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ϵ-Weak Cauchy sequences and a quantitative version of Rosenthal's ℓ_1-theorem ☆

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ABSTRACT

A bounded sequence (x_n) in a Banach space is called ϵ-weak Cauchy, for some $\epsilon > 0$, if for all $x^* \in B_{X^*}$ there exists some $n_0 \in \mathbb{N}$ such that $|x^*(x_n) - x^*(x_m)| < \epsilon$ for all $n \geq n_0$ and $m \geq n_0$. It is shown that given $\epsilon > 0$ and a bounded sequence (x_n) in a Banach space then either (x_n) admits an ϵ-weak Cauchy subsequence or, for all $\delta > 0$, there exists a subsequence (x_{m_n}) with the following property. If I is a finite subset of $\mathbb{N}$ and $\phi\colon I \to \mathbb{N} \setminus I$ is any map then

$$\Big\| \sum_{n\in I} \lambda_n (x_{m_n} - x_{m_{\phi(n)}}) \Big\| \geq (\frac{\epsilon}{\pi} - \delta) \sum_{n \in I} |\lambda_n|$$

for every sequence of complex scalars $(\lambda_n)_{n\in I}$. This provides an alternative proof for Rosenthal's ℓ_1-theorem and strengthens its quantitative version due to Behrends. As a corollary we obtain that for any uniformly bounded sequence (f_n) of complex-valued functions, continuous on the compact Hausdorff space K and satisfying $\limsup_{n,m\to\infty} |f_n(t) - f_m(t)| \leq \epsilon$, for some $\epsilon > 0$ and all $t \in K$, there exists a subsequence (f_{j_n}) satisfying $\limsup_{n,m\to\infty} |\int_K (f_{j_n} - f_{j_m}) d\mu| \leq 2\epsilon$, for every Radon measure μ on K.

1. Introduction

One of the most important results in Banach space theory is Rosenthal's ℓ_1-theorem [16]:

Theorem 1.1. *Every bounded sequence in a (real or complex) Banach space admits a subsequence which is either weak Cauchy, or equivalent to the usual ℓ_1-basis.*

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Rosenthal proved his result for real Banach spaces and subsequently Dor [5] settled the complex case. Most proofs of the ℓ_1-theorem (see [7,14,4,10]) rely on the infinite Ramsey theorem [6] which states that every analytic subset of $[\mathbb{N}]$ is Ramsey (see also [8]). A proof of the ℓ_1-theorem that avoids the use of Ramsey theory is given in [2]. We recall here that for an infinite subset L of $\mathbb{N}$, $[L]$ stands for the set of all of its infinite subsets. $[\mathbb{N}]$ is endowed with the topology of pointwise convergence. A subset $\mathcal{A}$ of $[\mathbb{N}]$ is a Ramsey set if for every $N \in [\mathbb{N}]$ there exists $M \in [N]$ such that either $[M] \subset \mathcal{A}$, or $[M] \cap \mathcal{A} = \emptyset$.

The purpose of this article is to provide a quantified version of the ℓ_1-theorem in the spirit of Behrends [4]. To explain our results, we first fix a compact Hausdorff space K and a bounded sequence (f_n) of complex-valued functions, continuous on K. Given $\epsilon > 0$, let us call (f_n) *ϵ-weak Cauchy on K* provided that for every $t \in K$ there exists $n_0 \in \mathbb{N}$ such that $|f_n(t) - f_m(t)| < \epsilon$ for all $n \geq n_0$ and $m \geq n_0$. We note that the concept of an ϵ-weak Cauchy sequence is implicit in [4]. Behrends' result may be restated as follows.

Theorem 1.2. *Let (f_n) be a bounded sequence in a complex $C(K)$ space and let $\epsilon > 0$. Then either (f_n) admits a subsequence which is ϵ-weak Cauchy on K or, for every $\delta > 0$, there is a subsequence (f_{m_n}) satisfying*

$$\Big\|\sum_n \lambda_n f_{m_n}\Big\| \geq \Big(\frac{\epsilon\sqrt{2}}{8} - \delta\Big) \sum_n |\lambda_n|$$

for every finitely supported sequence of complex scalars (λ_n).

Our main result is as follows.

Theorem 1.3. *Let (f_n) be a bounded sequence in a complex $C(K)$ space and let $\epsilon > 0$. Then either (f_n) admits a subsequence which is ϵ-weak Cauchy on K or, for every $\delta > 0$, there is a subsequence (f_{m_n}) with the following property. If I is a finite subset of $\mathbb{N}$ and $\phi\colon I \to \mathbb{N} \setminus I$ is any map then*

$$\Big\|\sum_{n\in I} \lambda_n (f_{m_n} - f_{m_{\phi(n)}})\Big\| \geq \Big(\frac{\epsilon}{\pi} - \delta\Big) \sum_{n\in I} |\lambda_n|$$

for every sequence of complex scalars $(\lambda_n)_{n\in I}$.

Two obvious choices of the map ϕ yield the next corollary.

Corollary 1.4. *Let (f_n) be a bounded sequence in a complex $C(K)$ space and let $\epsilon > 0$. Assume that none of the subsequences of (f_n) is ϵ-weak Cauchy on K. Then for every $\delta > 0$, there is a subsequence (f_{m_n}) with the following properties.*

$$\Big\|\sum_{n=2}^{\infty} \lambda_n (f_{m_n} - f_{m_1})\Big\| \geq \Big(\frac{\epsilon}{\pi} - \delta\Big) \sum_{n=2}^{\infty} |\lambda_n| \tag{1.1}$$

for all finitely supported sequences of complex scalars $(\lambda_n)_{n=2}^{\infty}$.

$$\Big\|\sum_n \lambda_n (f_{l_{2n}} - f_{l_{2n-1}})\Big\| \geq \Big(\frac{\epsilon}{\pi} - \delta\Big) \sum_n |\lambda_n| \tag{1.2}$$

for all finitely supported sequences of complex scalars (λ_n) and every infinite subset $L = (l_n)$ of $M = (m_n)$.

It follows now that, under the assumptions of Corollary 1.4, the resulting subsequence (f_{m_n}) has the property that $(f_{m_n} - f_{m_1})_{n=2}^{\infty}$ C-dominates the usual ℓ_1-basis, where $C = \frac{\epsilon}{\pi} - \delta$ for sufficiently small $\delta > 0$,

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and therefore $(f_{m_n} - f_{m_1})_{n=2}^{\infty}$ is an ℓ_1-sequence (i.e., equivalent to the usual ℓ_1-basis). It is not hard to see that any translate of an ℓ_1-sequence admits a tail subsequence which is an ℓ_1-sequence too. We conclude that if (f_n) is a bounded sequence in $C(K)$ admitting no ℓ_1-subsequence, then every subsequence of (f_n) admits, for all $\epsilon > 0$, a subsequence which is ϵ-weak Cauchy on K and thus, by an easy diagonalization argument, a weak Cauchy subsequence. We have thus recaptured the ℓ_1-theorem.

Moreover, if (f_{m_n}) satisfies the conclusion of Corollary 1.4 for a sufficiently small δ then it follows, by a result of Knaust and Odell (Proposition 4.2 of [11] which holds for complex Banach spaces as well), that there is some $k \geq 2$ so that $(f_{m_n})_{n=k}^{\infty}$ C-dominates the usual ℓ_1-basis, where $C = \frac{\epsilon}{\pi} - \delta$. We deduce from this that if (f_n) lacks subsequences which are ϵ-weak Cauchy on K for some fixed $\epsilon > 0$, then the subsequence (f_{m_n}) resulting from Corollary 1.4 for a given $\delta > 0$, admits a further subsequence satisfying the conclusion of Theorem 1.2 with the constant $\sqrt{2}/8$ being replaced by $1/\pi$. We have thus obtained a modest strengthening of Behrends' quantified version of the ℓ_1-theorem for complex Banach spaces.

We note here that in [4] it is shown that for real $C(K)$ spaces Theorem 1.2 holds with the constant $\sqrt{2}/8$ being replaced by $1/2$. Direct modifications of our arguments show that for real $C(K)$ spaces, the constant $1/\pi$ which appears in Theorem 1.3 may also be replaced by $1/2$.

Our final result may be viewed as a Rainwater type of result [15] about ϵ-weak Cauchy sequences. Let X be a Banach space and endow B_{X^*} with the w^*-topology. We naturally identify X with a closed subspace of $C(B_{X^*})$. Call a bounded sequence (x_n) in X *ϵ-weak Cauchy*, for some fixed $\epsilon > 0$, if (x_n) is ϵ-weak Cauchy on B_{X^*}. By combining results from the theory of Schreier families [1] and transfinite averages [3] with Corollary 1.4, we obtain the following.

Corollary 1.5. *Let X be a Banach space and K a w^*-compact subset of B_{X^*} which norms X isometrically (i.e., $\|x\| = \sup_{x^* \in K} |x^*(x)|$ for all $x \in X$). Let (x_n) be a bounded sequence in X which is ϵ-weak Cauchy on K for some $\epsilon > 0$. Then (x_n) admits, for all $\delta > 0$, a $(2\epsilon + \delta)$-weak Cauchy subsequence.*

We use standard Banach space facts and terminology as may be found in [12]. If $M \in [\mathbb{N}]$ then $[M]^{<\infty}$ stands for the set of all finite subsets of M.

2. Proofs of the results

We fix a compact Hausdorff space K and a bounded sequence (f_n) in $C(K)$.

Notation. Let E be a closed subset of $\mathbb{C}^2$ and $P \in [\mathbb{N}]$. Define

$$\mathcal{D}(P, E) = \{L \in [P],\, L = (l_n) : \exists\, t \in K,\, (f_{l_{2n-1}}(t), f_{l_{2n}}(t)) \in E,\, \forall n \in \mathbb{N}\}.$$

Notation. For $\epsilon > 0$ we set $F_\epsilon = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1 - z_2| \geq \epsilon\}$.

Lemma 2.1. *$\mathcal{D}(P, E)$ is pointwise closed in $[\mathbb{N}]$.*

Proof. Let (L_m) be a sequence in $\mathcal{D}(P, E)$ converging pointwise to some $L \in [P]$, $L = (l_n)$. Fix $n \in \mathbb{N}$ and choose $m_n \in \mathbb{N}$ such that $\{l_k : k \leq 2n\}$ is an initial segment of L_{m_n}. Next choose $t_n \in K$ so that $(f_{l_{2j-1}}(t_n), f_{l_{2j}}(t_n)) \in E$ for all $j \leq n$. Let $t \in K$ be a cluster point of the sequence (t_n). Since the f_j's are continuous and E is closed, we obtain that $(f_{l_{2n-1}}(t), f_{l_{2n}}(t)) \in E$ for all $n \in \mathbb{N}$ and so $L \in \mathcal{D}(P, E)$ completing the proof of the lemma. □

Remark 1. Assume that $\mathcal{D}(P, F_\epsilon)$ is a proper subset of $[P]$ for all $P \in [\mathbb{N}]$ and $\epsilon > 0$. Since $\mathcal{D}(P, F_\epsilon)$ is Ramsey, an easy diagonalization argument shows that (f_n) admits a weak Cauchy subsequence.

Lemma 2.2. *Assume that* $[\mathbb{N}] = \mathcal{D}(\mathbb{N}, F_\epsilon)$ *for some* $\epsilon > 0$. *Let* $\delta_1 > 0$ *and choose an integer multiple* A *of* δ_1, $A > \delta_1$, *so that* $\|f_n\| \leq A$ *for all* $n \in \mathbb{N}$. *Then there exist* $M \in [\mathbb{N}]$ *and two squares* Δ_1 *and* Δ_2 *contained in* $[-A, A]^2$ *with sides parallel to the axes and equal to* δ_1 *so that*

(1) $(\Delta_1 \times \Delta_2) \cap F_\epsilon \neq \emptyset$.
(2) *For every choice* J_1 *and* J_2 *of pairwise disjoint subsets of* M *there exists* $t \in K$ *so that* $f_n(t) \in \Delta_s$ *for all* $n \in J_s$ *and* $s \leq 2$.

Proof. Let Π be a finite partition of $[-A, A]^2$ into pairwise non-overlapping squares having sides parallel to the axes and equal to δ_1. Define

$$\mathcal{D} = \cup_{(\Delta_1,\Delta_2)\in\Pi^2} \mathcal{D}[\mathbb{N}, (\Delta_1 \times \Delta_2) \cap F_\epsilon].$$

This is a pointwise closed subset of $[\mathbb{N}]$, thanks to Lemma 2.1, and therefore it is Ramsey. Let $L \in [\mathbb{N}]$, $L = (l_n)$. Since $L \in \mathcal{D}(\mathbb{N}, F_\epsilon)$ there exist $t \in K$, $I \in [\mathbb{N}]$ and $(\Delta_1, \Delta_2) \in \Pi^2$ so that $(f_{l_{2j-1}}(t), f_{l_{2j}}(t)) \in (\Delta_1 \times \Delta_2) \cap F_\epsilon$ for all $j \in I$. It follows that $L_1 \in [L] \cap \mathcal{D}$, where $L_1 = \cup_{j\in I}\{l_{2j-1}, l_{2j}\}$. Hence, $[L] \cap \mathcal{D} \neq \emptyset$ for all $L \in [\mathbb{N}]$. The infinite Ramsey theorem now yields $N \in [\mathbb{N}]$ so that $[N] \subset \mathcal{D}$. A second application of the infinite Ramsey theorem provides us some $P \in [N]$ and $(\Delta_1, \Delta_2) \in \Pi^2$ so that $[P] \subset \mathcal{D}[\mathbb{N}, (\Delta_1 \times \Delta_2) \cap F_\epsilon]$. Finally, if $P = (p_n)$, let $M = \{p_{3n-1} : n \in \mathbb{N}\}$. Clearly, M, Δ_1 and Δ_2 satisfy (1) and (2). $\square$

Lemma 2.3. *Let* $\epsilon > 0$, $\delta > 0$ *and* $0 < \delta_1 < \delta/4$. *Let* Δ_1, Δ_2 *be subsets of* $\mathbb{C}$ *of diameter at most equal to* δ_1 *with* $0 \in \Delta_1$. *Assume that the following conditions are fulfilled:*

(1) $(\Delta_1 \times \Delta_2) \cap F_\epsilon \neq \emptyset$.
(2) *For every choice* J_1 *and* J_2 *of pairwise disjoint subsets of* $\mathbb{N}$ *there exists* $t \in K$ *so that* $f_n(t) \in \Delta_s$ *for all* $n \in J_s$ *and* $s \leq 2$.

Then for every $I \in [\mathbb{N}]^{<\infty}$, *every map* $\phi\colon I \to \mathbb{N} \setminus I$ *and every sequence of complex scalars* $(\lambda_n)_{n\in I}$ *the following inequality holds.*

$$\Big\|\sum_{n\in I} \lambda_n(f_n - f_{\phi(n)})\Big\| \geq (\frac{\epsilon}{\pi} - \delta)\sum_{n\in I} |\lambda_n|.$$

Proof. Since $(\Delta_1 \times \Delta_2) \cap F_\epsilon \neq \emptyset$ there exists $z_2 \in \Delta_2$ so that $|z_2| \geq \epsilon - \delta_1$. Fix $I \in [\mathbb{N}]^{<\infty}$ and a map $\phi\colon I \to \mathbb{N} \setminus I$. Let $(\lambda_n)_{n\in I}$ be a sequence of complex scalars. By applying Lemma 6.3 in [17] we may choose $J_2 \subset I$ such that

$$\Big|\sum_{n\in J_2} \lambda_n\Big| \geq \frac{1}{\pi}\sum_{n\in I} |\lambda_n|.$$

Let $J_1 = (I \setminus J_2) \cup \phi(I)$. By our hypothesis, there is some $t \in K$ such that $f_n(t) \in \Delta_1$ for all $n \in J_1$, while $f_n(t) \in \Delta_2$ for all $n \in J_2$. Note that

$$\begin{aligned}\Big\|\sum_{n\in I} \lambda_n(f_n - f_{\phi(n)})\Big\| &\geq \Big|\sum_{n\in J_2} \lambda_n f_n(t)\Big| - \sum_{n\in I\setminus J_2} |\lambda_n||f_n(t)| - \sum_{n\in I} |\lambda_n||f_{\phi(n)}(t)| \\ &\geq \Big|\sum_{n\in J_2} \lambda_n f_n(t)\Big| - 2\delta_1 \sum_{n\in I} |\lambda_n|\end{aligned}$$

$$\geq \Big|\sum_{n\in J_2} \lambda_n z_2\Big| - \sum_{n\in J_2} |\lambda_n||f_n(t) - z_2| - 2\delta_1 \sum_{n\in I} |\lambda_n|$$
$$\geq \Big|\sum_{n\in J_2} \lambda_n\Big||z_2| - 3\delta_1 \sum_{n\in I} |\lambda_n|$$
$$\geq \frac{\epsilon-\delta_1}{\pi} \sum_{n\in I} |\lambda_n| - 3\delta_1 \sum_{n\in I} |\lambda_n| \geq (\frac{\epsilon}{\pi} - \delta) \sum_{n\in I} |\lambda_n|. \qquad \square$$

Proof of Theorem 1.3. Suppose that (f_n) has no subsequence which is ϵ-weak Cauchy on K. It follows that $[P] \cap \mathcal{D}(\mathbb{N}, F_\epsilon) = \emptyset$ for no $P \in [\mathbb{N}]$. Since $\mathcal{D}(\mathbb{N}, F_\epsilon)$ is pointwise closed, by Lemma 2.1, the infinite Ramsey theorem implies that $\mathcal{D}(N, F_\epsilon) = [N]$ for some $N \in [\mathbb{N}]$. Without loss of generality, by relabeling if necessary, we may assume that $N = \mathbb{N}$. Let $\delta > 0$ and choose $0 < \delta_1 < \frac{\delta}{4\sqrt{2}}$. Applying Lemma 2.2, passing to a subsequence and relabeling if necessary, we may also assume that there exist two squares Δ_1 and Δ_2 with sides parallel to the axes and equal to δ_1 so that

(1) $(\Delta_1 \times \Delta_2) \cap F_\epsilon \neq \emptyset$.
(2) For every choice J_1 and J_2 of pairwise disjoint subsets of $\mathbb{N}$ there exists $t \in K$ so that $f_n(t) \in \Delta_s$ for all $n \in J_s$ and $s \leq 2$.

By replacing (f_n) by (g_n), where for all $n \in \mathbb{N}$ $g_n = f_n + z$ for a suitable choice of $z \in \mathbb{C}$, we may assume that $\Delta_1 = [0, \delta_1]^2$. The assertion of the theorem now follows by applying Lemma 2.3. $\square$

We remark here that in case $\lambda_n \in \mathbb{R}$ for all $n \in I$ then we may replace the constant $1/\pi$ by $1/2$ in the conclusion of Theorem 1.3.

Proof of Corollary 1.5. Let (x_n) be ϵ-weak Cauchy on K and let $\rho > 0$ be such that (x_n) admits no ρ-weak Cauchy subsequence. It will suffice showing that $\rho \leq 2\epsilon$. Let $\delta > 0$. By applying Corollary 1.4, passing to an appropriate subsequence and relabeling, there is no loss of generality in assuming that

$$\Big\|\sum_{n\in I} \lambda_n (x_{2n} - x_{2n-1})\Big\| \geq (\frac{\rho}{2} - \delta) \sum_{n\in I} |\lambda_n| \tag{2.1}$$

for all $I \in [\mathbb{N}]^{<\infty}$ and every sequence of real scalars $(\lambda_n)_{n\in I}$.

We next define

$$\mathcal{F} = \{F \in [\mathbb{N}]^{<\infty} : \exists\, x^* \in K \text{ with } |x^*(x_{2n} - x_{2n-1})| \geq \epsilon, \forall\, n \in F\}.$$

Clearly, $\mathcal{F}$ is a hereditary family of finite subsets of $\mathbb{N}$ (i.e., $G \in \mathcal{F}$ whenever $G \subset F$ and $F \in \mathcal{F}$). Endow $\mathcal{P}(\mathbb{N})$ (i.e., the powerset of $\mathbb{N}$) with the topology of pointwise convergence. By applying a compactness argument similar to the one in the proof of Lemma 2.1, based on the fact that K is w^*-compact and (x_n) is ϵ-weak Cauchy on K, we infer that $\mathcal{F}$ is pointwise closed in $\mathcal{P}(\mathbb{N})$.

It follows that $\mathcal{F}$ is a countable, compact metric space in the topology of pointwise convergence. A classical result [13] now yields a countable ordinal ξ such that $\mathcal{F}^{(\xi)} = \emptyset$. We next deduce from the dichotomy theorem of [9] that there exists $M \in [\mathbb{N}]$ so that $\mathcal{F} \cap [M]^{<\infty} \subset \mathcal{S}_\xi$, the latter denoting the Schreier family of order ξ [1]. It is shown in [3] that there exist $F \in [M]^{<\infty}$ and positive scalars $(\lambda_n)_{n\in F}$ so that $\sum_{n\in F} \lambda_n = 1$, while $\sum_{n\in G} \lambda_n < \delta$ for all $G \subset F$ with $G \in \mathcal{S}_\xi$.

We apply (2.1) to obtain

$$\Big\|\sum_{n\in F} \lambda_n (x_{2n} - x_{2n-1})\Big\| \geq \frac{\rho}{2} - \delta. \tag{2.2}$$

Let $x^* \in K$ and set $G = \{n \in F : |x^*(x_{2n} - x_{2n-1})| \geq \epsilon\}$. Our preceding choices ensure that $G \in \mathcal{F} \cap [M]^{<\infty}$ and so $G \in \mathcal{S}_\xi$. Hence,

$$\left| \sum_{n \in G} \lambda_n x^*(x_{2n} - x_{2n-1}) \right| < 2\delta C, \tag{2.3}$$

where $C = \sup_n \|x_n\|$. It is also clear that

$$\left| \sum_{n \in F \setminus G} \lambda_n x^*(x_{2n} - x_{2n-1}) \right| < \epsilon. \tag{2.4}$$

Since K isometrically norms X and $x^* \in K$ is arbitrary, we deduce from (2.3) and (2.4) that

$$\left\| \sum_{n \in F} \lambda_n (x_{2n} - x_{2n-1}) \right\| \leq \epsilon + 2\delta C. \tag{2.5}$$

We finally conclude, by combining (2.2) with (2.5), that

$$\frac{\rho}{2} - \delta \leq \epsilon + 2\delta C$$

for all $\delta > 0$. Therefore, $\rho \leq 2\epsilon$. $\square$

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