J. Math. Anal. Appl. 444 (2016) 464–474

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A triple Mertens evaluation

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A R T I C L E I N F O

Article history:
Received 10 January 2016
Available online 30 June 2016
Submitted by B.C. Berndt

Keywords:
Arithmetic functions
Asymptotic results
Mertens type evaluations
Dilogarithm
Trilogarithm

A B S T R A C T

We prove the following triple Mertens evaluation

$$\sum_{pqr\leq x,\ p,q,r \text{ prime}} \frac{1}{pqr} = (\ln(\ln x) + B)^3 - \frac{\pi^2}{2}(\ln(\ln x) + B) + 2\zeta(3) + O\left(\frac{\ln^2(\ln x)}{\ln x}\right)$$

where B is the Mertens constant and ζ is the Riemann zeta function.

1. Introduction and notation

The main purpose of this paper is to prove the result stated in the abstract, the paper being a natural continuation of the paper [9]. The famous Mertens's theorem, sometimes called Mertens's second Theorem, asserts that there exists a constant B, called the Mertens constant, such that $\sum_{p\leq x} \frac{1}{p} = \ln(\ln x) + B + O\left(\frac{1}{\ln x}\right)$, see [1,3,4,11] and the more recent [6]. In his book, G. Tenenbaum, see [11, Problem 12, page 22], states the problem of evaluating the sum $\sum_{pq\leq x} \frac{1}{pq}$, where p and q denote primes and generalizing the result. In [9,10], an answer to the first part of Tenenbaum's problem, namely the following double Mertens type evaluation was provided

$$\sum_{pq\leq x} \frac{1}{pq} = (\ln(\ln x) + B)^2 - \frac{\pi^2}{6} + O\left(\frac{\ln(\ln x)}{\ln x}\right).$$

To get the above double Mertens evaluation from that proved in [9] we use the following well-known equality $-\ln^2 2 + 2\int_{0+0}^{\frac{1}{2}} \frac{\ln(1-x)}{x} dx = -\frac{\pi^2}{6}$, see [5, page 5, (1.11)]. Less precise formulas known in the literature are: $\sum_{pq\leq x} \frac{1}{pq} = (\log\log x)^2 + 2B\log\log x + O(1)$, see [8, page 23; solution on pages 60–62] and

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http://dx.doi.org/10.1016/j.jmaa.2016.06.054
0022-247X/

$\sum\limits_{pq\le x}\frac{1}{pq}\backsim(\log\log x)^2$, see [7, page 315]. Our paper provides a first step toward answering the second part of Tenenbaum's problem. Let us fix some notations and notions. By e we denote the Euler number and $\ln x=\log_e x$, $x>0$. Let $a\in\mathbb{R}\cup\{-\infty\}$, $g:(a,\infty)\to[0,\infty)$ be fixed. If $f:(a,\infty)\to\mathbb{R}$ is a function, we write $f(x)=O(g(x))$ if and only if there exists $M>0$, $b\ge a$ such that $|f(x)|\le Mg(x)$ for all $x\ge b$. We need also the Riemann zeta function $\zeta:(1,\infty)\to(0,\infty)$, $\zeta(s)=\sum\limits_{n=1}^{\infty}\frac{1}{n^s}$ and recall that $\zeta(2)=\frac{\pi^2}{6}$.

Throughout this paper, we use the notation $\sum\limits_{i\le x}$ to mean $\sum\limits_{i\le x;\ i\in\mathbb{N}}$, the notation $\sum\limits_{p\le x}$ to mean $\sum\limits_{p\le x;\ p\text{ prime}}$ and the notation $\sum\limits_{pq\le x}$ to mean $\sum\limits_{pq\le x;\ p,\ q\text{ prime}}$, etc. All notation and notion used and not defined in this paper are standard (see [1,3,4,11]).

2. The dilogarithm and trilogarithm functions

In our proofs for the Mertens triple evaluation the dilogarithm and trilogarithm functions appear in a natural way. Recall, see [5], that the dilogarithm $Li_2:[0,1]\to\mathbb{R}$ is defined by

$$Li_2(x)=-\int\limits_0^x\frac{\ln(1-t)}{t}dt=\sum_{n=1}^{\infty}\frac{x^n}{n^2}$$

and the trilogarithm $Li_3:[0,1]\to\mathbb{R}$ is defined by

$$Li_3(x)=\int\limits_0^x\frac{Li_2(t)}{t}dt=\sum_{n=1}^{\infty}\frac{x^n}{n^3}.$$

We need the following well-known results, see [5, page 5, (1.11) and page 155, (6.12)].

Proposition 1. *(i) For all $x\in(0,1)$ the following relation holds* $Li_2(x)+Li_2(1-x)=-\ln x\cdot\ln(1-x)+\frac{\pi^2}{6}$.

(ii) $Li_2\left(\frac{1}{2}\right)=\frac{\pi^2}{12}-\frac{\ln^2 2}{2}$.

(iii) $Li_3\left(\frac{1}{2}\right)=\frac{7\zeta(3)}{8}-\frac{(\ln 2)\zeta(2)}{2}+\frac{\ln^3 2}{6}$.

We will need to make use of the following integral later.

Proposition 2. $\int_{0+0}^{\frac{1}{2}}\frac{\ln^2(1-x)}{x}dx=\frac{\zeta(3)}{4}-\frac{\ln^3 2}{3}$.

Proof. Integrating by parts and then making an obvious change of variables, we have

$$\begin{aligned}\int\limits_{0+0}^{\frac{1}{2}}\frac{\ln^2(1-x)}{x}dx&=\ln x\cdot\ln^2(1-x)\,\Big|_{0+0}^{\frac{1}{2}}+2\int\limits_{0+0}^{\frac{1}{2}}\frac{(\ln x)\ln(1-x)}{1-x}dx\\&=-\ln^3 2+2\int\limits_{\frac{1}{2}}^{1-0}\frac{(\ln t)\ln(1-t)}{t}dt.\end{aligned}$$

Also

$$\int\limits_{\frac{1}{2}}^{1-0}\frac{(\ln t)\ln(1-t)}{t}dt=-\sum_{n=1}^{\infty}\frac{1}{n}\int\limits_{\frac{1}{2}}^{1-0}t^{n-1}(\ln t)\,dt$$

$$= -\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{t^n \ln t}{n} \Big|_{\frac{1}{2}}^{1} - \frac{1}{n} \int_{\frac{1}{2}}^{1} t^{n-1} dt \right)$$

$$= -\ln 2 \sum_{n=1}^{\infty} \frac{1}{2^n n^2} + \sum_{n=1}^{\infty} \frac{1}{n^3} \left(1 - \frac{1}{2^n}\right)$$

$$= -(\ln 2)\, Li_2\left(\frac{1}{2}\right) + \zeta(3) - Li_3\left(\frac{1}{2}\right).$$

Replacing the values of $Li_2\left(\frac{1}{2}\right)$ and $Li_3\left(\frac{1}{2}\right)$ we get the statement of the proposition. □

3. The hyperbola method of Dirichlet for a triple sum

Proposition 3. *(i) Let $\psi : \mathbb{N}^3 \to \mathbb{R}$ be an arbitrary function. For each $0 < y < x$ the following equality holds*

$$\sum_{ijk \le x} \psi(i,j,k) = \sum_{k \le y} \sum_{ij \le \frac{x}{k}} \psi(i,j,k) + \sum_{ij \le \frac{x}{y}} \sum_{y < k \le \frac{x}{ij}} \psi(i,j,k).$$

(ii) Let $f, g, h : \mathbb{N} \to \mathbb{R}$ be arbitrary functions and define $S_{f,g} : (0,\infty) \to \mathbb{R}$, $S_h : (0,\infty) \to \mathbb{R}$, $S_{f,g}(x) = \sum_{ij \le x} f(i)\, g(j)$ and $S_h(x) = \sum_{i \le x} h(i)$. For each $0 < y < x$ the following equality holds

$$\sum_{ijk \le x} f(i)\, g(j)\, h(k) = \sum_{k \le y} h(k)\, S_{f,g}\left(\frac{x}{k}\right) + \sum_{ij \le \frac{x}{y}} f(i)\, g(j)\, S_h\left(\frac{x}{ij}\right) - S_h(y)\, S_{f,g}\left(\frac{x}{y}\right).$$

Proof. (i) Let us note the following equality

$$\left\{(i,j,k) \in \mathbb{N}^3 \mid ijk \le x\right\} = \left\{(i,j,k) \in \mathbb{N}^3 \mid ijk \le x,\ k \le y\right\} \cup \left\{(i,j,k) \in \mathbb{N}^3 \mid ijk \le x,\ y < k\right\}$$

and that the sets from the right member are disjoint. Then

$$\sum_{ijk \le x} \psi(i,j,k) = \sum_{ijk \le x,\ k \le y} \psi(i,j,k) + \sum_{ijk \le x,\ y < k} \psi(i,j,k)$$

and, from the obvious equality $\sum_{ijk \le x,\ k \le y} \psi(i,j,k) = \sum_{k \le y} \sum_{ij \le \frac{x}{k}} \psi(i,j,k)$ we get

$$\sum_{ijk \le x} \psi(i,j,k) = \sum_{k \le y} \sum_{ij \le \frac{x}{k}} \psi(i,j,k) + \sum_{ijk \le x,\ y < k} \psi(i,j,k). \tag{1}$$

We show that

$$\left\{(i,j,k) \in \mathbb{N}^3 \mid ijk \le x,\ y < k\right\} = \left\{(i,j,k) \in \mathbb{N}^3 \mid ij \le \frac{x}{y},\ y < k \le \frac{x}{ij}\right\}.$$

Indeed, let $(i,j,k) \in \mathbb{N}^3$ be such that $ijk \le x$, $y < k$. Then $y < k \le \frac{x}{ij}$, $ijy < ijk$ and since $ijk \le x$ it follows that $ijy < x$ i.e. $ij < \frac{x}{y}$. The reverse inclusions is obvious. From this equality we deduce

$$\sum_{ijk \le x,\ y < k} \psi(i,j,k) = \sum_{ij \le \frac{x}{y},\ y < k \le \frac{x}{ij}} h(i,j,k). \tag{2}$$

From (1) and (2) we obtain the equality from the statement.

(ii) By (i) we have

$$\begin{aligned}\sum_{ijk\le x} f(i)\,g(j)\,h(k) &= \sum_{k\le y}\sum_{ij\le \frac{x}{k}} f(i)\,g(j)\,h(k) + \sum_{ij\le\frac{x}{y}}\sum_{y<k\le\frac{x}{ij}} f(i)\,g(j)\,h(k)\\ &= \sum_{k\le y} h(k)\sum_{ij\le\frac{x}{k}} f(i)\,g(j) + \sum_{ij\le\frac{x}{y}} f(i)\,g(j)\left(\sum_{k\le\frac{x}{ij}} h(k) - \sum_{k\le y} h(k)\right)\\ &= \sum_{k\le y} h(k)\,S_{f,g}\left(\frac{x}{k}\right) + \sum_{ij\le\frac{x}{y}} f(i)\,g(j)\left(S_h\left(\frac{x}{ij}\right) - S_h(y)\right)\\ &= \sum_{k\le y} h(k)\,S_{f,g}\left(\frac{x}{k}\right) + \sum_{ij\le\frac{x}{y}} f(i)\,g(j)\,S_h\left(\frac{x}{ij}\right) - S_h(y)\,S_{f,g}\left(\frac{x}{y}\right). \quad\square\end{aligned}$$

Corollary 4. *(i) Let $f:\mathbb{N}\to\mathbb{R}$ be an arbitrary function and define S_1, $S_2:(0,\infty)\to\mathbb{R}$ by $S_1(x)=\sum_{i\le x} f(i)$ and $S_2(x)=\sum_{ij\le x} f(i)\,f(j)$. For each $x>1$ the following equality holds*

$$\sum_{ijk\le x} f(i)\,f(j)\,f(k) = \sum_{i\le\sqrt{x}} f(i)\,S_2\left(\frac{x}{i}\right) + \sum_{ij\le\sqrt{x}} f(i)\,f(j)\,S_1\left(\frac{x}{ij}\right) - S_1\left(\sqrt{x}\right)S_2\left(\sqrt{x}\right).$$

(ii) Let $\mathbb{P}$ be the set of the all prime numbers, $u:\mathbb{P}\to\mathbb{R}$ be an arbitrary function and define V_1, $V_2:(0,\infty)\to\mathbb{R}$ by $V_1(x)=\sum_{p\le x} u(p)$, $V_2(x)=\sum_{pq\le x} u(p)\,u(q)$. For each $x>1$ the following equality holds

$$\sum_{pqr\le x} u(p)\,u(q)\,u(r) = \sum_{p\le\sqrt{x}} u(p)\,V_2\left(\frac{x}{p}\right) + \sum_{pq\le\sqrt{x}} u(p)\,u(q)\,V_1\left(\frac{x}{pq}\right) - V_1\left(\sqrt{x}\right)V_2\left(\sqrt{x}\right).$$

Proof. (i) Take in Proposition 3 (ii) $y=\sqrt{x}$.

(ii) Take in (i) $f=u\chi_{\mathbb{P}}$, that is $f(n)=\begin{cases} u(n) \text{ for } n\in\mathbb{P}\\ 0 \text{ for } n\notin\mathbb{P}\end{cases}$. $\square$

4. The basic result

In the rest of the paper we consider the polynomials $P_1(y)=y+B$, $P_2(y)=(y+B)^2-\frac{\pi^2}{6}=(y+B)^2-\zeta(2)$, where B is the Mertens constant.

Proposition 5. *The following evaluation holds*

$$\begin{aligned}\sum_{pqr\le x}\frac{1}{pqr} &= \sum_{p\le\sqrt{x}}\frac{P_2\left(\ln\left(\ln\frac{x}{p}\right)\right)}{p} + \sum_{pq\le\sqrt{x}}\frac{P_1\left(\ln\left(\ln\frac{x}{pq}\right)\right)}{pq}\\ &\quad - P_1\left(\ln\left(\ln\sqrt{x}\right)\right)P_2\left(\ln\left(\ln\sqrt{x}\right)\right) + O\left(\frac{\ln^2(\ln x)}{\ln x}\right)\end{aligned}$$

Proof. Let us define V_1, $V_2:(0,\infty)\to\mathbb{R}$ by $V_1(x)=\sum_{p\le x,}\frac{1}{p}$, $V_2(x)=\sum_{pq\le x,}\frac{1}{pq}$. By Corollary 4 for all $x>1$ the following equality holds

$$\sum_{pqr\leq x}\frac{1}{pqr}=\sum_{p\leq\sqrt{x}}\frac{1}{p}V_2\left(\frac{x}{p}\right)+\sum_{pq\leq\sqrt{x}}\frac{1}{pq}V_1\left(\frac{x}{pq}\right)-V_1\left(\sqrt{x}\right)V_2\left(\sqrt{x}\right).$$

From the Mertens and double Mertens evaluations we have

$$V_1(x)=P_1(\ln(\ln x))+R_1(x),R_1(x)=O\left(\frac{1}{\ln x}\right),$$
$$V_2(x)=P_2(\ln(\ln x))+R_2(x),R_2(x)=O\left(\frac{\ln(\ln x)}{\ln x}\right).$$

We deduce

$$\sum_{pqr\leq x}\frac{1}{pqr}=\sum_{p\leq\sqrt{x}}\frac{1}{p}P_2\left(\ln\left(\ln\frac{x}{p}\right)\right)+\sum_{pq\leq\sqrt{x}}\frac{1}{pq}P_1\left(\ln\left(\ln\frac{x}{pq}\right)\right)-V_1\left(\sqrt{x}\right)V_2\left(\sqrt{x}\right)+R_3(x)$$

where

$$R_3(x)=\sum_{p\leq\sqrt{x}}\frac{1}{p}R_2\left(\frac{x}{p}\right)+\sum_{pq\leq\sqrt{x}}\frac{1}{pq}R_1\left(\frac{x}{pq}\right).$$

Since $p\geq 2$ and $0\leq\sum_{p\leq\sqrt{x}}\frac{1}{p}\cdot\frac{1}{\ln\frac{x}{p}}\leq\sum_{p\leq\sqrt{x}}\frac{1}{p}\cdot\frac{1}{\ln\frac{x}{\sqrt{x}}}=O\left(\frac{\ln(\ln x)}{\ln x}\right)$ (by Mertens's theorem), we deduce

$$\sum_{p\leq\sqrt{x}}\frac{1}{p}R_2\left(\frac{x}{p}\right)=\sum_{p\leq\sqrt{x}}\frac{1}{p}O\left(\frac{\ln\left(\ln\frac{x}{p}\right)}{\ln\frac{x}{p}}\right)=O\left(\ln\left(\ln\frac{x}{2}\right)\sum_{p\leq\sqrt{x}}\frac{1}{p}\cdot\frac{1}{\ln\frac{x}{p}}\right)=O\left(\frac{\ln^2(\ln x)}{\ln x}\right).$$

Also

$$\sum_{pq\leq\sqrt{x}}\frac{1}{pq}R_1\left(\frac{x}{pq}\right)=\sum_{pq\leq\sqrt{x}}\frac{1}{pq}O\left(\frac{1}{\ln\frac{x}{pq}}\right)=O\left(\sum_{pq\leq\sqrt{x}}\frac{1}{pq\ln\frac{x}{pq}}\right)=O\left(\sum_{pq\leq\sqrt{x}}\frac{1}{pq\ln\frac{x}{\sqrt{x}}}\right)$$
$$=O\left(\frac{1}{\ln x}\sum_{pq\leq\sqrt{x}}\frac{1}{pq}\right)=O\left(\frac{\ln^2(\ln x)}{\ln x}\right).$$

We have

$$V_1\left(\sqrt{x}\right)V_2\left(\sqrt{x}\right)=P_1\left(\ln\left(\ln\sqrt{x}\right)\right)P_2\left(\ln\left(\ln\sqrt{x}\right)\right)+P_1\left(\ln\left(\ln\sqrt{x}\right)\right)O\left(\frac{\ln(\ln x)}{\ln x}\right)$$
$$+P_2\left(\ln\left(\ln\sqrt{x}\right)\right)O\left(\frac{1}{\ln x}\right)+O\left(\frac{\ln(\ln x)}{\ln^2 x}\right)$$

and since $P_1(\ln(\ln\sqrt{x}))O\left(\frac{\ln(\ln x)}{\ln x}\right)=O\left(\frac{\ln^2(\ln x)}{\ln x}\right)$, $P_2(\ln(\ln\sqrt{x}))O\left(\frac{1}{\ln x}\right)=O\left(\frac{\ln^2(\ln x)}{\ln x}\right)$, $\frac{\ln(\ln x)}{\ln^2 x}=O\left(\frac{\ln^2(\ln x)}{\ln x}\right)$ we deduce

$$V_1\left(\sqrt{x}\right)V_2\left(\sqrt{x}\right)=P_1\left(\ln\left(\ln\sqrt{x}\right)\right)P_2\left(\ln\left(\ln\sqrt{x}\right)\right)+O\left(\frac{\ln^2(\ln x)}{\ln x}\right).$$

From all these evaluations we get the statement of the proposition. □

5. The evaluation of the sum $\sum\limits_{p\le\sqrt{x}} \frac{P_2\left(\ln\left(\ln\frac{x}{p}\right)\right)}{p}$

Proposition 6. *The following evaluations hold:*

(i) $\sum\limits_{p\le\sqrt{x}} \frac{1}{p}\left[\ln\left(1-\frac{\ln p}{\ln x}\right)\right] = \frac{a^2-\zeta(2)}{2} + O\left(\frac{1}{\ln x}\right)$*;*

(ii) $\sum\limits_{p\le\sqrt{x}} \frac{1}{p}\left[\ln\left(1-\frac{\ln p}{\ln x}\right)\right]^2 = \int_{0+0}^{\frac{1}{2}} \frac{\ln^2(1-x)}{x}dx = \frac{\zeta(3)}{4} - \frac{a^3}{3} + O\left(\frac{1}{\ln x}\right)$*, where* $a = \ln 2$*.*

Proof. (i) It follows from [9, Corollary 1] and $\int_{0+0}^{\frac{1}{2}} \frac{\ln(1-x)}{x}dx = -Li_2\left(\frac{1}{2}\right) = \frac{\ln^2 2}{2} - \frac{\pi^2}{12}$.

(ii) We have $[\ln(1-x)]^2 = \sum\limits_{k=1}^{\infty} a_k x^k$ for $0 \le x < 1$, where $a_1 = 0$, $a_{n+1} = \frac{2}{n+1}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)$, $n \ge 1$. Then

$$\sum_{p\le\sqrt{x}} \frac{1}{p}\left[\ln\left(1-\frac{\ln p}{\ln x}\right)\right]^2 = \sum_{k=1}^{\infty} a_k \sum_{p\le\sqrt{x}} \frac{1}{p}\left(\frac{\ln p}{\ln x}\right)^k.$$

By Proposition 1 in [9]

$$\sum_{p\le\sqrt{x}} \frac{1}{p}\left(\frac{\ln p}{\ln x}\right)^k = \frac{1}{k\cdot 2^k} + \frac{1}{2^{k-1}}O\left(\frac{1}{\ln x}\right)$$

and then

$$\begin{aligned}\sum_{p\le\sqrt{x}} \frac{1}{p}\left[\ln\left(1-\frac{\ln p}{\ln x}\right)\right]^2 &= \sum_{k=1}^{\infty}\frac{a_k}{k\cdot 2^k} + \left(\sum_{k=1}^{\infty}\frac{a_k}{2^{k-1}}\right)O\left(\frac{1}{\ln x}\right)\\ &= \sum_{k=1}^{\infty}\frac{a_k}{k\cdot 2^k} + O\left(\frac{1}{\ln x}\right),\end{aligned}$$

since the series $\sum\limits_{k=1}^{\infty}\frac{a_k}{2^{k-1}}$ is convergent. From $\frac{[\ln(1-x)]^2}{x} = \sum\limits_{k=1}^{\infty} a_k x^{k-1}$ for $0 < x \le \frac{1}{2}$ and $\int_0^{\frac{1}{2}} x^{k-1}dx = \frac{1}{k\cdot 2^k}$, we deduce $\sum\limits_{k=1}^{\infty}\frac{a_k}{k\cdot 2^k} = \int_{0+0}^{\frac{1}{2}}\frac{[\ln(1-x)]^2}{x}dx$. The evaluation from the statement follows from Proposition 2. □

Proposition 7. *The following evaluation holds*

$$\sum_{p\le\sqrt{x}} \frac{P_2\left(\ln\left(\ln\frac{x}{p}\right)\right)}{p} = U_2(\ln(\ln x)) + O\left(\frac{\ln^2(\ln x)}{\ln x}\right),$$

where

$$U_2(y) = P_2(y)P_1(y) - \zeta(2)P_1(y) + \frac{\zeta(3)}{4} - aP_2(y) + a^2P_1(y) - \frac{a^3}{3}$$

and $a = \ln 2$*.*

Proof. We have

$$\sum_{p\le\sqrt{x}} \frac{P_2\left(\ln\left(\ln\frac{x}{p}\right)\right)}{p} = \sum_{p\le\sqrt{x}} \frac{1}{p}\left[\ln\left(\ln\frac{x}{p}\right)\right]^2 + 2B\sum_{p\le\sqrt{x}} \frac{1}{p}\left[\ln\left(\ln\frac{x}{p}\right)\right] + \left(B^2-\zeta(2)\right)\sum_{p\le\sqrt{x}}\frac{1}{p}.$$

Now from the equalities

$$\sum_{p\le\sqrt{x}}\frac{1}{p}\left[\ln\left(\ln\frac{x}{p}\right)\right]^2=\sum_{p\le\sqrt{x}}\frac{1}{p}\left[\ln(\ln x)+\ln\left(1-\frac{\ln p}{\ln x}\right)\right]^2$$
$$=[\ln(\ln x)]^2\sum_{p\le\sqrt{x}}\frac{1}{p}+2[\ln(\ln x)]\sum_{p\le\sqrt{x}}\frac{1}{p}\ln\left(1-\frac{\ln p}{\ln x}\right)+\sum_{p\le\sqrt{x}}\frac{1}{p}\left[\ln\left(1-\frac{\ln p}{\ln x}\right)\right]^2$$

the Mertens evaluation $\sum\limits_{p\le\sqrt{x}}\frac{1}{p}=P_1(\ln(\ln\sqrt{x}))+O\left(\frac{1}{\ln x}\right)$, Proposition 6 and the evaluation

$$\sum_{p\le\sqrt{x}}\frac{1}{p}\left[\ln\left(\ln\frac{x}{p}\right)\right]=[\ln(\ln x)]\,P_1\left(\ln\left(\ln\sqrt{x}\right)\right)-Li_2\left(\frac{1}{2}\right)+O\left(\frac{\ln(\ln x)}{\ln x}\right)$$

(see the proof of Theorem 1 in [9, page 1163]), we get $\sum\limits_{p\le\sqrt{x}}\frac{P_2\left(\ln\left(\ln\frac{x}{p}\right)\right)}{p}=U_2(\ln(\ln x))+O\left(\frac{\ln^2(\ln x)}{\ln x}\right)$, where

$$U_2(y)=P_2(y)\,P_1(y-\ln 2)-2P_1(y)\,Li_2\left(\frac{1}{2}\right)+\int_0^{\frac{1}{2}}\frac{[\ln(1-x)]^2}{x}dx.$$

Using the values of $Li_2\left(\frac{1}{2}\right)$ and $\int_0^{\frac{1}{2}}\frac{[\ln(1-x)]^2}{x}dx$ (Proposition 2) by simple calculation we get the statement. □

6. Evaluation of the sum $\sum\limits_{pq\le\sqrt{x}}\frac{P_1\left(\ln\left(\ln\frac{x}{pq}\right)\right)}{pq}$

Proposition 8. *For all natural numbers k the following evaluation holds*

$$\frac{1}{(\ln x)^k}\sum_{pq\le x}\frac{[\ln(pq)]^k}{pq}=\frac{2\ln(\ln x)}{k}+\frac{2(kB-1)}{k^2}+O\left(\frac{\ln^2(\ln x)}{\ln x}\right)$$

where the constant which appear in the symbol O does not depend on k.

Proof. For $k=1$ it follows from the double Mertens evaluation

$$\sum_{pq\le x}\frac{\ln(pq)}{pq}=f(x)+A(x);\ f(x)=2[\ln(\ln x)](\ln x)+2(B-1)\ln x,\ \ A(x)=O\left(\ln^2(\ln x)\right),$$

see [2, Theorem 2]. Let $k\ge 2$. For $x\ge 2$, by the double Abel summation formula, see [2, Corollary 1], we have

$$\sum_{pq\le x}\frac{[\ln(pq)]^k}{pq}=\left(\sum_{pq\le x}\frac{\ln(pq)}{pq}\right)\left(\ln^{k-1}x\right)-\int_2^x\left(\sum_{pq\le t}\frac{\ln(pq)}{pq}\right)\left(\ln^{k-1}t\right)'dt$$

that is, integrating by parts,

$$\sum_{pq\le x}\frac{[\ln(pq)]^k}{pq}=(f(x)+A(x))\ln^{k-1}x-\int_2^x(f(t)+A(t))\left(\ln^{k-1}t\right)'dt$$
$$=\int_2^x f'(t)\ln^{k-1}tdt+f(2)\ln^{k-1}2+R(x),$$

where $R(x)=A(x)\ln^{k-1}x-(k-1)\int_2^x A(t)\cdot\frac{\ln^{k-2}t}{t}dt$. Now

$$|R(x)|\le|A(x)|\ln^{k-1}x+(k-1)\int_2^x|A(t)|\cdot\frac{\ln^{k-2}t}{t}dt$$
$$=\left[\ln^{k-1}x\right]O\left(\ln^2(\ln x)\right)+O\left((k-1)\int_2^x\frac{\left[\ln^{k-2}t\right]\ln^2(\ln t)}{t}dt\right)$$
$$=\left[\ln^{k-1}x\right]O\left(\ln^2(\ln x)\right);$$

we have used that

$$(k-1)\int_2^x\frac{\left[\ln^{k-2}t\right]\ln^2(\ln t)}{t}dt=(k-1)\int_{\ln 2}^{\ln x}v^{k-2}\ln^2 vdv$$
$$=\left[v^{k-1}\ln^2 v-\frac{2v^{k-1}\ln v}{k-1}+\frac{2v^{k-1}}{(k-1)^2}\right]_{\ln 2}^{\ln x}$$
$$=\left[\ln^{k-1}x\right]O\left(\ln^2(\ln x)\right),$$

where the constant which appear in the symbol O does not depend on k. We have

$$\int v^{k-1}(B+\ln v)\,dv=\frac{v^k(B+\ln v)}{k}-\frac{1}{k}\int v^{k-1}dv=\frac{v^k\ln v}{k}+\frac{(kB-1)v^k}{k^2}$$

and thus

$$\int_2^x\frac{B+\ln(\ln t)}{t}\cdot\left(\ln^{k-1}t\right)dt=\int_{\ln 2}^{\ln x}v^{k-1}(B+\ln v)\,dv=\frac{(\ln x)^k[\ln(\ln x)]}{k}+\frac{(kB-1)(\ln x)^k}{k^2}+A_k$$

where $A_k=\frac{(kB-1)(\ln 2)^k}{k^2}-\frac{(\ln 2)^k[\ln(\ln 2)]}{k}$. Then

$$\int_2^x f'(t)\ln^{k-1}tdt=2\int_2^x\frac{B+\ln(\ln t)}{t}\cdot\left(\ln^{k-1}t\right)dt=\frac{2(\ln x)^k[\ln(\ln x)]}{k}+\frac{2(kB-1)(\ln x)^k}{k^2}+A_k,$$

which gives us

$$\sum_{pq\le x}\frac{[\ln(pq)]^k}{pq}=\int_2^x f'(t)\ln^{k-1}tdt+f(2)\ln^{k-1}2+O\left(\ln^2(\ln x)\left[\ln^{k-1}x\right]\right)$$
$$=\frac{2(\ln x)^k[\ln(\ln x)]}{k}+\frac{2(kB-1)(\ln x)^k}{k^2}+A_k+f(2)\ln^{k-1}2+O\left(\ln^2(\ln x)\left[\ln^{k-1}x\right]\right)$$

that is,

$$\frac{1}{(\ln x)^k}\sum_{pq\le x}\frac{[\ln(pq)]^k}{pq}=\frac{2\ln(\ln x)}{k}+\frac{2(kB-1)}{k^2}+\frac{A_k+f(2)\ln^{k-1}2}{(\ln x)^k}+O\left(\frac{\ln^2(\ln x)}{\ln x}\right)$$
$$=\frac{2\ln(\ln x)}{k}+\frac{2(kB-1)}{k^2}+O\left(\frac{\ln^2(\ln x)}{\ln x}\right). \quad \square$$

Proposition 9. *The following evaluations hold:*

(i) $$\sum_{pq\le\sqrt{x}}\frac{1}{pq}\ln\left(1-\frac{\ln(pq)}{\ln x}\right)=V(\ln(\ln x))+O\left(\frac{\ln^2(\ln x)}{\ln x}\right),$$

where $V(y)=-P_1(y)\zeta(2)+a^2P_1(y)-\frac{2a^3}{3}$.

(ii) $$\sum_{pq\le\sqrt{x}}\frac{\ln\left(\ln\frac{x}{pq}\right)}{pq}=U(\ln(\ln x))+O\left(\frac{\ln^2(\ln x)}{\ln x}\right),$$

where $U(y)=yP_2(y-a)-P_1(y)\zeta(2)+a^2P_1(y)-\frac{2a^3}{3}$; $a=\ln 2$.

Proof. (i) For all natural numbers k by Proposition 8 we deduce

$$\frac{1}{(\ln x)^k}\sum_{pq\le\sqrt{x}}\frac{[\ln(pq)]^k}{pq}=\frac{2\ln(\ln\sqrt{x})}{2^k k}+\frac{2(kB-1)}{2^k k^2}+\frac{1}{2^k}O\left(\frac{\ln^2(\ln x)}{\ln x}\right).$$

Since $\ln(1-u)=-\sum_{k=1}^{\infty}\frac{u^k}{k}$, $-1<u<1$, we have

$$\sum_{pq\le\sqrt{x}}\frac{1}{pq}\ln\left(1-\frac{\ln(pq)}{\ln x}\right)=-\sum_{k=1}^{\infty}\frac{1}{k(\ln x)^k}\sum_{pq\le\sqrt{x}}\frac{(\ln(pq))^k}{pq}$$
$$=-\sum_{k=1}^{\infty}\left(\frac{2\ln(\ln\sqrt{x})}{2^k k^2}+\frac{2(kB-1)}{2^k k^3}\right)+O\left(\frac{\ln^2(\ln x)}{\ln x}\right)$$
$$=-2\ln\left(\ln\sqrt{x}\right)\sum_{k=1}^{\infty}\frac{1}{2^k k^2}-\sum_{k=1}^{\infty}\frac{2(kB-1)}{2^k k^3}+O\left(\frac{\ln^2(\ln x)}{\ln x}\right)$$
$$=-2\left(\left[\ln\left(\ln\sqrt{x}\right)\right]+B\right)Li_2\left(\frac{1}{2}\right)+2Li_3\left(\frac{1}{2}\right)+O\left(\frac{\ln^2(\ln x)}{\ln x}\right).$$

Thus

$$V(y)=-2P_1(y-\ln 2)Li_2\left(\frac{1}{2}\right)+2Li_3\left(\frac{1}{2}\right)$$
$$=-(P_1(y)-a)\left(\zeta(2)-a^2\right)+\frac{7\zeta(3)}{4}-a\zeta(2)+\frac{a^3}{3}$$
$$=-P_1(y)\zeta(2)+a^2P_1(y)-\frac{2a^3}{3}.$$

(ii) From the double Mertens evaluation in [9, Theorem 1], $\sum_{pq\le x}\frac{1}{pq}=P_2(\ln(\ln x))+O\left(\frac{\ln(\ln x)}{\ln x}\right)$ and (i) we have

$$\sum_{pq\le\sqrt{x}}\frac{1}{pq}\ln\left(\ln\frac{x}{pq}\right)=[\ln(\ln x)]\sum_{pq\le\sqrt{x}}\frac{1}{pq}+\sum_{pq\le\sqrt{x}}\frac{1}{pq}\ln\left(1-\frac{\ln(pq)}{\ln x}\right)$$
$$=[\ln(\ln x)]\,P_2(\ln(\ln x)-\ln 2)+V(\ln(\ln x))+O\left(\frac{\ln^2(\ln x)}{\ln x}\right).$$

Thus $U(y)=yP_2(y-a)-P_1(y)\zeta(2)+a^2P_1(y)-\frac{2a^3}{3}$. $\square$

Proposition 10. *The following evaluation holds*

$$\sum_{pq\le\sqrt{x}}\frac{P_1\left(\ln\left(\ln\frac{x}{pq}\right)\right)}{pq}=U_1(\ln(\ln x))+O\left(\frac{\ln^2(\ln x)}{\ln x}\right),$$

where $U_1(y)=P_1^3(y)-2\zeta(2)P_1(y)+\frac{7\zeta(3)}{4}-2aP_1^2(y)+2a^2P_1(y)-\frac{2a^3}{3}$.

Proof. From the double Mertens evaluation and Proposition 9 we have

$$\sum_{pq\le\sqrt{x}}\frac{P_1\left(\ln\left(\ln\frac{x}{pq}\right)\right)}{pq}=\sum_{pq\le\sqrt{x}}\frac{\ln\left(\ln\frac{x}{pq}\right)}{pq}+B\sum_{pq\le\sqrt{x}}\frac{1}{pq}$$
$$=U(\ln(\ln x))+O\left(\frac{\ln^2(\ln x)}{\ln x}\right)+BP_2(\ln(\ln x)-\ln 2)+O\left(\frac{\ln(\ln x)}{\ln x}\right)$$
$$=U_1(\ln(\ln x))+O\left(\frac{\ln^2(\ln x)}{\ln x}\right),$$

where

$$U_1(y)=U(y)+BP_2(y-a)=P_1(y)P_2(y-a)-P_1(y)\zeta(2)+a^2P_1(y)-\frac{2a^3}{3}$$
$$=P_1^3(y)-2aP_1^2(y)+a^2P_1(y)-\zeta(2)P_1(y)$$
$$=P_1^3(y)-2\zeta(2)P_1(y)+\frac{7\zeta(3)}{4}-2aP_1^2(y)+2a^2P_1(y)-\frac{2a^3}{3};$$

we have used that $P_2(y)=P_1^2(y)-\zeta(2)$ and

$$P_2(y-a)=(P_1(y)-a)^2-\zeta(2)=P_1^2(y)-2aP_1(y)+a^2-\zeta(2). \quad \square$$

7. A triple Mertens evaluation

The next result is a triple Mertens evaluation.

Theorem 11. *The following evaluation holds*

$$\sum_{pqr\le x}\frac{1}{pqr}=(\ln(\ln x)+B)^3-\frac{\pi^2}{2}(\ln(\ln x)+B)+2\zeta(3)+O\left(\frac{\ln^2(\ln x)}{\ln x}\right).$$

Proof. By Proposition 5 we have

$$\sum_{pqr\leq x}\frac{1}{pqr}=\sum_{p\leq\sqrt{x}}\frac{P_2\left(\ln\left(\ln\frac{x}{p}\right)\right)}{p}+\sum_{pq\leq\sqrt{x}}\frac{P_1\left(\ln\left(\ln\frac{x}{pq}\right)\right)}{pq}-P_1\left(\ln\left(\ln\sqrt{x}\right)\right)P_2\left(\ln\left(\ln\sqrt{x}\right)\right)$$
$$+O\left(\frac{\ln^2(\ln x)}{\ln x}\right).$$

Then, by Propositions 7 and 10 we deduce

$$\sum_{pqr\leq x}\frac{1}{pqr}=P_3\left(\ln\left(\ln x\right)\right)+O\left(\frac{\ln^2(\ln x)}{\ln x}\right)$$

where $P_3(y)=U_2(y)+U_1(y)-P_1(y-a)P_2(y-a)$ and $a=\ln 2$. Let us calculate the value of the polynomial P_3. We have

$$U_2(y)=P_2(y)P_1(y)-\zeta(2)P_1(y)+\frac{\zeta(3)}{4}-aP_2(y)+a^2P_1(y)-\frac{a^3}{3},$$
$$U_1(y)=P_1^3(y)-2\zeta(2)P_1(y)+\frac{7\zeta(3)}{4}-2aP_1^2(y)+2a^2P_1(y)-\frac{2a^3}{3}.$$

Also

$$\begin{aligned}P_1(y-a)P_2(y-a)&=(P_1(y)-a)\left(P_1^2(y)-2aP_1(y)+a^2-\zeta(2)\right)\\&=P_1^3(y)-\zeta(2)P_1(y)-a\left(3P_1^2(y)-\zeta(2)\right)+3a^2P_1(y)-a^3\end{aligned}$$

We deduce that

$$\begin{aligned}P_3(y)&=P_2(y)P_1(y)-2\zeta(2)P_1(y)+2\zeta(3)=P_1(y)(P_2(y)-2\zeta(2))+2\zeta(3)\\&=P_1^3(y)-3\zeta(2)P_1(y)+2\zeta(3)\end{aligned}$$

the coefficient of a is 0 since $P_2(y)=P_1^2(y)-\zeta(2)$. □

Acknowledgments

We thank the referee for his/her constructive comments and remarks which have improved the first version of the paper.

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