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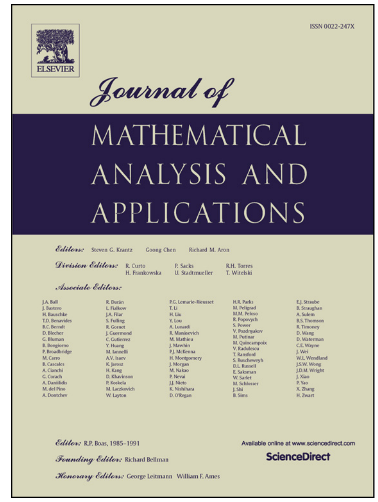

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A Unified Class of Integral Transforms Related to the Dunkl Transform

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Abstract

In the present paper, a new family of integral transforms depending on two parameters and related to the Dunkl transform is introduced. Well-known transforms, such as the fractional Dunkl transform, Dunkl transform, linear canonical transform, canonical Hankel transform, Fresnel transform etc, can be seen to be special cases of this general transform. Some useful properties of the considered transform such as Riemann-Lebesgue lemma, reversibility property, additivity property, operational formula, Plancherel formula, Bochner type identity and master formula are derived. The intimate connection that exists between this transformation and the quantum harmonic oscillator is developed.

Keywords: Canonical commutation relation, Dunkl transform, fractional Dunkl transform, Generalized Hermite polynomials and functions, semigroups of operators.

1 Intorduction

Integral transforms provide effective ways to solve a variety of problems arising in pure and applied mathematics. One example is the linear canonical transform (LCT) which represents a class of integral transforms indexed by a matrix parameter $M \in SL(2,\mathbb{R})$ [4]. Many well-known transforms such as Fourier transform, fractional Fourier transform, Weierstrass transform and Fresnel transform can be considred as special cases of this transformation (see [4, 32, 33]). While the theory of classical Fourier transform has a long and rich history, the growin interest in the theory of Dunkl transform, associated to a finite reflection groups and a multiplicity function k, is comparably recent. The Dunkl transform, which is a generalization of the Fourier and Hankel transforms, was introduced by C. F. Dunkl [9] and further studied by several authors (see [5, 9, 24]).

The primary aim of this article is to investigate a new integral transform that can unify all integral transforms stated in the previous paragraph. It seems desirable to have a more unified approach to all these integral transforms. According to literature M. Moshinsky and C. Quesne tackled this issue and considered that LCT is the group of unitary integral transforms that preserves the basic Heisenberg uncertainty relation of quantum mechanics in one or higher dimensions [22]. Furthermore, LCTs can be seen as the group of actions generated by the Lie algebra of quadratic Hamiltonian operators [32]. We briefly survey this mathematical framework. Let $\mathcal{H}$ be a Hilbert space. For a linear operator T on $\mathcal{H}$, we denote by $D(T)$ the domain of T. We say that a set $\{Q_j, P_j\}_{j=1}^N$ of self-adjoint operators on $\mathcal{H}$ is a representation of the canonical commutation relations (CCR) with N degrees of freedom [13], if there exists a dense subspace D of $\mathcal{H}$ such that
- $D \subset \cap_{j,k=1}^N [D(Q_jP_k) \cap D(P_kQ_j) \cap D(Q_jQ_k) \cap D(P_jP_k)]$
- Q_j and P_j satisfy on D the CCR

$$[Q_j, Q_k] = 0, \quad [P_j, P_k] = 0, \quad [Q_j, P_k] = \delta_k^j, \quad j,k = 1,\dots,N,$$

where δ_k^j is the Kronecker symbol.

It is well known that a standard representation of the CCR is the Schrödinger representation $\left\{q_{\xi_j}, p_{\xi_j}\right\}_{j=1}^N$ which is given as follows: $\{\xi_j\}_{j=1}^N$ is an orthonormal basis of $\mathbb{R}^N$ with respect to the standard inner product $\langle . \rangle$, $\mathcal{H} = L^2(\mathbb{R}^N, dx)$, $q_{\xi_j} = \langle ., \xi_j \rangle$ (the multiplication operator by the jth coordinate $\langle x, \xi_j \rangle$), $p_{\xi_j} = \frac{1}{i}\frac{\partial}{\partial \xi_j}$ (

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the directional derivative in the direction of ξ_j), $D = \mathcal{S}(\mathbb{R}^N)$ (the Schwartz space of rapidly decreasing $\mathcal{C}^\infty$ functions on $\mathbb{R}^N$).

In relation to the Schrödinger representation, consider the set $\{\tilde{q}_{\xi_j}, \tilde{p}_{\xi_j}\}_{j=1}^N$ of self-adjoint operators on $L^2(\mathbb{R}^N, dx)$ of the following type [32, 33]:

$$\tilde{q}_{\xi_j} = d\ q_{\xi_j} - b\ p_{\xi_j}, \quad \tilde{p}_{\xi_j} = -c\ q_{\xi_j} + a\ p_{\xi_j}, \quad j = 1, \ldots, N,$$

where a, d, c and d are real numbers such that $ad - bc = 1$. It is easy to see that the operators $\tilde{q}_{\xi_j}$ and $\tilde{p}_{\xi_j}$ are related to q_{ξ_j} and p_{ξ_j} through a canonical transform as follows:

$$\begin{bmatrix} \tilde{q}_{\xi_j} \\ \tilde{p}_{\xi_j} \end{bmatrix} = M^{-1} \begin{bmatrix} q_{\xi_j} \\ p_{\xi_j} \end{bmatrix},$$

where M^{-1} is the inverse of the unimodular matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Note that since q_{ξ_j} and p_{ξ_j} are self-adjoint, the family $\{\tilde{q}_{\xi_j}, \tilde{p}_{\xi_j}\}_{j=1}^N$ is a representation of the CCR unitarily equivalent to the Schrödinger representation on $L^2(\mathbb{R}^N, dx)$. More precisely, for each $M \in SL(2, \mathbb{R})$, the N-dimensional linear canonical transform

$$\mathcal{F}^M : L^2(\mathbb{R}^N, dx) \longrightarrow L^2(\mathbb{R}^N, dx),$$

which is defined by [4] :

$$\mathcal{F}^M f(x) = \begin{cases} \frac{1}{(2i\pi b)^{N/2}} \displaystyle\int_{\mathbb{R}^N} e^{\frac{i}{2}(\frac{d}{b}|x|^2 + \frac{a}{b}|y|^2) - \frac{i}{b}\langle x,y\rangle} f(y)\ dy, & b \neq 0 \\ \frac{e^{\frac{i}{2}\frac{c}{a}|x|^2}}{|a|^{N/2}}\ f(x/a), & b = 0, \end{cases} \tag{1.1}$$

is a unitary operator leaving invariant the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ and we have for $j = 1, \ldots, N$ the following identities:

$$\begin{aligned} \mathcal{F}^M \circ q_{\xi_j} \circ \left(\mathcal{F}^M\right)^{-1} &= \tilde{q}_{\xi_j} = d\ q_{\xi_j} - b\ p_{\xi_j}, \\ \mathcal{F}^M \circ p_{\xi_j} \circ \left(\mathcal{F}^M\right)^{-1} &= \tilde{p}_{\xi_j} = -c\ q_{\xi_j} + a\ p_{\xi_j}. \end{aligned}$$

We note that $\mathcal{F}^M$ is reduced to the classical Fourier transform if $M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$; to the fractional Fourier transform $\mathcal{F}^\alpha$ which is defined by [21]:

$$\mathcal{F}^\alpha f(x) = \begin{cases} \frac{e^{i(N/2)((\alpha - 2n\pi) - \hat{\alpha}\pi/2)}}{(2\pi|\sin(\alpha)|)^{N/2}} \displaystyle\int_{\mathbb{R}^N} e^{\frac{i}{2}(|x|^2 + |y|^2)\cot(\alpha) - \frac{i}{\sin(\alpha)}\langle x,y\rangle} f(y)\ dy, & (2n-1)\pi < \alpha < (2n+1)\pi, \\ f(x), & \alpha = 2n\pi, \\ f(-x), & \alpha = (2n+1)\pi, \end{cases}$$

where $n \in \mathbb{Z}$ and $\hat{\alpha} = \mathrm{sgn}(\sin(\alpha))$ if $M = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$.

In this paper, we adopt the same approach described above. We consider the set $\{Q_{\xi_j}, P_{\xi_j}\}_{j=1}^N$ of self-adjoint operators on $L^2(\mathbb{R}^N, \omega_k(x)dx)$ where $\{\xi_j\}_{j=1}^N$ is an orthonormal basis of $\mathbb{R}^N$ with respect to the standard inner product $\langle . \rangle$, ω_k is a family of weight functions invariant under a finite reflection group, Q_{ξ_j} is the multiplication operator by the function $x \mapsto \langle x, \xi_j\rangle$ and $iP_{\xi_j} = T_{\xi_j}$, $j = 1, \ldots, N$, are the Dunkl operators which were introduced by C.F. Dunkl in [7] as a differential-difference operators associated with a finite reflection group acting on some Euclidian space and can be regarded as a generalization of the directional derivative. We obtain the following commutation relations

$$[Q_{\xi_j}, Q_{\xi_k}] = 0, \quad [P_{\xi_j}, P_{\xi_k}] = 0, \quad [Q_{\xi_j}, P_{\xi_k}] = iE_{\xi_j, \xi_k}, \quad j, k = 1, \ldots, N, \tag{1.2}$$

where E_{ξ_j,ξ_k} is the bounded linear operator on $L^2(\mathbb{R}^N, \omega_k(x)dx)$ defined by

$$E_{\xi_j,\xi_k} f(x) =< \xi_j, \xi_k > f(x) + \sum_{\eta \in \mathcal{R}^+} k(\eta) < \eta, \xi_j >< \eta, \xi_k > f(\sigma_\eta x).$$

It is important to note that the above commutation relations are:
• a realization of the Heisenberg-Weyl algebra if $k \equiv 0$ [13],
• a realization of the deformed Heisenberg-Weyl algebra with reflection if $N = 1$ [25].
Next we consider the set $\left\{\tilde{Q}_{\xi_j}, \tilde{P}_{\xi_j}\right\}_{j=1}^{N}$ of self-adjoint operators on $L^2(\mathbb{R}^N, \omega_k(x)dx)$ defined by

$$\tilde{Q}_{\xi_j} = d\ Q_{\xi_j} - b\ P_{\xi_j}, \quad \tilde{P}_{\xi_j} = -c\ Q_{\xi_j} + a\ P_{\xi_j}, \quad j = 1, \dots, N,$$

where a, b, c and d are real numbers such that $ad - bc = 1$. We prove that these operators satisfy the same commutation relations as in (1.2). We introduce a new family $\left\{D_k^M\right\}$ of integral transforms depending on two parameters; one is a matrix $M \in SL(2, \mathbb{R})$ and the other is a multiplicity function k on root system, and preserving the commutation relations (1.2). More precisely, we prove the following identities:

$$\begin{aligned} D_k^M \circ Q_{\xi_j} \circ \left(D_k^M\right)^{-1} &= \tilde{Q}_{\xi_j} = d\ Q_{\xi_j} - b\ P_{\xi_j}, \\ D_k^M \circ P_{\xi_j} \circ \left(D_k^M\right)^{-1} &= \tilde{P}_{\xi_j} = -c\ Q_{\xi_j} - a\ P_{\xi_j}. \end{aligned}$$

This new family of integral transforms is interesting for several reasons such as when considering only the parameter k, D_k^M generalizes many operations such as canonical Hankel transform, linear canonical transform etc. Moreover, various choices of the matrix M yield different integral transforms such as:
• The Dunkl transform which was introduced and studied by Dunkl [9]. Dunkl's results were completed and extended later by de Jeu [5]. The Dunkl transform which is a generalization of the classical Fourier transform and Hankel transform, is defined, for $f \in L^1(\mathbb{R}^N, \omega_k(x)dx)$, by [9]:

$$D_k f(x) = \frac{c_k}{2^{\gamma+N/2}} \int_{\mathbb{R}^N} f(y) E_k(-ix, y)\omega_k(y)dy,$$

where E_k is the Dunkl kernel (see[8, 23]) and $c_k^{-1} = \int_{\mathbb{R}^N} e^{-|x|^2} w_k(x)dx$.
• The fractional Dunkl transform which is defined by [14, 15]

$$D_k^{\alpha} f(x) = \begin{cases} \frac{c_k e^{i(\gamma+N/2)((\alpha-2n\pi)-\hat{\alpha}\pi/2)}}{(2|\sin\alpha|)^{\gamma+N/2}} \int_{\mathbb{R}^N} e^{\frac{i}{2}(|x|^2+|y|^2)\cot\alpha} E_k(-ix/\sin(\alpha), y) f(y)\ dy, & (2n-1)\pi < \alpha < (2n+1)\pi, \\ f(x), & \alpha = 2n\pi, \\ f(-x), & \alpha = (2n+1)\pi, \end{cases} \tag{1.3}$$

where $n \in \mathbb{Z}$ and $\hat{\alpha} = \operatorname{sgn}(\sin\alpha)$.
• A Dunkl-type analogues of the classical Fresnel transform which was investigated by Rösler [29] in the study of the semi-group $\left(e^{it\overline{\Delta_k}}\right)_{t\geq 0}$ (Δ_k is the Dunkl Laplacian operator) as follows:

$$e^{it\overline{\Delta_k}} f(x) = \begin{cases} \frac{c_k}{(2it)^{\gamma+(N/2)}} \int_{\mathbb{R}^N} e^{\frac{i}{2t}(|x|^2+|y|^2)} E_k(-ix/t, y) f(y)\omega_k(y)\ dy, & t > 0 \\ f(x), & t = 0. \end{cases}$$

Also, D_k^M provides a unified framework for studying:
• Riemann-Lebesgue lemma.
• Reversibility property.
• Additivity property: Under certain conditions imposed on the functions f, we establish the following theorem:

$$D_k^{M_1} D_k^{M_2} f = e^{i\psi}\ D_k^{M_1 M_2} f,$$

where M_1 and M_2 are an arbitrary matrix of $SL(2, \mathbb{R})$ and ψ a constant phase.
• Operational formula: we prove that D_k^M leaves invariant the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ and satisfies on $\mathcal{S}(\mathbb{R}^N)$ the following identities:

$$\begin{aligned} D_k^M \circ Q_{\xi_j} \circ \left(D_k^M\right)^{-1} &= \tilde{Q}_{\xi_j} = d\ Q_{\xi_j} - b\ P_{\xi_j}, \\ D_k^M \circ P_{\xi_j} \circ \left(D_k^M\right)^{-1} &= \tilde{P}_{\xi_j} = -c\ Q_{\xi_j} - a\ P_{\xi_j}, \end{aligned}$$

where $j = 1, \dots, N$.
• Bochner type identity: by application of the Dunkl type Funk-Hecke formula for k-spherical harmonics which was established by Y. Xu [34], we derive the following identity

$$D_k^M f(x) = p(x)\mathcal{H}^M_{n+\gamma+(N/2)-1}\psi(|x|), \tag{1.4}$$

where f is of the form $f(x) = p(x)\psi(|x|)$ (p is a homogeneous polynomial of degree n and satisfies the Dunkl-Laplace equation $\Delta_k p(x) = 0$ and ψ is a one-dimensional function on $\mathbb{R}_+$) and where $\mathcal{H}^M_{n+\gamma+(N/2)-1}$ is the canonical Hankel transform of order $n + \gamma + (N/2) - 1$.
We note that the Bochner type identity (1.4) reduces to the Bochner identity for the Dunkl transform; which was proved by Dunkl in [9] and later in [1] using a representation theory approach, if $M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
As application we obtain the following results:

$$D_k^M \psi_{m,n,j}(x) \quad = \quad \lambda_{m,n,a,b} \quad e^{\frac{i(ac+bd)}{2(a^2+b^2)}|x|^2} \;\; \psi_{m,n,j}\left(\frac{x}{\sqrt{a^2+b^2}}\right),$$

where $\psi_{m,n,j}(x)$ is the generalized Laguerre functions and $\lambda_{m,n,a,b}$ is an appropriate constant.
• Master formula: we prove the following identity. Let P be a homogeneous polynomial of degree n. Then

$$D_k^M f_n(x) \quad = \quad \lambda_{n,a,b} \quad e^{\frac{i(ac+bd)}{2(a^2+b^2)}|x|^2} \;\; f_n\left(\frac{x}{\sqrt{a^2+b^2}}\right),$$

where f_n is of the form $f_n(x) = e^{-\frac{|x|^2}{2}} e^{-\frac{\Delta_k}{4}} p(x)$ and $\lambda_{n,a,b}$ is an appropriate constant.
• Plancherel formula: we prove that the generalized Dunkl transform D_k^M initially defined on $L^1(\mathbb{R}^N, \omega_k(x)dx)$ has a unique extension to a unitary operator of $L^2(\mathbb{R}^N, \omega_k(x)dx)$.
• A generalized Dunkl-Schrödinger operator: we prove the following result. Let $\left\{ M(\tau) = \begin{bmatrix} a(\tau) & b(\tau) \\ c(\tau) & d(\tau) \end{bmatrix} \right\}_{\tau\in\mathbb{R}}$ be a one-parameter subgroup of $SL(2,\mathbb{R})$. Then the family $\left\{ D_k^{M(\tau)} \right\}_{\tau\in\mathbb{R}}$ is a $\mathcal{C}_0$-group of unitary operators on $L^2(\mathbb{R}^N, \omega_k(x)dx)$ and we derive its generator $\mathcal{L}$ which is the self-adjoint extension of

$$\mathcal{L}_{|\mathcal{S}(\mathbb{R}^N)} = -a^{'}(0)\mathbb{H} + c^{'}(0)\mathbb{E} + b^{'}(0)\mathbb{F},$$

where

$$\mathbb{E} = i\frac{|x|^2}{2}, \quad \mathbb{F} = i\frac{\Delta_k}{2}, \quad \mathbb{H} = (\gamma + N/2) + \sum_{j=1}^{N} x_j \frac{\partial}{\partial x_j},$$

are the $sl(2)$ triple which was first introduced by Heckman in [18] and later in [1, 2, 3] where the authors showed that there exists an infinitesimal representation of the Lie algebra $sl(2,\mathbb{R})$ on the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ that can be used as a crucial (and surprising) tool to treat various problems related to the theory of Dunkl operators. As application we solve the following generalized Dunkl-Schrödinger equation

$$\begin{cases} i\dfrac{\partial}{\partial t}u(t,x) = -ia^{'}(0)\left((\gamma + N/2) + \displaystyle\sum_{j=1}^{N} x_j \frac{\partial}{\partial x_j}\right)u(t,x) - \left(c^{'}(0)\,\dfrac{|x|^2}{2} + b^{'}(0)\,\dfrac{\Delta_k}{2}\right)u(t,x), \\ u(0,x) = f(x) \in L^2(\mathbb{R}^N, \omega_k(x)dx). \end{cases}$$

This paper is organized as follows. Section 2 is devoted to an overview of the Dunkl theory. In section 3 we introduce the generalized Dunkl transform D_k^M on $\mathbb{R}^N$ with parameter $M \in SL(2,\mathbb{R})$. Riemann-Lebesgue lemma, reversibility property, additivity property, operational formula, Bochner type identity and master formula are derived in section 4. Section 5 and 6 are devoted to the extension of the generalized Dunkl transform D_k^M as an isometry from $L_k^2(\mathbb{R}^N)$ to itself and the intimate relationship between the generalized Dunkl transform and the quantum harmonic oscillator. In section 7 we present some interesting one-parameter subgroups of $SL(2,\mathbb{R})$ with the associated integral transform, its basic properties and the related Dunkl-Schrödinger operator and equation.

2 Background: Dunkl theory

In this section, we recall some notations and results on Dunkl operators, Dunkl transform, and generalized Hermite functions (see, [5, 6, 7, 8, 23, 27]).
Notation:
• We denote by $\mathbb{Z}_+$ the set of non-negative integers. For a multi-index $\nu = (\nu_1, \dots, \nu_N) \in \mathbb{Z}_+^N$, we write $|\nu| = \nu_1 + \cdots + \nu_N$. The $\mathbb{C}$-algebra of polynomial functions on $\mathbb{R}^N$ is denoted by $\mathcal{P} = \mathbb{C}[\mathbb{R}^N]$. It has a natural grading

$$\mathcal{P} = \bigoplus_{n\geq 0} \mathcal{P}_n,$$

where $\mathcal{P}_n$ is the subspace of homogenous polynomials of (total) degree n. $\mathcal{S}(\mathbb{R}^N)$ is the Schwartz space of rapidly decreasing functions on $\mathbb{R}^N$ and $\mathcal{C}_0(\mathbb{R}^N)$ is the space of continuous functions on $\mathbb{R}^N$ vanishing at infinity.

- For $z = x + iy$ $\Big((x, y) \in \mathbb{R}^2\backslash\{(x, 0); x \leq 0\}\Big)$, we denote by $z^a = e^{a\ln(z)}$ where

$$\ln(z) = \ln\sqrt{x^2+y^2} + 2i\arctan\left(\frac{y}{x+\sqrt{x^2+y^2}}\right)$$

is the principal branch of the complex logarithmic function. Then we can write:

$$\left(e^{i\alpha}\right)^{\gamma+\frac{N}{2}} = e^{i(\gamma+\frac{N}{2})(\alpha-2k\pi)} \text{ with } (2k-1)\pi < \alpha < (2k-1)\pi \text{ and } k \in \mathbb{Z}.$$

2.1 Dunkl operators and Dunkl Kernel

In $\mathbb{R}^N$, we consider the standard inner product

$$\langle x,\, y\rangle = \sum_{k=1}^N x_k y_k.$$

We shall use the same notation for its bilinear extension to $\mathbb{C}^N \times \mathbb{C}^N$. For $x \in \mathbb{R}^N$, denote $|x| = \sqrt{\langle x,\, x\rangle}$.
For $u \in \mathbb{R}^N\backslash\{0\}$, let σ_u be the reflection in the hyperplane $(\mathbb{R}u)^\perp$ orthogonal to u

$$\sigma_u(x) = x - 2\frac{\langle u,\, x\rangle}{|u|^2}u, \quad x \in \mathbb{R}^N. \tag{2.1}$$

A root system is a finite spanning set $\mathcal{R} \subset \mathbb{R}^N$ of nonzero vectors such that, for every $u \in \mathcal{R}$, σ_u preserves $\mathcal{R}$. We shall always assume that $\mathcal{R}$ is reduced, i.e. $\mathcal{R} \cap \mathbb{R}u = \pm u$, for all $u \in \mathcal{R}$. Each root system can be written as a disjoint union $\mathcal{R} = \mathcal{R}^+ \cup (-\mathcal{R}^+)$, where $\mathcal{R}^+$ and $(-\mathcal{R}^+)$ are separated by a hyperplane through the origin. The subgroup $G \subset O(N)$ generated by the reflections $\{\sigma_u;\ u \in \mathcal{R}\}$ is called the finite reflection group associated with $\mathcal{R}$. Henceforth, we shall normalize $\mathcal{R}$ so that $\langle u, u\rangle = 2$ for all $u \in \mathcal{R}$. This simplifies formulas, without loss of generality for our purposes. We refer to [19] for more details on the theory of root systems and reflection groups.
A multiplicity function on $\mathcal{R}$ is a G-invariant function $k : \mathcal{R} \to \mathbb{C}$, i.e. $k(\sigma u) = k(u)$, for fall $u \in \mathcal{R}$ and $\sigma \in G$. The $\mathbb{C}$-vector space of multiplicity functions on $\mathcal{R}$ is denoted by $\mathfrak{K}$. The dimension of $\mathfrak{K}$ is equal to the number of G-orbits in $\mathcal{R}$. We set $\mathfrak{K}^+$ to be the set of multiplicity functions k such that $k(u) \geq 0$ for all $u \in \mathcal{R}$.

For $\xi \in \mathbb{C}^N$ and $k \in \mathfrak{K}$, C. Dunkl [7] defined a family of first order differential-difference operators $T_\xi(k)$ that play the role of the usual partial differentiation. Dunkl's operators are defined by

$$T_\xi(k)f(x) := \partial_\xi f(x) + \sum_{\eta\in\mathcal{R}^+} k(\eta) < \eta, \xi > \frac{f(x) - f(\sigma_\eta x)}{\langle \eta, x\rangle}, \quad f \in \mathcal{C}^1(\mathbb{R}^N). \tag{2.2}$$

Here ∂_ξ denotes the derivative in the direction of ξ. Thanks to the G-invariance of the function k, this definition is independent of the choice of the positive subsystem $\mathcal{R}_+$. The operators $T_\xi(k)$ are homogeneous of degree (-1). Moreover, by the G-invariance of the multiplicity function k, the Dunkl operators satisfy

$$h \circ T_\xi(k) \circ h^{-1} = T_{h\xi}(k), \quad \forall\ h \in G,$$

where $h.f(x) = f(h^{-1}x)$. The most striking property of Dunkl operators $T_\xi(k)$, which is the foundation for rich analytic structures with them, is the following

Theorem 2.1 *For fixed k, $T_\xi(k) \circ T_\eta(k) = T_\eta(k) \circ T_\xi(k)$, $\quad \forall \xi, \eta \in \mathbb{R}^N$.*

This result was obtained in [7] by a clever direct argumentation. An alternative proof, relying on Koszul complex ideas, is given in [10].
The Dunkl operators T_ξ have the following regularity properties:

Theorem 2.2
(1) *If $f \in \mathcal{C}^m(\mathbb{R}^N)$ with $m \geq 1$, then $T_\xi f \in \mathcal{C}^{m-1}(\mathbb{R}^N)$.*
(2) *T_ξ leaves $\mathcal{C}_c^\infty(\mathbb{R}^N)$ and $\mathcal{S}(\mathbb{R}^N)$ invariant.*
(3) *(Cf. [9].) Let $k \geq 0$. For every $f \in \mathcal{S}(\mathbb{R}^N)$ and $g \in \mathcal{C}_b^1(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N} T_\xi f(x)\overline{g(x)}w_k(x)dx = -\int_{\mathbb{R}^N} f(x)\overline{T_\xi g(x)}w_k(x)dx. \tag{2.3}$$

For $k \in \mathfrak{K}^+$, there exists a generalization of the usual exponential kernel $e^{\langle .,. \rangle}$ by means of the Dunkl system of differential equations.

Theorem 2.3 *Assume that $k \in \mathfrak{K}^+$.*
(i) (Cf. [8, 23].) There exists a unique holomorphic function E_k on $\mathbb{C}^N \times \mathbb{C}^N$ characterized by

$$\begin{cases} T_\xi(k)E_k(z,w) = \langle \xi, w\rangle E_k(z,w), \quad \forall\, \xi \in \mathbb{C}^N, \\ E_k(0,w) = 1, \end{cases} \tag{2.4}$$

Further, the Dunkl kernel E_k is symmetric in its arguments and satisfies

$$E_k(\lambda z, w) = E_k(z, \lambda w), \quad \overline{E_k(z,w)} = E_k(\overline{z}, \overline{w}) \quad \text{and} \quad E_k(gz, gw) = E_k(z,w) \tag{2.5}$$

for all $z, w \in \mathbb{C}^N$, $\lambda \in \mathbb{C}$ and $g \in G$.
(ii) (Cf. [26].) For all $x \in \mathbb{R}^N$, $y \in \mathbb{C}^N$ and all multi-indices $\nu \in \mathbb{Z}_+^N$,

$$|\partial_y^\nu E_k(x,y)| \leq |x|^{|\nu|} \max_{g\in G} e^{Re\langle gx, y\rangle}.$$

In particular,

$$|\partial_y^\nu E_k(x,y)| \leq |x|^{|\nu|} e^{|x||Re y|}, \tag{2.6}$$

and for all $x, y \in \mathbb{R}^N$:

$$|E_k(ix,y)| \leq 1. \tag{2.7}$$

Remark 2.1
- *When $k = 0$, we have $E_0(z,w) = e^{\langle z,w\rangle}$ for $z, w \in \mathbb{C}^N$.*
- *For complex-valued k, there is a detailed investigation of (2.4) by Opdam [23]. Theorem 2.3 (i) is a weak version of Opdam's result.*
- *M. de Jeu had already an estimate on E_k with slightly weaker bounds in [5], differing by an additional factor $\sqrt{|G|}$.*

The counterpart of the usual Laplacian is the Dunkl-Laplacian operator defined by $\Delta_k := \sum_{i=0}^{N} T_{\xi_i}(k)^2$, where $\{\xi_j\}_{j=1}^N$ is an arbitrary orthonormal basis of $(\mathbb{R}^N, \langle .,, \rangle)$. It is homogeneous of degree -2. By the normalization $\langle u, u\rangle = 2$, we can rewrite Δ_k as

$$\Delta_k f(x) = \Delta f(x) + 2 \sum_{\eta\in\mathcal{R}^+} k(\eta) \left[\frac{\langle \nabla f(x), \eta\rangle}{\langle \eta, x\rangle} - \frac{f(x) - f(\sigma_\eta x)}{\langle \eta, x\rangle^2} \right], \tag{2.8}$$

where Δ and ∇ denote the usual Laplacian and gradient operators, respectively (cf.[7]).

2.2 Dunkl transform

For fixed $k \in \mathfrak{K}^+$, let ω_k be the weight function on $\mathbb{R}^N$ defined by

$$\omega_k(x) = \prod_{\eta\in\mathcal{R}^+} |\langle \eta, x\rangle|^{2k(\eta)}.$$

It is G-invariant and homogeneous of degree 2γ, with the index

$$\gamma = \gamma(k) = \sum_{\eta\in\mathcal{R}^+} k(\eta).$$

Let dx be the Lebesgue measure corresponding to $\langle .,. \rangle$ and set $L_k^p(\mathbb{R}^N)$ the space of measurable functions on $\mathbb{R}^N$ such that

$$\|f\|_{k,p} \quad = \quad \left(\int_{\mathbb{R}^N} |f(x)|^p \omega_k(x)\, dx \right)^{\frac{1}{p}} < +\infty, \quad \text{if } 1 \leq p < +\infty.$$

Following Dunkl [9], we define the Dunkl transform on the space $L^1_k(\mathbb{R}^N)$ by

$$D_k f(x) = \frac{c_k}{2^{\gamma+N/2}} \int_{\mathbb{R}^N} f(y) E_k(-ix, y)\omega_k(y)dy,$$

where c_k denotes the Mehta-type constant $c_k = \left(\int_{\mathbb{R}^N} e^{-|x|^2} w_k(x)dx \right)^{-1}$. Many properties of the Euclidean Fourier transform carry over to the Dunkl transform. In particular:

Theorem 2.4 *(Cf. [5, 9].)*
*a) **(Riemann-Lebesgue lemma)** For all $f \in L^1_k(\mathbb{R}^N)$, the Dunkl transform $D_k f$ belongs to $C_0(\mathbb{R}^N)$.*
*b) (L^1**-inversion**) For all $f \in L^1_k(\mathbb{R}^N)$ with $D_k f \in L^1_k(\mathbb{R}^N)$,*

$$D_k^2 f = \check{f}, \ \text{a.e}, \ \text{where} \ \check{f}(x) = f(-x). \tag{2.9}$$

c) The Dunkl transform $f \to D_k f$ is an automorphism of $\mathcal{S}(\mathbb{R}^N)$.
d) For all $f \in \mathcal{S}(\mathbb{R}^N)$, the Dunkl transform satisfies the following identities:

$$D_k T_\xi f(x) = i\langle \xi, x\rangle D_k f(x), \qquad T_\xi D_k f(x) = -iD_k \left[\langle \xi, y\rangle f(y)\right](x) \tag{2.10}$$

*e) **(Plancherel Theorem)***
i) If $f \in L^1_k(\mathbb{R}^N) \cap L^2_k(\mathbb{R}^N)$, then $D_k f \in L^2_k(\mathbb{R}^N)$ and $\|D_k f\|_{k,2} = \|f\|_{k,2}$.
ii) The Dunkl transform has a unique extension to an isometric isomorphism of $L^2_k(\mathbb{R}^N)$. The extension is also denoted by $f \to D_k f$.

We conclude this subsection with two important reproducing properties for the Dunkl kernel due to [9].

Theorem 2.5 *(Cf. [9].) For all $p \in \mathcal{P}$ and $y, z \in \mathbb{C}^N$,*
(1) $\frac{c_k}{2^{\gamma+N/2}} \int_{\mathbb{R}^N} e^{-\Delta_k/2} p(x) \ E_k(x, y) \ \omega_k(x) \ e^{-|x|^2/2}dx = e^{l(y)/2} p(y).$
(2) $\frac{c_k}{2^{\gamma+N/2}} \int_{\mathbb{R}^N} E_k(x, y)E_k(x, z) \ \omega_k(x) \ e^{-|x|^2/2}dx = e^{(l(y)+l(z))/2} E_k(y, z).$

2.3 Generalized Hermite functions

For an arbitrary finite reflection group G and for any non-negative multiplicity function k, Rösler [24] introduced a complete systems of orthogonal polynomials with respect to the weight function $\omega_k(x) \ e^{-|x|^2}dx$, called generalised Hermite polynomials. The key to their definition is the following bilinear form on $\mathcal{P}$, which was introduced in [8]:

$$[p, q]_k := (p(T)q)(0) \quad \text{for } p, q \in \mathcal{P}.$$

The homogeneity of the Dunkl operators implies that $\mathcal{P}_n \perp \mathcal{P}_m$ for $n \neq m$. Moreover, if $p, q \in \mathcal{P}_n$, then

$$[p, q]_k = 2^n c_k \int_{\mathbb{R}^N} e^{-\Delta_k/4} p(x) \ e^{-\Delta_k/4} q(x) \ \omega_k(x) \ e^{-|x|^2}dx.$$

This is obtained from Theorem 3.10 of [8] by rescaling, see lemma (2.1) in [24]. So in particular, $[.,.]_k$ is a scalar product on the vector space $\mathcal{P}_{\mathbb{R}} = \mathbb{R}[x_1, \ldots, x_N]$.

Now let $\{\varphi_\nu, \ \nu \in \mathbb{Z}^N_+\}$ be an (arbitrary) orthonormal basis of $\mathcal{P}_{\mathbb{R}}$ with respect to $[.,.]_k$ such that $\varphi_\nu \in \mathcal{P}_{|\nu|}$ (For details concerning the construction and canonical choices of such a basis, we refer to [24]). Then the generalised Hermite polynomials $\{H_\nu, \nu \in \mathbb{Z}^N_+\}$ and the (normalised) generalised Hermite functions $\{h_\nu, \nu \in \mathbb{Z}^N_+\}$ associated with G, k and $\{\varphi_\nu\}$ are defined by

$$H_\nu(x) := 2^{|\nu|} e^{-\Delta_k/4} \varphi_\nu(x) \quad \text{and} \quad h_\nu(x) := \frac{\sqrt{c_k}}{2^{|\nu|/2}} \ e^{-|x|^2/2} H_\nu(x) \quad (x \in \mathbb{R}^N). \tag{2.11}$$

We list some standard properties of generalised Hermite functions that we shall use in this article.

Theorem 2.6 *(Cf. [24].) Let $\{H_\nu\}$ and $\{h_\nu\}$ be the Hermite polynomials and Hermite functions associated with the basis $\{\varphi_\nu\}$ on $\mathbb{R}^N$ and let $x, y \in \mathbb{R}^N$. Then*
(1) The h_ν satisfy $h_\nu(-x) = (-1)^{|\nu|} h_\nu(x)$.
(2) $\{h_\nu, \nu \in \mathbb{Z}^N_+\}$ is an orthonormal basis of $L^2_k(\mathbb{R}^N)$.

(3) *The* h_ν *are eigenfunctions of the Dunkl transform on* $L^2_k(\mathbb{R}^N)$, *with* $D_k h_\nu = (-i)^{|\nu|} h_\nu$.
(4) *(Mehler formula) For* $r \in \mathbb{C}$ *with* $|r| < 1$,

$$\sum_{\nu\in\mathbb{Z}^N_+} \frac{H_\nu(x)H_\nu(y)}{2^{|\nu|}} r^{|\nu|} = \frac{e^{-\frac{r^2(|x|^2+|y|^2)}{1-r^2}}}{(1-r^2)^{\gamma+(N/2)}} E_k\left(\frac{2zx}{1-z^2}, y\right).$$

Throughout this paper, $\mathcal{R}$ denotes a root system in $\mathbb{R}^N$, $\mathcal{R}^+$ a fixed positive subsystem of $\mathcal{R}$ and k a nonnegative multiplicity function defined on $\mathcal{R}$.

3 A generalized Dunkl transform.

3.1 Some remarks on Dunkl operators

Consider the Hilbert space $L^2_k(\mathbb{R}^N)$ equipped with the inner product $\langle .,.\rangle_k$ given by

$$\langle f, g\rangle_k = \int_{\mathbb{R}^N} f(x)\overline{g(x)} w_k(x)dx.$$

For each $\xi \in \mathbb{R}^N$, we denote by Q_ξ the multiplication operator

$$Q_\xi f(x) = \langle \xi, x\rangle f(x)$$

acting in $L^2_k(\mathbb{R}^N)$ with domain $D(Q_\xi) = \left\{f \in L^2_k(\mathbb{R}^N);\ Q_\xi f \in L^2_k(\mathbb{R}^N)\right\}$ and P_ξ the operator defined on $\mathcal{S}(\mathbb{R}^N)$ by

$$P_\xi f(x) = -iT_\xi f(x).$$

3.1.1 Commutation relations for the Dunkl and multiplication operators

Let ξ and $\xi' \in \mathbb{R}^N$. We denote by $E_{\xi,\xi'}$ the bounded linear operator on $L^2_k(\mathbb{R}^N)$ defined by

$$E_{\xi,\xi'} f(x) =< \xi, \xi' > f(x) + \sum_{\eta\in\mathcal{R}^+} k(\eta) < \eta, \xi >< \eta, \xi' > f(\sigma_\eta x).$$

Remark 3.1
(i) *The operator* $E_{\xi,\xi'}$ *is symmetric with respect to* ξ *and* ξ'. *In other words, for all* ξ *and* $\xi' \in \mathbb{R}^N$, $E_{\xi,\xi'} = E_{\xi',\xi}$.
(ii) *When the multiplicity function* $k = 0$, *the operator* $E_{\xi,\xi'}$ *reduces to* $< \xi, \xi' > I$ *(I is the identity operator).*

The following lemma will be useful to our study of the commutator relations between the Dunkl and multiplication operators.

Lemma 3.1 *Let* ξ *and* $\xi' \in \mathbb{R}^N$. *The following equality hold in* $\mathcal{S}(\mathbb{R}^N)$:

$$[T_\xi, Q_{\xi'}] = E_{\xi,\xi'}. \tag{3.1}$$

Proof. Let $\xi, \xi' \in \mathbb{R}^N$. Since $\mathcal{S}(\mathbb{R}^N)$ is an invariant subspace for T_ξ and for $Q_{\xi'}$ (see [5]), then

$$\mathcal{S}(\mathbb{R}^N) \subset D(T_\xi Q_{\xi'}) \cap D(Q_{\xi'} T_\xi).$$

Let $f \in \mathcal{S}(\mathbb{R}^N)$. Obviously,

$$P_\xi Q_{\xi'} f(x) = -iT_\xi(\langle \xi', x\rangle f(x)).$$

Applying the Dunkl operator T_ξ to $\langle \xi', .\rangle f$, we obtain

$$\begin{aligned} T_\xi(\langle \xi', x\rangle f(x)) &= \partial_\xi(\langle \xi', x\rangle f(x)) \\ &+ \sum_{\eta\in R_+} k(\eta)\langle \eta, \xi\rangle \frac{\langle \xi', x\rangle f(x) - \langle \xi', \sigma_\eta x\rangle f(\sigma_\eta x)}{\langle \eta, x\rangle}. \end{aligned} \tag{3.2}$$

The product rule for the directional derivative of $\langle \xi^{'}, . \rangle f$, gives

$$\partial_\xi(\langle \xi^{'}, . \rangle f) = \langle \xi^{'}, \xi \rangle f + \langle \xi^{'}, . \rangle \partial_\xi f. \tag{3.3}$$

Now substituting (3.3) into (3.2) and replacing σ_η on the left-hand side (3.2) by their expression giving in (2.1), we find that

$$\begin{aligned} T_\xi(\langle \xi^{'}, x \rangle f(x)) &= \langle \xi^{'}, \xi \rangle f(x) + \langle \xi^{'}, x \rangle \left[\partial_\xi f(x) + \sum_{\eta \in R_+} k(\eta) \langle \eta, \xi \rangle \frac{f(x) - f(\sigma_\eta x)}{\langle x, \eta \rangle} \right] \\ &+ \sum_{\eta \in R_+} k(\eta) \langle \eta, \xi \rangle \langle \eta, \xi^{'} \rangle f(\sigma_\eta x), \\ &= \langle \xi^{'}, \xi \rangle f(x) + \langle \xi^{'} x \rangle T_\xi f(x) + \sum_{\eta \in R_+} k(\eta) \langle \eta, \xi \rangle \langle \eta, \xi^{'} \rangle f(\sigma_\eta x). \end{aligned}$$

Hence,

$$[T_\xi, Q_{\xi'}] = E_{\xi,\xi'}.$$

To each orthonormal basis $\{\xi_j\}_{j=1}^N$ from $\mathbb{R}^N$ we associate the following family $\{Q_{\xi_j}, P_{\xi_j}\}_{j=1}^N$ of operators. Then

Corollary 3.1 *The operators Q_{ξ_j} and P_{ξ_j}, $j = 1, \dots, N$ satisfy on $\mathcal{S}(\mathbb{R}^N)$ the commutation relations:*

$$[Q_{\xi_j}, Q_{\xi_k}] = 0, \quad [P_{\xi_j}, P_{\xi_k}] = 0, \quad [Q_{\xi_j}, P_{\xi_k}] = iE_{\xi_j,\xi_k}, \quad j, k = 1, \dots, N. \tag{3.4}$$

Proof. The first equality is clear, the second by Theorem 2.1, the third by (3.1).

Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be any reel unimodular matrix. Let $Q_\xi^{'}$ and $P_\xi^{'}$ be the two operators defined on $\mathcal{S}(\mathbb{R}^N)$ by

$$Q_\xi^{'} = d\ Q_\xi - b\ P_\xi, \quad P_\xi^{'} \;\; = \;\; -c\ Q_\xi + a\ P_\xi. \tag{3.5}$$

It is easily to see that these two operators can be written in matrix form as follows:

$$\begin{pmatrix} Q_\xi^{'} \\ P_\xi^{'} \end{pmatrix} = M^{-1} \begin{pmatrix} Q_\xi \\ P_\xi \end{pmatrix}.$$

By means of direct calculation one can verify that these operators satisfy the following commutation relations.

Proposition 3.1 *Let $\{\xi_j\}_{j=1}^N$ be any orthonormal basis of $\mathbb{R}^N$. The operators $Q_{\xi_j}^{'}$ and $P_{\xi_j}^{'}$, $j = 1, \dots, N$ satisfy on $\mathcal{S}(\mathbb{R}^N)$ the commutation relations:*

$$[Q_{\xi_j}^{'}, Q_{\xi_k}^{'}] = 0, \quad [P_{\xi_j}^{'}, P_{\xi_k}^{'}] = 0, \quad [Q_{\xi_j}^{'}, P_{\xi_k}^{'}] = iE_{\xi_j,\xi_k}, \quad j, k = 1, \dots, N.$$

Proof. Let ξ and $\xi^{'} \in \mathbb{R}^N$. The commutator of $Q_\xi^{'}$ and $Q_{\xi'}^{'}$ is

$$\begin{aligned} [Q_\xi^{'}, Q_{\xi'}^{'}] &= [dQ_\xi - bP_\xi, dQ_{\xi'} - bP_{\xi'}] \\ &= d^2[Q_\xi, Q_{\xi'}] + b^2[P_\xi, P_{\xi'}] + bd[P_{\xi'}, Q_\xi] - bd[P_\xi, Q_{\xi'}]. \end{aligned}$$

Since $[Q_\xi, Q_{\xi'}] = 0 = [P_\xi, P_{\xi'}]$ and $[P_{\xi'}, Q_\xi] = E_{\xi,\xi'} = [P_\xi, Q_{\xi'}]$, we conclude

$$[Q_\xi^{'}, Q_{\xi'}^{'}] = 0.$$

In a similar way, the commutator of $P_\xi^{'}$ and $P_{\xi'}^{'}$ is

$$\begin{aligned} [P_\xi^{'}, P_{\xi'}^{'}] &= [-cQ_\xi + aP_\xi, -cQ_{\xi'} + aP_{\xi'}] \\ &= c^2[Q_\xi, Q_{\xi'}] + a^2[P_\xi, P_{\xi'}] + ac[P_{\xi'}, Q_\xi] - ac[P_\xi, Q_{\xi'}] \\ &= 0. \end{aligned}$$

Proceeding as before, the commutator of $Q_\xi^{'}$ and $P_{\xi'}^{'}$ is

$$\begin{aligned} [Q_\xi^{'}, P_{\xi'}^{'}] &= [dQ_\xi - bP_\xi, -cQ_{\xi'} + aP_{\xi'}] \\ &= -dc[Q_\xi, Q_{\xi'}] - ab[P_\xi, P_{\xi'}] - ad[P_{\xi'}, Q_\xi] + bc[P_\xi, Q_{\xi'}] \\ &= i(ad - bc)E_{\xi,\xi'} \\ &= iE_{\xi,\xi'}. \end{aligned}$$

3.2 The generalized Dunkl kernel

In this subsection, we construct a family of a unitary operator $\mathcal{J}: L_k^2(\mathbb{R}^N) \longrightarrow L_k^2(\mathbb{R}^N)$ which preserves the commutation relations (3.4). More precisely, we look for a unitary operator $\mathcal{J}$ from $L_k^2(\mathbb{R}^N)$ onto $L_k^2(\mathbb{R}^N)$ such that:

$$\mathcal{J} \circ Q_\xi \circ \mathcal{J}^{-1} = Q'_\xi = d\ Q_\xi - b\ P_\xi, \tag{3.6}$$
$$\mathcal{J} \circ P_\xi \circ \mathcal{J}^{-1} = P'_\xi = -c\ Q_\xi + a\ P_\xi. \tag{3.7}$$

We shall denote the transform operator as $\mathcal{J}_M$ by the unimodular matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det M = 1$. We proceed as in the proof of Theorem 5.7 in [3]. By the Schwartz kernel theorem, the operator $\mathcal{J}_M$ can be expressed by means of a distribution kernel $K_M(x,y)$. If we adopt Gelfand's notation on a generalized functions, we may write the operator $\mathcal{J}_M$ on $L_k^2(\mathbb{R}^N)$ as an 'integral transform' against the measure $\omega_k(y)dy$:

$$\mathcal{J}_M(f)(x) = \int_{\mathbb{R}^N} f(y) K_M(x,y)\omega_k(y)dy. \tag{3.8}$$

According to (3.6) and (3.7), the $\mathcal{J}_M$ transform of $(Q_\xi f)(y)$ will be

$$\begin{aligned} \mathcal{J}_M(Q_\xi f)(x) &= (\mathcal{J}_M Q_\xi \mathcal{J}_M^{-1})(\mathcal{J}_M(f))(x) = (d\ Q_\xi - b\ P_\xi)\mathcal{J}_M(f)(x) \\ &= d\ Q_\xi(\mathcal{J}_M(f))(x) - b\ P_\xi(\mathcal{J}_M(f))(x). \end{aligned} \tag{3.9}$$

Similarly, the $\mathcal{J}_M$ transform of $(P_\xi f)(y)$ is

$$\begin{aligned} \mathcal{J}_M(P_\xi f)(x) &= (\mathcal{J}_M P_\xi \mathcal{J}_M^{-1})(\mathcal{J}_M(f))(x) = (-c\ Q_\xi + a\ P_\xi)(\mathcal{J}_M(f))(x) \\ &= -c\ Q_\xi(\mathcal{J}_M(f))(x) + a\ P_\xi(\mathcal{J}_M(f))(x). \end{aligned} \tag{3.10}$$

Rewriting the conditions (3.9) and (3.10) by means of the 'integral transform' (3.8):

$$\begin{aligned} \int_{\mathbb{R}^N} (Q_\xi f)(y) K_M(x,y)\omega_k(y)dy &= \int_{\mathbb{R}^N} f(y) Q_\xi^y K_M(x,y)\omega_k(y)dy \\ &= (d\ Q_\xi^x - b\ P_\xi^x) \int_{\mathbb{R}^N} f(y) K_M(x,y)\omega_k(y)dy, \\ &= \int_{\mathbb{R}^N} f(y)(d\ Q_\xi^x - b\ P_\xi^x) K_M(x,y)\omega_k(y)dy \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \int_{\mathbb{R}^N} P_\xi^y f(y) K_M(x,y)\omega_k(y)dy &= -\int_{\mathbb{R}^N} f(y) P_\xi^y K_M(x,y)\omega_k(y)dy \\ &= (-c\ Q_\xi^x + a\ P_\xi^x) \int_{\mathbb{R}^N} f(y) K_M(x,y)\omega_k(y)dy. \\ &= \int_{\mathbb{R}^N} f(y)(-c\ Q_\xi^x + a\ P_\xi^x) K_M(x,y)\omega_k(y)dy, \end{aligned} \tag{3.12}$$

where f is any test function (i.e. $f(x)\omega_k(x)^{\frac{1}{2}} \in \mathcal{S}(\mathbb{R}^n)$).
A sufficient condition for (3.11) and (3.12) to hold is that $K_M(x,y)$ satisfy the following differential-difference equations:

$$\begin{cases} Q_\xi^y K_M(x,y) = & (d\ Q_\xi^x - b\ P_\xi^x) K_M(x,y), \\ -P_\xi^y K_M(x,y) = & (-c\ Q_\xi^x + a\ P_\xi^x) K_M(x,y), \end{cases} \tag{3.13}$$

where Q_ξ^x and P_ξ^x act in the x variable.

Remark 3.2
• *In the case* $M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ *the system (3.13) reduces to*

$$\begin{cases} Q_\xi^y K_M(x,y) = & P_\xi^x K_M(x,y), \\ P_\xi^y K_M(x,y) = & Q_\xi^x K_M(x,y). \end{cases} \tag{3.14}$$

The Dunkl kernel $E_k(ix,y)$ is a solution of (3.14).
• *In the case $b=0$, $K_M(x,y)$ is a distribution kernel satisfies the differential equations*

$$\begin{cases} Q_\xi^y K_M(x,y) = & \frac{1}{a}\ Q_\xi^x K_M(x,y), \\ -P_\xi^y K_M(x,y) = & (-c\ Q_\xi^x + a\ P_\xi^x) K_M(x,y). \end{cases}$$

In this case we prove that the operator $\mathcal{J}_1 :\ L_k^2(\mathbb{R}^N) \longrightarrow L_k^2(\mathbb{R}^N)$ defined by

$$\mathcal{J}_1 f(x) = \frac{e^{i\frac{c}{2a}|x|^2}}{|a|^{\gamma+(N/2)}}\ f(x/a)$$

is unitary and we have on $\mathcal{S}(\mathbb{R}^N)$ the following identities:

$$\begin{aligned} \mathcal{J}_1 \circ Q_\xi \circ \mathcal{J}_1^{-1} &= \frac{1}{a}\ Q_\xi, \\ \mathcal{J}_1 \circ P_\xi \circ \mathcal{J}_1^{-1} &= -c\ Q_\xi + a\ P_\xi. \end{aligned}$$

Throughout this paper, we denote by $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ an arbitrary matrix in $SL(2,\mathbb{R})$.

Theorem 3.1 *Let $M \in SL(2,\mathbb{R})$ such that $b \neq 0$. Then the function*

$$E_k^M(x,y) = e^{\frac{i}{2}(\frac{d}{b}|x|^2 + \frac{a}{b}|y|^2)} E_k(-ix/b, y) \tag{3.15}$$

is a solution of

$$\begin{cases} Q_\xi^y E_k^M(x,y) = & (d\ Q_\xi^x - b\ P_\xi^x) E_k^M(x,y), \\ -P_\xi^y E_k^M(x,y) = & (-c\ Q_\xi^x + a\ P_\xi^x) E_k^M(x,y). \end{cases} \tag{3.16}$$

Proof. Clearly,

$$P_\xi^x E_k^M(x,y) = -ie^{\frac{i}{2}\frac{a}{b}|y|^2} T_\xi(e^{\frac{i}{2}\frac{d}{b}|x|^2} E_k(-ix/b, y)). \tag{3.17}$$

The product rule of the Dunkl operators T_ξ gives

$$\begin{aligned} T_\xi(e^{\frac{i}{2}\frac{d}{b}|x|^2} E_k(-ix/b, y)) &= T_\xi(e^{\frac{i}{2}\frac{d}{b}|x|^2}) E_k(-ix/b, y) + T_\xi(E_k(-ix/b, y)) e^{\frac{i}{2}\frac{d}{b}|x|^2} \\ &= \frac{id}{b}\langle x, \xi\rangle e^{\frac{i}{2}\frac{d}{b}|x|^2} E_k(-ix/b, y) - \frac{i}{b}\langle y, \xi\rangle e^{\frac{i}{2}\frac{d}{b}|x|^2} E_k(-ix/b, y). \end{aligned} \tag{3.18}$$

Hence by (3.17) and (3.18), we deduce

$$Q_\xi^y E_k^M(x,y) = (d\ Q_\xi^x - b\ P_\xi^x) E_k^M(x,y).$$

Similarly, we can show that

$$-P_\xi^y E_k^M(x,y) = (-c\ Q_\xi^x + a\ P_\xi^x) E_k^M(x,y). \tag{3.19}$$

We list some important properties of the kernel $E_k^M(x,y)$ in the following proposition.

Proposition 3.2 *Let $M \in SL(2,\mathbb{R})$ such that $b \neq 0$, $g \in G$ and $x,\ y \in \mathbb{R}^N$. Then*
1) $\overline{E_k^M(x,y)} = E_k^{M^{-1}}(y,x)$,
2) $E_k^M(gx,gy) = E_k^M(x,y)$,
3) $|E_k^M(x,y)| \leq 1$.

Proof. These statements are a direct consequence of Theorem 2.3.

3.3 The generalized Dunkl transfrom

Definition 3.1 *Let $M \in SL(2,\mathbb{R})$ such that $b \neq 0$. We define the generalized Dunkl transform D_k^M for $f \in L_k^1(\mathbb{R}^N)$ by*

$$D_k^M f(x) = \frac{c_k}{(2ib)^{\gamma+(N/2)}} \int_{\mathbb{R}^N} f(y) E_k^M(x,y) \omega_k(y) dy. \tag{3.20}$$

3.3.1 Case $b = 0$

In order to extend the Definition 3.1 for $b = 0$, we need another integral representation for D_k^M. We begin by the following lemma.

Lemma 3.2 *For $z \in \mathbb{C}^N$, let $l(z) = \sum_{i=1}^{N} z_i^2$. Then for all $z, \omega \in \mathbb{C}^N$,*

$$c_k \int_{\mathbb{R}^N} E_k(2z, x) E_k(2\omega, x) e^{-A|x|^2} \omega_k(x)\ dx = \frac{e^{\frac{l(z)+l(\omega)}{A}}}{A^{\gamma+N/2}} E_k(2z/A, \omega), \tag{3.21}$$

where A is a complex number such that $\Re(A) > 0$.

Proof. First compute this integral when $A > 0$.

$$\int_{\mathbb{R}^N} E_k(2z, x) E_k(2\omega, x) e^{-A|x|^2} \omega_k(x)\ dx = \int_{\mathbb{R}^N} E_k(2z, x) E_k(2\omega, x) e^{-|\sqrt{A}x|^2} \omega_k(x)\ dx.$$

By the change of variables $u = \sqrt{A}x$ and the homogeneity of ω_k, it follows that

$$\int_{\mathbb{R}^N} E_k(2z, x) E_k(2\omega, x) e^{-A|x|^2} \omega_k(x)\ dx = \frac{1}{A^{\gamma+N/2}} \int_{\mathbb{R}^N} E_k(2z/\sqrt{A}, x) E_k(2\omega/\sqrt{A}, x) e^{-|x|^2} \omega_k(x)\ dx.$$

Using Theorem 2.5 2), we find the equality (3.21) for $A > 0$. By analytic continuation, this holds for $\{A \in \mathbb{C} : \Re(A) > 0\}$.

Theorem 3.2 *Let $M \in SL(2, \mathbb{R})$ such that $a \neq 0$ and $b \neq 0$. Let $f \in L_k^1(\mathbb{R}^N) \cap L_k^2(\mathbb{R}^N)$ such that $D_k f \in L_k^1(\mathbb{R}^N)$. Then*
1)

$$D_k^M f(x) = \frac{c_k\ e^{i\varphi}}{|2a|^{\gamma+N/2}} \int_{\mathbb{R}^N} e^{\frac{i}{2}(\frac{c}{a}|x|^2 - \frac{b}{a}|y|^2)}\ E_k(ix/a, y) D_k f(y) \omega_k(y)\ dy, \tag{3.22}$$

where $\varphi = \frac{\pi}{2}\left(\gamma + \frac{N}{2}\right)(sgn(\frac{a}{b}) - sgn(b))$.
2) *If $a > 0$, then*

$$\lim_{b \to 0^+} D_k^M f(x) = \lim_{b \to 0^-} D_k^M f(x) = \frac{e^{i\frac{c}{2a}|x|^2}}{a^{\gamma+(N/2)}}\ f(x/a),\ \ a.e.$$

3) *If $a < 0$, then*

$$\begin{aligned} \lim_{b \to 0^+} D_k^M f(x) &= e^{-i\pi(\gamma+(N/2))}\ \frac{e^{i\frac{c}{2a}|x|^2}}{|a|^{\gamma+(N/2)}}\ f(x/a),\ \ a.e. \\ \lim_{b \to 0^-} D_k^M f(x) &= e^{i\pi(\gamma+(N/2))}\ \frac{e^{i\frac{c}{2a}|x|^2}}{|a|^{\gamma+(N/2)}}\ f(x/a),\ \ a.e. \end{aligned}$$

Definition 3.2 *We define the generalized Dunkl transform $D_k^M f$ for $b = 0$ by*

$$D_k^M f(x) = \frac{e^{i\frac{c}{2a}|x|^2}}{|a|^{\gamma+(N/2)}}\ f(x/a). \tag{3.23}$$

Proof of Theorem 3.2:
1) For any $\epsilon > 0$, define

$$F_\epsilon(x) = \int_{\mathbb{R}^N} f(y) g_\epsilon(y) \omega_k(y)\ dy,$$

where $g_\epsilon(y) =\ e^{-(\epsilon - \frac{i}{2}\frac{a}{b})|y|^2} E_k(-ix/b, y)$.
From (2.7), we deduce that $|g_\epsilon(y)| \leq 1$. Then $|f(y) g_\epsilon(y)| \leq |f(y)|$, so we can apply the dominated convergence theorem to get

$$\lim_{\epsilon \to 0} F_\epsilon(x) = c_k^{-1}\ e^{-\frac{i}{2}\frac{d}{b}|x|^2} D_k^M f(x). \tag{3.24}$$

Using Lemma 3.2, we can show

$$(2\epsilon - i(a/b))^{\gamma+N/2}\ D_k g_\epsilon(\xi) = e^{-\frac{|x|^2}{4\epsilon b^2 - 2iab}} e^{-\frac{|\xi|^2}{4\epsilon - 2i\frac{a}{b}}}\ E_k(-x/(2\epsilon b - ia), \xi). \tag{3.25}$$

Now applying the Parseval formula for the Dunkl transform (see Theorem 2.4, e)) and using (3.25), we obtain

$$(2\epsilon - i(a/b))^{\gamma+N/2}\ F_\epsilon(x) = e^{-\frac{|x|^2}{4\epsilon b^2 - 2iab}} \int_{\mathbb{R}^N} e^{-\frac{|\xi|^2}{4\epsilon - 2i\frac{a}{b}}} E_k(-x/(2\epsilon b - ia), \xi) D_k f(-\xi) \omega_k(\xi)\ d\xi.$$

(2.6) gives the following majorization:

$$|E_k(-x/(2\epsilon b - ia), \xi)| \leq e^{\frac{2|b|\epsilon|x||\xi|}{4\epsilon^2 b^2 + a^2}},$$

Hence,

$$\left| e^{-\frac{|\xi|^2}{4\epsilon - 2i\frac{a}{b}}} E_k(-x/(2\epsilon b - ia), \xi) \right| \quad \leq \quad e^{-r_1(\epsilon)|\xi|^2 + r_2(\epsilon)|\xi|}, \tag{3.26}$$

where

$$r_1(\epsilon) = \frac{\epsilon b^2}{4\epsilon^2 b^2 + a^2} \quad \text{and} \quad r_2(\epsilon) = \frac{2|b|\epsilon|x|}{4\epsilon^2 b^2 + a^2}.$$

As $r_1(\epsilon) > 0$, we deduce that

$$\sup_{s \geq 0}(-r_1(\epsilon)s^2 + r_2(\epsilon)s) = -\frac{r_2^2(\epsilon)}{4r_1(\epsilon)}. \tag{3.27}$$

Applying (3.26) and (3.27), we obtain

$$\left| e^{-\frac{|\xi|^2}{4\epsilon - 2i\frac{a}{b}}} E_k(-x/(2\epsilon b - ia), \xi) D_k f(-\xi) \right| \quad \leq \quad B_x |D_k f(-\xi)|.$$

where $B_x = \sup_{\epsilon \in]0,1]} e^{\frac{\epsilon\ |x|^2}{4\epsilon^2 b^2 + a^2}}$. The function $\xi \mapsto D_k f(-\xi)$ is in $L^1_k(\mathbb{R}^N)$, then the dominated convergence theorem implies

$$\left|\frac{a}{2b}\right|^{\gamma+N/2} e^{-i\frac{\pi}{2}(\gamma+N/2)\mathrm{sgn}(\frac{a}{b})} \lim_{\epsilon \to 0} F_\epsilon(x) = e^{\frac{-i|x|^2}{2ab}} \int_{\mathbb{R}^N} e^{-\frac{ib}{2a}|\xi|^2}\ E_k(-ix/a, \xi) D_k f(-\xi) \omega_k(\xi)\ d\xi. \tag{3.28}$$

Hence, (3.24) and (3.28) gives after simplification

$$D_k^M f(x) \quad = \quad \frac{c_k\ e^{i\varphi}}{|2a|^{\gamma+N/2}} \int_{\mathbb{R}^N} e^{\frac{i}{2}(\frac{c}{a}|x|^2 - \frac{b}{a}|y|^2)}\ E_k(-ix/a, y) D_k f(-y) \omega_k(y)\ dy. \tag{3.29}$$

Finally, if we make the change of variables $u = -y$ in (3.29), then we find (3.22).
2) and 3) follow from (3.22) together with the dominated convergence theorem and Theorem 2.4, b).

3.3.2 The generalized Dunkl transform in the rank-one case.

Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{R})$. We first observe that when the multiplicity function $k = 0$, the generalized Dunkl transform D_k^M coincides with the N-dimensional linear canonical transform $\mathcal{F}^M$ which is defined by (1.1). In the one-dimensional case ($N = 1$), the corresponding reflection group W is $\mathbb{Z}_2$ and the multiplicity function k is equal to $\mu + 1/2 \geq 0$. The kernel $E_k^M(x, y)$ defined by (3.15) becomes

$$E_\mu^M(x, y) \quad = \quad e^{\frac{i}{2}(\frac{d}{b}x^2 + \frac{a}{b}y^2)} E_\mu(-ix/b, y), \tag{3.30}$$

where $E_\mu(x, y)$ is the Dunkl kernel of type A_2 given by (see [27])

$$E_\mu(ix, y) = j_\mu(xy) + \frac{ixy}{2(\mu+1)} j_{\mu+1}(xy),$$

and j_μ denotes the normalized spherical Bessel function

$$j_\mu(x) := 2^\nu\Gamma(\mu+1)\frac{J_\mu(x)}{x^\mu} = \Gamma(\mu+1)\sum_{n=0}^{+\infty}\frac{(-1)^n(x/2)^{2n}}{n!\Gamma(n+\mu+1)}.$$

Here J_μ is the classical Bessel function (see, Watson [30]). The related generalized Dunkl transform D_μ^M in rank-one case takes the form

$$D_\mu^M f(x) = \begin{cases} \dfrac{1}{\Gamma(\mu+1)(2ib)^{\mu+1}}\displaystyle\int_{-\infty}^{+\infty} E_\mu^M(x,y)f(y)|y|^{2\mu+1}\,dy, & b\neq 0 \\ \frac{e^{\frac{i}{2}\frac{c}{a}x^2}}{|a|^{\mu+1}}\; f(x/a), & b=0. \end{cases} \tag{3.31}$$

Remark 3.3
• *The even part of the one-dimensional generalized Dunkl transform (3.31) coincides with the canonical Hankel transform which is defined by [32]:*

$$\mathcal{H}_\mu^M f(x) = \begin{cases} \dfrac{2}{\Gamma(\mu+1)(2ib)^{\mu+1}}\displaystyle\int_0^{+\infty} e^{\frac{i}{2}(\frac{d}{b}x^2+\frac{a}{b}y^2)}\; j_\mu\left(\frac{xy}{b}\right) f(y)y^{2\mu+1}\,dy, & b\neq 0, \\ \frac{e^{\frac{i}{2}\frac{c}{a}x^2}}{|a|^{\mu+1}}\; f(x/a), & b=0. \end{cases}$$

• *In the case where* $M = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$, $\alpha\in\mathbb{R}$, *the one-dimensional generalized Dunkl transform (3.31) becomes the fractional Hankel transform multiplied by the constant phase* $\left(e^{i\alpha}\right)^{\mu+1}$ *[20]*

$$\mathcal{H}_\mu^\alpha f(x) = \begin{cases} \dfrac{2e^{i(\mu+1)((\alpha-2n\pi)-\hat{\alpha}\pi/2)}}{\Gamma(\mu+1)(2|\sin(\alpha)|)^{\mu+1}}\displaystyle\int_0^{+\infty} e^{-\frac{i}{2}\cot(\alpha)(x^2+y^2)}\; j_\mu\left(\frac{xy}{\sin(\alpha)}\right) f(y)y^{2\mu+1}\,dy, & (2n-1)\pi<\alpha<(2n+1)\pi, \\ f(x), & \alpha=2n\pi, \\ f(-x), & \alpha=(2n+1)\pi. \end{cases}$$

4 Properties of generalized Dunkl transform.

In this section, we discuss basic properties of D_k^M for general M and k.

4.1 The reversibility property.

Theorem 4.1 *Let* $M\in SL(2,\mathbb{R})$.
1) *Suppose that* $b\neq 0$. *Then for all* $f\in L_k^1(\mathbb{R}^N)$, $D_k^M f$ *belongs to* $\mathcal{C}_0(\mathbb{R}^N)$ *and verifies*

$$\|D_k^M f\|_\infty \leq \frac{c_k}{(2|b|)^{\gamma+(N/2)}}\;\|f\|_{k,1}. \tag{4.1}$$

2) *For all* $f\in L_k^1(\mathbb{R}^N)$ *with* $D_k^M f\in L_k^1(\mathbb{R}^N)$,

$$(D_k^{M^{-1}}\circ D_k^M)f = f,\; a.e, \quad and \quad (D_k^M\circ D_k^{M^{-1}})f = f,\; a.e.$$

2) *The generalized Dunkl transform* D_k^M *is a one-to-one and onto mapping from* $\mathcal{S}(\mathbb{R}^N)$ *into* $\mathcal{S}(\mathbb{R}^N)$. *Moreover,*

$$(D_k^M)^{-1}f = D_k^{M^{-1}}f, \quad f\in\mathcal{S}(\mathbb{R}^N). \tag{4.2}$$

Proof.
1) The first statement follows immediately from (3.15) and Riemann-Lebesgue lemma for the Dunkl transform (see Theorem 2.4, a)).
2) It is clear when $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\in SL(2,\mathbb{R})$ then $M^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
Assume $b=0$. In view of (3.23) and $ad=1$, a simple computation shows

$$\begin{aligned} D_k^M\circ D_k^{M^{-1}}f(x) &= \frac{e^{i\frac{c}{2a}|x|^2}}{|a|^{\gamma+(N/2)}}D_k^{M^{-1}}f(x/a) \\ &= \frac{e^{i\frac{c}{2a}|x|^2}e^{-i\frac{c}{2da^2}|x|^2}}{|ad|^{\gamma+(N/2)}}f(x/(ad)) \\ &= f(x). \end{aligned}$$

When $b \neq 0$, we have

$$\begin{aligned} D_k^M \circ D_k^{M^{-1}} f(x) &= \frac{c_k^2}{(2ib)^{\gamma+\frac{N}{2}}(-2ib)^{\gamma+\frac{N}{2}}} e^{i\frac{d}{2b}|x|^2} \int_{\mathbb{R}^N} E_k(-ix/b, y) \\ &\times \left(\int_{\mathbb{R}^N} e^{-i\frac{d}{2b}|z|^2} f(z) E_k(iy/b, z)\omega_k(z)\ dz \right) \omega_k(y)\ dy. \end{aligned}$$

By the change of variables $u = y/b$ and the homogeneity of ω_k, we obtain

$$\begin{aligned} D_k^M \circ D_k^{M^{-1}} f(x) &= e^{i\frac{d}{2b}|x|^2} \frac{c_k^2}{4^{\gamma+N/2}} \int_{\mathbb{R}^N} E_k(-ix, u) \\ &\times \left(\int_{\mathbb{R}^N} e^{-i\frac{d}{2b}|z|^2} f(z) E_k(iu, z)\omega_k(z)\ dz \right) \omega_k(u) du \\ &= e^{i\frac{d}{2b}|x|^2} D_k \left(D_k \left[e^{-i\frac{d}{2b}|z|^2} f(-z) \right] \right)(x), \\ &= f(x), \text{ a.e.} \end{aligned}$$

3) That $D_k^M : \mathcal{S}(\mathbb{R}^N) \longrightarrow \mathcal{S}(\mathbb{R}^N)$ is an homeomorphism follows from Theorem 2.4 and the fact that the mapping M_λ defined by

$$(M_\lambda f)(x) = e^{\frac{i}{2}\lambda|x|^2}\ ,\ f \in \mathcal{S}(\mathbb{R}^N)$$

is an automorphism on $\mathcal{S}(\mathbb{R}^N)$ for each $\lambda \in \mathbb{R}$. The statement $(D_k^M)^{-1} = D_k^{M^{-1}}$ follows from part 2).

4.2 An additivity property.

Throughout this subsection, we denote by $M_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} (i = 1, 2)$ an arbitrary matrix in $SL(2, \mathbb{R})$. We begin by following lemmas

Lemma 4.1 *Let $\epsilon > 0$, $b_i \neq 0$ $(i = 1, 2)$ and $(x, z) \in \mathbb{R}^N \times \mathbb{R}^N$. Then*

$$\begin{aligned} c_k \int_{\mathbb{R}^N} e^{-\epsilon|y|^2} E_k^{M_1}(x, y) E_k^{M_2}(y, z)\omega_k(y)\ dy &= \frac{1}{c(\epsilon)} \exp\left(\frac{i}{2} \left(\frac{d_1}{b_1}|x|^2 + \frac{a_2}{b_2}|z|^2 \right) \right) \\ &\times \exp\left(-(r_1(\epsilon)|x|^2 + r_2(\epsilon)|z|^2) \right) E_\mu\left(-r_3(\epsilon)x, z \right), \end{aligned}$$

where

$$r_1(\epsilon) = \frac{b_2}{4\epsilon b_2 b_1^2 - 2ib_1(a_1b_2 + b_1d_2)}, \quad r_2(\epsilon) = \frac{b_1}{4\epsilon b_1 b_2^2 - 2ib_2(a_1b_2 + b_1d_2)},$$

$$r_3(\epsilon) = \frac{1}{2\epsilon b_1 b_2 - i(a_1b_2 + b_1d_2)} \quad \text{and} \quad c(\epsilon) = \left(\epsilon - \frac{i(a_1b_2 + b_1d_2)}{2b_1b_2} \right)^{\gamma+(N/2)}.$$

Proof. Replacing $E_k^{M_1}(x, y)$ and $E_k^{M_2}(y, z)$ by their definitions, we get

$$\begin{aligned} &\int_{\mathbb{R}^N} e^{-\epsilon|y|^2} E_k^{M_1}(x, y) E_k^{M_2}(y, z)\omega_k(y)\ dy = \exp\left(\frac{i}{2} \left(\frac{d_1}{b_1}|x|^2 + \frac{a_2}{b_2}|z|^2 \right) \right) \\ &\times \int_{\mathbb{R}^N} e^{-\left(\epsilon - \frac{i(a_1b_2+b_1d_2)}{2b_1b_2}\right)|y|^2} E_k(-ix/b_1, y) E_k(-iy/b_2, z)\omega_k(y)\ dy. \end{aligned}$$

The desired result follows from Lemma 3.2

Lemma 4.2 *Suppose that $b_i \neq 0$ $(i = 1, 2)$ and $a_1b_2 + b_1d_2 \neq 0$. Let f in $L_k^1(\mathbb{R}^N)$ with $D_k^{M_2} f \in L_k^1(\mathbb{R}^N)$. Then*

$$\begin{aligned} &c_k \int_{\mathbb{R}^N} E_k^{M_1}(x, y) \left(\int_{\mathbb{R}^N} f(z) E_k^{M_2}(y, z)\omega_k(z) dz \right) \omega_k(y) dy = e^{i\varphi_1} \left| \frac{2b_1b_2}{a_1b_2 + b_1d_2} \right|^{\gamma+(N/2)} \\ &\times \int_{\mathbb{R}^N} f(z)\ E_k^{M_1M_2}(x, z)\omega_k(z)\ dz, \end{aligned}$$

where

$$\varphi_1 = \frac{\pi}{2}(\gamma + (N/2))\ sgn\left(\frac{a_1b_2 + b_1d_2}{b_1b_2} \right).$$

Proof. For any positive number ϵ, we define the function I_ϵ on $\mathbb{R}$ by

$$I_\epsilon(x) \quad = \quad c_k \int_{\mathbb{R}^N} e^{-\epsilon|y|^2} E_k^{M_1}(x,y) \left(\int_{\mathbb{R}^N} f(z) E_k^{M_2}(y,z) \omega_k(z) dz \right) \omega_k(y) dy.$$

Since $D_k^{M_2} f \in L_k^1(\mathbb{R}^N)$, it follows from the dominated convergence theorem that

$$\lim_{\epsilon \to 0} I_\epsilon(x) = c_k \int_{\mathbb{R}^N} E_k^{M_1}(x,y) \left(\int_{\mathbb{R}^N} f(z) E_k^{M_2}(y,z) \omega_k(z) dz \right) \omega_k(y) dy.$$

Using Fubini's Theorem and Lemma 4.1, we obtain

$$I_\epsilon(x) \quad = \quad \frac{e^{\left(\frac{i}{2}\frac{d_1}{b_1} - r_1(\epsilon)\right)|x|^2}}{c(\epsilon)} \int_{\mathbb{R}^N} e^{\left(\frac{i}{2}\frac{a_2}{b_2} - r_2(\epsilon)\right)|z|^2} f(z) E_k\left(-r_3(\epsilon)x, z\right) \omega_k(z) dz.$$

Using the fact that $a_1 d_1 - b_1 c_1 = 1$ and $a_2 d_2 - b_2 c_2 = 1$, we can show

$$\begin{aligned}
\lim_{\epsilon \to 0} c(\epsilon) \quad &= \quad e^{-i\varphi_1} \left| \frac{a_1 b_2 + b_1 d_2}{2 b_1 b_2} \right|^{\gamma + (N/2)}, \\
\lim_{\epsilon \to 0} e^{\left(\frac{i}{2}\frac{d_1}{b_1} - r_1(\epsilon)\right)|x|^2} \quad &= \quad e^{\frac{i}{2}\frac{b_2 c_1 + d_1 d_2}{a_1 b_2 + b_1 d_2}|x|^2}, \\
\lim_{\epsilon \to 0} e^{\left(\frac{i}{2}\frac{a_2}{b_2} - r_2(\epsilon)\right)|z|^2} f(z) E_k\left(-r_3(\epsilon)x, z\right) \quad &= \quad e^{\frac{i}{2}\frac{a_1 a_2 + b_1 c_2}{a_1 b_2 + b_1 d_2}|z|^2} f(z) E_k(-ix/(a_1 b_2 + b_1 d_2), z).
\end{aligned}$$

From (2.6), the following majorization holds:

$$\left| e^{(\frac{i a_2}{2 b_2} - r_2(\epsilon))|z|^2} E_k(-r_3(\epsilon)x, z) \right| = |e^{-r_2(\epsilon))|z|^2} E_k(-r_3(\epsilon)x, z)| \le e^{-\Re(r_2(\epsilon))|z|^2 + |\Re(r_3(\epsilon))||x||z|}, \tag{4.3}$$

where

$$\Re(r_2(\epsilon)) = \frac{\epsilon b_1^2}{(2\epsilon b_1 b_2)^2 + (a_1 b_2 + b_1 d_2)^2}, \; \Re(r_3(\epsilon)) = \frac{2\epsilon b_1 b_2}{(2\epsilon b_1 b_2)^2 + (a_1 b_2 + b_1 d_2)^2}.$$

As $\Re(r_2(\epsilon)) > 0$, we obtain

$$\begin{aligned}
e^{-\Re(r_2(\epsilon))|z|^2 + |\Re(r_3(\epsilon))||x||z|} \quad &\le \quad e^{-\frac{\Re^2(r_3(\epsilon))|x|^2}{4\Re(r_2(\epsilon))}} \\
&= \quad e^{\frac{\epsilon b_2^2}{(2\epsilon b_1 b_2)^2 + (a_1 b_2 + b_1 d_2)^2}|x|^2}.
\end{aligned} \tag{4.4}$$

By means of (4.3) and (4.4), we can write

$$\left| e^{-r_2(\epsilon)|z|^2} E_k(-r_3(\epsilon)x, z) f(z) \right| \quad \le \quad r_x |f(z)|,$$

where $r_x = \sup\limits_{\epsilon \in]0,1]} e^{\frac{\epsilon b_2^2}{(2\epsilon b_1 b_2)^2 + (a_1 b_2 + b_1 d_2)^2}|x|^2}$. Thus, the dominated convergence theorem leads to

$$\lim_{\epsilon \to 0} I_\epsilon(x) = e^{i\varphi_1} \left| \frac{2 b_1 b_2}{a_1 b_2 + b_1 d_2} \right|^{\gamma + (N/2)} \int_{\mathbb{R}^N} f(z) \; E_k^{M_1 M_2}(x,z) \omega_k(z) \; dz.$$

This completes the proof.

Theorem 4.2 *Let $f \in L_k^1(\mathbb{R}^N)$ with $D_k^{M_2} f \in L_k^1(\mathbb{R}^N)$. Then*

$$D_k^{M_1} D_k^{M_2} f = e^{i\psi} \; D_k^{M_1 M_2} f,$$

where the constant phase ψ is given by

$$\psi = \begin{cases}
0, & b_1 = 0, \quad b_2 = 0, \\
\frac{\pi}{2}(\gamma + (N/2))(sgn(a_1 b_2) - sgn(b_2)), & b_1 = 0, \quad b_2 \neq 0, \\
\frac{\pi}{2}(\gamma + (N/2))(sgn(a_2 b_1) - sgn(b_1)), & b_1 \neq 0, \quad b_2 = 0, \\
-\frac{\pi}{2}(\gamma + (N/2))(sgn(b_1) + sgn(b_2)), & b_1 \neq 0, \quad b_2 \neq 0, \quad a_1 b_2 + b_1 d_2 = 0, \\
\frac{\pi}{2}(\gamma + (N/2)) \left(sgn(a_1 b_2 + b_1 d_2) + sgn\left(\frac{a_1 b_2 + b_1 d_2}{b_1 b_2} \right) - sgn(b_1) - sgn(b_2) \right), & b_1 \neq 0, \quad b_2 \neq 0, \quad a_1 b_2 + b_1 d_2 \neq 0,
\end{cases}$$

with equality a. e when $b_1 \neq 0, \; b_2 \neq 0$ and $a_1 b_2 + b_1 d_2 = 0$.

Proof. We shall divide the proof into five steps.
Step I. Suppose that $b_1 = 0$ and $b_2 = 0$. By virtue of (3.23), we have

$$\begin{aligned} D_k^{M_1} D_k^{M_2} f(x) &= \frac{e^{i\frac{c_1}{2a_1}|x|^2}}{|a_1|^{\gamma+(N/2)}} D_k^{M_2} f(x/a_1) \\ &= \frac{e^{i\frac{c_1}{2a_1}|x|^2}}{|a_1|^{\gamma+(N/2)}} \frac{e^{i\frac{c_2}{2a_2}|\frac{x}{a_1}|^2}}{|a_2|^{\gamma+(N/2)}} f(x/(a_1a_2)) \\ &= \frac{e^{\frac{i}{2}\frac{c_1a_2+c_2d_1}{a_1a_2}|x|^2}}{|a_1a_2|^{\gamma+(N/2)}} f(x/(a_1a_2)) \\ &= D_k^{M_1M_2} f(x). \end{aligned}$$

Step II. Suppose that $b_1 = 0$ and $b_2 \neq 0$. It is clear that

$$\frac{c_k}{|a_1|^{\gamma+(N/2)}(2ib_2)^{\gamma+(N/2)}} = \frac{c_k e^{i\psi_2}}{(2ia_1b_2)^{\gamma+(N/2)}},$$

where $\psi_2 = \frac{\pi}{2}(\gamma + (N/2))(\mathrm{sgn}(a_1b_2) - \mathrm{sgn}(b_2))$. By virtue of (3.23) and (3.20), we have

$$\begin{aligned} D_k^{M_1} D_k^{M_2} f(x) &= \frac{e^{i\frac{c_1}{2a_1}|x|^2}}{|a_1|^{\gamma+(N/2)}} D_k^{M_2} f(x/a_1) \\ &= \frac{c_k e^{i\psi_2}}{(2ia_1b_2)^{\gamma+(N/2)}}\ e^{i\frac{c_1}{2a_1}|x|^2} \int_{\mathbb{R}^N} f(y) e^{\frac{i}{2}(\frac{d_2}{b_2}|\frac{x}{a_1}|^2 + \frac{a_2}{b_2}|y|^2)} E_k(-ix/(a_1b_2), y)\omega_k(y)\ dy \\ &= e^{i\psi_2}\ D_k^{M_1M_2} f(x). \end{aligned}$$

Step III. Suppose that $b_1 \neq 0$ and $b_2 = 0$. By (4.2) and step II,

$$D_k^{M_2^{-1}} D_k^{M_1^{-1}} = D_k^{M_2^{-1}M_1^{-1}} = D_k^{(M_1M_2)^{-1}}.$$

The desired result follows upon taking inverses.
Step IV. Suppose $b_1 \neq 0,\ b_2 \neq 0$ and $a_1b_2 + b_1d_2 = 0$. We have

$$\frac{c_k}{(2ib_1)^{\gamma+(N/2)}} \frac{c_k}{(2ib_2)^{\gamma+(N/2)}} = \frac{c_k^2 e^{i\psi_4}}{2^{2\gamma+N}(|b_1b_2|)^{\gamma+(N/2)}},$$

where $\psi_4 = -\frac{\pi}{2}(\gamma + (N/2))(\mathrm{sgn}(b_1) + \mathrm{sgn}(b_2))$. By (3.20), we have

$$\begin{aligned} 2^{\gamma+(N/2)} D_k^{M_1} D_k^{M_2} f(x) &= \frac{c_k^2 e^{i\psi_4}}{(2|b_1b_2|)^{\gamma+(N/2)}} e^{i\frac{d_1}{2b_1}|x|^2} \int_{\mathbb{R}^N} D_k^{M_2} f(y) e^{\frac{i}{2}\frac{a_1}{b_1}|y|^2} E_k(-ix/b_1, y)\omega_k(y)\ dy \\ &= \frac{c_k^2 e^{i\psi_4}}{(2|b_1b_2|)^{\gamma+(N/2)}} e^{i\frac{d_1}{2b_1}|x|^2} \int_{\mathbb{R}^N} e^{\frac{i}{2}\frac{a_1b_2+b_1d_2}{b_1b_2}|y|^2} \left(\int_{\mathbb{R}^N} f(z) e^{\frac{i}{2}\frac{a_2}{b_2}|z|^2} E_k(-iy/b_2, z)\omega_k(z)\ dz \right) \\ &\times\ E_k(-ix/b_1, y)\omega_k(y)\ dy \\ &= \frac{c_k^2 e^{i\psi_4}}{(2|b_1b_2|)^{\gamma+(N/2)}} e^{i\frac{d_1}{2b_1}|x|^2} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} f(z) e^{\frac{i}{2}\frac{a_2}{b_2}|z|^2} E_k(-iy/b_2, z)\omega_k(z)\ dz \right) E_k(-ix/b_1, y)\omega_k(y)\ dy. \end{aligned}$$

By using the change of variables $u = \frac{y}{b_1},\ v = \frac{b_1}{b_2}z$ together with homogeneity of ω_k, one gets

$$\begin{aligned} D_k^{M_1} D_k^{M_2} f(x) &= \frac{|2b_2|^{2\gamma+N}}{c_k^2} \frac{c_k^2 e^{i\psi_4}}{2^{2\gamma+N}(|b_1b_2|)^{\gamma+(N/2)}} e^{i\frac{d_1}{2b_1}|x|^2}\ D_k^2 \left[f((b_2/b_1)z) e^{i\frac{a_2b_2}{2b_1^2}|z|^2} \right](x) \\ &= \frac{e^{i\psi_4}}{|b_1/b_2|^{\gamma+(N/2)}}\ e^{\frac{i}{2}(\frac{d_1}{b_1} + \frac{a_2b_2}{b_1^2})|x|^2}\ \ f((-b_2/b_1)x),\ \text{a. e.} \end{aligned}$$

Since $a_1b_2 + b_1d_2 = 0$ and $a_2d_2 - b_2c_2 = 1$, it follows that $-\frac{b_1}{b_2} = a_1a_2 + b_1c_2$ and using $a_1d_1 - b_1c_1 = 1$, we obtain

$$\frac{b_1d_1 + a_2b_2}{b_1^2} = \frac{a_2c_1 + d_1c_2}{a_1a_2 + b_1c_2}.$$

Hence

$$D_k^{M_1} D_k^{M_2} f(x) \quad = \quad e^{i\psi_4} D_k^{M_1 M_2} f(x), \text{ a. e.}$$

Step V. Suppose $b_1 \neq 0,\ b_2 \neq 0$ and $a_1 b_2 + b_1 d_2 \neq 0$. By lemma 4.2, we have

$$\begin{aligned} D_k^{M_1} D_k^{M_2} f(x) &= \frac{c_k^2 e^{i\psi_4}}{2^{2\gamma+N}(|b_1 b_2|)^{\gamma+(N/2)}} \int_{\mathbb{R}^N} E_k^{M_1}(x,y) \left(\int_{\mathbb{R}^N} f(z) E_k^{M_2}(y,z)\omega_k(z)dz \right) \omega_k(y)dy \\ &= \frac{c_k e^{i(\psi_4+\varphi_1)}}{2^{2\gamma+N}(|b_1 b_2|)^{\gamma+(N/2)}} \left| \frac{2b_1 b_2}{a_1 b_2 + b_1 d_2} \right|^{\gamma+(N/2)} \int_{\mathbb{R}^N} f(z)\ E_k^{M_1 M_2}(x,z)\omega_k(z)\ dz \\ &= e^{i\psi_5} D_k^{M_1 M_2} f(x), \end{aligned}$$

where $\psi_5 = \frac{\pi}{2}(\gamma + (N/2)) \left(\operatorname{sgn}(a_1 b_2 + b_1 d_2) + \operatorname{sgn}\left(\frac{a_1 b_2 + b_1 d_2}{b_1 b_2}\right) - \operatorname{sgn}(b_1) - \operatorname{sgn}(b_2) \right)$.

4.3 Operational Formula.

Proposition 4.1 *Let $M \in SL(2,\mathbb{R})$. Then the following properties hold on $\mathcal{S}(\mathbb{R}^N)$.*
(1) $D_k^M \circ Q_\xi = [d\ Q_\xi - b\ P_\xi] \circ D_k^M$,
(2) $D_k^M \circ P_\xi = [-c\ Q_\xi + a\ P_\xi] \circ D_k^M$,
(3) $D_k^M \circ E_{\xi,\xi'} = E_{\xi,\xi'} \circ D_k^M$.

Proof.
Case $b = 0$. From (3.23), we have

$$\begin{aligned} (D_k^M M_\xi f)(x) &= \frac{e^{i\frac{c}{2a}|x|^2}}{a^{\gamma+(N/2)}} M_\xi f(x/a) \\ &= \frac{1}{a}\langle \xi, x\rangle D_k^M f(x) \\ &= d(M_\xi D_k^M f)(x). \end{aligned}$$

To prove (2), using the product rule of the Dunkl operators T_ξ to get

$$\begin{aligned} P_\xi(D_k^M f)(x) &= -\frac{i}{|a|^{\gamma+\frac{N}{2}}} T_\xi(e^{i\frac{c}{2a}|x|^2} f(x/a)) \\ &= \frac{c}{a}\langle x, \xi\rangle D_k^M f(x) - \frac{ie^{i\frac{c}{2a}|x|^2}}{|a|^{\gamma+\frac{N}{2}}} T_\xi(x \longmapsto f(x/a)). \end{aligned}$$

In view of (2.2), a simple computation schows

$$T_\xi(x \longmapsto f(x/a)) = \frac{1}{a}(T_\xi f)(x/a).$$

Then

$$aP_\xi(D_k^M f)(x) = cM_\xi(D_k^M f)(x) + D_k^M(P_\xi f)(x).$$

Case $b \neq 0$. Let $f \in \mathcal{S}(\mathbb{R}^N)$. Using the anti-symmetry of the Dunkl operators T_ξ, we obtain

$$\begin{aligned} D_k^M\left[P_\xi f(y)\right](x) &= \frac{c_k}{(2ib)^{\gamma+(N/2)}} \int_{\mathbb{R}^N} E_k^M(x,y) P_\xi f(y)\omega_k(y)dy \\ &= \frac{c_k}{(2ib)^{\gamma+(N/2)}} \int_{\mathbb{R}^N} -P_\xi^y E_k^M(x,y) P_\xi f(y)\omega_k(y)dy. \end{aligned} \tag{4.5}$$

Substituting (3.19) in (4.5), we get

$$D_k^M\left[P_\xi f(y)\right](x) \quad = \quad -\frac{a}{b} D_k^M\left[\langle y, \xi\rangle f(y)\right](x) + \frac{1}{b}\langle x, \xi\rangle D_k^M f(x). \tag{4.6}$$

To compute $P_\xi(D_k^M f)$, write $D_k^M f = f_1 f_2$, where $f_1(x) = e^{\frac{i}{2}\frac{d}{b}|x|^2}$ and $f_2(x) = \frac{1}{(ib)^{\gamma+\frac{N}{2}}} D_k\left[e^{\frac{i}{2}\frac{a}{b}|y|^2} f(y)\right](\frac{x}{b})$. The product rule of the Dunkl operators T_ξ shows that

$$P_\xi(D_k^M f) = P_\xi(f_1) f_1 + f_1 P_\xi(f_2).$$

By (2.10), it follows that

$$P_\xi(f_2)(x) = -\frac{1}{(ib)^{\gamma+\frac{N}{2}}}\frac{1}{b}D_k\left[e^{\frac{i}{2}\frac{a}{b}|y|^2}\langle y,\xi\rangle f(y)\right]\left(\frac{x}{b}\right).$$

Hence,

$$P_\xi(D_k^M f)(x) = \frac{d}{b}\langle x,\xi\rangle D_k^M f(x) - \frac{1}{b}D_k^M[\langle y,\xi\rangle f(y)](x). \tag{4.7}$$

Finally, (4.6) and (4.7) together with the relation $ad-bc=1$, gives (1) and (2).
(3) Let $f\in\mathcal{S}(\mathbb{R}^N)$ and $\eta\in R_+$. We denote by $\mathbb{S}_\eta$ the N-dimensional symmetry operator defined by

$$\mathbb{S}_\eta f(y) := f(\sigma_\eta y).$$

By the change of variables $u=\sigma_\eta y$, the G-invariance of ω_k and according to (2.5), we obtain

$$\begin{aligned} D_k^M(\mathbb{S}_\eta f)(x) &= \frac{c_k}{(2ib)^{\gamma+N/2}}\int_{\mathbb{R}^N} e^{\frac{i}{2}(\frac{d}{b}|x|^2+\frac{a}{b}|y|^2)}E_\mu\left(-ix/b,y\right)f(\sigma_\eta y)\omega_k(y)dy \\ &= \frac{c_k}{(2ib)^{\gamma+N/2}}\int_{\mathbb{R}^N} e^{\frac{i}{2}(\frac{d}{b}|x|^2+\frac{a}{b}|y|^2)}E_\mu\left(-i\sigma_\eta x/b,y\right)f(y)\omega_k(y)dy, \\ &= \mathbb{S}_\eta(D_k^M f)(x). \end{aligned}$$

Hence $D_k^M\mathbb{S}_\eta = D_k^M\mathbb{S}_\eta$. As $E_{\xi,\xi'}$ is a finite linear combination of $\mathbb{S}_\eta$ $(\eta\in R_+)$, we deduce the desired result.

Remark 4.1
- *The two properties* (1) *and* (2) *can be written as*

$$D_k^M\begin{bmatrix} Q_\xi \\ P_\xi\end{bmatrix} = \begin{bmatrix} Q_\xi' \\ P_\xi'\end{bmatrix}D_k^M \quad \textit{where} \quad \begin{bmatrix} Q_\xi' \\ P_\xi'\end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a\end{bmatrix}\begin{bmatrix} Q_\xi \\ P_\xi\end{bmatrix}.$$

- *One can give an alternative proof of* (3)*:*

$$\begin{aligned} D_k^M E_{\xi,\xi'} D_k^{M^{-1}} &= iD_k^M[P_\xi, Q_{\xi'}]D_k^{M^{-1}} \\ &= i\left[P_\xi', Q_{\xi'}'\right] \\ &= i(-i)E_{\xi,\xi'} \\ &= E_{\xi,\xi'}. \end{aligned}$$

4.4 Bochner type identity for the generalized Dunkl transform.

In this section, we begin with a brief summary on the theory of k-spherical harmonics. A introduction to this subject can be found in the monograph [11]. The space of k-spherical harmonics of degree $n\geq 0$ is defined by

$$H_n^k = \mathrm{Ker}\Delta_k \cap \mathcal{P}_n.$$

Let $S^{N-1} = \left\{x\in\mathbb{R}^N;\ |x|=1\right\}$ be the unit sphere in $\mathbb{R}^N$ with normalized Lebesgue surface mesure $d\sigma$ and $L^2(S^{N-1},\ \omega_k(x)\ d\sigma(x))$ be the Hilbert space with the following inner product given by

$$\langle\langle f,g\rangle\rangle_k = \int_{S^{N-1}} f(\omega)\overline{g(\omega)}\omega_k(\omega)\ d\sigma(\omega).$$

As in the theory of ordinary spherical harmonics, the space $L^2(S^{N-1},\ \omega_k(x)\ d\sigma(x))$ decomposes as an orthogonal Hilbert space sum

$$L^2(S^{N-1},\ \omega_k(x)\ d\sigma(x)) = \bigoplus_{n=0}^{\infty} H_n^k.$$

In [34], Y. Xu gives an analogue of the Funk-Hecke formula for k-spherical harmonics. The well-known special case of the Dunkl-type Funk-Hecke formula is the following (see [26]):

Proposition 4.2 *Let $N \geq 2$ and put $\lambda = \gamma + (N/2) - 1$. Then for all $Y \in H_n^k$ and $x \in \mathbb{R}^N$,*

$$\frac{1}{d_k}\int_{S^{N-1}} E_k(ix, y)Y(y)\omega_k(y)\ d\sigma(y) = \frac{\Gamma(\lambda+1)}{2^n\Gamma(n+\lambda+1)} j_{n+\lambda}(|x|)Y(ix), \tag{4.8}$$

where

$$d_k = \int_{S^{N-1}} \omega_k(y)\ d\sigma(y).$$

In particular

$$\frac{1}{d_k}\int_{S^{N-1}} E_k(ix, y)\omega_k(y)\ d\sigma(y) = j_\lambda(|x|). \tag{4.9}$$

An application of the Dunkl-type Funk-Hecke formula is the following:

Theorem 4.3 *(Bochner type identity) Let $M \in SL(2, \mathbb{R})$ such that $b \neq 0$. If $f \in L_k^1(\mathbb{R}^N) \cap L_k^2(\mathbb{R}^N)$ is of the form $f(x) = p(x)\psi(|x|)$ for some $p \in H_n^k$ and a one-variable ψ on $\mathbb{R}_+$, then*

$$D_k^M f(x) = p(x)\mathcal{H}^M_{n+\gamma+(N/2)-1}\psi(|x|). \tag{4.10}$$

In particular, if f is radial, then

$$D_k^M f(x) = \mathcal{H}^M_{\gamma+(N/2)-1}\psi(|x|).$$

Proof. By spherical polar coordinates, we have

$$\begin{aligned} D_k^M f(x) &= \frac{c_k}{(2ib)^{\gamma+\frac{N}{2}}}\int_{\mathbb{R}^N} f(y)E_k^M(x, y)\omega_k(y)dy \\ &= \frac{c_k}{(2ib)^{\gamma+\frac{N}{2}}}\int_0^{+\infty} r^{N-1}F(r, x)\ dr, \end{aligned} \tag{4.11}$$

where

$$F(r, x) = \frac{2\pi^{N/2}}{\Gamma(N/2)}\int_{S^{N-1}} E_k^M(x, ry)p(ry)\psi(r|y|)\omega_k(ry)\ d\sigma(y).$$

From (3.15) and the homogeneity of ω_k and p, we obtain

$$F(r, x) = \frac{2\pi^{N/2}}{\Gamma(N/2)} e^{\frac{i}{2}(\frac{d}{b}|x|^2+\frac{a}{b}r^2)}\psi(r)r^{2\gamma+n}\int_{S^{N-1}} p(y)E_k(-irx/b, y)\omega_k(y)\ d\sigma(y).$$

Using (4.8), we get

$$\begin{aligned} F(r, x) &= \frac{2\pi^{N/2}d_k}{\Gamma(N/2)}\frac{\Gamma(\lambda+1)}{2^n\Gamma(\lambda+n+1)} \\ &\times\ e^{\frac{i}{2}(\frac{d}{b}|x|^2+\frac{a}{b}r^2)}\psi(r)r^{2\gamma+n}p\left(-\frac{irx}{b}\right)j_{\lambda+n}\left(\frac{r|x|}{b}\right), \end{aligned}$$

where

$$\lambda = \gamma + (N/2) - 1.$$

Using again the homogeneity of p, we get

$$\begin{aligned} F(r, x) &= \frac{2\pi^{N/2}d_k}{\Gamma(N/2)}\frac{\Gamma(\lambda+1)}{2^n\Gamma(\lambda+n+1)}\left(-\frac{i}{b}\right)^n \\ &\times\ e^{\frac{i}{2}(\frac{d}{b}|x|^2+\frac{a}{b}r^2)}\psi(r)r^{2\gamma+2n}p(x)j_{\lambda+n}\left(\frac{r|x|}{b}\right). \end{aligned}$$

Now we can express a relationship between d_k and c_k. In fact

$$\begin{aligned} c_k^{-1} &= \int_{\mathbb{R}^N} e^{-|y|^2}\omega_k(y)\ dy \\ &= \frac{2\pi^{N/2}}{\Gamma(N/2)}\int_0^{+\infty} r^{N-1}e^{-r^2}\int_{S^{N-1}}\omega_k(ry)\ d\sigma(y)\ dr \\ &= \frac{2\pi^{N/2}}{\Gamma(N/2)}\int_0^{+\infty} r^{2\gamma+N-1}e^{-r^2}\int_{S^{N-1}}\omega_k(y)\ d\sigma(y)\ dr \\ &= \frac{\pi^{N/2}\Gamma(\lambda+1)d_k}{\Gamma(N/2)}. \end{aligned} \tag{4.12}$$

By the use of (4.12), we obtain

$$\frac{c_k}{(2ib)^{\gamma+\frac{N}{2}}}\frac{2\pi^{N/2}d_k}{\Gamma(N/2)}\frac{\Gamma(\lambda+1)}{2^n\Gamma(\lambda+n+1)}\left(-\frac{i}{b}\right)^n = \frac{2}{\Gamma(\lambda+n+1)(2ib)^{\lambda+n+1}}.$$

Hence

$$F(r,x) = \frac{2e^{\frac{i}{2}(\frac{d}{b}|x|^2+\frac{a}{b}r^2)}r^{2\gamma+2n}}{\Gamma(\lambda+n+1)(2ib)^{\lambda+n+1}}\psi(r)p(x)j_{\lambda+n}\left(\frac{r|x|}{b}\right). \tag{4.13}$$

Substituting (4.13) in (4.11) to get

$$\begin{aligned} D_k^M f(x) &= \frac{2}{\Gamma(\lambda+n+1)(2ib)^{\lambda+n+1}}p(x) \\ &\times \int_0^{+\infty} e^{\frac{i}{2}(\frac{d}{b}|x|^2+\frac{a}{b}r^2)}\psi(r)r^{2(\lambda+n)+1}j_{\lambda+n}\left(\frac{r|x|}{b}\right)\ dr \\ &= p(x)\mathcal{H}^M_{n+\lambda}\psi(|x|) \\ &= p(x)\mathcal{H}^M_{n+\gamma+(N/2)-1}\psi(|x|). \end{aligned}$$

Now, we give the material needed for an application of Bochner type identity. Let $\{p_{n,j}\}_{1\le j\le a_n^k}$ be an orthonormal basis of H_n^k ($a_n^k = \dim(H_n^k)$). Let $m,\ n$ be non-negative integers and $1\le j\le a_n^k$. Define

$$c_{m,n} = \left[\frac{m!\ \Gamma(N/2)}{\pi^{N/2}\Gamma(\gamma+n+m+(N/2))}\right]^{1/2}$$

and

$$\psi_{m,n,j}(x) = c_{m,n}\ p_{n,j}(x)\ L_m^{(n+\gamma+N/2-1)}(|x|^2)\ e^{-|x|^2/2}, \tag{4.14}$$

where $L_m^{(a)}$ denote the Laguerre polynomial in the standard notation. It follows from Proposition 2.4 and Theorem 2.5 of Dunkl [9] that

$$\{\psi_{m,n,j} : m,\ n = 0,\ 1,\ 2,\dots,\ j = 1,\dots,a_n^k\}$$

forms an orthonormal basis of $L_k^2(\mathbb{R}^N)$.

Theorem 4.4 *Let $M\in SL(2,\mathbb{R})$ such that $b\neq 0$. The generalized Dunkl transform of the generalized Laguerre functions are*

$$D_k^M\psi_{m,n,j}(x) = \lambda_{m,n,a,b}\ e^{\frac{i(ac+bd)}{2(a^2+b^2)}|x|^2}\ \psi_{m,n,j}\left(\frac{x}{\sqrt{a^2+b^2}}\right),$$

where

$$\lambda_{m,n,a,b} = \left(\frac{a-ib}{a+ib}\right)^m\frac{(a-ib)^{\frac{n}{2}}}{(a+ib)^{\frac{n}{2}}}\frac{e^{i\theta}}{(a+ib)^{\gamma+(N/2)}}$$

and

$$\theta = 2(\gamma+(N/2)+n-1)\left\{sgn(b)\arctan\left(\frac{a}{|b|+\sqrt{a^2+b^2}}\right)-\frac{\pi}{4}sgn(b)+\arctan\left(\frac{b}{a+\sqrt{a^2+b^2}}\right)\right\}. \tag{4.15}$$

Proof. Applying Theorem 4.3 with p replaced by $p_{n,j}$ and with $\psi(r) = L_m^{(n+\gamma+N/2-1)}(r^2)\ e^{-r^2/2}$, we obtain

$$D_k^M \psi_{m,n,j}(x) = c_{m,n} p_{n,j}(x) \mathcal{H}_\nu^M \psi(|x|),$$

where

$$\nu = n + \gamma + (N/2) - 1,$$

and

$$\mathcal{H}_\nu^M \psi(|x|) \quad = \quad \frac{2}{\Gamma(\nu+1)(2ib)^{\nu+1}} \int_0^{+\infty} e^{\frac{i}{2}(\frac{d}{b}|x|^2 + \frac{a}{b}r^2)} j_\nu\left(\frac{r|x|}{b}\right) L_m^{(\nu)}(r^2) e^{-\frac{r^2}{2}} r^{2\nu+1}\ dr.$$

Observe that

$$\mathcal{H}_\nu^M \psi(|x|) \quad = \quad \frac{2}{\Gamma(\nu+1)(2ib)^{\nu+1}} e^{\frac{i}{2}\frac{d}{b}|x|^2} I_\nu(x),$$

where

$$\begin{aligned} I_\nu(x) \quad &= \quad \int_0^{+\infty} r^{2\nu+1} L_m^{(\nu)}(r^2) e^{-(\frac{1}{2} - \frac{i}{2}\frac{a}{b})r^2} j_\nu\left(\frac{r|x|}{b}\right)\ dr \\ &= \quad 2^\nu \Gamma(\nu+1) \left(\frac{b}{|x|}\right)^\nu \int_0^{+\infty} r^{\nu+1} L_m^{(\nu)}(r^2) e^{-(\frac{1}{2} - \frac{i}{2}\frac{a}{b})r^2} J_\nu\left(\frac{r|x|}{b}\right)\ dr. \end{aligned}$$

To compute $I_\nu(x)$, we need the following formulas (see 7.4.21 (4) in [17])

$$\int_0^{+\infty} y^{\nu+1} e^{-\beta y^2} L_m^\nu(ay^2) J_\nu(zy)\ dy = d_m z^\nu e^{-z^2/(4\beta)} L_m^\nu\left[\frac{az^2}{4\beta(a-\beta)}\right]$$

where $d_m = ((\beta - a)^m/(2^{\nu+1}\beta^{\nu+m+1})),\ a,\ \Re\beta > 0,\ \Re\nu > -1.$
Let us take $\beta = \frac{1}{2} - \frac{i}{2}\frac{a}{b} = \frac{b-ia}{2b},\ a = 1$ and $z = \frac{|x|}{b}$, then

$$\begin{aligned} d_m \quad &= \quad \left(\frac{ia+b}{ia-b}\right)^m \left(\frac{b}{b-ia}\right)^{\nu+1}, \\ \frac{az^2}{4\beta(a-\beta)} \quad &= \quad \frac{|x|^2}{a^2+b^2}, \\ -\frac{z^2}{4\beta} \quad &= \quad -\frac{|x|^2}{2(a^2+b^2)} - \frac{i}{2}\frac{a}{b}\frac{|x|^2}{(a^2+b^2)}. \end{aligned}$$

Hence

$$\begin{aligned} \int_0^{+\infty} r^{\nu+1} L_m^{(\nu)}(r^2) e^{-(\frac{1}{2} - \frac{i}{2}\frac{a}{b})r^2} J_\nu\left(\frac{r|x|}{b}\right)\ dr \quad &= \quad \left(\frac{ia+b}{ia-b}\right)^m \left(\frac{b}{b-ia}\right)^{\nu+1} \left(\frac{|x|}{b}\right)^\nu \\ &\times \quad e^{-\frac{|x|^2}{2(a^2+b^2)}} e^{-\frac{i}{2}\frac{a}{b}\frac{|x|^2}{(a^2+b^2)}} L_m^{(\nu)}\left(\frac{|x|^2}{a^2+b^2}\right) \end{aligned}$$

and therefore

$$\begin{aligned} \mathcal{H}_\nu^M \psi(|x|) \quad &= \quad \left(\frac{ia+b}{ia-b}\right)^m \left(\frac{2ib}{a+ib}\right)^{\nu+1} \frac{1}{(2ib)^{\nu+1}} \\ &\times \quad e^{\frac{i(ac+bd)}{2(a^2+b^2)}|x|^2} e^{-\frac{|x|^2}{2(a^2+b^2)}} L_m^{(\nu)}\left(\frac{|x|^2}{a^2+b^2}\right). \end{aligned}$$

Since

$$\left(\frac{ib}{a+ib}\right)^{\nu+1} = e^{i\theta}\frac{(ib)^{\nu+1}}{(a+ib)^{\nu+1}}, \qquad \left(\sqrt{a^2+b^2}\right)^n = (a+ib)^{(n/2)}(a-ib)^{(n/2)}$$

and $p_{n,j}$ is homogeneous of degree n, then

$$\begin{aligned} D_k^M \psi_{m,n,j}(x) \quad &= \quad \frac{e^{\frac{i}{2}\frac{(ac+bd)}{(a^2+b^2)}|x|^2}}{(a+ib)^{n+\gamma+(N/2)}} \left(\frac{a-ib}{a+ib}\right)^m p_{n,j}(x) L_m^{(\nu)}\left(\frac{|x|^2}{a^2+b^2}\right) \\ &= \quad \left(\frac{a-ib}{a+ib}\right)^m \frac{(a-ib)^{\frac{n}{2}}}{(a+ib)^{\frac{n}{2}}} \frac{e^{i\theta}}{(a+ib)^{\gamma+(N/2)}} \\ &\times \quad e^{\frac{i(ac+bd)}{2(a^2+b^2)}|x|^2}\ \psi_{m,n,j}\left(\frac{x}{\sqrt{a^2+b^2}}\right). \end{aligned}$$

4.5 Master formula for the generalized Dunkl transform.

In this section, we shall derive a Master formula for the generalized Dunkl transform. For this we need the following lemma.

Lemma 4.3 *Let $p \in \mathcal{P}_n$ and $x \in \mathbb{C}^N$. Then for $\omega \in \mathbb{C}$ and $\Re(\omega) > 0$,*

$$c_k \int_{\mathbb{R}^N} p(y) E_k(x, 2y) e^{-\omega|y|^2} \omega_k(y)\, dy = \frac{e^{\frac{l(x)}{\omega}}}{\omega^{\gamma+n+(N/2)}} e^{\frac{\omega}{4}\Delta_k} p(x). \tag{4.16}$$

Proof. First compute this integral when $\omega > 0$.

$$c_k \int_{\mathbb{R}^N} p(y) E_k(x, 2y) e^{-\omega|y|^2} \omega_k(y)\, dy = c_k \int_{\mathbb{R}^N} p(y) E_k(x, 2y) e^{-|\sqrt{\omega} y|^2} \omega_k(y)\, dy.$$

By the change of variables $u = \sqrt{\omega} y$ and the homogeneity of ω_k and p, we obtain

$$c_k \int_{\mathbb{R}^N} p(y) E_k(x, 2y) e^{-\omega|y|^2} \omega_k(y)\, dy = \frac{c_k}{\omega^{\gamma+(n+N)/2}} \int_{\mathbb{R}^N} p(y) E_k(x/\sqrt{\omega}, 2y) e^{-|y|^2} \omega_k(y)\, dy. \tag{4.17}$$

Using Theorem 2.5,1) we deduce an equivalent identity:

$$c_k \int_{\mathbb{R}^N} p(y) E_k(x, 2y) e^{-|y|^2} \omega_k(y)\, dy = e^{l(x)} e^{\frac{\Delta_k}{4}} p(x). \tag{4.18}$$

Combine (4.17) and (4.18) to get

$$c_k \int_{\mathbb{R}^N} p(y) E_k(x, 2y) e^{-\omega|y|^2} \omega_k(y)\, dy = \frac{e^{\frac{l(x)}{\omega}}}{\omega^{\gamma+(n+N)/2}} e^{\frac{\Delta_k}{4}} p\left(\frac{x}{\sqrt{\omega}}\right).$$

Now use Lemma 2.1 from [24] to obtain

$$e^{\frac{\Delta_k}{4}} p\left(\frac{x}{\sqrt{\omega}}\right) = \frac{1}{\omega^{n/2}} e^{\frac{\omega}{4}\Delta_k} p(x).$$

Hence, we find the equality (4.16) for $\omega > 0$. By analytic continuation, this holds for $\{\omega \in \mathbb{C} :\ \Re(\omega) > 0\}$.
We are now in a position to give the Master formula.

Theorem 4.5 *Let $M \in SL(2, \mathbb{R})$ such that $b \neq 0$. Let f_n is of the form $f_n(x) = e^{-\frac{|x|^2}{2}} e^{-\frac{\Delta_k}{4}} p(x)$ for some $p \in \mathcal{P}_n$, Then*

$$D_k^M f_n(x) \;=\; \lambda_{n,a,b} \quad e^{\frac{i(ac+bd)}{2(a^2+b^2)}|x|^2} \; f_n\left(\frac{x}{\sqrt{a^2+b^2}}\right), \tag{4.19}$$

where

$$\lambda_{n,a,b} = \frac{(a-ib)^{\frac{n}{2}}}{(a+ib)^{\frac{n}{2}}} \frac{e^{i\theta}}{(a+ib)^{\gamma+(N/2)}}$$

and θ as in (4.15).

Proof. It follows easily from (3.20) that

$$D_k^M \left[e^{-\frac{|y|^2}{2}} e^{-\frac{\Delta_k}{4}} p(y)\right](x) = \frac{c_k}{(2ib)^{\gamma+(N/2)}} e^{\frac{i}{2}\frac{d}{b}|x|^2} \int_{\mathbb{R}^N} e^{-\frac{\Delta_k}{4}} p(y) E_k(-ix/b, y) e^{-\omega|y|^2} \omega_k(y)\, dy,$$

where

$$\omega = \frac{1}{2} - \frac{i}{2}\frac{a}{b}. \tag{4.20}$$

Since

$$e^{-\frac{\Delta_k}{4}} p(y) = \sum_{s=0}^{[\frac{n}{2}]} \frac{(-1)^s}{s! 4^s} \Delta_k^s p(y),$$

we conclude that

$$\int_{\mathbb{R}^N} e^{-\frac{\Delta_k}{4}} p(y) E_k(-ix/b, y) e^{-\omega|y|^2} \omega_k(y)\ dy = \sum_{s=0}^{[\frac{n}{2}]} \frac{(-1)^s}{s!4^s} \int_{\mathbb{R}^N} \Delta_k^s p(y) E_k(-ix/b, y) e^{-\omega|y|^2} \omega_k(y)\ dy. \tag{4.21}$$

For $s \in \mathbb{Z}_+$ with $2s \leq n$, the polynomial $\Delta_k^s p$ is homogeneous of degree $n - 2s$. Hence by the previous Lemma, we obtain

$$c_k \int_{\mathbb{R}^N} \Delta_k^s p(y) E_k(-ix/b, y) e^{-\omega|y|^2} \omega_k(y)\ dy = \frac{e^{\frac{l(X_b)}{\omega}}}{\omega^{\gamma+n+(N/2)}} e^{\frac{\omega}{4}\Delta_k} \left[\omega^{2s} \Delta_k^s p\right](X_b), \tag{4.22}$$

where

$$X_b = -\frac{ix}{2b}. \tag{4.23}$$

Substitute (4.22) in (4.21) to get

$$\begin{aligned} c_k \int_{\mathbb{R}^N} e^{-\frac{\Delta_k}{4}} p(y) E_k(-ix/b, y) e^{-\omega|y|^2} \omega_k(y)\ dy &= \frac{e^{\frac{l(X_b)}{\omega}}}{\omega^{\gamma+n+(N/2)}} e^{\frac{\omega}{4}\Delta_k} \sum_{s=0}^{[\frac{n}{2}]} \frac{(-1)^s \omega^{2s}}{s!4^s} \Delta_k^s p(X_b) \\ &= \frac{e^{\frac{l(X_b)}{\omega}}}{\omega^{\gamma+n+(N/2)}} e^{\frac{\omega}{4}\Delta_k} e^{-\frac{\omega^2}{4}\Delta_k} p(X_b) \\ &= \frac{e^{\frac{l(X_b)}{\omega}}}{\omega^{\gamma+n+(N/2)}} e^{\frac{\omega-\omega^2}{4}\Delta_k} p(X_b). \end{aligned}$$

Replace ω and X_b by their values given in (4.20) and (4.23) and use Lemma 2.1 in [24], we obtain

$$\begin{aligned} e^{\frac{\omega-\omega^2}{4}\Delta_k} p(X_b) &= \left(-\frac{i}{2b}\right)^n e^{-b^2(\omega-\omega^2)\Delta_k} p(x) \\ &= \left(-\frac{i}{2b}\right)^n e^{-\frac{a^2+b^2}{4}\Delta_k} p(x). \end{aligned}$$

Also,

$$\begin{aligned} \omega^{n+\gamma+(N/2)} &= \left(\frac{b-ia}{2b}\right)^{n+\gamma+(N/2)} \\ e^{\frac{l(X_b)}{\omega}} &= e^{-\frac{|x|^2}{2b(b-ia)}}. \end{aligned}$$

Then

$$\frac{c_k}{(2ib)^{\gamma+(N/2)}} \int_{\mathbb{R}^N} e^{-\frac{\Delta_k}{4}} p(y) E_k(-ix/b, y) e^{-\omega|y|^2} \omega_k(y)\ dy = e^{i\theta}\ \frac{e^{-\frac{|x|^2}{2b(b-ia)}}}{(a+ib)^{n+\gamma+(N/2)}}\ e^{-\frac{a^2+b^2}{4}\Delta_k} p(x). \tag{4.24}$$

Now, if we multiply equation (4.24) by $e^{\frac{id}{2b}|x|^2}$, we obtain:

$$\begin{aligned} D_k^M \left[e^{-\frac{|y|^2}{2}} e^{-\frac{\Delta_k}{4}} p(y)\right](x) &= e^{i\theta}\ \frac{e^{-\frac{d-ic}{2b(b-ia)}|x|^2}}{(a+ib)^{n+\gamma+(N/2)}}\ e^{-\frac{a^2+b^2}{4}\Delta_k} p(x) \\ &= e^{i\theta}\ \frac{e^{i\frac{db+ac}{2(a^2+b^2)}|x|^2}}{(a+ib)^{n+\gamma+(N/2)}}\ e^{-\frac{|x|^2}{2(a^2+b^2)}} e^{-\frac{a^2+b^2}{4}\Delta_k} p(x). \end{aligned}$$

Use again Lemma 2.1 in [24], we deduce

$$e^{-\frac{a^2+b^2}{4}\Delta_k} p(x) = (a^2+b^2)^{n/2}\ e^{-\frac{\Delta_k}{4}} p\left(\frac{x}{\sqrt{a^2+b^2}}\right).$$

Therefore

$$\begin{aligned} D_k^M \left[e^{-\frac{|y|^2}{2}} e^{-\frac{\Delta_k}{4}} p(y)\right](x) &= \frac{e^{i\theta}}{(a+ib)^{\gamma+(N/2)}} \frac{(a-ib)^{(n/2)}}{(a+ib)^{(n/2)}} \\ &\times\ e^{\frac{i}{2}\frac{ac+bd}{(a^2+b^2)}|x|^2}\ e^{-\frac{|x|^2}{2(a^2+b^2)}} \left(e^{-\frac{\Delta_k}{4}} p\right)\left(\frac{x}{\sqrt{a^2+b^2}}\right). \end{aligned}$$

As an immediate consequence of the Master formula (4.19), we have

Corollary 4.1 *(Hecke type identity)If in addition to the assumption in Theorem 4.5, the polynomial* $p \in H_n^k$, *then (4.19) becomes*

$$D_k^M \left[e^{-\frac{|.|^2}{2}} p \right](x) \quad = \quad \lambda_{n,a,b} \quad e^{\frac{i(ac+bd)}{2(a^2+b^2)}|x|^2} \quad e^{-\frac{|x|^2}{2(a^2+b^2)}} p\left(\frac{x}{\sqrt{a^2+b^2}} \right) \tag{4.25}$$

where

$$\lambda_{n,a,b} = \frac{(a-ib)^{\frac{n}{2}}}{(a+ib)^{\frac{n}{2}}} \frac{e^{i\theta}}{(a+ib)^{\gamma+(N/2)}}$$

and

$$\theta = 2(\gamma + (N/2) + n - 1) \left\{ sgn(b) \arctan\left(\frac{a}{|b| + \sqrt{a^2+b^2}} \right) - \frac{\pi}{4} sgn(b) + \arctan\left(\frac{b}{a + \sqrt{a^2+b^2}} \right) \right\}$$

5 Plancherel Theorem.

We begin with the following Proposition.

Proposition 5.1 *Let* f *and* g *be in* $L_k^1(\mathbb{R}^N)$ *and* $M \in SL(2,\mathbb{R})$ *such that* $b \neq 0$. *Then*

$$\int_{\mathbb{R}^N} D_k^M f(x) \overline{g(x)} \omega_k(x)\ dx = \int_{\mathbb{R}^N} f(x) \overline{D_k^{M^{-1}} g(x)} \omega_k(x)\ dx. \tag{5.1}$$

Proof. Let f and $g \in L_k^1(\mathbb{R}^N)$. Using Fubini's theorem we write

$$\begin{aligned}
\int_{\mathbb{R}^N} D_k^M f(x) \overline{g(x)} \omega_k(x)\ dx &= \frac{c_k}{(2ib)^{\gamma+(N/2)}} \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} E_k(x,y) f(y) \omega_k(y)\ dy \right) \overline{g(x)} \omega_k(x)\ dx, \\
&= \int_{\mathbb{R}^N} f(y) \overline{\frac{c_k}{(-2ib)^{\gamma+(N/2)}} \int_{\mathbb{R}^N} g(x) E_k^{M^{-1}}(y,x) \omega_k(x)\ dx} \omega_k(y)\ dy, \\
&= \int_{\mathbb{R}^N} f(y) \overline{D_k^{M^{-1}} g(y)} \omega_k(y)\ dy.
\end{aligned}$$

This complete the proof.

Corollary 5.1 *Let* $f \in \mathcal{S}(\mathbb{R}^N)$ *and* $M \in SL(2,\mathbb{R})$. *Then*

$$\left\| D_k^M f \right\|_{k,2} = \|f\|_{k,2}.$$

Proof. It is easy to check that Corollary 5.1 holds for $b = 0$. Now suppose $b \neq 0$. By Proposition 5.1 and Theorem 4.1, 3), we have

$$\begin{aligned}
\left\| D_k^M f \right\|_{k,2}^2 &= \int_{\mathbb{R}^N} D_k^M f(x) \overline{D_k^M f(x)} \omega_k(x)\ dx, \\
&= \int_{\mathbb{R}^N} f(x) \overline{D_k^{M^{-1}} D_k^M f(x)} \omega_k(x)\ dx, \\
&= \|f\|_{k,2}^2.
\end{aligned}$$

Theorem 5.1 *Let* $M \in SL(2,\mathbb{R})$.
1) *If* $f \in L_k^1(\mathbb{R}^N) \cap L_k^2(\mathbb{R}^N)$, *then* $D_k^M f \in L_k^2(\mathbb{R}^N)$ *and* $\left\| D_k^M f \right\|_{k,2} = \|f\|_{k,2}$.
2) *There exists a unique unitary operator on* $L_k^2(\mathbb{R}^N)$ *that coincides with* D_k^M *on* $L_k^1(\mathbb{R}^N) \cap L_k^2(\mathbb{R}^N)$. *The extension is also denoted by* $f \longrightarrow D_k^M f$.

Proof. It suffices to assume that $b \neq 0$. From Corollary 5.1 and the density of $\mathcal{S}(\mathbb{R}^N)$ in $L_k^2(\mathbb{R}^N)$, we deduce the existence of a unique continuous operator $\hat{D}_k^M$ on $L_k^2(\mathbb{R}^N)$ that coincides with D_k^M on $\mathcal{S}(\mathbb{R}^N)$. If $f, g \in \mathcal{S}(\mathbb{R}^N)$ then

$$\begin{aligned}
\int_{\mathbb{R}^N} \hat{D}_k^M f(x) \overline{g(x)} \omega_k(x)\ dx &= \int_{\mathbb{R}^N} D_k^M f(x) \overline{g(x)} \omega_k(x)\ dx \\
&= \int_{\mathbb{R}^N} f(x) \overline{D_k^{M^{-1}} g(x)} \omega_k(x)\ dx \\
&= \int_{\mathbb{R}^N} f(x) \overline{\hat{D}_k^{M^{-1}} g(x)} \omega_k(x)\ dx.
\end{aligned}$$

Let $f, g \in L_k^2(\mathbb{R}^N)$. By the density of $\mathcal{S}(\mathbb{R}^N)$ in $L_k^2(\mathbb{R}^N)$, we conclude that

$$\int_{\mathbb{R}^N} \hat{D}_k^M f(x)\overline{g(x)}\omega_k(x)\,dx = \int_{\mathbb{R}^N} f(x)\overline{\hat{D}_k^{M^{-1}} g(x)}\omega_k(x)\,dx.$$

Now, if $f \in L_k^1(\mathbb{R}^N) \cap L_k^2(\mathbb{R}^N)$ and $g \in \mathcal{S}(\mathbb{R}^N)$, then

$$\begin{aligned}\int_{\mathbb{R}^N} D_k^M f(x)\overline{g(x)}\omega_k(x)\,dx &= \int_{\mathbb{R}^N} f(x)\overline{D_k^{M^{-1}} g(x)}\omega_k(x)\,dx \\ &= \int_{\mathbb{R}^N} f(x)\overline{\hat{D}_k^{M^{-1}} g(x)}\omega_k(x)\,dx \\ &= \int_{\mathbb{R}^N} \hat{D}_k^M f(x)\overline{g(x)}\omega_k(x)\,dx.\end{aligned}$$

Hence $D_k^M f = \hat{D}_k^M f$, a.e, which proves the first statement in part 1). The second statement of part 1) follows from Corollary 5.1. Part 2) follows from part 1), Corollary 5.1 and Theorem 2.6, 2).

Corollary 5.2 *For each $f \in L_k^2(\mathbb{R}^N)$ and $M_1,\ M_2 \in SL(2,\mathbb{R})$, we have*

$$D_k^{M_1} \circ D_k^{M_2}(f) = e^{i\psi} D_k^{M_1 M_2}(f), \tag{5.2}$$

with ψ as in Theorem 4.2

6 A generalized Dunkl-Schrödinger operator

Let $\left\{M(\tau) = \begin{bmatrix} a(\tau) & b(\tau) \\ c(\tau) & d(\tau) \end{bmatrix};\ \tau \in \mathbb{R}\right\}$ be a continuous one-parameter subgroup of $SL(2,\mathbb{R})$ such that the $e^{i\psi}$ which appears in $D_k^{M(\tau_1)} \circ D_k^{M(\tau_2)} = e^{i\psi} D_k^{M(\tau_1+\tau_2)}$ is equal to 1.

6.1 The $\mathcal{C}_0$-group $\left\{D_k^{M(\tau)}\right\}_{\tau\in\mathbb{R}}$

We begin with the following lemma:

Lemma 6.1 *Let $f \in L_k^2(\mathbb{R}^N)$. Then*

$$\lim_{a\to 1} \|f(ay) - f(y)\|_{k,2} = 0. \tag{6.1}$$

Proof. First we prove the lemma in the case $f \in \mathcal{C}_c^\infty(\mathbb{R}^N)$. Choose $r > 0$ such that $\mathrm{supp}(f) \subset B(0,r)$. Here $B(0,r)$ denote the closed ball centered at 0 with radius r. It is clear that if $a > 1$, then $\mathrm{supp}(f(a.)) \subset B(0,r)$. Applying the dominated convergence theorem one gets:

$$\lim_{a\downarrow 1} \|f(ay) - f(y)\|_{k,2} = 0.$$

By the change of variable $u = ay$ and the homogeneity of ω_k, we have:

$$\|f(ay) - f(y)\|_{k,2} = \frac{1}{|a|^{\gamma+(N/2)}} \|f(y) - f(y/a)\|_{k,2}\,.$$

Then

$$\lim_{a\uparrow 1} \|f(ay) - f(y)\|_{k,2} = 0.$$

A more general result can be obtained by the density of $\mathcal{C}_c^\infty(\mathbb{R}^N)$ in $L_k^2(\mathbb{R}^N)$.

Theorem 6.1 *Let $f \in L_k^2(\mathbb{R}^N)$. Then*

$$\lim_{\tau\to 0} \left\|D_k^{M(\tau)} f - f\right\|_{k,2} = 0. \tag{6.2}$$

Proof. First we prove the theorem in the case $f \in \mathcal{S}(\mathbb{R}^N)$. By the change of variable $u = \frac{y}{a(\tau)}$ and the homogeneity of ω_k, equation (3.22) becomes

$$D_K^{M(\tau)} f(x) = a(\tau)^{\gamma+N/2} \frac{c_k}{2^{\gamma+N/2}} \int_{\mathbb{R}^N} e^{\frac{i}{2}(\frac{c(\tau)}{a(\tau)}|x|^2 - a(\tau)b(\tau)|y|^2)} E_k(ix,y) \mathcal{D}_k f(a(\tau)y)\omega_k(y)\, dy.$$

Using this and the inverse formula for the Dunkl transform (2.9)

$$f(x) = \frac{c_k}{2^{\gamma+N/2}} \int_{\mathbb{R}^N} E_k(ix,y)\mathcal{D}_k f(y)\omega_k(y)\, dy, \tag{6.3}$$

we obtain

$$\begin{aligned} D_K^{M(\tau)} f(x) - f(x) &= \frac{c_k}{2^{\gamma+N/2}} \int_{\mathbb{R}^N} E_k(ix,y) \left[a(\tau)^{\gamma+N/2} e^{\frac{i}{2}(\frac{c(\tau)}{a(\tau)}|x|^2 - a(\tau)b(\tau)|y|^2)} \mathcal{D}_k f(a(\tau)y) - \mathcal{D}_k f(y)\right] \omega_k(y)\, dy \\ &= F_1(x) + F_2(x), \end{aligned}$$

where

$$\begin{aligned} F_1(x) &= a(\tau)^{\gamma+N/2} \frac{c_k}{2^{\gamma+N/2}} \int_{\mathbb{R}^N} E_k(ix,y) \left[e^{\frac{i}{2}(\frac{c(\tau)}{a(\tau)}|x|^2 - a(\tau)b(\tau)|y|^2)} \mathcal{D}_k f(a(\tau)y) - \mathcal{D}_k f(y)\right] \omega_k(y)\, dy \\ F_2(x) &= \left[a(\tau)^{\gamma+N/2} - 1\right] \frac{c_k}{2^{\gamma+N/2}} \int_{\mathbb{R}^N} E_k(ix,y)\mathcal{D}_k f(y)\omega_k(y)\, dy \\ &= \left[a(\tau)^{\gamma+N/2} - 1\right] f(x). \end{aligned}$$

Clearly

$$\lim_{\tau \to 0} \|F_2\|_{k,2} = 0.$$

From the relation

$$\begin{aligned} e^{\frac{i}{2}(\frac{c(\tau)}{a(\tau)}|x|^2 - a(\tau)b(\tau)|y|^2)} \mathcal{D}_k f(a(\tau)y) - \mathcal{D}_k f(y) &= e^{\frac{i}{2}\frac{c(\tau)}{a(\tau)}|x|^2} \left[e^{-\frac{i}{2}a(\tau)b(\tau)|y|^2} \mathcal{D}_k f(a(\tau)y)\right. \\ &- \left.\mathcal{D}_k f(y)\right] + \left[e^{\frac{i}{2}\frac{c(\tau)}{a(\tau)}|x|^2} - 1\right] \mathcal{D}_k f(y), \end{aligned}$$

we can write

$$\begin{aligned} F_1(x) &= \left[e^{\frac{i}{2}\frac{c(\tau)}{a(\tau)}|x|^2} - 1\right] a(\tau)^{\gamma+N/2} \frac{c_k}{2^{\gamma+N/2}} \int_{\mathbb{R}^N} E_k(ix,y)\mathcal{D}_k f(y)\omega_k(y)\, d + F_3(x) \\ &= \left[e^{\frac{i}{2}\frac{c(\tau)}{a(\tau)}|x|^2} - 1\right] a(\tau)^{\gamma+N/2} f(x) + F_3(x), \end{aligned}$$

where

$$F_3(x) = a(\tau)^{\gamma+N/2} \frac{c_k}{2^{\gamma+N/2}} \int_{\mathbb{R}^N} E_k(ix,y)\; e^{\frac{i}{2}\frac{c(\tau)}{a(\tau)}|x|^2} \left[e^{-\frac{i}{2}a(\tau)b(\tau)|y|^2} \mathcal{D}_k f(a(\tau)y) - \mathcal{D}_k f(y)\right] \omega_k(y)\, dy.$$

Using the dominated convergence theorem, we get

$$\begin{aligned} \lim_{\tau\to 0} \left\| \left(e^{\frac{i}{2}\frac{c(\tau)}{a(\tau)}|x|^2} - 1\right) f(x)\right\|_2^2 &= \lim_{\tau \to 0} \int_{\mathbb{R}^N} \left|e^{\frac{i}{2}\frac{c(\tau)}{a(\tau)}|x|^2} - 1\right|^2 |f(x)|^2 \omega_k(x)\, dx \\ &= 0. \end{aligned}$$

As $\mathcal{D}_k$ is an isometric isomorphism of $L^2_k(\mathbb{R}^N)$ we deduce

$$\|F_3\|_2 = a(\tau)^{\gamma+N/2} \left\|e^{-\frac{i}{2}a(\tau)b(\tau)|y|^2} \mathcal{D}_k f(a(\tau)y) - \mathcal{D}_k f(y)\right\|_{k,2}.$$

The triangle inequality shows that:

$$\begin{aligned} \|F_3\|_{k,2} &= a(\tau)^{\gamma+N/2} \left\|e^{-\frac{i}{2}a(\tau)b(\tau)|y|^2} \left(\mathcal{D}_k f(a(\tau)y) - \mathcal{D}_k f(y)\right) + (e^{-\frac{i}{2}a(\tau)b(\tau)|y|^2} - 1)\mathcal{D}_k f(y)\right\|_{k,2} \\ &\leq a(\tau)^{\gamma+N/2} \left\|e^{-\frac{i}{2}a(\tau)b(\tau)|y|^2} \left(\mathcal{D}_k f(a(\tau)y) - \mathcal{D}_k f(y)\right)\right\|_{k,2} + a(\tau)^{\gamma+N/2} \left\|\left(e^{-\frac{i}{2}a(\tau)b(\tau)|y|^2} - 1\right) \mathcal{D}_k f(y)\right\|_{k,2} \\ &= a(\tau)^{\gamma+N/2} \left\|\mathcal{D}_k f(a(\tau)y) - \mathcal{D}_k f(y)\right\|_{k,2} + a(\tau)^{\gamma+N/2} \left\|\left(e^{-\frac{i}{2}a(\tau)b(\tau)|y|^2} - 1\right) \mathcal{D}_k f(y)\right\|_{k,2}. \end{aligned}$$

By the dominated convergence theorem,

$$\begin{aligned}\lim_{\tau\to 0}\left\|\left(e^{-\frac{i}{2}a(\tau)b(\tau)|y|^2}-1\right)D_kf\right\|_{k,2}^2 &= \lim_{\tau\to 0}\int_{\mathbb{R}^N}\left|e^{-\frac{i}{2}a(\tau)b(\tau)|y|^2}-1\right|^2|D_kf(y)|^2\omega_k(y)\,dy\\ &= 0.\end{aligned}$$

By Lemma 6.1,

$$\lim_{\tau\to 0}\|D_kf(a(\tau)y)-D_kf(y)\|_{k,2}=0.$$

Hence

$$\lim_{\tau\to 0}\|F_3\|_{k,2}=0$$

and therefore (6.2) holds for each $f\in\mathcal{S}(\mathbb{R}^N)$.
Next, in the case $f\in L_k^2(\mathbb{R}^N)$. Let $\epsilon>0$ be arbitrary. Since $\mathcal{S}(\mathbb{R}^N)$ is dense in $L_k^2(\mathbb{R}^N)$, there exists $\psi\in\mathcal{S}(\mathbb{R}^N)$ such that $\|f-\psi\|_{k,2}\le\frac{\epsilon}{2}$. Then

$$\left\|D_k^{M(\tau)}f-D_k^{M(\tau)}\psi\right\|_{k,2}=\left\|D_k^{M(\tau)}(f-\psi)\right\|_{k,2}=\|f-\psi\|_{k,2}\le\frac{\epsilon}{2}.$$

From this, we can deduce that

$$\begin{aligned}\left\|D_k^{M(\tau)}f-f\right\|_{k,2} &\le \left\|D_k^{M(\tau)}f-D_k^{M(\tau)}\psi\right\|_{k,2}+\left\|D_k^{M(\tau)}\psi-\psi\right\|_{k,2}+\|\psi-f\|_{k,2}\\ &\le \left\|D_k^{M(\tau)}\psi-\psi\right\|_{k,2}+\epsilon,\end{aligned}$$

so that, since ϵ was arbitrary and $\psi\in\mathcal{S}(\mathbb{R}^N)$,

$$\lim_{\tau\to 0}\left\|D_k^{M(\tau)}f-f\right\|_{k,2}=0.$$

Corollary 6.1 *The family of operators* $\left\{D_k^{M(\tau)}\right\}_{\tau\in\mathbb{R}}$ *is a* $\mathcal{C}_0$*-group of unitary operators on* $L_k^2(\mathbb{R}^N)$*.*

Proof. It is clear that the family $\left\{D_k^{M(\tau)}\right\}_{\tau\in\mathbb{R}}$ satisfies the algebraic properties of a group:

$$D_k^{M(0)}=I,\quad D_k^{M(\tau_1)}\circ D_k^{M(\tau_2)}=D_k^{M(\tau_1+\tau_2)}=D_k^{M(\tau_2)}\circ D_k^{M(\tau_1)}\ ;\quad \tau_1,\ \tau_2\in\mathbb{R}.$$

For the strong continuity, we use theorem 6.1.

6.2 The generator of the $\mathcal{C}_0$-group $\{D_k^{M(\tau)}\}_{\tau\in\mathbb{R}}$.

The infinitesimal generator $\mathcal{L}$ of $\left\{D_k^{M(\tau)}\right\}_{\tau\in\mathbb{R}}$ is defined by

$$\begin{aligned}\mathcal{L}:D(\mathcal{L}) &\longrightarrow L_k^2(\mathbb{R}^N),\\ f &\longmapsto \mathcal{L}f\end{aligned}$$

where

$$\begin{aligned}D(\mathcal{L}) &= \left\{f\in L_k^2(\mathbb{R}^N):\ \lim_{\tau\to 0}(1/\tau)[D_k^{M(\tau)}f-f]\in L_k^2(\mathbb{R}^N)\right\},\\ \mathcal{L}f &= \lim_{\tau\to 0}(1/\tau)[D_k^{M(\tau)}f-f],\quad f\in D(\mathcal{L}).\end{aligned}$$

From the Hille-Yosida Theorem (see[[16], p. 15]), the operator $\mathcal{L}$ is closed and densely defined and since $\left\{D_k^{M(\tau)}\right\}_{\tau\in\mathbb{R}}$ is unitary, it follows from Stone's Theorem [[16], p. 32] that $\mathcal{L}$ is skew-adjoint ($\mathcal{L}^*=-\mathcal{L}$) and therefore $i\mathcal{L}$ is self-adjoint. Since it is often difficult to determine $D(\mathcal{L})$, it is important to know a core of $\mathcal{L}$ (any dense subspace with respect to the graph-norm $\|f\|_{\mathcal{L}}:=\|f\|_{k,2}+\|\mathcal{L}f\|_{k,2}$ on $D(\mathcal{L})$). For this purpose, we need some Lemmas.

Lemma 6.2 *Let f be a function in $\mathcal{C}^1(\mathbb{R}^N)$ such that f and $\rho(f)$ in $L^2_k(\mathbb{R}^N)$. Then*

$$\lim_{\tau\to 0}\left\|\frac{1}{\tau}[f(x/a(\tau))-f(x)]+a^{'}(0)\rho(f)(x)\right\|_{k,2}=0, \tag{6.4}$$

where $\rho(f)(x)=\sum_{j=1}^{N}x_j\dfrac{\partial}{\partial x_j}f(x)$ is the Euler operator.

Proof. Applying Taylor's formula to the function $t\longmapsto f(x/a(t))$, we obtain

$$f(x/a(\tau))=f(x)+\tau\int_0^1 g_x(s\tau)\,ds,\quad \text{where}\quad g_x(s\tau)=-\frac{a^{'}(s\tau)}{a^2(s\tau)}\sum_{j=1}^{N}x_j\frac{\partial}{\partial x_j}f(x/a(s\tau)).$$

Hence

$$\frac{1}{\tau}[f(x/a(\tau))-f(x)]+a^{'}(0)\rho(f)(x)=\int_0^1(g_x(s\tau)-g_x(0))\,ds.$$

Minkowski's inequality for integrals implies that

$$\begin{aligned}\left\|\frac{1}{\tau}[f(x/a(\tau))-f(x)]+a^{'}(0)\rho(f)(x)\right\|_{k,2} &= \left\|\int_0^1(g_x(s\tau)-g_x(0))\,ds\right\|_{k,2}\\ &\le \int_0^1\|g_x(s\tau)-g_x(0)\|_{k,2}\,ds.\end{aligned}$$

By Lemma 6.1,

$$\lim_{\tau\to 0}\|g_x(s\tau)-g_x(0)\|_{k,2}=0.$$

Under this condition, the dominated convergence theorem implies

$$\lim_{\tau\to 0}\int_0^1\|g_x(s\tau)-g_x(0)\|_{k,2}\,ds=0.$$

Lemma 6.3 *Let f be a function such that f and $|x|^2 f$ in $L^2_k(\mathbb{R}^N)$. Then*

$$\lim_{\tau\to 0}\left\|\frac{1}{\tau}\left[e^{\frac{i}{2}\frac{c(\tau)}{a(\tau)}|x|^2}-1\right]f(x/a(\tau))-\frac{i}{2}c^{'}(0)|x|^2f(x)\right\|_{k,2}=0.$$

Proof. Let G_1^τ the function defined by

$$G_1^\tau(x)=\frac{1}{\tau}\left[e^{\frac{i}{2}\frac{c(\tau)}{a(\tau)}|x|^2}-1\right]f(x/a(\tau)).$$

Clearly, the change of variable $u=x/a(\tau)$ gives

$$\begin{aligned}\|G_1^\tau(x)-\frac{i}{2}c^{'}(0)|x|^2f(x/a(\tau))\|_{k,2}^2 &= \int_{\mathbb{R}^N}\left|\frac{1}{\tau}\left[e^{\frac{i}{2}\frac{c(\tau)}{a(\tau)}|x|^2}-1\right]-\frac{i}{2}c^{'}(0)|x|^2\right|^2|f(x/a(\tau))|^2\omega_k(x)\,dx\\ &= (a(\tau))^{2\gamma+N}\int_{\mathbb{R}^N}\left|\frac{1}{\tau}\left[e^{\frac{i}{2}a(\tau)c(\tau)|x|^2}-1\right]-\frac{i}{2}c^{'}(0)a^2(\tau)|x|^2\right|^2|f(x)|^2\omega_k(x)\,dx.\end{aligned}$$

Using the Taylor's formula, we can show

$$\frac{1}{\tau}\left[e^{\frac{i}{2}a(\tau)c(\tau)|x|^2}-1\right]=\frac{i}{2}|x|^2\int_0^1(ac)^{'}(s\tau)e^{\frac{i}{2}a(s\tau)c(s\tau)|x|^2}\,ds$$

and therefore, there is a constant $M_1\ge 0$ and $\tau_0\ge 0$ such that

$$\left|\frac{1}{\tau}\left[e^{\frac{i}{2}a(\tau)c(\tau)|x|^2}-1\right]\right|\le M|x|^2$$

for all $\tau \in [0, \tau_0]$. Hence, for all $\tau \in [0, \tau_0]$, we have

$$\left| \frac{1}{\tau} \left[e^{\frac{i}{2} a(\tau)c(\tau)|x|^2} - 1 \right] - \frac{i}{2} c^{'}(0) a^2(\tau)|x|^2 \right|^2 |f(x)|^2 \leq M_2 \left| |x|^2 f(x) \right|^2,$$

where $M_2 \geq 0$. As $x \longmapsto |x|^2 f(x) \in L^2_k(\mathbb{R}^N)$, the dominated convergence theorem gives

$$\lim_{\tau \to 0} \int_{\mathbb{R}^N} \left| \frac{1}{\tau} \left[e^{\frac{i}{2} a(\tau)c(\tau)|x|^2} - 1 \right] - \frac{i}{2} c^{'}(0) a^2(\tau)|x|^2 \right|^2 |f(x)|^2 \omega_k(x)\ dx = 0.$$

Therefore

$$\lim_{\tau \to 0} \left\| G_1^\tau(x) - \frac{i}{2} c^{'}(0)|x|^2 f(x/a(\tau)) \right\|_{k,2} = 0. \tag{6.5}$$

Now, using lemma 4.1, we get

$$\lim_{\tau \to 0} \left\| \frac{|x|^2}{a^2(\tau)} f(x/a(\tau)) - |x|^2 f(x) \right\|_{k,2} = 0. \tag{6.6}$$

Hence, the desired result is an immediate consequence of (6.5) and (6.6).

Lemma 6.4 *Let f be a function such that f and $|x|^2 D_k f$ in $L^1_k(\mathbb{R}^N) \cap L^2_k(\mathbb{R}^N)$ and put*

$$G_2^\tau(x) = \frac{c_k e^{\frac{i}{2} \frac{c(\tau)}{a(\tau)} |x|^2}}{(2a(\tau))^{\gamma + N/2}} \int_{\mathbb{R}^N} \frac{1}{\tau} \left[e^{-\frac{i}{2} \frac{b(\tau)}{a(\tau)} |y|^2} - 1 \right] E_k(ix/a(\tau), y) D_k f(y) \omega_k(y)\ dy,$$

where $x \in \mathbb{R}^N$. Then

$$\lim_{\tau \to 0} \left\| G_2^\tau(x) + \frac{i}{2} b^{'}(0) D_k \left[|y|^2 D_k f(y) \right](-x) \right\|_{k,2} = 0.$$

Proof. Let

$$G_3^\tau(x) = \frac{i}{2} b^{'}(0) \frac{c_k e^{\frac{i}{2} \frac{c(\tau)}{a(\tau)} |x|^2}}{(2a(\tau))^{\gamma + N/2}} \int_{\mathbb{R}^N} E_k(ix/a(\tau), y) |y|^2 D_k f(y) \omega_k(y)\ dy.$$

Then

$$\begin{aligned} G_2^\tau(x) + G_3^\tau(x) &= \frac{c_k e^{\frac{i}{2} \frac{c(\tau)}{a(\tau)} |x|^2}}{(2a(\tau))^{\gamma + N/2}} \int_{\mathbb{R}^N} \left\{ \frac{1}{\tau} \left[e^{-\frac{i}{2} \frac{b(\tau)}{a(\tau)} |y|^2} - 1 \right] + \frac{i}{2}\ b^{'}(0)|y|^2 \right\} E_k(ix/a(\tau), y) D_k f(y) \omega_k(y)\ dy \\ &= \frac{e^{\frac{i}{2} \frac{c(\tau)}{a(\tau)} |x|^2}}{(a(\tau))^{\gamma + N/2}} D_k h(-x/a(\tau)), \end{aligned}$$

where

$$h(y) = \left\{ \frac{1}{\tau} \left[e^{-\frac{i}{2} \frac{b(\tau)}{a(\tau)} |y|^2} - 1 \right] + \frac{i}{2}\ b^{'}(0)|y|^2 \right\} D_k f(y).$$

According to hypothesis, the function $h \in L^1_k(\mathbb{R}^N) \cap L^2_k(\mathbb{R}^N)$. Then

$$G_2^\tau + G_3^\tau \in L^2_k(\mathbb{R}^N)$$

and

$$\|G_2^\tau + G_3^\tau\|_{k,2} = \frac{1}{(a(\tau))^{\gamma + N/2}}\ \|D_k h(-x/a(\tau))\|_{k,2}.$$

Now, using the change of variable $u = -x/a(\tau)$ to get

$$\|D_k h(-x/a(\tau))\|_{k,2} = (a(\tau))^{\gamma + N/2} \|D_k h\|_{k,2}.$$

Since D_k is an isometric isomorphism of $L^2_k(\mathbb{R}^N)$, we deduce

$$\|G_2^\tau + G_3^\tau\|_{k,2}^2 = \|h\|_{k,2}^2 = \int_{\mathbb{R}^N} \left| \frac{1}{\tau}\left[e^{-\frac{i}{2}\frac{b(\tau)}{a(\tau)}|y|^2} - 1\right] + \frac{i}{2}\, b^{'}(0)|y|^2 \right|^2 |D_k f(y)|^2 \omega_k(y)\; dy.$$

By means of a similar technic used in the proof of Lemma 6.3, we have

$$\lim_{\tau\to 0} \int_{\mathbb{R}^N} \left| \frac{1}{\tau}\left[e^{-\frac{i}{2}\frac{b(\tau)}{a(\tau)}|y|^2} - 1\right] + \frac{i}{2}\, b^{'}(0)|y|^2 \right|^2 |D_k f(y)|^2 \omega_k(y)\; dy = 0,$$

hence

$$\lim_{\tau\to 0} \|G_2^\tau + G_3^\tau\|_{k,2} = 0. \tag{6.7}$$

Clearly

$$G_3^\tau(x) = \frac{i}{2} b^{'}(0) \frac{e^{\frac{i}{2}\frac{c(\tau)}{a(\tau)}|x|^2}}{(a(\tau))^{\gamma+N/2}} G_4(x/a(\tau)),$$

where

$$G_4(x/a(\tau)) = D_k[|y|^2 D_k f(y)](-x/a(\tau)).$$

Using again Lemma 6.1, we get

$$\lim_{\tau\to 0} \|G_4(x/a(\tau)) - G_4(x)\|_{k,2} = 0.$$

This establishes

$$\lim_{\tau\to 0} \left\| G_3^\tau(x) - \frac{i}{2} b^{'}(0) D_k[|y|^2 D_k f(y)](-x) \right\|_{k,2} = 0. \tag{6.8}$$

Finally, the desired result is an immediate consequence of (6.7) and (6.8).

Theorem 6.2 *Let*

$$W = \left\{ f \in \mathcal{C}^1(\mathbb{R}^N);\; f, |x|^2 D_k f \in L^1_k(\mathbb{R}^N) \cap L^2_k(\mathbb{R}^N) \text{ and } \rho(f) \in L^2_k(\mathbb{R}^N) \right\}.$$

Then $W \subset D(\mathcal{L})$ and for all $f \in W$,

$$\mathcal{L}f(x) = -a^{'}(0)\left[(\gamma + N/2) + \rho\right] f(x) + ic^{'}(0)\; \frac{|x|^2 f(x)}{2} \;- \frac{i}{2} b^{'}(0) D_k \left[|y|^2 D_k f(y)\right](-x), \tag{6.9}$$

where

$$\rho(f)(x) = \sum_{j=1}^N x_j \frac{\partial}{\partial x_j} f(x).$$

Proof. Let $f \in W$. By (3.22) and (6.3), we get

$$\frac{1}{\tau}\left[D_k^{M(\tau)} f(x) - f(x)\right] = \frac{1}{\tau}\left[(a(\tau))^{-(\gamma+\frac{N}{2})} - 1\right] f(x) + H_1^\tau(x),$$

where

$$H_1^\tau(x) = \frac{c_k}{(2a(\tau))^{\gamma+\frac{N}{2}}} \int_{\mathbb{R}^N} \frac{1}{\tau}\left[e^{\frac{i}{2}(\frac{c(\tau)}{a(\tau)}|x|^2 - \frac{b(\tau)}{a(\tau)}|y|^2)}\; E_k(ix/a(\tau), y) - E_k(ix, y)\right]\; D_k f(y) \omega_k(y)\; dy.$$

Now, writing

$$\begin{aligned}
\frac{1}{\tau}\left[e^{\frac{i}{2}(\frac{c(\tau)}{a(\tau)}|x|^2 - \frac{b(\tau)}{a(\tau)}|y|^2)}\; E_k(ix/a(\tau), y) - E_k(ix, y)\right] \;&=\; e^{\frac{i}{2}\frac{c(\tau)}{a(\tau)}|x|^2} \frac{1}{\tau}\left[e^{-\frac{i}{2}\frac{b(\tau)}{a(\tau)}|y|^2} - 1\right] E_k(ix/a(\tau), y) \\
&+\; \frac{1}{\tau}\left[e^{\frac{i}{2}\frac{c(\tau)}{a(\tau)}|x|^2} - 1\right]\; E_k(ix/a(\tau), y) \\
&+\; \frac{1}{\tau}\left[E_k(ix/a(\tau), y) - E_k(ix, y)\right]
\end{aligned}$$

and using (2.9), then we have

$$H_1^\tau(x) \quad = \quad \frac{1}{(a(\tau))^{\gamma+\frac{N}{2}}}\left[\frac{1}{\tau}\left[f(x/a(\tau))-f(x)\right]+\frac{1}{\tau}\left[e^{\frac{i}{2}\frac{c(\tau)}{a(\tau)}|x|^2}-1\right]f(x/a(\tau))\right]+G_2^\tau(x),\ \text{a. e.}$$

By applying respectively lemmas 6.2, 6.3 and 6.4, we deduce

$$\begin{aligned}\lim_{\tau\to 0}\frac{1}{\tau}\left[D_k^{M(\tau)}f(x)-f(x)\right] &= -a^{'}(0)(\gamma+(N/2))f(x)-a^{'}(0)\rho(f)(x)\\ &+ ic^{'}(0)\,\frac{|x|^2}{2}f(x)\ -\frac{i}{2}b^{'}(0)D_k\left[|y|^2D_kf(y)\right](-x)\end{aligned}$$

with respect the $\|.\|_{k,2}$. This proves that $f\in D(\mathcal{L})$ and

$$\mathcal{L}f(x)=-a^{'}(0)\left[(\gamma+N/2)+\rho\right]f(x)+ic^{'}(0)\,\frac{|x|^2}{2}f(x)\ -\frac{i}{2}b^{'}(0)D_k\left[|y|^2D_kf(y)\right](-x).$$

In the following theorem we establish the main relations between the $\mathcal{C}_0$-group $\{D_k^{M(\tau)}\}_{\tau\in\mathbb{R}}$ and a generalized Dunkl-Schrödinger equation.

Theorem 6.3 *The following properties holds.*
(1) *The Schwartz space $\mathcal{S}(\mathbb{R}^N)$ is a core of the generator $\mathcal{L}$ of the $\mathcal{C}_0$-group $\left\{D_k^{M(\tau)}\right\}_{\tau\in\mathbb{R}}$ and*

$$\mathcal{L}_{|\mathcal{S}(\mathbb{R}^N)}f=-a^{'}(0)\left((\gamma+N/2)+\sum_{j=1}^{N}x_j\frac{\partial}{\partial x_j}\right)f+i\left(c^{'}(0)\,\frac{|x|^2}{2}+b^{'}(0)\,\frac{\Delta_k}{2}\right)f. \tag{6.10}$$

(2) *For each $f\in D(\mathcal{L})$ the function $u(t,x)=D_k^{M(t)}f(x)$ is the unique classical solution of the problem*

$$\begin{cases}\frac{\partial}{\partial t}u(t,x)=\mathcal{L}u(t,x)\ \text{on}\ \mathbb{R}\times\mathbb{R}^N,\\ u(0,.)=f\in D(\mathcal{L});\end{cases} \tag{6.11}$$

here "classical" means $u\in\mathcal{C}^1(\mathbb{R},\ L_k^2(\mathbb{R}^N))$ with $u(t,.)\in D(\mathcal{L})$ for all $t\in\mathbb{R}$.
(3) *(i) Let $f\in\mathcal{S}(\mathbb{R}^N)$. The function*

$$\begin{aligned}u(t,x) &= D_k^{M(t)}f(x)\\ &= \frac{c_k}{(2ib(t))^{\gamma+(N/2)}}\int_{\mathbb{R}^N}e^{\frac{i}{2}(\frac{d(t)}{b(t)}|x|^2+\frac{a(t)}{b(t)}|y|^2)}E_k(-ix/b(t),y)f(y)\omega_k(y)dy\end{aligned}$$

is the unique solution of the generalized Dunkl-Schrödinger equation

$$\begin{cases}i\frac{\partial}{\partial t}u(t,x)=-ia^{'}(0)\left((\gamma+N/2)+\sum_{j=1}^{N}x_j\frac{\partial}{\partial x_j}\right)u(t,x)-\left(c^{'}(0)\ \frac{|x|^2}{2}+b^{'}(0)\ \frac{\Delta_k}{2}\right)u(t,x)\ \text{on}\ \mathbb{R}\times\mathbb{R}^N,\\ u(0,.)=f\in\mathcal{S}(\mathbb{R}^N).\end{cases} \tag{6.12}$$

Moreover, $u(t,x)$ has the following properties:
(ii) $u(t,.)\in\mathcal{S}(\mathbb{R}^N)$ for all $t\in\mathbb{R}$.
(iii) $u(t+s,x)=D_k^{M(t)}(u(s,.))(x)$ for all $t,s\in\mathbb{R}$ and $x\in\mathbb{R}^N$.
(iv) For all $t\in\mathbb{R}$ such that $b(t)\neq 0$,

$$\|u(t,.)\|_\infty\le\frac{c_k}{(2|b(t)|)^{\gamma+(N/2)}}\|u(0,.)\|_{1,k}.$$

(v) $\|u(t,.)\|_{k,2}=\|u(0,.)\|_{k,2}$ for all $t\in\mathbb{R}$.

Proof.
(1) It is easy to see that $\mathcal{S}(\mathbb{R}^N)\subset W\subset D(\mathcal{L})$. To prove (6.10) it suffices to show that

$$D_k\left[|y|^2D_k(y)\right](-x)\quad=\quad-\Delta_kf(x).$$

Let $f\in\mathcal{S}(\mathbb{R}^N)$. By (2.10), we obtain

$$-y_j^2D_kf(y)=D_k[T_j^2f](y);\quad j\in\{1,2,\dots,N\}.$$

As a consequence of this, we deduce

$$-|y|^2 D_k f(y) = D_k[\Delta_k f](y).$$

Therefore

$$-D_k\left[|y|^2 D_k(y)\right](-x) \quad = \quad D_k^2[\Delta_k f(y)](-x) = \Delta_k f(x).$$

The fact that $\mathcal{S}(\mathbb{R}^N)$ is a core of $\mathcal{L}$ can be proved by Proposition 1.7 of [12] since $\mathcal{S}(\mathbb{R}^N)$ is $\|.\|_{k,2}$-dense in $L_k^2(\mathbb{R}^N)$ and invariant under the semigroup $\left\{D_k^{M(\tau)}\right\}_{\tau\in\mathbb{R}}$.
(2) follows from the Theorem 1.2 of [16] since $\mathcal{L}$ generates the $\mathcal{C}_0$-group$\left\{D_k^{M(\tau)}\right\}_{\tau\in\mathbb{R}}$.
(3)(i) is a direct application of (6.11) and the integral representation (3.20) of $D_k^{M(t)}f(x)$.
(3)(ii) follows immediately from Theorem 4.1, 2).
(3)(iii) follows from Theorem 4.2.
(3)(iv) follows from the estimates (4.1), while (3)(v) is obtained by the Plancheral Theorem 5.1.

Remark 6.1 *The first statement of the previous Theorem shows that the so called generalized Dunkl-Schrödinger operator*

$$-a^{'}(0)\left((\gamma+N/2)+\sum_{j=1}^{N}x_j\frac{\partial}{\partial x_j}\right)+i\left(c^{'}(0)\ \frac{|x|^2}{2}+b^{'}(0)\ \frac{\Delta_k}{2}\right)$$

is closable and its closure is the generator of the $\mathcal{C}_0$-group$\left\{D_k^{M(\tau)}\right\}_{\tau\in\mathbb{R}}$.

7 One-parameter subgroups of $SL(2,\mathbb{R})$ and the associated generalized Dunkl transform.

We conclude this paper by mentioning some interesting one-parameter subgroups of $SL(2,\mathbb{R})$ with the associated integral transform, its basic properties and the related Dunkl-Schrödinger operator and equation.

7.1 Basic properties of the generalized Fresnel transform.

In this subsection, the one-parameter subgroup of $SL(2,\mathbb{R})$ is $\left\{M(\tau)=\begin{bmatrix}1 & \tau\\ 0 & 1\end{bmatrix};\ \tau\in\mathbb{R}\right\}$.

7.1.1 The Fresnel transform associated with the Dunkl transform

The Fresnel transform of a function $f\in L^1(\mathbb{R},dx)$ is defined by [4]

$$\mathcal{E}_\tau(f)(x)=\frac{1}{\sqrt{2i\pi\tau}}\int_{-\infty}^{+\infty}e^{\frac{i}{2\tau}(x-y)^2}f(y)\ dy. \tag{7.1}$$

It corresponds to the one dimension linear canonical transform parameter matrix $M=\begin{bmatrix}1 & \tau\\ 0 & 1\end{bmatrix}$. It is perfectly reasonable to generalize (7.1) in the Dunkl setting as follows:

Definition 7.1 *We define the Fresnel transform in the Dunkl setting $\mathcal{E}_{k,\tau}$ on the space $L_k^1(\mathbb{R}^N)$ by setting*

$$\begin{aligned}\mathcal{E}_{k,\tau}(f)(x) &= D_k^{M(\tau)}(f)(x)\\ &= \begin{cases}\frac{c_k}{(2i\tau)^{\gamma+(N/2)}}\displaystyle\int_{\mathbb{R}^N}e^{\frac{i}{2\tau}(|x|^2+|y|^2)}E_k(-ix/\tau,y)f(y)\omega_k(y)\ dy, & \tau\neq 0\\ f(x), & \tau=0,\end{cases}\end{aligned} \tag{7.2}$$

where $M(\tau)=\begin{bmatrix}1 & \tau\\ 0 & 1\end{bmatrix}$.

Remark 7.1
1) *The integral transform (7.2) is an extension for all* $\tau \in \mathbb{R}$ *of the one given by Rösler in ([29]).*
2) *When the multiplicity function* $k \equiv 0$*, the Fresnel transform in the Dunkl setting* $\mathcal{E}_{k,\tau}$ *coincides with the* N*-dimensional Fresnel transform*

$$\mathcal{E}_\tau(f)(x) = \frac{1}{(2i\pi\tau)^{N/2}} \int_{\mathbb{R}^N} e^{\frac{i}{2\tau}|x-y|^2} f(y)\ dy.$$

3) *Let* $f \in L^1_k(\mathbb{R}^N) \cap L^2_k(\mathbb{R}^N)$ *be a radial function and put* $f(x) = \psi(|x|)$*. According to Theorem 4.3, it follows that:*

$$\mathcal{E}_{k,\tau}(f)(x) = \frac{2}{\Gamma(\lambda+1)(2i\tau)^{\lambda+1}} \int_0^{+\infty} e^{\frac{i}{2\tau}(|x|^2+r^2)} \psi(r) j_\lambda\left(\frac{rx}{\tau}\right)\ r^{2\lambda+1} dr,$$

where $\lambda = \gamma + (N/2) - 1$.

Thanks to Remark 7.1, 3), we can state the following definition:

Definition 7.2 *We define the Fresnel transform* $\mathcal{W}_{\mu,\tau}$ *associated with the Hankel transform* $\mathcal{H}_\mu$ *for suitable function* f *on* $\mathbb{R}_+$ *and* $\mu \geq -1/2$ *by*

$$\mathcal{W}_{\mu,\tau}(f)(x) = \frac{2}{\Gamma(\mu+1)(2i\tau)^{\mu+1}} \int_0^{+\infty} e^{\frac{i}{2\tau}(|x|^2+r^2)} f(r) j_\mu\left(\frac{rx}{\tau}\right)\ r^{2\mu+1} dr. \tag{7.3}$$

7.1.2 Basic properties

Here we list some properties of the generalized Fresnel transform.

Proposition 7.1 *(Riemann-Lebesgue lemma): Suppose that* $\tau \neq 0$*. Then for all* $f \in L^1_k(\mathbb{R}^N)$*,* $\mathcal{E}_{k,\tau}$ *belongs to* $\mathcal{C}_0(\mathbb{R}^N)$ *and verifies*

$$\|\mathcal{E}_{k,\tau} f\|_\infty \leq \frac{c_k}{(2|\tau|)^{\gamma+(N/2)}}\ \|f\|_{k,1}.$$

Proposition 7.2 *(The reversibility property:)*
1) *For all* $f \in L^1_k(\mathbb{R}^N)$ *with* $\mathcal{E}_{k,\tau} f \in L^1_k(\mathbb{R}^N)$*,*

$$(\mathcal{E}_{k,-\tau} \circ \mathcal{E}_{k,\tau}) f = f,\ a.e, \quad and \quad (\mathcal{E}_{k,\tau} \circ \mathcal{E}_{k,-\tau}) f = f,\ a.e.$$

2) *The generalized Fresnel transform* $\mathcal{E}_{k,\tau}$ *is a one-to-one and onto mapping from* $\mathcal{S}(\mathbb{R}^N)$ *into* $\mathcal{S}(\mathbb{R}^N)$*. Moreover,*

$$(\mathcal{E}_{k,\tau})^{-1} f = \mathcal{E}_{k,-\tau} f, \quad f \in \mathcal{S}(\mathbb{R}^N).$$

Proposition 7.3 *(An additivity property) Let* τ_1 *and* τ_2 *be real numbers and let* $f \in L^1_k(\mathbb{R}^N)$ *with* $\mathcal{E}_{k,\tau_2} f \in L^1_k(\mathbb{R}^N)$*. Then*

$$\mathcal{E}_{k,\tau_1}\ \mathcal{E}_{k,\tau_2} f = e^{i\psi}\ \mathcal{E}_{k,\tau_1+\tau_2} f, \tag{7.4}$$

where the constant phase ψ *is given by*

$$\psi = \begin{cases} \frac{\pi}{2}(\gamma + (N/2))\left(sgn(\tau_1+\tau_2) + sgn\left(\frac{\tau_1+\tau_2}{\tau_1\tau_2}\right) - sgn(\tau_1) - sgn(\tau_2)\right), & \tau_1 \neq 0, \quad \tau_2 \neq 0, \quad \tau_1+\tau_2 \neq 0, \\ 0, & if\ not\ , \end{cases}$$

with equality a. e when $\tau_1 \neq 0$*,* $\tau_2 \neq 0$ *and* $\tau_1 + \tau_2 = 0$*.*
In particular if $\tau_1\tau_2 \geq 0$*, then*

$$\mathcal{E}_{k,\tau_1}\ \mathcal{E}_{k,\tau_2} f = \mathcal{E}_{k,\tau_1+\tau_2} f. \tag{7.5}$$

Proposition 7.4 *(Operational formula) Let* $\tau \in \mathbb{R}$*. Then the following properties hold on* $\mathcal{S}(\mathbb{R}^N)$*.*
(1) $\mathcal{E}_{k,\tau} \circ Q_\xi = (Q_\xi - \tau\ P_\xi) \circ \mathcal{E}_{k,\tau}$.
(2) $\mathcal{E}_{k,\tau} P_\xi = P_\xi \circ \mathcal{E}_{k,\tau}$.

Proposition 7.5 *(Bochner type identity) Let τ be a real number such that $\tau \neq 0$. If $f \in L^1_k(\mathbb{R}^N) \cap L^2_k(\mathbb{R}^N)$ is of the form $f(x) = p(x)\psi(|x|)$ for some $p \in H^k_n$ and a one-variable ψ on $\mathbb{R}_+$, then*

$$\mathcal{E}_{k,\tau} f(x) = p(x)\mathcal{W}_{n+\gamma+(N/2)-1,\tau}\ \psi(|x|). \tag{7.6}$$

In particular, if f is radial, then

$$\mathcal{E}_{k,\tau} f(x) = \mathcal{W}_{\gamma+(N/2)-1,\tau}\ \psi(|x|).$$

Proposition 7.6 *Let $\tau \in \mathbb{R}$ such that $\tau \neq 0$. The generalized Fresenel transform of the generalized Laguerre functions are*

$$\mathcal{E}_{k,\tau}\psi_{m,n,j}(x) \quad = \quad \lambda_{m,n,\tau} \quad e^{\frac{i\ \tau}{2(1+\tau^2)}|x|^2}\ \psi_{m,n,j}\left(\frac{x}{\sqrt{1+\tau^2}}\right),$$

where

$$\lambda_{m,n,\tau} = \left(\frac{1-i\tau}{1+i\tau}\right)^m \frac{(1-i\tau)^{\frac{n}{2}}}{(1+i\tau)^{\frac{n}{2}}} \frac{e^{i\theta}}{(1+i\tau)^{\gamma+(N/2)}}$$

and

$$\theta = 2(\gamma + (N/2) + n - 1)\left\{ sgn(\tau) \arctan\left(\frac{1}{|\tau| + \sqrt{1+\tau^2}}\right) - \frac{\pi}{4} sgn(\tau) + \arctan\left(\frac{\tau}{1+\sqrt{1+\tau^2}}\right)\right\}.$$

Proposition 7.7 *(Master formula) Let τ be a real number such that $\tau \neq 0$. Let f_n is of the form $f_n(x) = e^{-\frac{|x|^2}{2}} e^{-\frac{\Delta_k}{4}} p(x)$ for some $p \in \mathcal{P}_n$, Then*

$$\mathcal{E}_{k,\tau} f_n(x) \quad = \quad \lambda_{n,\tau} \quad e^{\frac{i\ \tau}{2(1+\tau^2)}|x|^2}\ f_n\left(\frac{x}{\sqrt{1+\tau^2}}\right),$$

where

$$\lambda_{n,\tau} = \frac{(1-i\tau)^{\frac{n}{2}}}{(1+i\tau)^{\frac{n}{2}}} \frac{e^{i\theta}}{(1+i\tau)^{\gamma+(N/2)}}$$

and

$$\theta = 2(\gamma + (N/2) + n - 1)\left\{ sgn(\tau) \arctan\left(\frac{1}{|\tau| + \sqrt{1+\tau^2}}\right) - \frac{\pi}{4} sgn(\tau) + \arctan\left(\frac{\tau}{1+\sqrt{1+\tau^2}}\right)\right\}$$

Proposition 7.8
(1) *The generalized Fresnel transform $\mathcal{E}_{k,\tau}$ have a unique extension to an unitary operator on $L^2_k(\mathbb{R}^N)$.*
(2) *The family $\{\mathcal{E}_{k,\tau}\}_{\tau \geq 0}$ is a $\mathcal{C}_0$-group of unitary operators on $L^2_k(\mathbb{R}^N)$. The Schwartz space $\mathcal{S}(\mathbb{R}^N)$ is a core of its generator A and $A_{|\mathcal{S}(\mathbb{R}^N)} = i\Delta_k$*
(3) *For each $f \in D(A)$ the function $u(t,x) = \mathcal{E}_{k,t} f(x)$ is the unique classical solution of the problem*

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) = Au(t,x) \ \text{on}\ \mathbb{R}_+ \times \mathbb{R}^N, \\ u(0,.) = f \in D(A); \end{cases}$$

(4) (*i*) *Let $f \in \mathcal{S}(\mathbb{R}^N)$. The function $u(t,x) = \mathcal{E}_{k,t} f(x)$ is the unique solution of the Dunkl-Schrödinger equation*

$$\begin{cases} i\frac{\partial}{\partial t} u(t,x) = -\Delta_k u(t,x) \ \text{on}\ \mathbb{R}_+ \times \mathbb{R}^N, \\ u(0,.) = f \in \mathcal{S}(\mathbb{R}^N). \end{cases} \tag{7.7}$$

Moreover, $u(t,x)$ has the following properties:
(*ii*) *$u(t,.) \in \mathcal{S}(\mathbb{R}^N)$ for all $t \in \mathbb{R}_+$.*
(*iii*) *$u(t+s,x) = \mathcal{E}_{k,t}(u(s,.))(x)$ for all $t,s \in \mathbb{R}_+$ and $x \in \mathbb{R}^N$.*
(*iv*) *For all $t > 0$,*

$$\|u(t,.)\|_\infty \leq \frac{c_k}{(2t)^{\gamma+(N/2)}} \|u(0,.)\|_{k,1}.$$

(*v*) *$\|u(t,.)\|_{k,2} = \|u(0,.)\|_{k,2}$ for all $t \in \mathbb{R}_+$.*

7.2 Basic properties of the fractional Dunkl transform

In this subsection, the one-parameter subgroup of $SL(2,\mathbb{R})$ is $\left\{ M(\alpha) = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix};\ \alpha \in \mathbb{R} \right\}$.

7.2.1 The fractional Dunkl transform

The fractional Dunkl transform D_k^α of real order $\alpha \in \mathbb{R}$ which is defined by[14, 15]:

$$D_k^\alpha f(x) \quad = \quad \begin{cases} \frac{c_k e^{i(\gamma+N/2)((\alpha-2n\pi)-\hat{\alpha}\pi/2)}}{(2|\sin\alpha|)^{\gamma+N/2}} \displaystyle\int_{\mathbb{R}^N} e^{\frac{i}{2}(|x|^2+|y|^2)\cot\alpha} E_k(-ix/\sin(\alpha), y) f(y)\, dy, & (2n-1)\pi < \alpha < (2n+1)\pi, \\ f(x), & \alpha = 2n\pi, \\ f(-x), & \alpha = (2n+1)\pi, \end{cases}$$

can be considered (except for a constant unimodular factor $\left(e^{i\alpha}\right)^{\gamma+\frac{N}{2}}$) a special cases of the generalized Dunkl transform D_k^M with parameter matrix $M(\alpha) = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$.

7.2.2 Basic properties

Here we list some properties of the fractional Dunkl transform.

Proposition 7.9 *(Riemann-Lebesgue lemma): Suppose that $\alpha \in \mathbb{R}\backslash\pi\mathbb{Z}$. Then for all $f \in L_k^1(\mathbb{R}^N)$, D_k^α belongs to $\mathcal{C}_0(\mathbb{R}^N)$ and verifies*

$$\|D_k^\alpha f\|_\infty \leq \frac{c_k}{(2|\sin(\alpha)|)^{\gamma+(N/2)}}\ \|f\|_{k,1}.$$

Proposition 7.10 *(The reversibility property:)*
1) *For all $f \in L_k^1(\mathbb{R}^N)$ with $D_k^\alpha f \in L_k^1(\mathbb{R}^N)$,*

$$(D_k^{-\alpha} \circ D_k^\alpha) f = f, \ \ a.e, \quad and \quad (D_k^\alpha \circ D_k^{-\alpha}) f = f, \ \ a.e.$$

2) *The fractional Dunkl transform D_k^α is a one-to-one and onto mapping from $\mathcal{S}(\mathbb{R}^N)$ into $\mathcal{S}(\mathbb{R}^N)$. Moreover,*

$$(D_k^\alpha)^{-1} f = D_k^{-\alpha} f, \quad f \in \mathcal{S}(\mathbb{R}^N).$$

Proposition 7.11 *(An additivity property) Let α and β be real numbers and let $f \in L_k^1(\mathbb{R}^N)$ with $D_k^\beta f \in L_k^1(\mathbb{R}^N)$. Then*

$$D_k^\alpha D_k^\beta f = D_k^{\alpha+\beta} f, \tag{7.8}$$

with equality a. e when $\alpha + \beta = 0$.

Proposition 7.12 *(Operational formula)*
Let $\alpha \in \mathbb{R}$ and $\{\xi_j\}_{j=1}^N$ is an orthonormal basis of $\mathbb{R}^N$. For $j = 1, \ldots, N$, define A_{ξ_j} and $A_{\xi_j}^*$ by

$$A_{\xi_j} = 2^{-1/2}\left[Q_{\xi_j} - T_{\xi_j}\right] \quad and \quad A_{\xi_j}^* = 2^{-1/2}\left[Q_{\xi_j} + T_{\xi_j}\right].$$

Then the following relations hold on $\mathcal{S}(\mathbb{R}^N)$:
(1) $D_k^\alpha \circ Q_\xi = (\cos(\alpha) Q_\xi + \sin(\alpha) P_\xi) \circ D_k^\alpha$.
(2) $D_k^\alpha \circ P_\xi = (-\sin(\alpha) Q_\xi + \cos(\alpha) P_\xi) \circ D_k^\alpha$.
(3) $\langle A_{\xi_j} f, g\rangle_k = \langle f, A_{\xi_j}^* g\rangle_k$; $f, g \in \mathcal{S}(\mathbb{R}^N)$.
(4) $D_k^\alpha \circ A_{\xi_j} = e^{i\alpha}(A_{\xi_j} \circ D_k^\alpha)$.
(5) $D_k^\alpha \circ A_{\xi_j}^* = e^{-i\alpha}(A_{\xi_j}^* \circ D_k^\alpha)$.
(6) $D_k^\alpha \circ \mathbb{H}_k = \mathbb{H}_k \circ D_k^\alpha$ *where $\mathbb{H}_k$ is the generalized Hermite operator which is defined by [24]:*

$$\mathbb{H}_k = \frac{1}{2}\sum_{j=1}^N A_{\xi_j}^* A_{\xi_j} + A_{\xi_j} A_{\xi_j}^* = \frac{1}{2}(|x|^2 - \Delta_k).$$

Proposition 7.13 *(Bochner type identity) Let α be a real number. If $f \in L_k^1(\mathbb{R}^N) \cap L_k^2(\mathbb{R}^N)$ is of the form $f(x) = p(x)\psi(|x|)$ for some $p \in H_n^k$ and a one-variable ψ on $\mathbb{R}_+$, then*

$$D_k^\alpha f(x) = p(x)\mathcal{H}^\alpha_{n+\gamma+(N/2)-1}\psi(|x|). \tag{7.9}$$

In particular, if f is radial, then

$$D_k^\alpha f(x) = \mathcal{H}^\alpha_{\gamma+(N/2)-1}\psi(|x|).$$

Proposition 7.14 *Let $\alpha \in \mathbb{R}$. The fractional Dunkl transform of the generalized Laguerre functions are*

$$D_k^\alpha \psi_{m,n,j}(x) \quad = \quad e^{i\alpha(n+2m)}\psi_{m,n,j}(x).$$

Proposition 7.15 *(Master formula)*
Let α be a real number such that $\alpha \neq 0$. Let f_n is of the form $f_n(x) = e^{-\frac{|x|^2}{2}} e^{-\frac{\Delta_k}{4}} p(x)$ for some $p \in \mathcal{P}_n$, Then:
(1)

$$D_k^\alpha f_n(x) = e^{in\alpha} f_n(x). \tag{7.10}$$

(2) *In particular,*

$$D_k^\alpha h_\nu(x) = e^{i|\nu|\alpha}\ h_\nu(x), \tag{7.11}$$

where $h_\nu(x) = \sqrt{c_k 2^{|\nu|}} e^{-\frac{|x|^2}{2}} e^{-\frac{\Delta_k}{4}} \varphi_\nu(x)$ is the generalized Hermite functions [24].

Proposition 7.16
(1) *The fractional Dunkl transform D_k^α have a unique extension to an unitary operator on $L_k^2(\mathbb{R}^N)$.*
(2) *The generalized Hermite functions $\{h_\nu, \nu \in \mathbb{Z}_+^N\}$ are an orthonormal basis of eigenfunctions of the fractional Dunkl transform D_k^α on $L_k^2(\mathbb{R}^N)$ satisfying $D_k^\alpha h_\nu(x) = e^{i|\nu|\alpha}\ h_\nu(x)$.*
(3) *The family of operators $\{D_k^\alpha\}_{\alpha\in\mathbb{R}}$ is a $\mathcal{C}_0$-group of unitary operators on $L_k^2(\mathbb{R}^N)$. The Schwartz space $\mathcal{S}(\mathbb{R}^N)$ is a core of its generator T and*

$$T_{|\mathcal{S}(\mathbb{R}^N)} = -i(\gamma + (N/2)) + \frac{i}{2}(|x|^2 - \Delta_k).$$

(4) *For each $f \in D(T)$ the function $u(t,x) = D_k^t f(x)$ is the unique classical solution of the problem*

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) = Tu(t,x) \text{ on } \mathbb{R}\times\mathbb{R}^N, \\ u(0,.) = f \in D(T); \end{cases}$$

(5) (*i*) *Let $f \in \mathcal{S}(\mathbb{R}^N)$. The function $u(t,x) = D_k^t f(x)$ is the unique solution of the Dunkl-Schrödinger equation*

$$\begin{cases} i\frac{\partial}{\partial t} u(t,x) = \left((\gamma + N/2) - \frac{1}{2}(|x|^2 - \Delta_k)\right) u(t,x) \text{ on } \mathbb{R}\times\mathbb{R}^N, \\ u(0,.) = f \in \mathcal{S}(\mathbb{R}^N). \end{cases} \tag{7.12}$$

Moreover, $u(t,x)$ has the following properties:
(*ii*) *$u(t,.) \in \mathcal{S}(\mathbb{R}^N)$ for all $t \in \mathbb{R}$.*
(*iii*) *$u(t+s,x) = D_k^t(u(s,.))(x)$ for all $t,s \in \mathbb{R}$ and $x \in \mathbb{R}^N$.*
(*iv*) *For all $t \in \mathbb{R}/\pi\mathbb{Z}$,*

$$\|u(t,.)\|_\infty \leq \frac{c_k}{(2|\sin(t)|)^{\gamma+(N/2)}}\|u(0,.)\|_{k,1}.$$

(*v*) *$\|u(t,.)\|_{k,2} = \|u(0,.)\|_{k,2}$ for all $t \in \mathbb{R}$.*

We conclude this subsection by an alternative proof of the following result established by Rösler in [24]

Corollary 7.1 *(see [14]) For $n \in \mathbb{N}$ and $p \in \mathcal{P}_n$, the function $f_n(x) = e^{-\frac{|x|^2}{2}} e^{-\frac{\Delta_k}{4}} p(x)$ satisfies*

$$\frac{1}{2}(|x|^2 - \Delta_k) f_n = (n + \gamma + (N/2)) f_n. \tag{7.13}$$

In particular, for $\nu \in \mathbb{Z}_+^N$

$$\frac{1}{2}(|x|^2 - \Delta_k) h_\nu = (|\nu| + \gamma + (N/2)) h_\nu. \tag{7.14}$$

Proof. Since $f_n \in \mathcal{S}(\mathbb{R}^N) \subset D(T)$, then

$$\begin{aligned} \lim_{\alpha\to 0} \frac{D_k^\alpha f_n - f_n}{\alpha} &= T(f_n) \\ &= -i(\gamma + (N/2)) f_n + \frac{i}{2}(|x|^2 - \Delta_k) f_n. \end{aligned}$$

Using (7.10) we obtain

$$\begin{aligned} \lim_{\alpha\to 0}\frac{D_k^\alpha f_n - f_n}{\alpha} &= \lim_{\alpha\to 0}\frac{e^{in\alpha}-1}{\alpha} f_n \\ &= in\alpha f_n. \end{aligned}$$

Hence

$$\frac{1}{2}(|x|^2-\Delta_k)f_n = (n+\gamma+(N/2))f_n.$$

7.3 Basic properties of the generalized Dunkl transform associated to the hyperbolic subgroup of $SL(2,\mathbb{R})$

In this subsection, the one-parameter subgroup of $SL(2,\mathbb{R})$ is $\left\{M(\alpha)=\begin{bmatrix}\cosh(\alpha) & \sinh(\alpha)\\ \sinh(\alpha) & \cosh(\alpha)\end{bmatrix};\ \alpha\in\mathbb{R}\right\}$.

Definition 7.3 *For $f\in L^1_k(\mathbb{R}^N)$ and $\alpha\in\mathbb{R}$, we define*

$$O_k^\alpha f(x) = D_k^{M(\alpha)}f(x) = \begin{cases} \frac{c_k}{(2i\sinh\alpha)^{\gamma+(N/2)}}\displaystyle\int_{\mathbb{R}^N} e^{\frac{i}{2}(|x|^2+|y|^2)\coth\alpha}\ E_k\left(-\frac{ix}{\sinh\alpha},y\right)f(y)\omega_k(y)\ dy, & \alpha\neq 0\\ f(x), & \alpha=0,\end{cases}$$

where $M(\alpha)=\begin{bmatrix}\cosh(\alpha) & \sinh(\alpha)\\ \sinh(\alpha) & \cosh(\alpha)\end{bmatrix}$.

7.3.1 Basic properties

Here we list some properties of O_k^α.

Proposition 7.17 *(Riemann-Lebesgue lemma): Suppose that $\alpha\neq 0$. Then for all $f\in L^1_k(\mathbb{R}^N)$, $O_k^\alpha f$ belongs to $\mathcal{C}_0(\mathbb{R}^N)$ and verifies*

$$\|O_k^\alpha f\|_\infty \leq \frac{c_k}{(2|\sinh(\alpha)|)^{\gamma+(N/2)}}\ \|f\|_{k,1}.$$

Proposition 7.18 *(The reversibility property:)*
1) *For all $f\in L^1_k(\mathbb{R}^N)$ with $O_k^\alpha f\in L^1_k(\mathbb{R}^N)$,*

$$(O_k^{-\alpha}\circ O_k^\alpha)f = f,\ a.e, \quad and \quad (O_k^\alpha\circ O_k^{-\alpha})f=f,\ a.e.$$

2) *O_k^α is a one-to-one and onto mapping from $\mathcal{S}(\mathbb{R}^N)$ into $\mathcal{S}(\mathbb{R}^N)$. Moreover,*

$$(O_k^\alpha)^{-1}f = O_k^{-\alpha}f, \quad f\in\mathcal{S}(\mathbb{R}^N).$$

Proposition 7.19 *(An additivity property) Let α and β be real numbers and let $f\in L^1_k(\mathbb{R}^N)$ with $O_k^\beta f\in L^1_k(\mathbb{R}^N)$. Then*

$$O_k^\alpha\ O_k^\beta f = e^{i\psi}\ O_k^{\alpha+\beta}f, \tag{7.15}$$

where the constant phase ψ is given by

$$\psi = \begin{cases} \frac{\pi}{2}(\gamma+(N/2))\left(sgn(\alpha+\beta)+sgn\left(\frac{\alpha+\beta}{\alpha\beta}\right)-sgn(\alpha)-sgn(\beta)\right), & \alpha\neq 0,\quad \beta\neq 0,\quad \alpha\neq 0,\\ 0, & if\ not\ ,\end{cases}$$

with equality a. e when $\alpha\neq 0$, $\beta\neq 0$ and $\alpha+\beta=0$.
In particular if $\alpha\beta\geq 0$, then

$$O_k^\alpha\ O_k^\beta f = O_k^{\alpha+\beta}f. \tag{7.16}$$

Proposition 7.20 *(Operational formula)*
Let $\alpha\in\mathbb{R}$ and $\{\xi_j\}_{j=1}^N$ is an orthonormal basis of $\mathbb{R}^N$. For $j=1,\dots,N$, define B_{ξ_j} and $C^*_{\xi_j}$ by

$$B_{\xi_j} = 2^{-1/2}\left[Q_{\xi_j}-P_{\xi_j}\right] \quad and \quad C_{\xi_j} = 2^{-1/2}\left[Q_{\xi_j}+P_{\xi_j}\right].$$

Then the following relations hold on $\mathcal{S}(\mathbb{R}^N)$*:*
(1) $O_k^\alpha \circ Q_\xi = (\cosh(\alpha) Q_\xi - \sinh(\alpha)\ P_\xi) \circ O_k^\alpha$.
(2) $O_k^\alpha \circ P_\xi = (-\sinh(\alpha) Q_\xi + \cosh(\alpha) P_\xi) \circ O_k^\alpha$.
(3) $O_k^\alpha \circ B_{\xi_j} = e^\alpha (B_{\xi_j} \circ O_k^\alpha)$.
(4) $O_k^\alpha \circ C_{\xi_j} = e^{-\alpha} (C_{\xi_j} \circ O_k^\alpha)$.
(5) $O_k^\alpha \circ \mathbb{B}_k = \mathbb{B}_k \circ O_k^\alpha$ *where* $\mathbb{B}_k$ *is the operator defined by:*

$$\mathbb{B}_k = \frac{1}{2} \sum_{j=1}^N B_{\xi_j} C_{\xi_j} + C_{\xi_j} B_{\xi_j} = \frac{1}{2}(|x|^2 + \Delta_k).$$

Proposition 7.21 *(Bochner type identity) Let* α *be a real number. If* $f \in L_k^1(\mathbb{R}^N) \cap L_k^2(\mathbb{R}^N)$ *is of the form* $f(x) = p(x)\psi(|x|)$ *for some* $p \in H_n^k$ *and a one-variable* ψ *on* $\mathbb{R}_+$*, then*

$$O_k^\alpha f(x) = p(x) \mathcal{V}_\lambda^\alpha\ \psi(|x|),$$

where $\lambda = n + \gamma + (N/2) - 1$ *and*

$$\mathcal{V}_\lambda^\alpha \psi(|x|) \quad = \quad \frac{2}{\Gamma(\lambda+1)(2ib)^{\lambda+1}} \int_0^{+\infty} e^{\frac{i}{2}\coth(\alpha)(|x|^2+y^2)}\ j_\lambda\left(\frac{|x|y}{\sinh(\alpha)}\right) f(y) y^{2\lambda+1}\ dy.$$

In particular, if f *is radial, then*

$$O_k^\alpha f(x) = \mathcal{V}_{\gamma+(N/2)-1}^\alpha\ \psi(|x|).$$

Remark 7.2 *According to the previous Proposition, we can put:*

$$\mathcal{V}_\mu^\alpha f(x) \quad = \quad \begin{cases} \dfrac{2}{\Gamma(\mu+1)(2ib)^{\mu+1}} \displaystyle\int_0^{+\infty} e^{\frac{i}{2}\coth(\alpha)(|x|^2+y^2)}\ j_\mu\left(\frac{|x|y}{\sinh(\alpha)}\right) f(y) y^{2\mu+1}\ dy, & \alpha \neq 0, \\ f(x), & \alpha = 0, \end{cases}$$

where $\mu \geq -\frac{1}{2}$.

Proposition 7.22 *Let* $\alpha \in \mathbb{R}$ *such that* $\alpha \neq 0$*. Then:*

$$O_k^\alpha \psi_{m,n,j}(x) \quad = \quad \lambda_{m,n,\alpha} \quad e^{\frac{i}{2}\coth(2\alpha)|x|^2}\ \psi_{m,n,j}\left(\frac{x}{\sqrt{\cosh(2\alpha)}}\right),$$

where

$$\lambda_{m,n,\alpha} = \left(\frac{\cosh(\alpha) - i\sinh(\alpha)}{\cosh(\alpha) + i\sinh(\alpha)}\right)^m \frac{(\cosh(\alpha) - i\sinh(\alpha))^{\frac{n}{2}}}{(\cosh(\alpha) + i\sinh(\alpha))^{\frac{n}{2}}} \frac{e^{i\theta}}{(\cosh(\alpha) + i\sinh(\alpha))^{\gamma+(N/2)}}$$

and

$$\theta = 2(\gamma+(N/2)+n-1)\left\{ sgn(\alpha)\arctan\left(\frac{\cosh(\alpha)}{|\sinh(\alpha)| + \sqrt{\cosh(2\alpha)}}\right) - \frac{\pi}{4} sgn(\alpha) + \arctan\left(\frac{\sinh(\alpha)}{\cosh(\alpha) + \sqrt{\cosh(2\alpha)}}\right)\right\}.$$

Proposition 7.23 *(Master formula) Let* α *be a real number such that* $\alpha \neq 0$*. Let* f_n *is of the form* $f_n(x) = e^{-\frac{|x|^2}{2}} e^{-\frac{\Delta_k}{4}} p(x)$ *for some* $p \in \mathcal{P}_n$*, Then*

$$O_k^\alpha f_n(x) \quad = \quad \lambda_{n,\alpha} \quad e^{\frac{i}{2}\coth(2\alpha)|x|^2}\ f_n(x)\left(\frac{x}{\sqrt{\cosh(2\alpha)}}\right),$$

where

$$\lambda_{n,\alpha} = \frac{(\cosh(\alpha) - i\sinh(\alpha))^{\frac{n}{2}}}{(\cosh(\alpha) + i\sinh(\alpha))^{\frac{n}{2}}} \frac{e^{i\theta}}{(\cosh(\alpha) + i\sinh(\alpha))^{\gamma+(N/2)}}$$

and

$$\theta = 2(\gamma+(N/2)+n-1)\left\{ sgn(\alpha)\arctan\left(\frac{\cosh(\alpha)}{|\sinh(\alpha)| + \sqrt{\cosh(2\alpha)}}\right) - \frac{\pi}{4} sgn(\alpha) + \arctan\left(\frac{\sinh(\alpha)}{\cosh(\alpha) + \sqrt{\cosh(2\alpha)}}\right)\right\}.$$

Proposition 7.24
(1) *The operator* O_k^α *have a unique extension to an unitary operator on* $L_k^2(\mathbb{R}^N)$.
(2) *The family of operators* $\{O_k^\alpha\}_{\alpha\in\mathbb{R}_+}$ *is a* $\mathcal{C}_0$*-group of unitary operators on* $L_k^2(\mathbb{R}^N)$. *The Schwartz space* $\mathcal{S}(\mathbb{R}^N)$ *is a core of its generator* A *and* $A_{|\mathcal{S}(\mathbb{R}^N)} = \frac{i}{2}(|x|^2 + \Delta_k)$.
(3) *For each* $f \in D(A)$ *the function* $u(t,x) = O_k^t f(x)$ *is the unique classical solution of the problem*

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = Au(t,x) \ \text{on } \mathbb{R}_+ \times \mathbb{R}^N, \\ u(0,.) = f \in D(A); \end{cases}$$

(4) (*i*) *Let* $f \in \mathcal{S}(\mathbb{R}^N)$. *The function* $u(t,x) = O_k^t f(x)$ *is the unique solution of the Dunkl-Schrödinger equation*

$$\begin{cases} i\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}(|x|^2 + \Delta_k)u(t,x) \ \text{on } \mathbb{R}_+ \times \mathbb{R}^N, \\ u(0,.) = f \in \mathcal{S}(\mathbb{R}^N). \end{cases} \tag{7.17}$$

Moreover, $u(t,x)$ *has the following properties:*
(*ii*) $u(t,.) \in \mathcal{S}(\mathbb{R}^N)$ *for all* $t \in \mathbb{R}_+$.
(*iii*) $u(t+s,x) = O_k^t\,(u(s,.))\,(x)$ *for all* $t,s \in \mathbb{R}_+$ *and* $x \in \mathbb{R}^N$.
(*iv*) *For all* $t \in \mathbb{R}$,

$$\|u(t,.)\|_\infty \leq \frac{c_k}{(2|\sinh(t)|)^{\gamma+(N/2)}}\|u(0,.)\|_{k,1}.$$

(*v*) $\|u(t,.)\|_{k,2} = \|u(0,.)\|_{k,2}$ *for all* $t \in \mathbb{R}_+$.

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