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Damped wave equation with a critical nonlinearity in higher space dimensions

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ABSTRACT

We study the Cauchy problem for nonlinear damped wave equations with a critical defocusing power nonlinearity $|u|^{\frac{2}{n}} u$, where n denotes the space dimension. For $n = 1, 2, 3$, global in time existence of small solutions was shown in [4]. In this paper, we generalize the results to any spatial dimension via the method of decomposition of the equation into the high and low frequency components under the assumption that the initial data are small and decay rapidly at infinity. Furthermore we present a sharp time decay estimate of solutions with a logarithmic correction.

1. Introduction

We study the large time asymptotics of solutions to the Cauchy problem for the nonlinear damped wave equation

$$\begin{cases} \partial_t^2 u + 2\partial_t u - \Delta u + |u|^{\frac{2}{n}} u = 0, \ x \in \mathbf{R}^n, \ t > 0 \\ u(0, x) = u_0(x), \ \partial_t u(0, x) = u_1(x), \ x \in \mathbf{R}^n, \end{cases} \tag{1.1}$$

in higher space dimensions $n \geq 4$.

The power of the nonlinearity $p_F = 1 + \frac{2}{n}$ is well-known Fujita critical exponent (see [1]). In the subcritical case $p < p_F$ the solution may blow up in a finite time even for small initial data (see [7,13,16,22–24]). In the supercritical case $p > p_F$ there exists global solution, which asymptotically behaves as a heat kernel (see [2,5,6,8–12,15,17–21]).

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In [4] or the book [5], the Cauchy problem (1.1) was considered in the case of $n = 1, 2, 3$. We obtained the following result. If the initial data $u_0 \in \mathbf{H}^{\delta,0} \cap \mathbf{H}^{0,\delta}$, $u_1 \in \mathbf{H}^{\delta-1,0} \cap \mathbf{H}^{-1,\delta}$ with $\delta > \frac{n}{2}$ are small and such that

$$\theta = \int_{\mathbf{R}^n} 2u_0(x) + u_1(x)\, dx > 0, \int_{\mathbf{R}^n} u_0(x)\, dx > 0,$$

then the Cauchy problem (1.1) has a unique global solution $u \in \mathbf{C}\left([0,\infty); \mathbf{H}^{\delta,0}\right)$ satisfying the following asymptotic property

$$\left\| u(t) - \theta G(t) g^{-\frac{n}{2}}(t) \right\|_{\mathbf{L}^p} \le C g^{-1-\frac{n}{2}}(t) \langle t \rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)} \log g(t)$$

for all $t > 0$, where $1 \le p \le \infty$, $g(t) = 1 + \varkappa \log \langle t \rangle$, $\varkappa = \frac{\theta^{\frac{2}{n}}}{n\pi}\left(\frac{n}{n+2}\right)^{\frac{n}{2}}$, $G(t,x) = (2\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2t}}$ is the heat kernel. Note that the nonlinearity $|u|^{\frac{2}{n}} u$ in the Cauchy problem (1.1) has not sufficient regularity, so we can not apply the methods of [4] or [5] to the higher space dimensions $n \ge 4$. In the present paper we apply a different approach based on the direct decomposition of equation (1.1) into the high and low frequency parts. It is known from the previous works, that in the high frequency part the solution has an exponential time decay, so that the solution is a remainder in this part. In the low frequency part we decompose the nonlinear damped wave equation into a system of two equations with the first order time derivative. One of these equations has exponential time decay. Another one is responsible for the large time asymptotics of solutions which is similar to that of the nonlinear heat equation. Our method in this paper works well for any dimension and it makes a proof much simpler than the previous one.

To state our result precisely we introduce some notations. The usual Lebesgue space is denoted by $\mathbf{L}^p$, $1 \le p \le \infty$. Define by

$$\mathbf{H}^{l,m} = \left\{ \phi \in \mathbf{L}^2; \left\| \langle x \rangle^m \langle i\nabla \rangle^l \phi(x) \right\|_{\mathbf{L}^2} < \infty \right\}$$

the weighted Sobolev space, where $\langle x \rangle = \sqrt{1 + |x|^2}$, $\langle i\nabla \rangle = \sqrt{1 - \Delta}$. We also use the notation $\mathbf{H}^l = \mathbf{H}^{l,0}$. We denote by $\mathcal{F}$ the Fourier transformation

$$\widehat{u}(\xi) \equiv \mathcal{F}u = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{-i\xi\cdot x} u(x)\, dx$$

and $\mathcal{F}^{-1}$ is the inverse Fourier transformation

$$\mathcal{F}^{-1}u = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{i\xi\cdot x} u(\xi)\, d\xi.$$

By $\mathbf{C}(\mathbf{I}; \mathbf{B})$ we denote the space of continuous functions from a time interval $\mathbf{I}$ to the Banach space $\mathbf{B}$. In what follows we denote by C different positive constants.

Our main result is the following.

Theorem 1.1. *Let the initial data* $u_0 \in \mathbf{H}^2 \cap \mathbf{H}^{1,\delta}$, $u_1 \in \mathbf{H}^1 \cap \mathbf{H}^{0,\delta}$ *with* $\delta > \frac{n}{2}$, $n \ge 4$ *and*

$$\theta = \int_{\mathbf{R}^n} 2u_0(x) + u_1(x)\, dx > 0.$$

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Then there exists a positive ε depending on

$$\varepsilon_0 = \|u_0\|_{\mathbf{H}^2} + \|u_0\|_{\mathbf{H}^{1,\delta}} + \|u_1\|_{\mathbf{H}^1} + \|u_1\|_{\mathbf{H}^{0,\delta}}$$

and $\theta > 0$ such that the Cauchy problem (1.1) has a unique global solution $u \in \mathbf{C}\left([0,\infty);\mathbf{H}^2 \cap \mathbf{H}^{1,\delta}\right)$ for any data satisfying $\varepsilon^{1+\frac{1}{n^2}} \leq \theta \leq \varepsilon_0 \leq \varepsilon$. Moreover the following asymptotic property

$$\left\|u(t) - \theta G(t)\, g^{-\frac{n}{2}}(t)\right\|_{\mathbf{L}^p} \leq C g^{-1-\frac{n}{2}}(t)\,\langle t\rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)} \log g(t)$$

holds for all $t > 0$, where $2 \leq p < \infty$ for $n = 4$ and $2 \leq p \leq \frac{2n}{n-4}$ for $n \geq 5$, and $g(t)$, $G(t)$ are the same as defined above.

Remark 1.1. We can also replace the nonlinear term $|u|^{\frac{2}{n}} u$ by $|u|^{1+\frac{2}{n}}$. The restriction on p such that $2 \leq p < \infty$ for $n = 4$ and $2 \leq p \leq \frac{2n}{n-4}$ for $n \geq 5$ comes from the $\mathbf{H}^2$ – regularity of solutions since by the Sobolev embedding theorem $\mathbf{H}^2 \subset \mathbf{L}^p$. It seems that higher regularity for solutions can not be obtained due to the lack of regularity of the nonlinearity.

The rest of the paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.1. In Section 3 we show some estimates of the Green operator solving the linearized Cauchy problem corresponding to (1.1).

2. Proof of Theorem 1.1

The local existence of solutions for the Cauchy problem (1.1) can be obtained by standard methods (see, e.g. [14]).

Theorem 2.1. *Let the initial data u_0, u_1 be such that $u_0 \in \mathbf{H}^2 \cap \mathbf{H}^{1,\delta}$, $u_1 \in \mathbf{H}^1 \cap \mathbf{H}^{0,\delta}$, where $\delta > \frac{n}{2}$, $n \geq 4$. Then there exists a unique solution $u \in \mathbf{C}\left([0,T];\mathbf{H}^2 \cap \mathbf{H}^{1,\delta}\right)$ of the Cauchy problem (1.1), where $T = O\left(\varepsilon_0^{-\frac{2}{n}}\right)$. Moreover if $\varepsilon_0 \leq \varepsilon$ and ε is small, then there exists a time $T > 1$ such that the solution u of the Cauchy problem (1.1) satisfy the estimates $\|u\|_{\mathbf{H}^2} + \|u\|_{\mathbf{H}^{1,\delta}} + \|u_t\|_{\mathbf{H}^1} + \|u_t\|_{\mathbf{H}^{0,\delta}} \leq C\varepsilon$ for all $t \in [0,T]$.*

We define the projectors on the high and low frequency parts

$$\mathcal{P}_h = \mathcal{F}^{-1}\left(1 - \chi(\xi)\right)\mathcal{F}$$

and

$$\mathcal{P}_l = \mathcal{F}^{-1}\chi(\xi)\,\mathcal{F}$$

respectively, where $\chi \in \mathbf{C}_0^\infty(\mathbf{R}^n)$ is such that $\chi(\xi) = 1$ for $|\xi| \leq \frac{1}{3}$ and $\chi(\xi) = 0$ for $|\xi| \geq \frac{2}{3}$. We have $u = \mathcal{P}_l u + \mathcal{P}_h u$, and by equation (1.1) we find

$$\begin{cases} \left(\partial_t^2 + 2\partial_t - \Delta\right)\mathcal{P}_l u + \mathcal{P}_l\left(|u|^{\frac{2}{n}} u\right) = 0,\ x \in \mathbf{R}^n,\ t > 0 \\ \mathcal{P}_l u(0,x) = \mathcal{P}_l u_0(x),\ \partial_t \mathcal{P}_l u(0,x) = \mathcal{P}_l u_1(x),\ x \in \mathbf{R}^n \end{cases} \tag{2.1}$$

and

$$\begin{cases} \left(\partial_t^2 + 2\partial_t - \Delta\right)\mathcal{P}_h u + \mathcal{P}_h\left(|u|^{\frac{2}{n}} u\right) = 0,\ x \in \mathbf{R}^n,\ t > 0 \\ \mathcal{P}_h u(0,x) = \mathcal{P}_h u_0(x),\ \partial_t \mathcal{P}_h u(0,x) = \mathcal{P}_h u_1(x),\ x \in \mathbf{R}^n. \end{cases} \tag{2.2}$$

In the low frequency part we apply the following factorization

$$\partial_t^2 + 2\partial_t - \Delta = \mathcal{L}_1\mathcal{L}_2 = \mathcal{L}_2\mathcal{L}_1,$$

where the operators $\mathcal{L}_1 = \partial_t + 1 - \sqrt{1+\Delta}$, and $\mathcal{L}_2 = \partial_t + 1 + \sqrt{1+\Delta}$. Then we change the dependent variables

$$\mathcal{L}_2\mathcal{P}_l u = v_1, \ \mathcal{L}_1\mathcal{P}_l u = v_2,$$

and applying equation $\left(\partial_t^2 + 2\partial_t - \Delta\right)\mathcal{P}_l u = \mathcal{L}_1\mathcal{L}_2\mathcal{P}_l u = \mathcal{L}_2\mathcal{L}_1\mathcal{P}_l u = -\mathcal{P}_l\left(|u|^{\frac{2}{n}} u\right)$, we get

$$\mathcal{L}_1 v_1 = -\mathcal{P}_l\left(|u|^{\frac{2}{n}} u\right)$$

and

$$\mathcal{L}_2 v_2 = -\mathcal{P}_l\left(|u|^{\frac{2}{n}} u\right).$$

Also by equations $\mathcal{L}_2\mathcal{P}_l u = v_1$ and $\mathcal{L}_1\mathcal{P}_l u = v_2$, we get $2\sqrt{1+\Delta}\mathcal{P}_l u = v_1 - v_2$. Hence

$$\mathcal{P}_l u = \mathcal{Q}\left(v_1 - v_2\right),$$

where $\mathcal{Q} = \frac{1}{2}\left(1+\Delta\right)^{-\frac{1}{2}}$ is defined well in the low frequency part. Thus we get a system of equations

$$\begin{cases} \mathcal{L}_1 v_1 = -\mathcal{P}_l\left(|u|^{\frac{2}{n}} u\right), \ \mathcal{L}_2 v_2 = -\mathcal{P}_l\left(|u|^{\frac{2}{n}} u\right), \\ \left(\partial_t^2 + 2\partial_t - \Delta\right) v_3 = -\mathcal{P}_h\left(|u|^{\frac{2}{n}} u\right), \\ v_1(0) = v_{10}, \ v_2(0) = v_{20}, \\ v_3(0) = v_{30}, \ \partial_t v_3(0) = v_{31}, \end{cases} \tag{2.3}$$

with $u = \mathcal{Q}\left(v_1 - v_2\right) + v_3$, $v_3 = \mathcal{P}_h u$, where

$$v_{10} = \left(1 + \sqrt{1+\Delta}\right)\mathcal{P}_l u_0 + \mathcal{P}_l u_1,$$
$$v_{20} = \left(1 - \sqrt{1+\Delta}\right)\mathcal{P}_l u_0 + \mathcal{P}_l u_1,$$
$$v_{30} = \mathcal{P}_h u_0, \ v_{31} = \mathcal{P}_h u_1.$$

Next we follow the method of paper [3] and make a change of the dependent variable $v_1(t,x) = e^{-\varphi(t)} w_1(t,x)$ to get for the new unknown function w_1

$$\mathcal{L}_1 w_1 = -e^{-\frac{2}{n}\varphi}\mathcal{P}_l\left(|e^{\varphi} u|^{\frac{2}{n}} e^{\varphi} u\right) + \varphi' w_1.$$

We now choose the auxiliary function $\varphi(t)$ by the following condition

$$\int_{\mathbf{R}^n}\left(-e^{-\frac{2}{n}\varphi}\mathcal{P}_l\left(|e^{\varphi} u|^{\frac{2}{n}} e^{\varphi} u\right) + \varphi' w_1\right) dx = 0.$$

Thus from the first equation of system (2.3) we obtain a conservation law

$$\frac{d}{dt}\int_{\mathbf{R}^n} w_1(t,x)\, dx = 0.$$

Hence

$$\int_{\mathbf{R}^n} w_1(t,x)\,dx = \int_{\mathbf{R}^n} w_1(0,x)\,dx$$

for all $t>0$. Also we choose $\varphi(0)=0$ so that

$$\begin{aligned}\int_{\mathbf{R}^n} w_1(t,x)\,dx &= \int_{\mathbf{R}^n} v_1(0,x)\,dx = \int_{\mathbf{R}^n} v_{10}(x)\,dx\\ &= \int_{\mathbf{R}^n} \left(\left(1+\sqrt{1+\Delta}\right)\mathcal{P}_l u_0 + \mathcal{P}_l u_1\right)dx\\ &= \int_{\mathbf{R}^n} \left(2u_0(x)+u_1(x)\right)dx = \theta > 0\end{aligned}$$

since $\int_{\mathbf{R}^n} \mathcal{P}_l f dx = \widehat{\mathcal{P}_l f}(0) = \widehat{f}(0) = \int_{\mathbf{R}^n} f dx$ and

$$\varphi' = \frac{1}{\theta} e^{-\frac{2}{n}\varphi} \int_{\mathbf{R}^n} \mathcal{P}_l\left(\left|e^{\varphi}u\right|^{\frac{2}{n}} e^{\varphi}u\right)dx = \frac{1}{\theta} e^{-\frac{2}{n}\varphi} \int_{\mathbf{R}^n} \left|e^{\varphi}u\right|^{\frac{2}{n}} e^{\varphi}u dx,$$

where $e^{\varphi}u = \mathcal{Q}w_1 - e^{\varphi}\mathcal{Q}v_2 + e^{\varphi}v_3$. We denote $h(t) = e^{\frac{2}{n}\varphi}$, then we get

$$h'(t) = \frac{2}{\theta n} \int_{\mathbf{R}^n} \left|e^{\varphi}u\right|^{\frac{2}{n}} e^{\varphi}u dx,\ \ h(0)=1.$$

Integration with respect to time yields

$$h(t) = 1 + \frac{2}{\theta n} \int_0^t \int_{\mathbf{R}^n} \left|e^{\varphi}u\right|^{\frac{2}{n}} e^{\varphi}u dx d\tau.$$

Define the Green operator

$$\mathcal{G}_1(t) = \mathcal{F}^{-1} L_1(t,\xi)\,\mathcal{F},$$

where

$$L_1(t,\xi) = e^{-t\left(1-\sqrt{1-|\xi|^2}\right)} \chi_1(\xi)$$

and $\chi_1 \in \mathbf{C}_0^{\infty}(\mathbf{R}^n)$ is such that $\chi_1(\xi)=1$ for $|\xi| \le \frac{2}{3}$ and $\chi_1(\xi)=0$ for $|\xi| \ge \frac{5}{6}$. Thus the operator $\mathcal{G}_1(t)$ is defined for the low frequency part. Also the Green operator

$$\mathcal{G}_2(t) = \mathcal{F}^{-1} e^{-t\left(1+\sqrt{1-|\xi|^2}\right)} \chi_1(\xi)\,\mathcal{F}$$

is defined for the functions in the low frequency part. Finally the Green operator

$$\mathcal{G}_3(t) = \mathcal{F}^{-1} L_3(t,\xi)\,\chi_2(\xi)\,\mathcal{F}$$

is defined for the functions in the high frequency part, where

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$$L_3(t,\xi) = e^{-t}\frac{\sin\left(t\sqrt{|\xi|^2-1}\right)}{\sqrt{|\xi|^2-1}},$$

and $\chi_2 \in \mathbf{C}^\infty(\mathbf{R}^n)$ is such that $\chi_2(\xi) = 1$ for $|\xi| \geq \frac{1}{3}$ and $\chi_2(\xi) = 0$ for $|\xi| \leq \frac{1}{6}$. Also denote

$$\mathcal{N}_1(u,h) = \mathcal{P}_l\left(|e^\varphi u|^{\frac{2}{n}} e^\varphi u\right) - \frac{1}{\theta} w_1 \int_{\mathbf{R}^n} |e^\varphi u|^{\frac{2}{n}} e^\varphi u dx,$$
$$\mathcal{N}_2(u) = \mathcal{P}_l\left(|u|^{\frac{2}{n}} u\right),\ \mathcal{N}_3(u) = \mathcal{P}_h\left(|u|^{\frac{2}{n}} u\right),$$
$$\mathcal{N}_4(u,h) = \frac{2}{\theta n}\int_{\mathbf{R}^n} |e^\varphi u|^{\frac{2}{n}} e^\varphi u dx.$$

Now the integral equations associated with system (2.3) can be written as

$$\begin{cases} w_1(t) = \mathcal{G}_1(t) v_{10} - \int_0^t \mathcal{G}_1(t-\tau)\mathcal{N}_1(u,h)\frac{d\tau}{h(\tau)}, \\ v_2(t) = \mathcal{G}_2(t) v_{20} - \int_0^t \mathcal{G}_2(t-\tau)\mathcal{N}_2(u)\, d\tau, \\ v_3(t) = (\partial_t + 1)\mathcal{G}_3(t) v_{30} + \mathcal{G}_3(t) v_{31} - \int_0^t \mathcal{G}_3(t-\tau)\mathcal{N}_3(u)\, d\tau, \\ h(t) = 1 + \int_0^t \mathcal{N}_4(u,h)\, d\tau, \end{cases}$$

where $e^\varphi u = \mathcal{Q}w_1 - h^{\frac{n}{2}}\mathcal{Q}v_2 + h^{\frac{n}{2}} v_3$.

Let us prove the following estimates

$$\|w_1\|_{\mathbf{X}_T} + \left\|\langle t\rangle^{\frac{1}{2}} v_2\right\|_{\mathbf{X}_T} + \left\|\langle t\rangle^{\frac{1}{2}} v_3\right\|_{\mathbf{X}_T} < C\varepsilon,$$
$$\|g(w_1 - \mathcal{G}_1(t) v_{10})\|_{\mathbf{X}_T} < C\varepsilon^{1+\frac{2}{n}-\frac{1}{n^2}},\ \frac{1}{3}g(t) < h(t) < \frac{4}{3}g(t)$$

for all $t \in [0,T]$, where we define the norms

$$\|\phi\|_{\mathbf{X}_T} = \sup_{t\in[0,T]} \langle t\rangle^{\frac{n}{4}} \left\|\left\langle \langle t\rangle^{\frac{1}{2}} i\nabla\right\rangle^2 \phi(t)\right\|_{\mathbf{L}^2} + \sup_{t\in[0,T]} \langle t\rangle^{\frac{n}{4}-\frac{\delta}{2}} \left\|\langle\cdot\rangle^\delta \left\langle \langle t\rangle^{\frac{1}{2}} i\nabla\right\rangle \phi(t)\right\|_{\mathbf{L}^2}$$

and

$$\|\phi\|_{\mathbf{Y}_T} = \sup_{t\in[0,T]} \langle t\rangle^{1+\frac{n}{4}} \left\|\left\langle \langle t\rangle^{\frac{1}{2}} \nabla\right\rangle \phi(t)\right\|_{\mathbf{L}^2} + \sup_{t\in[0,T]} \langle t\rangle^{1+\frac{n}{4}-\frac{\delta}{2}} \left\|\langle\cdot\rangle^\delta \phi(t)\right\|_{\mathbf{L}^2},$$

here $\delta > \frac{n}{2}$. We argue by the contradiction. Since for $T = 0$ the estimates are true

$$\|w_1\|_{\mathbf{X}_0} + \left\|\langle t\rangle^{\frac{1}{2}} v_2\right\|_{\mathbf{X}_0} + \left\|\langle t\rangle^{\frac{1}{2}} v_3\right\|_{\mathbf{X}_0}$$
$$\leq \|v_{10}\|_{\mathbf{H}^2} + \|v_{10}\|_{\mathbf{H}^{1,\delta}} + \|v_{20}\|_{\mathbf{H}^2} + \|v_{20}\|_{\mathbf{H}^{1,\delta}} + \|v_{30}\|_{\mathbf{H}^2} + \|v_{30}\|_{\mathbf{H}^{1,\delta}} \leq C\varepsilon,$$

in view of the continuity of the norm $\mathbf{X}_T$ with respect to time T, we can find the first time $T_1 \in [0,T]$ such that

$$\|w_1\|_{\mathbf{X}_{T_1}} + \left\|\langle t\rangle^{\frac{1}{2}} v_2\right\|_{\mathbf{X}_{T_1}} + \left\|\langle t\rangle^{\frac{1}{2}} v_3\right\|_{\mathbf{X}_{T_1}} \leq C\varepsilon,$$
$$\|g(w_1 - \mathcal{G}_1(t) v_{10})\|_{\mathbf{X}_{T_1}} \leq C\varepsilon^{1+\frac{2}{n}-\frac{1}{n^2}},\ \frac{1}{3}g(t) \leq h(t) \leq Cg(t)$$

for all $t \in [0,T_1]$. Below we write $T = T_1$ for simplicity.

By Lemma 3.5 we get

$$\begin{aligned}\left\|\mathcal{N}_1(u,h)\right\|_{\mathbf{Y}_T} &= \left\|\mathcal{P}_l\left(\left|e^{\varphi}u\right|^{\frac{2}{n}}e^{\varphi}u\right)-\frac{1}{\theta}w_1\int_{\mathbf{R}^n}\left|e^{\varphi}u\right|^{\frac{2}{n}}e^{\varphi}udx\right\|_{\mathbf{Y}_T}\\ &\le C\left\|\mathcal{P}_l\left(\left|e^{\varphi}u\right|^{\frac{2}{n}}e^{\varphi}u\right)\right\|_{\mathbf{Y}_T}+\frac{1}{\theta}C\left\|w_1\right\|_{\mathbf{X}_T}\left|\langle t\rangle\int_{\mathbf{R}^n}\left|e^{\varphi}u\right|^{\frac{2}{n}}e^{\varphi}udx\right|\\ &\le C\left\|e^{\varphi}u\right\|_{\mathbf{X}_T}^{1+\frac{2}{n}}+\frac{C}{\theta}\left\|w_1\right\|_{\mathbf{X}_T}\left\|e^{\varphi}u\right\|_{\mathbf{X}_T}^{1+\frac{2}{n}}\\ &\le C\varepsilon^{1+\frac{2}{n}}+C\theta^{-1}\varepsilon^{2+\frac{2}{n}}\le C\varepsilon^{1+\frac{2}{n}-\frac{1}{n^2}},\end{aligned}$$

$$\begin{aligned}\left\|\mathcal{N}_2(u)\right\|_{\mathbf{Y}_T}+\left\|\mathcal{N}_3(u)\right\|_{\mathbf{Y}_T} &= \left\|\mathcal{P}_l\left(|u|^{\frac{2}{n}}u\right)\right\|_{\mathbf{Y}_T}+\left\|\mathcal{P}_h\left(|u|^{\frac{2}{n}}u\right)\right\|_{\mathbf{Y}_T}\\ &\le\left\||u|^{\frac{2}{n}}u\right\|_{\mathbf{Y}_T}\le C\left\|u\right\|_{\mathbf{X}_T}^{1+\frac{2}{n}}\le C\varepsilon^{1+\frac{2}{n}},\end{aligned}$$

and

$$\begin{aligned}\left|\mathcal{N}_4(u,h)\right| &= \left|\frac{2}{\theta n}\int_{\mathbf{R}^n}\left|e^{\varphi}u\right|^{\frac{2}{n}}e^{\varphi}udx\right|\le\frac{2}{\theta n}\left\|\left|e^{\varphi}u\right|^{\frac{2}{n}}e^{\varphi}u\right\|_{\mathbf{L}^1}\\ &\le C\theta^{-1}\langle t\rangle^{-1}\left\|e^{\varphi}u\right\|_{\mathbf{X}_T}^{1+\frac{2}{n}}\le C\theta^{-1}\langle t\rangle^{-1}\varepsilon^{1+\frac{2}{n}}\le C\langle t\rangle^{-1}\varepsilon^{\frac{2}{n}-\frac{1}{n^2}},\end{aligned}$$

since

$$\left\|e^{\varphi}u\right\|_{\mathbf{X}_T}\le\left\|\mathcal{Q}w_1\right\|_{\mathbf{X}_T}+\left\|h^{\frac{n}{2}}\mathcal{Q}v_2\right\|_{\mathbf{X}_T}+\left\|h^{\frac{n}{2}}v_3\right\|_{\mathbf{X}_T}\le C\varepsilon.$$

Next by Lemma 3.1 and Lemma 3.3 we find

$$\begin{aligned}\left\|g\left(w_1-\mathcal{G}_1(t)v_{10}\right)\right\|_{\mathbf{X}_T} &= \left\|g(t)\int_0^t\mathcal{G}_1(t-\tau)\mathcal{N}_1(u,h)\frac{d\tau}{h(\tau)}\right\|_{\mathbf{X}_T}\\ &\le C\left\|\frac{g}{h}\mathcal{N}_1(u,h)\right\|_{\mathbf{Y}_T}\le C\left\|e^{\varphi}u\right\|_{\mathbf{X}_T}^{1+\frac{2}{n}}+\frac{1}{\theta}C\left\|e^{\varphi}u\right\|_{\mathbf{X}_T}^{2+\frac{2}{n}}\\ &\le C\varepsilon^{1+\frac{2}{n}-\frac{1}{n^2}}.\end{aligned}$$

By the definition of the kernel $G_1(t,x)=\mathcal{F}^{-1}\left(e^{-\frac{1}{2}\langle t\rangle|\xi|^2}\right)=(2\pi\langle t\rangle)^{-\frac{n}{2}}e^{-\frac{1}{2\langle t\rangle}|x|^2}$ we find

$$\left\|\widehat{v_{10}}(0)G_1(t)\right\|_{\mathbf{X}_T}\le C\left\|v_{10}\right\|_{\mathbf{H}^{0,\delta}}\left\|\langle t\rangle^{-\frac{n}{2}}e^{-\frac{1}{2\langle t\rangle}|x|^2}\right\|_{\mathbf{X}_T}\le C\left\|v_{10}\right\|_{\mathbf{H}^{0,\delta}}.$$

Then by Lemma 3.3

$$\begin{aligned}\left\|\mathcal{G}_1(t)v_{10}\right\|_{\mathbf{X}_T} &\le\langle t\rangle^{-\frac{\gamma}{2}}\left\|\langle t\rangle^{\frac{\gamma}{2}}\left(\mathcal{G}_1(t)v_{10}-\widehat{v_{10}}(0)G(t)\right)\right\|_{\mathbf{X}_T}\\ &\quad+\left\|\widehat{v_{10}}(0)G_1(t)\right\|_{\mathbf{X}_T}\le C\left\|v_{10}\right\|_{\mathbf{H}^{0,\delta}}.\end{aligned}$$

Hence

$$\begin{aligned}\|w_1\|_{\mathbf{X}_T} &\leq \|\mathcal{G}_1(t)\, v_{10}\|_{\mathbf{X}_T} + \|g(w_1 - \mathcal{G}_1(t)\, v_{10})\|_{\mathbf{X}_T} \\ &\leq C\|v_{10}\|_{\mathbf{H}^{0,\delta}} + C\varepsilon^{1+\frac{2}{n}-\frac{1}{n^2}} < C\varepsilon.\end{aligned}$$

Next by Lemma 3.2 and Lemma 3.4 we obtain

$$\begin{aligned}\left\|\langle t\rangle^{\frac{1}{2}} v_2\right\|_{\mathbf{X}_T} &\leq \left\|\langle t\rangle^{\frac{1}{2}} \mathcal{G}_2(t)\, v_{20}\right\|_{\mathbf{X}_T} + \left\|\langle t\rangle^{\frac{1}{2}} (v_2 - \mathcal{G}_2(t)\, v_{20})\right\|_{\mathbf{X}_T} \\ &\leq C\|v_{20}\|_{\mathbf{H}^1} + C\|v_{20}\|_{\mathbf{H}^{0,\delta}} + \left\|\langle t\rangle^{\frac{1}{2}} \int_0^t \mathcal{G}_2(t-\tau)\, \mathcal{N}_2(u)\, d\tau\right\|_{\mathbf{X}_T} \\ &\leq C\|v_{20}\|_{\mathbf{H}^1} + C\|v_{20}\|_{\mathbf{H}^{0,\delta}} + C\|\mathcal{N}_2(u)\|_{\mathbf{Y}_T} \\ &\leq C\varepsilon + C\varepsilon^{1+\frac{2}{n}} < C\varepsilon.\end{aligned}$$

In the same manner

$$\begin{aligned}\left\|\langle t\rangle^{\frac{1}{2}} v_3\right\|_{\mathbf{X}_T} &\leq \left\|\langle t\rangle^{\frac{1}{2}} (\partial_t + 1)\, \mathcal{G}_3(t)\, v_{30}\right\|_{\mathbf{X}_T} + \left\|\langle t\rangle^{\frac{1}{2}} \mathcal{G}_3(t)\, v_{31}\right\|_{\mathbf{X}_T} + \left\|\langle t\rangle^{\frac{1}{2}} \int_0^t \mathcal{G}_3(t-\tau)\, \mathcal{N}_3(u)\, d\tau\right\|_{\mathbf{X}_T} \\ &\leq C\|v_{30}\|_{\mathbf{H}^2} + C\|v_{30}\|_{\mathbf{H}^{1,\delta}} + C\|v_{31}\|_{\mathbf{H}^1} + C\|v_{31}\|_{\mathbf{H}^{0,\delta}} + C\|\mathcal{N}_3(u)\|_{\mathbf{Y}_T} \\ &\leq C\varepsilon + C\varepsilon^{1+\frac{2}{n}} < C\varepsilon.\end{aligned}$$

We now consider the estimate for h. We substitute $e^{\varphi}u = \mathcal{Q}w_1 - h^{\frac{n}{2}}\mathcal{Q}v_2 + h^{\frac{n}{2}}v_3$ into

$$h = 1 + \int_0^t \frac{2}{\theta n} \int_{\mathbf{R}^n} |e^{\varphi}u|^{\frac{2}{n}} e^{\varphi}u\, dx\, d\tau.$$

By Lemma 3.3 we have

$$w_1 = \mathcal{G}_1(t)\, v_{10} + r_1 = \widehat{v_{10}}(0)\, G_1(t) + r_2 + r_1,$$

where

$$\|gr_1\|_{\mathbf{X}_T} \leq C\varepsilon^{1+\frac{2}{n}-\frac{1}{n^2}}, \left\|\langle t\rangle^{\frac{\gamma}{2}} r_2\right\|_{\mathbf{X}_T} \leq C\|v_{10}\|_{\mathbf{H}^{0,\delta}}.$$

Also

$$\left\|\langle t\rangle^{\frac{1}{2}} v_2\right\|_{\mathbf{X}_{T_1}} + \left\|\langle t\rangle^{\frac{1}{2}} v_3\right\|_{\mathbf{X}_{T_1}} \leq C\varepsilon.$$

Hence

$$\begin{aligned}e^{\varphi}u &= \mathcal{Q}w_1 - h^{\frac{n}{2}}\mathcal{Q}v_2 + h^{\frac{n}{2}}v_3 \\ &= \widehat{v_{10}}(0)\, G_1(t) + \mathcal{Q}r_2 + \mathcal{Q}r_1 - h^{\frac{n}{2}}\mathcal{Q}v_2 + h^{\frac{n}{2}}v_3 \\ &= \widehat{v_{10}}(0)\, G_1(t) + R,\end{aligned}$$

where

$$\|gR\|_{\mathbf{X}_T} \le C\varepsilon^{1+\frac{2}{n}-\frac{1}{n^2}}.$$

Then we get

$$\int_{\mathbf{R}^n} |e^{\varphi}u|^{\frac{2}{n}} e^{\varphi}udx = \theta^{1+\frac{2}{n}} \int_{\mathbf{R}^n} (G_1(t))^{1+\frac{2}{n}} dx + R_1,$$

where

$$\begin{aligned} |R_1| &= \left| \int_{\mathbf{R}^n} |e^{\varphi}u|^{\frac{2}{n}} e^{\varphi}udx - (\widehat{v_{10}}(0))^{1+\frac{2}{n}} \int_{\mathbf{R}^n} (G_1(t))^{1+\frac{2}{n}} dx \right| \\ &\le \int_{\mathbf{R}^n} \left(|e^{\varphi}u|^{\frac{2}{n}} + |\widehat{v_{10}}(0) G_1(t)|^{\frac{2}{n}} \right) |R| \, dx. \end{aligned}$$

In the same way as in the proof of Lemma 3.4 we obtain the estimate

$$\left\| |\psi|^{\frac{2}{n}} \phi \right\|_{\mathbf{L}^1} \le C \langle t \rangle^{-1} \|\psi\|_{\mathbf{X}_T}^{\frac{2}{n}} \|\phi\|_{\mathbf{X}_T}.$$

Hence

$$\begin{aligned} |R_1| &\le \int_{\mathbf{R}^n} \left(|e^{\varphi}u|^{\frac{2}{n}} + |\widehat{v_{10}}(0) G_1(t)|^{\frac{2}{n}} \right) |R| \, dx \\ &\le C \langle t \rangle^{-1} g^{-1} \left(\|e^{\varphi}u\|_{\mathbf{X}_T}^{\frac{2}{n}} + \|\widehat{v_{10}}(0) G_1\|_{\mathbf{X}_T}^{\frac{2}{n}} \right) \|gR\|_{\mathbf{X}_T} \\ &\le C\varepsilon^{1+\frac{4}{n}-\frac{1}{n^2}} \langle t \rangle^{-1} g^{-1}. \end{aligned}$$

Therefore

$$\int_{\mathbf{R}^n} |e^{\varphi}u|^{\frac{2}{n}} e^{\varphi}udx = \theta^{1+\frac{2}{n}} \int_{\mathbf{R}^n} (G_1(t))^{1+\frac{2}{n}} dx + O\left(\varepsilon^{1+\frac{4}{n}-\frac{1}{n^2}} \langle t \rangle^{-1} g^{-1} \right)$$

with $\theta = \widehat{v_{10}}(0)$. By the definition of $G_1(t,x) = (2\pi \langle t \rangle)^{-\frac{n}{2}} e^{-\frac{1}{2\langle t \rangle}|x|^2}$ we find

$$\int_{\mathbf{R}^n} (G_1(t))^{1+\frac{2}{n}} dx = (2\pi)^{-1} \left(1 + \frac{2}{n}\right)^{-\frac{n}{2}} \langle t \rangle^{-1}.$$

Hence

$$\int_{\mathbf{R}^n} |e^{\varphi}u|^{\frac{2}{n}} e^{\varphi}udx = \theta^{1+\frac{2}{n}} (2\pi)^{-1} \left(1 + \frac{2}{n}\right)^{-\frac{n}{2}} \langle t \rangle^{-1} + O\left(\varepsilon^{1+\frac{4}{n}-\frac{1}{n^2}} \langle t \rangle^{-1} g^{-1} \right).$$

Then

$$h = 1 + \int_0^t \frac{2}{\theta n} \int_{\mathbf{R}^n} |e^{\varphi}u|^{\frac{2}{n}} e^{\varphi}udxd\tau$$

$$
\begin{aligned}
&= 1 + \frac{\theta^{\frac{2}{n}}}{\pi n}\left(1+\frac{2}{n}\right)^{-\frac{n}{2}} \int_0^t \langle \tau \rangle^{-1} d\tau + O\left(\theta^{-1}\varepsilon^{1+\frac{4}{n}-\frac{1}{n^2}} \int_0^t \langle \tau \rangle^{-1} g^{-1}(\tau)\, d\tau\right) \\
&= 1 + \frac{\theta^{\frac{2}{n}}}{\pi n}\left(1+\frac{2}{n}\right)^{-\frac{n}{2}} \log\langle t\rangle + O\left(\varepsilon^{1+\frac{4}{n}-\frac{1}{n^2}}\theta^{-1-\frac{2}{n}} \log g(t)\right) \\
&= g(t) + O\left(\varepsilon^{1+\frac{4}{n}-\frac{1}{n^2}-\left(1+\frac{2}{n}\right)\left(1+\frac{1}{n^2}\right)} \log g(t)\right) \\
&= g(t) + O\left(\varepsilon^{\frac{2}{n}\left(1-\frac{1}{n}-\frac{1}{n^2}\right)} \log g(t)\right)
\end{aligned}
$$

where $g(t) = 1+\varkappa \log\langle t\rangle$, $\varkappa = \frac{\theta^{\frac{2}{n}}}{n\pi}\left(\frac{n}{n+2}\right)^{\frac{n}{2}}$. Thus estimates are true for $T = T_1$. We obtained a contradiction, hence the estimates are true until the existence time T. By a standard continuation argument we find existence of a global solution $u \in \mathbf{C}\left([0,\infty); \mathbf{H}^2 \cap \mathbf{H}^{0,\delta}\right)$ which satisfies the estimates

$$
\begin{aligned}
&\|w_1\|_{\mathbf{X}_\infty} + \left\|\langle t\rangle^{\frac{1}{2}} v_2\right\|_{\mathbf{X}_\infty} + \left\|\langle t\rangle^{\frac{1}{2}} v_3\right\|_{\mathbf{X}_\infty} < C\varepsilon, \\
&\|g(w_1 - \mathcal{G}_1(t)\, v_{10})\|_{\mathbf{X}_\infty} < C\varepsilon^{1+\frac{2}{n}-\frac{1}{n^2}}, \\
&\frac{1}{3}g(t) < h(t) < \frac{4}{3}g(t).
\end{aligned}
$$

We now consider the asymptotics of the solution. We have

$$
u = e^{-\varphi}\theta G_1(t) + e^{-\varphi}R,
$$

where $\|gR\|_{\mathbf{X}_\infty} \le C\varepsilon^{1+\frac{2}{n}-\frac{1}{n^2}}$. Since

$$
e^{-\varphi} = h^{-\frac{n}{2}}(t) = g^{-\frac{n}{2}}(t) + O\left(\varepsilon^{\frac{2}{n}\left(1-\frac{1}{n}-\frac{1}{n^2}\right)} g^{-\frac{n}{2}-1}(t) \log g(t)\right),
$$

then

$$
\left\|u - g^{-\frac{n}{2}}(t)\,\theta G_1(t)\right\|_{\mathbf{X}_\infty} \le C\varepsilon^{\frac{2}{n}\left(1-\frac{1}{n}-\frac{1}{n^2}\right)} g^{-\frac{n}{2}-1}(t) \log g(t).
$$

By the Sobolev embedding theorem

$$
\|\phi\|_{\mathbf{L}^p} \le C \left\||\nabla|^{\alpha}\phi\right\|_{\mathbf{L}^2} \le C\|\phi\|_{\mathbf{L}^2}^{1-\frac{\alpha}{2}} \|\Delta\phi\|_{\mathbf{L}^2}^{\frac{\alpha}{2}},
$$

with $\alpha = \frac{n}{2} - \frac{n}{p}$, $2 \le p < \infty$ for $n = 4$ and $2 \le p \le \frac{2n}{n-4}$ for $n \ge 5$. Therefore we get

$$
\left\|u - \theta G_1(t)\, g^{-\frac{n}{2}}(t)\right\|_{\mathbf{L}^p} \le C\langle t\rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)} \varepsilon^{\frac{2}{n}\left(1-\frac{1}{n}-\frac{1}{n^2}\right)} g^{-\frac{n}{2}-1}(t) \log g(t).
$$

Theorem 1.1 is proved.

3. Lemmas

Consider the Green operator

$$
\mathcal{G}_1(t) = \mathcal{F}^{-1}L_1(t,\xi)\,\mathcal{F},
$$

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where $L_1(t,\xi) = e^{-t\left(1-\sqrt{1-|\xi|^2}\right)}\chi_1(\xi)$ and $\chi_1 \in \mathbf{C}_0^\infty(\mathbf{R}^n)$ is such that $\chi_1(\xi) = 1$ for $|\xi| \le \frac{2}{3}$ and $\chi_1(\xi) = 0$ for $|\xi| \ge \frac{5}{6}$. Thus the operator $\mathcal{G}_1(t)$ is defined for the low frequency part. Define a kernel $G_1(t,x) = \mathcal{F}^{-1}\left(e^{-\frac{1}{2}\langle t\rangle|\xi|^2}\right) = (2\pi\langle t\rangle)^{-\frac{n}{2}} e^{-\frac{1}{2\langle t\rangle}|x|^2}$.

We first prove some preliminary estimates for the Green operator $\mathcal{G}_1(t)$.

Lemma 3.1. *The inequalities*

$$\left\||\cdot|^\omega \nabla^j \mathcal{G}_1(t)\phi\right\|_{\mathbf{L}^2} \le C\langle t\rangle^{\frac{\omega-j}{2}}\|\phi\|_{\mathbf{L}^2} + C\langle t\rangle^{-\frac{j}{2}}\||\cdot|^\omega\phi\|_{\mathbf{L}^2},$$
$$\left\|\nabla^j\left(\mathcal{G}_1(t)\phi - \widehat{\phi}(0)G_1(t)\right)\right\|_{\mathbf{L}^2} \le C\langle t\rangle^{-\frac{\gamma+j}{2}-\frac{n}{4}}\|\langle\cdot\rangle^\gamma\phi\|_{\mathbf{L}^1}$$

and

$$\left\||\cdot|^\delta\nabla^j\left(\mathcal{G}_1(t)\phi - \widehat{\phi}(0)G_1(t)\right)\right\|_{\mathbf{L}^2} \le C\langle t\rangle^{\frac{\delta-\gamma-j}{2}-\frac{n}{4}}\|\langle\cdot\rangle^\gamma\phi\|_{\mathbf{L}^1} + C\langle t\rangle^{-\frac{j}{2}}\left\||\cdot|^\delta\phi\right\|_{\mathbf{L}^2}$$

are true for all $t > 0$, where $\omega, \delta \ge 0$, $\gamma \in [0,1]$, $j = 0, 1, 2$.

Proof. Note that there exists a smooth and rapidly decaying kernel

$$K_j(t,x) = \mathcal{F}^{-1}\xi^j L_1(t,\xi).$$

So that by the Young inequality we have

$$\begin{aligned}
\left\||\cdot|^\omega\nabla^j\mathcal{G}_1(t)\phi\right\|_{\mathbf{L}^2} &= \left\||\cdot|^\omega\mathcal{F}^{-1}\xi^j L_1(t,\xi)\widehat{\phi}(\xi)\right\|_{\mathbf{L}^2} \\
&= \left\||x|^\omega\int_{\mathbf{R}^n} K_j(t,x-y)\phi(y)\,dy\right\|_{\mathbf{L}^2} \\
&\le C\left\|\int_{\mathbf{R}^n}\left(|x-y|^\omega|K_j(t,x-y)| + |K_j(t,x-y)||y|^\omega\right)|\phi(y)|\,dy\right\|_{\mathbf{L}^2} \\
&\le C\||\cdot|^\omega K_j(t)\|_{\mathbf{L}^1}\|\phi\|_{\mathbf{L}^2} + C\|K_j(t)\|_{\mathbf{L}^1}\||\cdot|^\omega\phi\|_{\mathbf{L}^2}.
\end{aligned}$$

By the estimate

$$\left|(-\Delta)^k\left(\xi^j L_1(t,\xi)\right)\right| \le C\langle t\rangle^k|\xi|^j e^{-Ct|\xi|^2}$$

for all $t > 0$, $|\xi| \le \frac{5}{6}$, $k \ge 0$, we have

$$\begin{aligned}
\left\||\cdot|^{2k}K_j(t)\right\|_{\mathbf{L}^2} &\le C\left\|(-\Delta)^k\left(\xi^j L_1(t,\xi)\right)\right\|_{\mathbf{L}^2} \\
&\le C\langle t\rangle^k\left\||\xi|^j e^{-Ct|\xi|^2}\right\|_{\mathbf{L}^2(|\xi|\le\frac{5}{6})} \le C\langle t\rangle^{k-\frac{n}{4}-\frac{j}{2}}.
\end{aligned}$$

Hence taking $2k > \omega \ge 0$ by the Hölder inequality we find

$$\||\cdot|^\omega K_j(t)\|_{\mathbf{L}^2} \le \|K_j(t)\|_{\mathbf{L}^2}^{1-\frac{\omega}{2k}}\left\||\cdot|^{2k}K_j(t)\right\|_{\mathbf{L}^2}^{\frac{\omega}{2k}} \le C\langle t\rangle^{\frac{\omega-j}{2}-\frac{n}{4}}.$$

Then choosing $a = \left\| |\cdot|^{\omega+\delta} K_j(t) \right\|_{\mathbf{L}^2}^{\frac{1}{\delta}} \left\| |\cdot|^{\omega} K_j(t) \right\|_{\mathbf{L}^2}^{-\frac{1}{\delta}}$ and $\delta > \frac{n}{2}$ by the Cauchy–Schwarz inequality we obtain

$$
\begin{aligned}
\left\| |\cdot|^{\omega} K_j(t) \right\|_{\mathbf{L}^1} &= \int\limits_{|x|\le a} \left| |x|^{\omega} K_j(t,x) \right| dx + \int\limits_{|x|>a} |x|^{-\delta} |x|^{\omega+\delta} \left| K_j(t,x) \right| dx \\
&\le C a^{\frac{n}{2}} \left\| |\cdot|^{\omega} K_j(t) \right\|_{\mathbf{L}^2} + C a^{\frac{n}{2}-\delta} \left\| |\cdot|^{\omega+\delta} K_j(t) \right\|_{\mathbf{L}^2} \\
&\le C \left\| |\cdot|^{\omega} K_j(t) \right\|_{\mathbf{L}^2}^{1-\frac{n}{2\delta}} \left\| |\cdot|^{\omega+\delta} K_j(t) \right\|_{\mathbf{L}^2}^{\frac{n}{2\delta}} \le C \langle t \rangle^{\frac{\omega-j}{2}}.
\end{aligned}
$$

Therefore the first estimate of the lemma follows. To prove the second estimate we write

$$
\begin{aligned}
\left\| \nabla^j \left(\mathcal{G}_1(t)\phi - \widehat{\phi}(0) G_1(t) \right) \right\|_{\mathbf{L}^2} &\le C \left\| \xi^j \left(L_1(t,\xi) - e^{-\frac{1}{2}\langle t \rangle |\xi|^2} \right) \widehat{\phi}(\xi) \right\|_{\mathbf{L}^2} \\
&\quad + C \left\| \xi^j e^{-\frac{1}{2}\langle t \rangle |\xi|^2} \left(\widehat{\phi}(\xi) - \widehat{\phi}(0) \right) \right\|_{\mathbf{L}^2}.
\end{aligned}
$$

We have

$$
\left\| \xi^j \left(L_1(t,\xi) - e^{-\frac{1}{2}\langle t \rangle |\xi|^2} \right) \right\|_{\mathbf{L}^2} \le C \langle t \rangle^{-1} \left\| |\xi|^j e^{-C\langle t \rangle |\xi|^2} \right\|_{\mathbf{L}^2} \le C \langle t \rangle^{-1-\frac{n}{4}-\frac{j}{2}}
$$

since $e^x = 1 + x e^{ax}$ and so

$$
\begin{aligned}
\left| L_1(t,\xi) - e^{-\frac{1}{2}\langle t \rangle |\xi|^2} \right| &= \left| e^{-\frac{t|\xi|^2}{1+\sqrt{1-|\xi|^2}}} \chi_1(\xi) - e^{-\frac{1}{2}\langle t \rangle |\xi|^2} \right| \\
&\le e^{-\frac{1}{2}t|\xi|^2} \left| e^{-\left(\frac{1}{1+\sqrt{1-|\xi|^2}} - \frac{1}{2} \right) t|\xi|^2} - 1 \right| |\chi_1(\xi)| + e^{-Ct} \\
&\le C \left| |\xi|^2 t |\xi|^2 e^{-a\left(\frac{1}{1+\sqrt{1-|\xi|^2}} \right) t|\xi|^2} \right| |\chi_1(\xi)| + e^{-Ct}.
\end{aligned}
$$

Hence we find for the first summand

$$
\begin{aligned}
\left\| \xi^j \left(L_1(t,\xi) - e^{-\frac{1}{2}\langle t \rangle |\xi|^2} \right) \widehat{\phi}(\xi) \right\|_{\mathbf{L}^2} &\le \left\| \xi^j \left(L_1(t,\xi) - e^{-\frac{1}{2}\langle t \rangle |\xi|^2} \right) \right\|_{\mathbf{L}^2} \left\| \widehat{\phi} \right\|_{\mathbf{L}^\infty} \\
&\le C \langle t \rangle^{-1-\frac{n}{4}-\frac{j}{2}} \|\phi\|_{\mathbf{L}^1}.
\end{aligned}
$$

The second term is estimated as follows

$$
\begin{aligned}
\left\| \xi^j e^{-\frac{1}{2}\langle t \rangle |\xi|^2} \left(\widehat{\phi}(\xi) - \widehat{\phi}(0) \right) \right\|_{\mathbf{L}^2} &\le \left\| |\xi|^{\gamma+j} e^{-\frac{1}{2}\langle t \rangle |\xi|^2} \right\|_{\mathbf{L}^2} \left\| |\xi|^{-\gamma} \left(\widehat{\phi}(\xi) - \widehat{\phi}(0) \right) \right\|_{\mathbf{L}^\infty} \\
&\le C \left\| |\xi|^{\gamma+j} e^{-\frac{1}{2}\langle t \rangle |\xi|^2} \right\|_{\mathbf{L}^2} \left\| \int\limits_{\mathbf{R}^n} \phi(x) |\xi|^{-\gamma} \left(e^{-ix\cdot\xi} - 1 \right) dx \right\|_{\mathbf{L}^\infty} \\
&\le C \langle t \rangle^{-\frac{\gamma+j}{2}-\frac{n}{4}} \left\| |\cdot|^{\gamma} \phi \right\|_{\mathbf{L}^1}
\end{aligned}
$$

if $\gamma \in [0,1]$ since $\left| e^{-ix\cdot\xi} - 1 \right| \le C |x|^{\gamma} |\xi|^{\gamma}$. Thus the second estimate of the lemma is true. We prove the third estimate of the lemma

$$\left\| |\cdot|^{\delta} \nabla^{j} \left(\mathcal{G}_1(t)\phi - \widehat{\phi}(0) G_1(t) \right) \right\|_{\mathbf{L}^2} \leq C \left\| (-\Delta)^{\frac{\delta}{2}} \xi^{j} \left(L_1(t,\xi) - e^{-\frac{1}{2}\langle t\rangle |\xi|^2} \right) \widehat{\phi}(\xi) \right\|_{\mathbf{L}^2} + C \left\| (-\Delta)^{\frac{\delta}{2}} \xi^{j} e^{-\frac{1}{2}\langle t\rangle |\xi|^2} \left(\widehat{\phi}(\xi) - \widehat{\phi}(0) \right) \right\|_{\mathbf{L}^2}.$$

We define

$$\widetilde{K}(t,x) = \mathcal{F}^{-1} \left(\xi^{j} \left(L_1(t,\xi) - e^{-\frac{1}{2}\langle t\rangle |\xi|^2} \right) \right),$$

then by the Young inequality we have

$$\begin{aligned} &\left\| (-\Delta)^{\frac{\delta}{2}} \xi^{j} \left(L_1(t,\xi) - e^{-\frac{1}{2}\langle t\rangle |\xi|^2} \right) \widehat{\phi}(\xi) \right\|_{\mathbf{L}^2} \\ &= \left\| |\cdot|^{\delta} \mathcal{F}^{-1} \xi^{j} \left(L_1(t,\xi) - e^{-\frac{1}{2}\langle t\rangle |\xi|^2} \right) \widehat{\phi}(\xi) \right\|_{\mathbf{L}^2} \\ &= \left\| |x|^{\delta} \int_{\mathbf{R}^n} \widetilde{K}(t,x-y)\phi(y)\,dy \right\|_{\mathbf{L}^2} \\ &\leq C \left\| \int_{\mathbf{R}^n} |x-y|^{\delta} \left| \widetilde{K}(t,x-y) \right| |\phi(y)|\,dy \right\|_{\mathbf{L}^2} + C \left\| \int_{\mathbf{R}^n} \left| \widetilde{K}(t,x-y) \right| |y|^{\delta} |\phi(y)|\,dy \right\|_{\mathbf{L}^2} \\ &\leq C \left\| |\cdot|^{\delta} \widetilde{K}(t) \right\|_{\mathbf{L}^2} \|\phi\|_{\mathbf{L}^1} + C \left\| \widetilde{K}(t) \right\|_{\mathbf{L}^1} \left\| |\cdot|^{\delta} \phi \right\|_{\mathbf{L}^2}. \end{aligned}$$

By the estimate

$$\left| (-\Delta)^{k} \left(\xi^{j} \left(L_1(t,\xi) - e^{-\frac{1}{2}\langle t\rangle |\xi|^2} \right) \right) \right| \leq C \langle t\rangle^{k-1} |\xi|^{j} e^{-C\langle t\rangle |\xi|^2}$$

for all $t > 0$, $j = 0, 1, 2$, $k \geq 0$, we have

$$\begin{aligned} \left\| |\cdot|^{2k} \widetilde{K}(t) \right\|_{\mathbf{L}^2} &\leq C \left\| (-\Delta)^{k} \left(\xi^{j} \left(L_1(t,\xi) - e^{-\frac{1}{2}\langle t\rangle |\xi|^2} \right) \right) \right\|_{\mathbf{L}^2} \\ &\leq C \langle t\rangle^{k-1} \left\| |\xi|^{j} e^{-C\langle t\rangle |\xi|^2} \right\|_{\mathbf{L}^2} \leq C \langle t\rangle^{k-1-\frac{n}{4}-\frac{j}{2}}. \end{aligned}$$

Hence

$$\left\| |\cdot|^{\delta} \widetilde{K}(t) \right\|_{\mathbf{L}^2} \leq C \langle t\rangle^{\frac{\delta}{2}-1-\frac{n}{4}-\frac{j}{2}}$$

and

$$\left\| \widetilde{K}(t) \right\|_{\mathbf{L}^1} \leq C \langle t\rangle^{-1-\frac{j}{2}}.$$

Therefore

$$\left\| (-\Delta)^{\frac{\delta}{2}} \xi^{j} \left(L_1(t,\xi) - e^{-\frac{1}{2}\langle t\rangle |\xi|^2} \right) \widehat{\phi}(\xi) \right\|_{\mathbf{L}^2} \leq C \langle t\rangle^{\frac{\delta}{2}-1-\frac{n}{4}-\frac{j}{2}} \|\phi\|_{\mathbf{L}^1} + C \langle t\rangle^{-1-\frac{j}{2}} \left\| |\cdot|^{\delta} \phi \right\|_{\mathbf{L}^2}.$$

We define $\mathcal{G}_{heat}(t) = \mathcal{F}^{-1} e^{-\frac{1}{2}\langle t\rangle |\xi|^2} \mathcal{F}$ and consider the second summand

$$\left\| (-\Delta)^{\frac{\delta}{2}} \xi^{j} e^{-\frac{1}{2}\langle t\rangle |\xi|^2} \left(\widehat{\phi}(\xi) - \widehat{\phi}(0) \right) \right\|_{\mathbf{L}^2} = \left\| |x|^{\delta} \nabla^{j} \left(\mathcal{G}_{heat}(t)\phi - G_1(t,x)\widehat{\phi}(0) \right) \right\|_{\mathbf{L}^2}.$$

By explicit computation we have

$$
\begin{aligned}
& |x|^{\delta} \nabla^{j} \left(\mathcal{G}_{heat}(t)\phi - G_1(t,x)\widehat{\phi}(0) \right) \\
&= (2\pi \langle t \rangle)^{-\frac{n}{2}} \int_{\mathbf{R}^n} |x|^{\delta} \left(-\left(\frac{x-y}{\langle t \rangle} \right)^{j} e^{-\frac{(x-y)^2}{2\langle t \rangle}} + \left(\frac{x}{\langle t \rangle} \right)^{j} e^{-\frac{x^2}{2\langle t \rangle}} \right) \phi(y)\, dy \\
&= (2\pi \langle t \rangle)^{-\frac{n}{2}} \left(\int_{|y| \le \sqrt{\langle t \rangle}} + \int_{|y| \ge \sqrt{\langle t \rangle}} \right) \\
&\quad \times |x|^{\delta} \left(-\left(\frac{x-y}{\langle t \rangle} \right)^{j} e^{-\frac{(x-y)^2}{2\langle t \rangle}} + \left(\frac{x}{\langle t \rangle} \right)^{j} e^{-\frac{x^2}{2\langle t \rangle}} \right) \phi(y)\, dy.
\end{aligned}
$$

In the case of $|y| \ge \sqrt{\langle t \rangle}$ we estimate

$$
\begin{aligned}
& |x|^{\delta} \left| \left(\frac{x}{\langle t \rangle} \right)^{j} e^{-\frac{x^2}{2\langle t \rangle}} - \left(\frac{x-y}{\langle t \rangle} \right)^{j} e^{-\frac{(x-y)^2}{2\langle t \rangle}} \right| \\
&\le C \left(|x-y|^{\delta} + |y|^{\delta} \right) \left(\frac{|x-y|}{\langle t \rangle} \right)^{j} e^{-\frac{(x-y)^2}{2\langle t \rangle}} + C|x|^{\delta} \left(\frac{|x|}{\langle t \rangle} \right)^{j} e^{-\frac{x^2}{2\langle t \rangle}} \\
&\le C \langle t \rangle^{\frac{\delta - j}{2}} \left(e^{-C(x-y)^2 \langle t \rangle^{-1}} + e^{-Cx^2 \langle t \rangle^{-1}} \right) + C|y|^{\delta} \left(\frac{|x-y|}{\langle t \rangle} \right)^{j} e^{-\frac{(x-y)^2}{2\langle t \rangle}}
\end{aligned}
$$

and for all $|y| \le \sqrt{\langle t \rangle}$ we find

$$
\begin{aligned}
& |x|^{\delta} \left| \left(\frac{x}{\langle t \rangle} \right)^{j} e^{-\frac{x^2}{2\langle t \rangle}} - \left(\frac{x-y}{\langle t \rangle} \right)^{j} e^{-\frac{(x-y)^2}{2\langle t \rangle}} \right| \\
&\le C|x|^{\delta} |y| \langle t \rangle^{-\frac{1}{2} - \frac{j}{2}} e^{-\frac{C(x-\theta y)^2}{\langle t \rangle}} \le C|y| \langle t \rangle^{\frac{\delta - j}{2} - \frac{1}{2}} e^{-C(x-y)^2 \langle t \rangle^{-1} - Cx^2 \langle t \rangle^{-1}}.
\end{aligned}
$$

Hence we get

$$
\begin{aligned}
& \left\| |x|^{\delta} \nabla^{j} \left(\mathcal{G}_{heat}(t)\phi - G_1(t,x) \int \phi(y)\, dy \right) \right\|_{\mathbf{L}^2} \\
&\le C \langle t \rangle^{\frac{\delta - j}{2} - \frac{1}{2} - \frac{n}{2}} \left\| \int_{|y| \le \sqrt{\langle t \rangle}} e^{-C(x-y)^2 \langle t \rangle^{-1} - Cx^2 \langle t \rangle^{-1}} |y| \phi(y)\, dy \right\|_{\mathbf{L}^2} \\
&\quad + C \langle t \rangle^{\frac{\delta - j}{2} - \frac{n}{2}} \left\| \int_{|y| \ge \sqrt{\langle t \rangle}} \left(e^{-C(x-y)^2 \langle t \rangle^{-1}} + C e^{-Cx^2 \langle t \rangle^{-1}} \right) \phi(y)\, dy \right\|_{\mathbf{L}^2} \\
&\quad + C \langle t \rangle^{-\frac{n}{2} - \frac{j}{2}} \left\| \int_{|y| \ge \sqrt{\langle t \rangle}} e^{-C(x-y)^2 \langle t \rangle^{-1}} |y|^{\delta} \phi(y)\, dy \right\|_{\mathbf{L}^2} \\
&\le C \langle t \rangle^{\frac{\delta - j}{2} - \frac{1}{2} - \frac{n}{4}} \| |\cdot| \phi \|_{\mathbf{L}^1\left(|x| \le \sqrt{\langle t \rangle}\right)} + C \langle t \rangle^{\frac{\delta - j}{2} - \frac{n}{4}} \|\phi\|_{\mathbf{L}^1\left(|x| \ge \sqrt{\langle t \rangle}\right)}
\end{aligned}
$$

$$+ C \langle t\rangle^{-\frac{n}{2}-\frac{j}{2}} \left\| e^{-Cx^2\langle t\rangle^{-1}} \right\|_{\mathbf{L}^1} \left\| |\cdot|^{\delta} \phi \right\|_{\mathbf{L}^2}$$
$$\leq C \langle t\rangle^{\frac{\delta-\gamma-j}{2}-\frac{n}{4}} \left\| \langle\cdot\rangle^{\gamma} \phi \right\|_{\mathbf{L}^1} + C \langle t\rangle^{-\frac{j}{2}} \left\| |\cdot|^{\delta} \phi \right\|_{\mathbf{L}^2}$$

for all $t > 0$ and $0 \leq \gamma \leq 1$. Lemma 3.1 is proved. □

Next we consider the linear Cauchy problem

$$\begin{cases} \mathcal{L}_2 v = f(t,x), \ x \in \mathbf{R}^n, \ t > 0, \\ v(0,x) = v_0(x), \ x \in \mathbf{R}^n, \end{cases}$$

where $\mathcal{L}_2 = \partial_t + 1 + \sqrt{1+\Delta}$. The solution can be written by the Duhamel formula

$$v(t) = \mathcal{G}_2(t) v_0 + \int_0^t \mathcal{G}_2(t-\tau) f(\tau) d\tau,$$

where the Green operator $\mathcal{G}_2(t) = \mathcal{F}^{-1} L_2(t,\xi) \chi_1(\xi) \mathcal{F}$ with $L_2(t,\xi) = e^{-t\left(1+\sqrt{1-|\xi|^2}\right)}$ is defined for the functions in the low frequency part.

We see that the linear operator $\mathcal{L}_2$ satisfies the dissipation condition which in terms of the symbol $1 + \sqrt{1-|\xi|^2}$ has the form

$$1 + \sqrt{1-|\xi|^2} \geq \frac{5}{3}$$

for all $|\xi| \leq \frac{2}{3}$, also the symbol $L_2(t,\xi)$ satisfies

$$|L_2(t,\xi)| \leq C \langle\xi\rangle^{-1} e^{-\frac{5}{3}t}$$

and obeys the estimate

$$\left| \partial_{\xi_j}^{l} \langle\xi\rangle L_2(t,\xi) \right| \leq C \langle t\rangle^{\frac{l}{2}} e^{-\frac{5}{3}t}$$

for all $|\xi| \leq \frac{2}{3}$, $l = 0,1,...,N$, $j = 1,2,...,n$. Also we consider the linear Cauchy problem

$$\begin{cases} \left(\partial_t^2 + 2\partial_t - \Delta\right) v = f(t,x), \ x \in \mathbf{R}^n, \ t > 0, \\ v(0,x) = v_0(x), \ v_t(0,x) = v_1(x), \ x \in \mathbf{R}^n. \end{cases}$$

The solution can be written by the Duhamel formula

$$v(t) = (1+\partial_t) \mathcal{G}_3(t) v_0 + \mathcal{G}_3(t) v_1 + \int_0^t \mathcal{G}_3(t-\tau) f(\tau) d\tau,$$

where the Green operator $\mathcal{G}_3(t) = \mathcal{F}^{-1} L_3(t,\xi) \chi_2(\xi) \mathcal{F}$ is defined for the functions in the high frequency part $|\xi| \geq \frac{1}{3}$, $L_3(t,\xi) = e^{-t} \frac{\sin\left(t\sqrt{|\xi|^2-1}\right)}{\sqrt{|\xi|^2-1}}$, i.e. $\chi_2(\xi) = 1$ for $|\xi| \geq \frac{1}{3}$ and $\chi_2(\xi) = 0$ for $|\xi| \leq \frac{1}{6}$. The symbol $L_3(t,\xi)$ satisfies estimates

$$|L_3(t,\xi)| \leq \langle\xi\rangle^{-1} e^{-\frac{t}{18}}$$

for all $|\xi| \geq \frac{1}{3}$. Also the symbol $L_3(t,\xi)$ is smooth and obeys the estimate

$$\left|\partial_{\xi_j}^{l}\left(\langle\xi\rangle L_3(t,\xi)\right)\right| \leq C\langle t\rangle^{\frac{l}{2}} e^{-\frac{t}{18}}$$

for all $t>0$, $|\xi| \geq \frac{1}{3}$, $l=0,1,...,N$, $j=1,2,...,n$.

In the next lemma we estimate the Green operator

$$\mathcal{G}(t) = \mathcal{F}^{-1} L(t,\xi)\mathcal{F}.$$

Lemma 3.2. *Suppose that the estimate*

$$\left|(-\Delta)^k\left(|\xi|^j\langle\xi\rangle^{1-j} L(t,\xi)\right)\right| \leq Ce^{-\beta t}$$

is valid for $t>0$, $k \geq 0$, where $\beta>0$, $j \geq 0$. Then the inequality

$$\left\||\cdot|^{\delta}\nabla^j\mathcal{G}(t)\phi\right\|_{\mathbf{L}^2} \leq Ce^{-\beta t}\left\|\langle i\nabla\rangle^{j-1}\langle\cdot\rangle^{\delta}\phi\right\|_{\mathbf{L}^2}$$

is true for all $t>0$, where $k \geq \delta \geq 0$.

Proof. Using the estimate for the commutator

$$\begin{aligned}\left\|\left[\partial_{\xi_m}^{\alpha},\psi\right]\phi\right\|_{\mathbf{L}^2} &\leq C\left\|\int_0^{\infty}\left(\psi(\xi)-\psi\left(\breve{\xi}_\eta\right)\right)\phi\left(\breve{\xi}_\eta\right)\eta^{-1-\alpha}d\eta\right\|_{\mathbf{L}^2}\\ &\leq C\left\|\langle i\nabla\rangle\psi\right\|_{\mathbf{L}^\infty}\|\phi\|_{\mathbf{L}^2},\end{aligned}$$

where $\breve{\xi}_\eta \equiv (\xi_1,...,\xi_m+\eta,...,\xi_n)$, if $\alpha \in (0,1)$. Then by the Leibnitz rule, taking $\alpha=\delta-[\delta]$ and $k=[\delta]+1$, we get

$$\begin{aligned}\left\||\cdot|^{\delta}\nabla^j\mathcal{G}(t)\phi\right\|_{\mathbf{L}^2} &\leq C\sum_{m=1}^{n}\sum_{l=0}^{[\delta]}\left\|\partial_{\xi_m}^{\delta-[\delta]}\left(\left(\partial_{\xi_m}^{[\delta]-l}\xi^j\langle\xi\rangle^{1-j}L(t,\xi)\right)\left(\partial_{\xi_m}^{l}\langle\xi\rangle^{j-1}\widehat{\phi}(\xi)\right)\right)\right\|_{\mathbf{L}^2}\\ &\leq C\sum_{m=1}^{n}\sum_{l=0}^{[\delta]}\left\|\left(\partial_{\xi_m}^{[\delta]-l}\xi^j\langle\xi\rangle^{1-j}L(t,\xi)\right)\left(\partial_{\xi_m}^{\delta-[\delta]+l}\langle\xi\rangle^{j-1}\widehat{\phi}(\xi)\right)\right\|_{\mathbf{L}^2}\\ &\quad+C\sum_{m=1}^{n}\sum_{l=0}^{[\delta]}\left\|\left[\partial_{\xi_m}^{\delta-[\delta]},\left(\partial_{\xi_m}^{[\delta]-l}\xi^j\langle\xi\rangle^{1-j}L(t,\xi)\right)\right]\left(\partial_{\xi_m}^{l}\langle\xi\rangle^{j-1}\widehat{\phi}(\xi)\right)\right\|_{\mathbf{L}^2}\\ &\leq C\sum_{m=1}^{n}\sum_{l=0}^{[\delta]}\left\|\partial_{\xi_m}^{[\delta]-l}\xi^j\langle\xi\rangle^{1-j}L(t,\xi)\right\|_{\mathbf{L}^\infty}\left\|\partial_{\xi_m}^{\delta-[\delta]+l}\langle\xi\rangle^{j-1}\widehat{\phi}(\xi)\right\|_{\mathbf{L}^2}\\ &\quad+C\sum_{m=1}^{n}\sum_{l=0}^{[\delta]}\left\|\langle i\nabla\rangle^{k}\xi^j\langle\xi\rangle^{1-j}L(t,\xi)\right\|_{\mathbf{L}^\infty}\left\|\partial_{\xi_m}^{l}\langle\xi\rangle^{j-1}\widehat{\phi}(\xi)\right\|_{\mathbf{L}^2}\\ &\leq Ce^{-\beta t}\left\|\langle i\nabla\rangle^{\delta}\langle\xi\rangle^{j-1}\widehat{\phi}(\xi)\right\|_{\mathbf{L}^2} \leq Ce^{-\beta t}\left\|\langle\cdot\rangle^{\delta}\langle i\nabla\rangle^{j-1}\phi\right\|_{\mathbf{L}^2}.\end{aligned}$$

Lemma 3.2 is proved. □

We define the norms

$$\|\phi\|_{\mathbf{X}_T} = \sup_{t\in[0,T]} \langle t\rangle^{\frac{n}{4}} \left\| \left\langle \langle t\rangle^{\frac{1}{2}} i\nabla \right\rangle^2 \phi(t) \right\|_{\mathbf{L}^2} + \sup_{t\in[0,T]} \langle t\rangle^{\frac{n}{4}-\frac{\delta}{2}} \left\| \langle\cdot\rangle^{\delta} \left\langle \langle t\rangle^{\frac{1}{2}} i\nabla \right\rangle \phi(t) \right\|_{\mathbf{L}^2},$$

and

$$\|\phi\|_{\mathbf{Y}_T} = \sup_{t\in[0,T]} \langle t\rangle^{1+\frac{n}{4}} \left\| \left\langle \langle t\rangle^{\frac{1}{2}} i\nabla \right\rangle \phi(t) \right\|_{\mathbf{L}^2} + \sup_{t\in[0,T]} \langle t\rangle^{1+\frac{n}{4}-\frac{\delta}{2}} \left\| \langle\cdot\rangle^{\delta} \phi(t) \right\|_{\mathbf{L}^2},$$

where $\delta > \frac{n}{2}$.

Lemma 3.3. *Suppose that the inequalities*

$$\left\| |\cdot|^{\omega} \nabla^j \mathcal{G}(t)\phi \right\|_{\mathbf{L}^2} \le C \langle t\rangle^{\frac{\omega-j}{2}} \|\phi\|_{\mathbf{L}^2} + C\langle t\rangle^{-\frac{j}{2}} \left\| |\cdot|^{\omega} \phi \right\|_{\mathbf{L}^2}, \tag{3.1}$$

$$\left\| \nabla^j \left(\mathcal{G}(t)\phi - \widehat{\phi}(0) G(t) \right) \right\|_{\mathbf{L}^2} \le C\langle t\rangle^{-\frac{\gamma+j}{2}-\frac{n}{4}} \left\| \langle\cdot\rangle^{\gamma} \phi \right\|_{\mathbf{L}^1} \tag{3.2}$$

and

$$\left\| |\cdot|^{\delta} \nabla^j \left(\mathcal{G}(t)\phi - \widehat{\phi}(0) G(t) \right) \right\|_{\mathbf{L}^2} \le C\langle t\rangle^{\frac{\delta-\gamma-j}{2}-\frac{n}{4}} \left\| \langle\cdot\rangle^{\gamma} \phi \right\|_{\mathbf{L}^1} + C\langle t\rangle^{-\frac{j}{2}} \left\| |\cdot|^{\delta} \phi \right\|_{\mathbf{L}^2} \tag{3.3}$$

are true for all $t \in [0,T]$, where $\omega, \delta \ge 0$, $\gamma \in [0,1]$, $j = 0,1,2$. Then the estimate is true

$$\left\| \langle t\rangle^{\frac{\gamma}{2}} \left(\mathcal{G}(t)\phi - \widehat{\phi}(0) G(t) \right) \right\|_{\mathbf{X}_T} \le C \|\phi\|_{\mathbf{H}^{0,\delta}}$$

with $0 < \gamma < \min\left(1, \delta - \frac{n}{2}\right)$. Moreover let the function $f(t,x)$ have a zero mean value $\widehat{f}(t,0) = 0$. Then the following inequality

$$\left\| g(t) \int_0^t \mathcal{G}(t-\tau) f(\tau)\, d\tau \right\|_{\mathbf{X}_T} \le C \|gf\|_{\mathbf{Y}_T}$$

is valid, where C does not depend on $T > 0$.

Proof. By condition (3.2) with $j = 0,2$ we get

$$\begin{aligned} &\left\| \nabla^j \left(\mathcal{G}(t)\phi - \widehat{\phi}(0) G(t) \right) \right\|_{\mathbf{L}^2} \\ &\le C\langle t\rangle^{-\frac{\gamma+j}{2}-\frac{n}{4}} \left\| \langle\cdot\rangle^{\gamma} \phi \right\|_{\mathbf{L}^1} \le C\langle t\rangle^{-\frac{\gamma+j}{2}-\frac{n}{4}} \left\| \langle\cdot\rangle^{\delta} \phi \right\|_{\mathbf{L}^2}. \end{aligned}$$

Next we apply condition (3.3) with $j = 0,1$

$$\begin{aligned} &\left\| |\cdot|^{\delta} \nabla^j \left(\mathcal{G}(t)\phi - \widehat{\phi}(0) G(t) \right) \right\|_{\mathbf{L}^2} \\ &\le C\langle t\rangle^{\frac{\delta-\gamma-j}{2}-\frac{n}{4}} \left\| \langle\cdot\rangle^{\gamma} \phi \right\|_{\mathbf{L}^1} + C\langle t\rangle^{-\frac{j}{2}} \left\| |\cdot|^{\delta} \phi \right\|_{\mathbf{L}^2} \\ &\le C\langle t\rangle^{\frac{\delta-\gamma-j}{2}-\frac{n}{4}} \left\| \langle\cdot\rangle^{\delta} \phi \right\|_{\mathbf{L}^2}. \end{aligned}$$

By the Cauchy–Schwarz inequality we find

$$\begin{aligned}\left\|\langle\cdot\rangle^{\gamma}\phi\right\|_{\mathbf{L}^1} &= \int\limits_{|x|\le\langle\tau\rangle^{\frac12}}\langle x\rangle^{\gamma}|\phi(x)|\,dx+\int\limits_{|x|>\langle\tau\rangle^{\frac12}}\langle x\rangle^{\gamma-\delta}\langle x\rangle^{\delta}|\phi(x)|\,dx\\ &\le C\langle\tau\rangle^{\frac n4+\frac{\gamma}{2}}\|\phi\|_{\mathbf{L}^2}+C\langle\tau\rangle^{\frac n4+\frac{\gamma-\delta}{2}}\left\|\langle\cdot\rangle^{\delta}\phi\right\|_{\mathbf{L}^2}\le C\langle\tau\rangle^{\frac{\gamma}{2}-1}\|\phi\|_{\mathbf{Y}_T}\end{aligned}$$

with $0<\gamma<\min\left(1,\delta-\frac n2\right)$. Then applying condition (3.2) with $j=0$ in the domain $0\le\tau\le\frac t2$ and condition (3.1) with $\omega=0$, $j=0$ in the domain $\frac t2\le\tau\le t$, we get

$$\begin{aligned}&\left\|\int_0^t\mathcal{G}(t-\tau)f(\tau)\,d\tau\right\|_{\mathbf{L}^2}\\ &\le C\int_0^{\frac t2}\langle t-\tau\rangle^{-\frac{\gamma}{2}-\frac n4}\left\|\langle\cdot\rangle^{\gamma}f(\tau)\right\|_{\mathbf{L}^1}d\tau+C\int_{\frac t2}^t\|f(\tau)\|_{\mathbf{L}^2}\,d\tau\\ &\le C\|gf\|_{\mathbf{Y}_T}\int_0^{\frac t2}\langle t-\tau\rangle^{-\frac{\gamma}{2}-\frac n4}\langle\tau\rangle^{\frac{\gamma}{2}-1}g^{-1}(\tau)\,d\tau+C\|gf\|_{\mathbf{Y}_T}\int_{\frac t2}^t\langle\tau\rangle^{-\frac n4-1}g^{-1}(\tau)\,d\tau\\ &\le C\langle t\rangle^{-\frac{\gamma}{2}-\frac n4}\|gf\|_{\mathbf{Y}_T}\int_0^{\frac t2}\langle\tau\rangle^{\frac{\gamma}{2}-1}g^{-1}(\tau)\,d\tau+C\langle t\rangle^{-\frac n4-1}g^{-1}(t)\|gf\|_{\mathbf{Y}_T}\int_{\frac t2}^t d\tau\\ &\le C\langle t\rangle^{-\frac n4}g^{-1}(t)\|gf\|_{\mathbf{Y}_T}.\end{aligned}$$

Next we use condition (3.2) with $j=2$ in the domain $0\le\tau\le\frac t2$ and condition (3.1) with $\omega=0$, $j=1$ in the domain $\frac t2\le\tau\le t$

$$\begin{aligned}&\left\|\Delta\int_0^t\mathcal{G}(t-\tau)f(\tau)\,d\tau\right\|_{\mathbf{L}^2}\\ &\le\int_0^{\frac t2}\|\Delta\mathcal{G}(t-\tau)f(\tau)\|_{\mathbf{L}^2}\,d\tau+\int_{\frac t2}^t\|\nabla\mathcal{G}(t-\tau)\nabla f(\tau)\|_{\mathbf{L}^2}\,d\tau\\ &\le C\int_0^{\frac t2}\langle t-\tau\rangle^{-\frac{\gamma}{2}-\frac n4-1}\left\|\langle\cdot\rangle^{\gamma}f(\tau)\right\|_{\mathbf{L}^1}d\tau+C\int_{\frac t2}^t\langle t-\tau\rangle^{-\frac12}\|\nabla f(\tau)\|_{\mathbf{L}^2}\,d\tau\\ &\le C\|gf\|_{\mathbf{Y}_T}\int_0^{\frac t2}\langle t-\tau\rangle^{-\frac{\gamma}{2}-\frac n4-1}\langle\tau\rangle^{\frac{\gamma}{2}-1}g^{-1}(\tau)\,d\tau+C\|gf\|_{\mathbf{Y}_T}\int_{\frac t2}^t\langle t-\tau\rangle^{-\frac12}\langle\tau\rangle^{-\frac n4-\frac32}g^{-1}(\tau)\,d\tau\\ &\le C\langle t\rangle^{-\frac{\gamma}{2}-\frac n4-1}\|gf\|_{\mathbf{Y}_T}\int_0^{\frac t2}\langle\tau\rangle^{\frac{\gamma}{2}-1}g^{-1}(\tau)\,d\tau+C\langle t\rangle^{-\frac n4-\frac32}g^{-1}(t)\|gf\|_{\mathbf{Y}_T}\int_{\frac t2}^t\langle t-\tau\rangle^{-\frac12}\,d\tau\\ &\le C\langle t\rangle^{-\frac n4-1}g^{-1}(t)\|gf\|_{\mathbf{Y}_T}.\end{aligned}$$

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Finally we apply condition (3.3) with $j = 0, 1$ in the domain $0 \le \tau \le \frac{t}{2}$ and condition (3.1) with $\omega = \delta$, $j = 0, 1$ in the domain $\frac{t}{2} \le \tau \le t$

$$\begin{aligned}
&\left\| |\cdot|^{\delta} \nabla^{j} \int_{0}^{t} \mathcal{G}(t-\tau) f(\tau) d\tau \right\|_{\mathbf{L}^2} \\
&\le C \int_{0}^{\frac{t}{2}} \left(\langle t-\tau \rangle^{\frac{\delta-\gamma-j}{2}-\frac{n}{4}} \left\| \langle \cdot \rangle^{\gamma} f(\tau) \right\|_{\mathbf{L}^1} + \langle t-\tau \rangle^{-\frac{j}{2}} \left\| |\cdot|^{\delta} f(\tau) \right\|_{\mathbf{L}^2} \right) d\tau \\
&\quad + C \int_{\frac{t}{2}}^{t} \left(\langle t-\tau \rangle^{\frac{\delta-j}{2}} \left\| f(\tau) \right\|_{\mathbf{L}^2} + \langle t-\tau \rangle^{-\frac{j}{2}} \left\| |\cdot|^{\delta} f(\tau) \right\|_{\mathbf{L}^2} \right) d\tau \\
&\le C \left\| g f \right\|_{\mathbf{Y}_T} \int_{0}^{\frac{t}{2}} \left(\langle t-\tau \rangle^{\frac{\delta-\gamma-j}{2}-\frac{n}{4}} + \langle t-\tau \rangle^{-\frac{j}{2}} \langle \tau \rangle^{\frac{\delta-\gamma}{2}-\frac{n}{4}} \right) \langle \tau \rangle^{\frac{\gamma}{2}-1} g^{-1}(\tau) d\tau \\
&\quad + C \left\| g f \right\|_{\mathbf{Y}_T} \int_{\frac{t}{2}}^{t} \left(\langle t-\tau \rangle^{\frac{\delta}{2}} \langle \tau \rangle^{-\frac{n}{4}-1} + \langle \tau \rangle^{\frac{\delta}{2}-\frac{n}{4}-1} \right) \langle t-\tau \rangle^{-\frac{j}{2}} g^{-1}(\tau) d\tau \\
&\le C \langle t \rangle^{\frac{\delta-\gamma-j}{2}-\frac{n}{4}} \left\| g f \right\|_{\mathbf{Y}_T} \int_{0}^{\frac{t}{2}} \langle \tau \rangle^{\frac{\gamma}{2}-1} g^{-1}(\tau) d\tau \\
&\quad + C \langle t \rangle^{\frac{\delta}{2}-\frac{n}{4}-1} g^{-1}(t) \left\| g f \right\|_{\mathbf{Y}_T} \int_{\frac{t}{2}}^{t} \langle t-\tau \rangle^{-\frac{j}{2}} d\tau \le C \langle t \rangle^{\frac{\delta-j}{2}-\frac{n}{4}} g^{-1}(t) \left\| g f \right\|_{\mathbf{Y}_T}.
\end{aligned}$$

This completes the proof of the lemma. □

Lemma 3.4. *Suppose that the inequality*

$$\left\| |\cdot|^{\delta} \nabla^{j} \mathcal{G}(t) \phi \right\|_{\mathbf{L}^2} \le C e^{-\beta t} \left\| \langle i \nabla \rangle^{j-1} \langle \cdot \rangle^{\delta} \phi \right\|_{\mathbf{L}^2} \tag{3.4}$$

is true for all $t > 0$, where $j \ge 0$, $\delta \ge 0$, $\beta > 0$. Then the following inequalities

$$\left\| \langle t \rangle \mathcal{G}(t) \phi \right\|_{\mathbf{X}_T} \le C \left\| \phi \right\|_{\mathbf{H}^1} + C \left\| \phi \right\|_{\mathbf{H}^{0,\delta}}$$

and

$$\left\| \langle t \rangle^{\frac{1}{2}} \int_{0}^{t} \mathcal{G}(t-\tau) f(\tau) d\tau \right\|_{\mathbf{X}_T} \le C \left\| f \right\|_{\mathbf{Y}_T}$$

are valid, where C does not depend on $T > 0$.

Proof. We have by (3.4) with $j = 0, 2$, $\delta = 0$

$$\left\| \nabla^{j} \mathcal{G}(t) \phi \right\|_{\mathbf{L}^2} \le C e^{-\beta t} \left\| \phi \right\|_{\mathbf{H}^1},$$

and by (3.4) with $j = 0, 1$, $\delta > 0$

$$\left\| |\cdot|^{\delta} \nabla^{j} \mathcal{G}(t) \phi \right\|_{\mathbf{L}^2} \leq Ce^{-\beta t} \|\phi\|_{\mathbf{H}^{0,\delta}}.$$

Thus the first estimate of the lemma is true. We have by (3.4) with $j = 0$, $\delta = 0$

$$\begin{aligned}\left\| \int_0^t \mathcal{G}(t-\tau) f(\tau) d\tau \right\|_{\mathbf{L}^2} &\leq \int_0^t \|\mathcal{G}(t-\tau) f(\tau)\|_{\mathbf{L}^2} d\tau \\ &\leq C \int_0^t e^{-\beta(t-\tau)} \|f(\tau)\|_{\mathbf{L}^2} d\tau \\ &\leq C \|f\|_{\mathbf{Y}_T} \int_0^t e^{-\beta(t-\tau)} \langle\tau\rangle^{-1-\frac{n}{4}} d\tau \leq C \langle t\rangle^{-1-\frac{n}{4}} \|f\|_{\mathbf{Y}_T},\end{aligned}$$

and by (3.4) with $j = 1$, $\delta = 0$

$$\begin{aligned}\left\| \Delta \int_0^t \mathcal{G}(t-\tau) f(\tau) d\tau \right\|_{\mathbf{L}^2} &\leq \int_0^t \|\nabla \mathcal{G}(t-\tau) \nabla f(\tau)\|_{\mathbf{L}^2} d\tau \\ &\leq C \int_0^t e^{-\beta(t-\tau)} \|\nabla f(\tau)\|_{\mathbf{L}^2} d\tau \leq C \|f\|_{\mathbf{Y}_T} \int_0^t e^{-\beta(t-\tau)} \langle\tau\rangle^{-\frac{3}{2}-\frac{n}{4}} d\tau \\ &\leq C \langle t\rangle^{-\frac{3}{2}-\frac{n}{4}} \|f\|_{\mathbf{Y}_T},\end{aligned}$$

since

$$\begin{aligned}\int_0^t e^{-\beta(t-\tau)} \langle\tau\rangle^{-b} d\tau &\leq Ce^{-\frac{1}{2}\beta t} \int_0^{\frac{t}{2}} \langle\tau\rangle^{-b} d\tau + C \langle t\rangle^{-b} e^{-\beta t} \int_{\frac{t}{2}}^t e^{C\tau} d\tau \\ &\leq Ce^{-\frac{1}{4}\beta t} + C \langle t\rangle^{-b} \leq C \langle t\rangle^{-b}\end{aligned}$$

for $b \geq 0$. Also we get by (3.4) with $j = 0, 1$, $\delta > \frac{n}{2}$

$$\begin{aligned}\left\| |\cdot|^{\delta} \nabla^{j} \int_0^t \mathcal{G}(t-\tau) f(\tau) d\tau \right\|_{\mathbf{L}^2} &\leq \int_0^t \left\| |\cdot|^{\delta} \mathcal{G}(t-\tau) \nabla^{j} f(\tau) \right\|_{\mathbf{L}^2} d\tau \\ &\leq C \int_0^t e^{-\beta(t-\tau)} \left\| \langle i\nabla\rangle^{-1} \langle\cdot\rangle^{\delta} \nabla^{j} f(\tau) \right\|_{\mathbf{L}^2} d\tau \\ &\leq C \int_0^t e^{-\beta(t-\tau)} \left\| \langle\cdot\rangle^{\delta} f(\tau) \right\|_{\mathbf{L}^2} d\tau \\ &\leq C \|f\|_{\mathbf{Y}_T} \int_0^t e^{-\beta(t-\tau)} \langle\tau\rangle^{-1-\frac{n}{4}+\frac{\delta}{2}} d\tau \leq C \langle t\rangle^{-\frac{1}{2}-\frac{j}{2}-\frac{n}{4}+\frac{\delta}{2}} \|f\|_{\mathbf{Y}_T}.\end{aligned}$$

This completes the proof of the lemma. □

Next we estimate the nonlinear terms.

Lemma 3.5. *The estimate*

$$\left\| |v|^{\frac{2}{n}} v \right\|_{\mathbf{Y}_T} + \sup_{t\in[0,T]} \langle t\rangle \left\| |v|^{\frac{2}{n}} v \right\|_{\mathbf{L}^1} \le C \|v\|_{\mathbf{X}_T}^{1+\frac{2}{n}}$$

is true.

Proof. By the Sobolev embedding theorem $\|\phi\|_{\mathbf{L}^p} \le C \left\| |\nabla|^\alpha \phi \right\|_{\mathbf{L}^2} \le C \|\phi\|_{\mathbf{L}^2}^{1-\frac{\alpha}{2}} \|\Delta\phi\|_{\mathbf{L}^2}^{\frac{\alpha}{2}}$, $\alpha = \frac{n}{2} - \frac{n}{p}$, $2 \le p < \infty$, we have

$$\left\| |v|^{\frac{2}{n}} v \right\|_{\mathbf{L}^2} = \|v\|_{\mathbf{L}^{2+\frac{4}{n}}}^{1+\frac{2}{n}} \le C \|v\|_{\mathbf{L}^2}^{\frac{1}{2}+\frac{2}{n}} \|\Delta v\|_{\mathbf{L}^2}^{\frac{1}{2}} \le \langle t\rangle^{-\frac{n}{4}-1} \|v\|_{\mathbf{X}_T}^{1+\frac{2}{n}}.$$

In the same manner by the Hölder inequality and the Sobolev embedding theorem $\|\nabla\phi\|_{\mathbf{L}^{\frac{2n}{n-2}}} \le C \|\Delta\phi\|_{\mathbf{L}^2}$

$$\begin{aligned}\left\| \nabla \left(|v|^{\frac{2}{n}} v \right) \right\|_{\mathbf{L}^2} &\le C \left\| |v|^{\frac{2}{n}} \nabla v \right\|_{\mathbf{L}^2} \le C \|v\|_{\mathbf{L}^2}^{\frac{2}{n}} \|\nabla v\|_{\mathbf{L}^{\frac{2n}{n-2}}} \\ &\le C \|v\|_{\mathbf{L}^2}^{\frac{2}{n}} \|\Delta v\|_{\mathbf{L}^2} \le C \langle t\rangle^{-\frac{n}{4}-\frac{3}{2}} \|v\|_{\mathbf{X}_T}^{1+\frac{2}{n}}.\end{aligned}$$

By the Hölder inequality

$$\begin{aligned}\left\| \langle\cdot\rangle^\delta |v|^{\frac{2}{n}} v \right\|_{\mathbf{L}^2} &\le C \left\| |v|^{\frac{2}{n}} \langle\cdot\rangle^\delta v \right\|_{\mathbf{L}^2} \le C \|v\|_{\mathbf{L}^2}^{\frac{2}{n}} \left\| \langle\cdot\rangle^\delta v \right\|_{\mathbf{L}^{\frac{2n}{n-2}}} \\ &\le C \|v\|_{\mathbf{L}^2}^{\frac{2}{n}} \left\| \nabla \langle\cdot\rangle^\delta v \right\|_{\mathbf{L}^2} \le C \langle t\rangle^{-1-\frac{n}{4}+\frac{\delta}{2}} \|v\|_{\mathbf{X}_T}^{1+\frac{2}{n}}.\end{aligned}$$

The second estimate follows by the Cauchy–Schwarz inequality

$$\begin{aligned}\|\phi\|_{\mathbf{L}^1} &= \int_{|x|\le\langle t\rangle^{\frac{1}{2}}} |\phi(x)|\, dx + \int_{|x|>\langle t\rangle^{\frac{1}{2}}} \langle x\rangle^{-\delta} \langle x\rangle^{\delta} |\phi(x)|\, dx \\ &\le C \langle t\rangle^{\frac{n}{4}} \|\phi\|_{\mathbf{L}^2} + C \langle t\rangle^{\frac{n}{4}-\frac{\delta}{2}} \left\| \langle\cdot\rangle^\delta \phi(t) \right\|_{\mathbf{L}^2} \le C \langle t\rangle^{-1} \|\phi\|_{\mathbf{Y}_T}.\end{aligned}$$

Hence

$$\left\| |v|^{\frac{2}{n}} v \right\|_{\mathbf{L}^1} \le C \langle t\rangle^{-1} \left\| |v|^{\frac{2}{n}} v \right\|_{\mathbf{Y}_T} \le C \langle t\rangle^{-1} \|v\|_{\mathbf{X}_T}^{1+\frac{2}{n}}.$$

Lemma 3.5 is proved. □

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