by induction, (2.2) holds for all $j \in \mathbb{N}$. So letting $N_j \to \infty$ completes the proof. □

The classical Kronecker approximation theorem (see for example [26, Lemma 1.8]) plays a crucial role in the proof of the following lemma.

Lemma 2.4. *Assume that $L(s)$ satisfies the hypothesis of Lemma 2.3. Then, for every z and $\delta > 0$, the set of real τ satisfying*

$$L(s + i\tau) = z \qquad \text{for some } 1 < \mathrm{Re}(s) < 1 + \delta,$$

has a positive lower density.

Proof. By Lemma 2.3, we choose $\sigma \in (1, 1 + \delta)$ and a sequence $\chi(p)$ with $|\chi(p)| = 1$ such that

$$\sum_p \sum_{k\geq 1} \frac{\chi(p)^k b(p^k)}{p^{k\sigma}} = z.$$

Next, since $F(s) = \sum_p \sum_{k\geq 1} \chi(p)^k b(p^k) p^{-ks}$ is analytic in the half-plane $\mathrm{Re}(s) > 1$, we can find r with $0 < r < \sigma - 1$ such that $F(s) - z \neq 0$ if $|s - \sigma| = r$. Then we put $\varepsilon := \min_{|s-z|=r} |F(s) - z|$.

Since the series $\sum_p \sum_{k=1}^{\infty} |b(p^k)| p^{-k(\sigma-r)}$ converges absolutely, we can take a positive integer M such that

$$\sum_{p\leq M} \sum_{k>M}^{\infty} \frac{|b(p^k)|}{p^{k(\sigma-r)}} + \sum_{p>M} \sum_{k=1}^{\infty} \frac{|b(p^k)|}{p^{k(\sigma-r)}} < \frac{\varepsilon}{4}.$$

Moreover, if we assume that

$$\max_{p\leq M} \left| p^{-i\tau} - \chi(p) \right| < \varepsilon_1 \tag{2.3}$$

for $\varepsilon_1 > 0$, then

$$\begin{aligned}&\left|p^{-ik\tau}-\chi(p)^k\right|\\&=\left|p^{-i\tau}-\chi(p)\right|\left|p^{-i(k-1)\tau}+p^{-i(k-2)\tau}\chi(p)+\cdots+p^{-i\tau}\chi(p)^{k-2}+\chi(p)^{k-1}\right|\\&<k\varepsilon_1\le M\varepsilon_1,\qquad 1\le k\le M.\end{aligned}$$

Therefore, for sufficiently small ε_1 and s satisfying $|s-\sigma|=r$, we obtain

$$\left|\sum_{p\le M}\sum_{k=1}^{M}\frac{b(p^k)}{p^{k(s+i\tau)}}-\sum_{p\le M}\sum_{k=1}^{M}\frac{b(p^k)\chi(p)^k}{p^{ks}}\right|<M\varepsilon_1\sum_{p\le M}\sum_{k=1}^{M}\frac{|b(p^k)|}{p^{k(\sigma-r)}}<\frac{\varepsilon}{2},$$

and

$$|L(s+i\tau)-z-(F(s)-z)|=|L(s+i\tau)-F(s)|<\varepsilon\le|F(s)-z|,$$

provided (2.3) holds.

Thus, by Rouché's theorem (see for example [26, Theorem 8.1]), for every τ satisfying (2.3) there is a complex number s with $|s-\sigma|\le r$ such that $L(s+i\tau)=z$. But, by the classical Kronecker approximation theorem, the set of τ satisfying (2.3) has a positive density, so the number of solutions of the equation $L(s+i\tau)=z$ with $1<\mathrm{Re}(s)<1+\delta$ and $\tau\in[0,T]$ is $\gg T$ for sufficiently large $T>0$. □

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Obviously, the case $m=0$ follows immediately from Lemma 2.4, since $a(p)=b(p)$ for every prime p. Thus it suffices to show that for every $m\ge1$ the function $(\log\mathcal{L}(s))^{(m)}$ satisfies the assumption of Lemma 2.4.

Note that

$$(-1)^m\big(\log\mathcal{L}(s)\big)^{(m)}=\sum_p\sum_{k=1}^{\infty}\frac{b(p^k)(k\log p)^m}{p^{ks}},\qquad \sigma>1,$$

$b(p)(\log p)^m=a(p)(\log p)^m\ll p^{\varepsilon}$ for every $\varepsilon>0$, and $b(p^k)(k\log p)^m\ll p^{k\theta_1}$ for some θ_1 with $\theta<\theta_1<1/2$ by the assumption $b(p^k)\ll p^{k\theta}$ for some $\theta<1/2$. Moreover, by partial summation and (1.2), we get

$$\sum_{p\le x}|b(p)|^2(\log p)^{2m}=\sum_{p\le x}|a(p)|^2(\log p)^{2m}=\kappa(\log x)^{2m}\pi(x)(1+o(1)),$$

which completes the proof. □

Remark 2.5. In Bohr's proof of Corollary 1.4 for $\mathcal{L}(s)=\zeta(s)$, the convexity of

$$-\log\big(1-p^{-s}\big)=\sum_{k=1}^{\infty}\frac{1}{kp^{ks}}$$

plays a crucial role (see also [26, Theorem 1.3] and [28, Theorem 11.6 (B)]). However, we prove Corollary 1.4 without using the convexity since the closed curve described by $\sum_{k=1}^{\infty}b(p^k)p^{-ks}$ is not always convex when t runs through the whole $\mathbb{R}$ (see also [17]).

Proof of Corollary 1.6. Put $L(s)=\log L_1(s)-\log L_2(s)$. Then $L(s)=\sum_p\sum_{k\ge1}(b_1(p^k)-b_2(p^k))p^{-ks}$, where the $b_j(p^k)$'s denote the coefficients in the Dirichlet series expansion of $\log L_j(s)$. Obviously

$b_1(p^k) - b_2(p^k) \ll p^{k\theta}$ for some $\theta < 1/2$, and $b_1(p) - b_2(p) \ll p^{\varepsilon}$ for every $\varepsilon > 0$. Thus, by (1.7), we can apply Lemma 2.4 to obtain that the set of real τ satisfying

$$L(s+i\tau) = \log(L_1(s+i\tau)/L_2(s+i\tau)) = \log(-c_2/c_1)$$

has a positive lower density. Therefore, the proof is complete. □

Remark 2.6. Note that

$$\sum_{p\le x} |a_1(p) - a_2(p)|^2 = \sum_{p\le x} |a_1(p)|^2 + \sum_{p\le x} |a_2(p)|^2 - 2\,\mathrm{Re} \sum_{p\le x} a_1(p)\overline{a_2(p)}. \tag{2.4}$$

Therefore, if the abscissa of absolute convergence for both L-functions $\mathcal{L}_1$ and $\mathcal{L}_2$ is 1, then the assumption (1.7) in Corollary 1.6 can be replaced by Selberg's orthonormality conjecture in the following stronger form

$$\forall_{j=1,2} \lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\le x} |a_j(p)|^2 = \kappa_j, \qquad \lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\le x} a_1(p)\overline{a_2(p)} = 0,$$

for some $\kappa_1, \kappa_2 > 0$.

On the other hand, if the abscissa of absolute convergence of one of them, say $\mathcal{L}_2$, is less than 1, then we get

$$\sum_{p\le x} |a_2(p)|^2 \le \sqrt{\sum_{p\le x} \frac{|a_2(p)|}{p^{\sigma_0}}} \sqrt{\sum_{p\le x} |a_2(p)|^3 p^{\sigma_0}} \ll x^{1/2+\sigma_0/2+\varepsilon}$$

for some $\sigma_0 < 1$ and every $\varepsilon > 0$. Moreover, by Cauchy–Schwarz inequality, we have

$$\mathrm{Re} \sum_{p\le x} a_1(p)\overline{a_2(p)} \le \sqrt{\sum_{p\le x} |a_1(p)|^2} \sqrt{\sum_{p\le x} |a_2(p)|^2} \ll x^{3/4+\sigma_0/4+\varepsilon}$$

for every $\varepsilon > 0$.

Therefore, by (2.4), we obtain

$$\sum_{p\le x} |a_1(p) - a_2(p)|^2 = \sum_{p\le x} |a_1(p)|^2 + O(x^{3/4+\sigma_0/4+\varepsilon}),$$

and assuming (1.2) for $\mathcal{L}_1$ implies Corollary 1.6.

3. Almost periodicity and Corollary 1.6

We follow the notion of almost periodicity in [26, Section 9.5]. In 1922, Bohr [4] proved that every Dirichlet series $f(s)$, having a finite abscissa of absolute convergence σ_a is almost periodic in the half-plane $\sigma > \sigma_a$. Namely, for any given $\delta > 0$ and $\varepsilon > 0$, there exists a length $l := l(f, \delta, \varepsilon)$ such that every interval of length l contains a number τ for which

$$\left| f(\sigma + \mathrm{i}t + \mathrm{i}\tau) - f(\sigma + \mathrm{i}t) \right| < \varepsilon$$

holds for any $\sigma \ge \sigma_a + \delta$ and for all $t \in \mathbb{R}$. From the Dirichlet series expression, the L-function $\mathcal{L}(s) \in \mathcal{S}_A$ is almost periodic when $\sigma > 1$. By using Corollary 1.6, we have the following corollary as a kind of analogue of almost periodicity.

Corollary 3.1. *Let $\mathcal{L}(s) := \sum_{n=1}^{\infty} a(n)n^{-s} \in \mathcal{S}_A$ satisfy (1.2). Suppose $c_1, c_2 \in \mathbb{C} \setminus \{0\}$ and $\mathrm{Re}(\eta) > 0$. Then one has*

$$\#\{s : \mathrm{Re}(s) > 1,\ \mathrm{Im}(s) \in [0,T] \ \text{and}\ c_1\mathcal{L}(s) + c_2\mathcal{L}(s+\eta) = 0\} \gg T$$

for sufficiently large T.

Proof. The corollary follows from Remark 2.6, since the abscissa of absolute convergence of $\mathcal{L}(s+\eta)$ is smaller than 1. □

On the contrary, we have the following proposition when $\mathrm{Re}(\eta) = 0$.

Proposition 3.2. *Let $\mathcal{L}(s) \in \mathcal{S}_A$. Then for any $\delta > 0$, there exists $\theta \in \mathbb{R} \setminus \{0\}$ such that the function*

$$\mathcal{L}(s) + \mathcal{L}(s + i\theta)$$

does not vanish in the region $\sigma \geq 1 + \delta$.

Proof. For any $\varepsilon > 0$, we can find $\theta \in \mathbb{R} \setminus \{0\}$ which satisfies

$$|\mathcal{L}(s) - \mathcal{L}(s+i\theta)| < \varepsilon, \qquad \mathrm{Re}(s) \geq 1 + \delta$$

from almost periodicity of $\mathcal{L}(s) \in \mathcal{S}_A$. Hence we have

$$\begin{aligned}
&\big|\mathcal{L}(s) + \mathcal{L}(s+i\theta)\big| \\
&= \big|2\mathcal{L}(s) + \mathcal{L}(s+i\theta) - \mathcal{L}(s)\big| \geq \big|2\mathcal{L}(s)\big| - \big|\mathcal{L}(s) - \mathcal{L}(s+i\theta)\big| \\
&> 2\prod_p \exp\left(-\sum_{k=1}^{\infty} \frac{|b(p^k)|}{p^{k(1+\delta)}}\right) - \varepsilon, \qquad \mathrm{Re}(s) \geq 1 + \delta.
\end{aligned}$$

From the assumption for $\mathcal{L}(s) \in \mathcal{S}_A$, the sum $\sum_p \sum_{k=1}^{\infty} |b(p^k)|p^{-k(1+\delta)}$ converges absolutely when $\delta > 0$. Hence, by taking a suitable $\varepsilon > 0$, we have

$$|\mathcal{L}(s) + \mathcal{L}(s+i\theta)| > 0, \qquad \mathrm{Re}(s) \geq 1 + \delta,$$

which implies Proposition 3.2. □

Remark 3.3. Proposition 3.2 should be compared with the following fact. Let $\theta \in \mathbb{R}\setminus\{0\}$ and $c_1, c_2 \in \mathbb{C}\setminus\{0\}$. Then the function

$$c_1\zeta(s) + c_2\zeta(s + i\theta)$$

vanishes in the strip $1/2 < \sigma < 1$. This is an easy consequence of [26, Theorem 10.7].

Hence, for any $\delta > 0$, there exists $\theta \in \mathbb{R} \setminus \{0\}$ such that the function

$$\zeta(s) + \zeta(s + i\theta)$$

does not vanish in the half-plane $\sigma \geq 1 + \delta$, but has infinitely many zeros in the vertical strip $1/2 < \sigma < 1$.

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