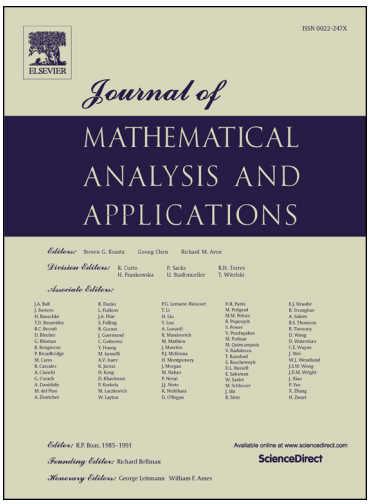
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Exact-approximate boundary controllability of the thermoelastic plate with a curved middle surface

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Abstract

Controllability properties of a thermoelastic plate with a curved middle surface are studied. By the Riemannian geometry approach and the multiplier technique, we obtain simultaneously exact controllability of the displacement of the plate and approximate controllability of the temperature, through three boundary controls under verifiable assumptions on the geometry of the middle surface.

Keywords:

exact-approximate boundary controllability, thermoelastic plate, curved middle surface, Riemannian geometry, multiplier method

1. Introduction and Main Results

Vibrations of beams, membranes, plates and shells and their controls are an active field of research in engineering, applied science and mathematics. The pioneering work of Timoshenko [25, 26] has established major models of structural elements as the foundations of structural dynamics for technological applications. For these models, modern distributed parameter control theory has found fertile applications. Among them, the multidimensional PDE models of plates and shells are particularly interesting. They require challenging techniques for estimation and control as the PDEs are universally of higher order and often also constitute coupled systems. There is also the added complexity of thermal effects as heat conduction in the material of the structure can affect structural behavior and performance.

There is a sizable amount of literature on thermoelastic beams, plates and shells. It would be arduous for anyone to give a comprehensive review of this subject. However, for the literature

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closely related to our present interest in this paper, namely, controllability and observability of thermoelastic plates, we may cite the work in [2–5, 9–11, 13, 14, 17–19, 23], and the references therein. In particular, Lagnese [18] showed that if the coupling parameter (α) is small enough and the boundary of the domain is star shaped, a thermoelastic plate is exactly controllable with respect to the mechanical variables. Lasiecka and Triggiani [19] derived the null controllability of a thermoelastic plate. Eller et al. [13, 14] studied the exact-approximate boundary controllability of thermoelastic plates with a variable thermal coefficient and with constant coefficient principal parts. Moreover, Chai, et al. [9, 10] studied stabilization where some decay rates of the energy of thermoelastic plates of the Euler-Bernoulli type with variable coefficients were obtained by a Riemannian geometric approach. For the Riemannian geometrical approach on some PDEs with variable coefficients, we refer to [6–8, 22, 27, 28] and the references therein.

If the plate is flat, namely, if the manifold of the plate under study is $M = \mathbb{R}^2$ with the Euclidean metric, then the thermoelastic plate PDE model is derived by Lagnese and Lions in [17] and the results of the exact-approximate controllability have been obtained by Avalos and Lasiecka [2].

Here, we will consider a "curved plate" whose middle surface Ω is part of a surface M as the underlying geometric manifold. We assume that the extension effects along the tangential direction are small and the strain tensor of the middle surface $\Psi(\zeta) = 0$. By the Principle of Virtual Work, we are able to derive a bending model of plate (as in (1.1) below). For the details, see Appendix at the end of the present paper. When one of the parameters (i.e., γ) is 0, the elasticity mechanical portion of the model has been studied by Guo and Yao in [15]. Our objective here is to study the exact-approximate controlabl1ity of the thermoelastic plate system, by extending and generalizing the work of Avalos and Lasiecka [2] and Yao [29], to the case when the middle-surface of the plate is curved.

The "curvedness"of the plate makes a new model that is *significantly more difficult* to treat than the ordinary flat plate model. Sophiticated differential geometric techniques in combination with advanced PDE estimates must be developed in order to make the curved thermoelastic plate model mathematically tractable. Indeed, this is the main thrust of this paper.

We now proceed to describe the mathematical model.

Let M be a surface in $\mathbb{R}^3$ with the induced metric g from the Euclidean metric of $\mathbb{R}^3$. Let $\Omega \subset M$ be an open, bounded set with a sufficiently smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ and $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$, where both Γ_0 and Γ_1 are open and nonempty. Further, let Γ_2 be any open and nonempty subset of Γ. We consider the following thermoelastic system on finite time $(0, T)$:

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$$\begin{cases} \begin{cases} w_{tt} - \gamma\Delta w_{tt} + \mathscr{A}w + \alpha\Delta\theta = 0 \\ \beta\theta_t - \eta\Delta\theta + \sigma\theta - \alpha\Delta w_t = 0 \end{cases} & \text{on } (0,T)\times\Omega, \\ w = \dfrac{\partial w}{\partial \nu} = 0 & \text{on } (0,T)\times\Gamma_0, \\ \begin{cases} \Delta w + (1-\mu)B_1 w + \alpha\theta = u_1 \\ \dfrac{\partial \Delta w}{\partial \nu} + (1-\mu)B_2 w - \gamma\dfrac{\partial w_{tt}}{\partial \nu} + \alpha\dfrac{\partial \theta}{\partial \nu} = u_2 \end{cases} & \text{on } (0,T)\times\Gamma_1, \\ \dfrac{\partial\theta}{\partial\nu} + \lambda\theta = \begin{cases} u_3 & \text{on } (0,T)\times\Gamma_2, \\ 0 & \text{on } (0,T)\times\Gamma\backslash\Gamma_2, \end{cases} & \\ w(0) = w_0, w_t(0) = w_1, \theta(0) = \theta_0 & \text{on } \Omega, \end{cases} \tag{1.1}$$

with three controls u_1, u_2 and u_3, where

$$\mathscr{A}w = \Delta^2 w + (1-\mu)\,\mathrm{div}\,(kDw),$$

B_1 and B_2 are boundary operators, defined by

$$B_1 w = -D^2 w(\tau,\tau) \text{ and } B_2 w = \frac{\partial}{\partial\tau}(D^2 w(\tau,\nu)) + k\frac{\partial w}{\partial\nu},$$

respectively. w and θ denote the vertical deflection and the temperature of the plate, respectively. The constant γ is proportional to the thickness of the plate and assumed to be small. β, η and σ are positive constants, $\alpha > 0$ is the *coupling parameter*, $0 < \mu < 1/2$ is the Poisson ratio, and λ is a nonnegative constant. Δ, div and D are the Laplace operator, the divergence and the Levi-Civita connection in the metric g, respectively. k is the Gaussian curvature of the surface M and $D^2 w$ is the hessian of w. ν is the outside normal along Γ in the metric g, τ is a unit tangential vector field on Γ such that ν, τ form an orthonormal basis of $(M_x, g(x))$ for each $x \in \Gamma$.

We recall that a vector field H on Ω is said to be an *escape vector field for plate* if there exists a positive function $b(x)$ on $\overline{\Omega}$ such that

$$DH(X,X) = b(x)|X|^2 \qquad \forall X \in M_x, x \in \Omega, \tag{1.2}$$

Here we may assume $\min\limits_{x\in\Omega} b(x) \geq 1$.

Escape vector field for plate was introduced by Yao [29] as a verifiable assumption for control of the plate with variable coefficients. From [29, 30], an escape vector field as in (1.2) always exists locally, but global existence of such vector fields depends on the Gaussian curvature of the middle surface Ω. For examples we refer to [29, 30].

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We denote by $\rho(x,\Gamma_2)$ the distance between x and Γ_2 in the induced metric g for $x\in\Omega$. Set

$$T^*=2max\left\{1+\frac{8sup\ |H|^2\gamma}{1-\mu},\ \sqrt{\gamma}sup\ \rho(x,\Gamma_2)\right\},\quad \Sigma_{2,T}=(0,T)\times\Gamma_2.$$

Denote

$$H^k_{\Gamma_0}(\Omega)=\left\{f\in H^k(\Omega):\frac{\partial^i f}{\partial\nu^i}\Big|_{\Gamma_0}=0,\ for\ \ i=0,1\cdots k-1\right\},$$

where

$$\Gamma_0=\{x\in\Gamma|\ \langle H\ ,\nu\rangle\le 0\}.$$

Our main results are the following.

Theorem 1.1. *Let H be an escape vector field for plate and let $T>T^*$ be given. Then for any initial data (w_0,w_1,θ_0) and terminal data $\left(w_0^T,w_1^T,\theta_0^T\right)$ in the space $H^2_{\Gamma_0}(\Omega)\times H^1_{\Gamma_0}(\Omega)\times L^2(\Omega)$, and arbitrary $\epsilon>0$, one can find control functions $\left(u_1^*,u_2^*,u_3^*\right)\in L^2(0,T;L^2(\Gamma_1)\times H^{-1}(\Gamma_1))\times C^r(\Sigma_{2,T})$ $(r\ge 0)$ such that the corresponding solution (w^*,w_t^*,θ^*) to system* (1.1) *satisfies the following steering property at terminal time T*

$$(w^*(T),w_t^*(T))=\left(w_0^T,w_1^T\right),\ \ \left\|\theta^*(T)-\theta_0^T\right\|_{L^2(\Omega)}<\epsilon. \tag{1.3}$$

As in [2], Theorem 1.1 can be derived from the following.

Theorem 1.2. *Let H be an escape vector field for plate and let $T>T^*$ be given. Then for any initial data (w_0,w_1,θ_0) and terminal data $\left(w_0^T,w_1^T,\theta_0^T\right)$ in the space $H^2_{\Gamma_0}(\Omega)\times H^1_{\Gamma_0}(\Omega)\times L^2(\Omega)$, there exists a control triple $(u_1,u_2,u_3)\in L^2(0,T;L^2(\Gamma_1)\times H^{-1}(\Gamma_1))\times H^s(\Sigma_{2,T})$ $(s\ge 0)$ such that the corresponding solution (w,w_t,θ) to system* (1.1) *satisfies $(w(T),w_t(T))=\left(w_0^T,w_1^T\right)$.*

The organization of our paper goes as follows. Our main approach is based on observability inequality, given in Section 2. The technical details of the proof for Theorem 1.2 are given in Section 3, and those for Theorem 1.1 are given in Section 4. Brief concluding remarks are given Section 5. The modeling of system (1.1) is given in the appendix.

2. Observability Inequality

Without loss of generality, we assume that the initial data is zero in system (1.1).

We shall use $\int_\Omega\cdots dg$ to denote the surface integral with respect to the induced metric g.

For each $x\in M$, we denote by $T^k(M_x)$ the kth order tensor field space on M_x with the following inner product:

$$\langle\alpha_1,\alpha_2\rangle_{T^k_x}=\sum_{i_1,\cdots,\ i_k=1}^{2}\alpha_1(e_{i_1},\cdots,e_{i_k})\,\alpha_2(e_{i_1},\cdots,e_{i_k}),\quad \alpha_1,\alpha_2\in T^k(M_x),\quad x\in M,$$

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where $\{ e_1, e_2 \}$ is an orthonormal basis of M_x.

Space $H^1_{\Gamma_0,\gamma}(\Omega)$ is equivalent to $H^1_{\Gamma_0}(\Omega)$ with the inner product

$$(f_1, f_2)_{H^1_{\Gamma_0,\gamma}(\Omega)} \equiv (f_1, f_2)_{L^2(\Omega)} + \gamma (Df_1, Df_2)_{T^1(\Omega)}, \quad f_1,\ f_2 \in H^1_{\Gamma_0}(\Omega),$$

where $(Df_1, Df_2)_{T^1(\Omega)} = \int_\Omega \langle Df_1, Df_2 \rangle_{T^1_x} \, \mathrm{d}g$. Its dual space is denoted as $H^{-1}_{\Gamma_0,\gamma}(\Omega)$.

Denote the state space by $\mathcal{H}_\gamma \equiv H^2_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0,\gamma}(\Omega) \times L^2(\Omega)$, and the control space by $\mathcal{U}_s \equiv L^2(0,T; L^2(\Gamma_1) \times H^{-1}(\Gamma_1)) \times H^s((0,T) \times \Gamma_2)$ for $s \geq 0$. We define the control-to-terminal-state map $P : D(P) \subset \mathcal{U}_s \to \mathcal{H}_\gamma$ as

$$P(u_1, u_2, u_3) = (w(T), w_t(T),\ \beta\theta(T)),$$

where (w, θ) is a solution to system (1.1) and

$$D(P) = \left\{ (u_1, u_2, u_3) \in \mathcal{U}_s : P(u_1, u_2, u_3) \in \mathcal{H}_\gamma \right\}.$$

P is a closed, densely defined and unbounded operator. Its adjoint is $P^* : D(P^*) \subset \mathcal{H}_\gamma \to \mathcal{U}_s^*$ with $D(P^*) = \left\{ (\varphi_0, \varphi_1, \psi_0) \in \mathcal{H}_\gamma : P^*(\varphi_0, \varphi_1, \psi_0) \in \mathcal{U}_s^* \right\}$, where $\mathcal{U}_s^* = L^2(0,T; L^2(\Gamma_1) \times H^1(\Gamma_1)) \times [H^s((0,T) \times \Gamma_2)]'$.

According to the relation

$$\langle (u_1, u_2, u_3), P^*(\varphi_0, \varphi_1, \psi_0) \rangle_{\mathcal{U}_s \times \mathcal{U}_s^*} = (P(u_1, u_2, u_3), (\varphi_0, \varphi_1, \psi_0))_{\mathcal{H}_\gamma},$$

we obtain the dual problem of system (1.1) as

$$\begin{cases} \begin{cases} \varphi_{tt} - \gamma \Delta \varphi_{tt} + \mathscr{A}\varphi + \alpha \Delta \psi = 0 \\ \beta \psi_t + \eta \Delta \psi - \sigma \psi - \alpha \Delta \varphi_t = 0 \end{cases} & \text{on } (0,T) \times \Omega, \\ \varphi = \dfrac{\partial \varphi}{\partial \nu} = 0 \qquad \text{on } (0,T) \times \Gamma_0, \\ \begin{cases} \Delta \varphi + (1-\mu) B_1 \varphi + \alpha \psi = 0 \\ \dfrac{\partial \Delta \varphi}{\partial \nu} + (1-\mu) B_2 \varphi - \gamma \dfrac{\partial \varphi_{tt}}{\partial \nu} + \alpha \dfrac{\partial \psi}{\partial \nu} = 0 \end{cases} & \text{on } (0,T) \times \Gamma_1, \\ \dfrac{\partial \psi}{\partial \nu} + \lambda \psi = 0 \qquad \text{on } (0,T) \times \Gamma, \\ \varphi(T) = \varphi_0,\ \varphi_t(T) = \varphi_1, \psi(T) = \psi_0 & \text{on } \Omega. \end{cases} \tag{2.1}$$

The adjoint operator P^* is found from the above to be

$$P^*(\varphi_0, \varphi_1, \psi_0) = \left(\frac{\partial \varphi_t}{\partial \nu} \Big|_{\Gamma_1}, -\varphi_t|_{\Gamma_1}, \eta \psi|_{\Gamma_2} \right), \tag{2.2}$$

where (φ, ψ) is the solution to problem (2.1).

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We will first be concerned with obtaining the exact controllability of the *displacement of the plate* only (without the thermal effect). To achieve this, we may consider system (2.1) with terminal thermal data $\psi_0 = 0$ in what follows. Furthermore, we define the projection operator $\mathbf{\Pi} : \mathcal{H}_\gamma \to H^2_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0,\gamma}(\Omega)$ by

$$\mathbf{\Pi}(w_0, w_1, \theta_0) = (w_0, w_1).$$

Then the asserted controllability result in Theorem 1.2 is equivalent to showing the surjectivity of the operator $\mathbf{\Pi}P$. In turn, by duality (see [12]), it is equivalent to establishing the following inequality

$$\|P^*\mathbf{\Pi}^*(\varphi_0, \varphi_1)\|^2_{\mathcal{U}_s^*} \geq C_T \|(\varphi_0, \varphi_1)\|^2_{H^2_{\Gamma_0}(\Omega)\times H^1_{\Gamma_0,\gamma}(\Omega)}, \tag{2.3}$$

where $P^*\mathbf{\Pi}^* : D(P^*\mathbf{\Pi}^*) \subset H^2_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0,\gamma}(\Omega) \to \mathcal{U}_s^*$ is a closed densely defined operator with $D(P^*\mathbf{\Pi}^*) = \left\{(\varphi_0, \varphi_1) \in H^2_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0,\gamma}(\Omega) : (\varphi_0, \varphi_1, 0) \in D(P^*)\right\}$.

We introduce the mechanical energy of system (2.1) as

$$2E_\varphi(t) = B(\varphi, \varphi) + \|\varphi_t\|^2_{H^1_{\Gamma_0,\gamma}(\Omega)},$$

where

$$B(\varphi, \varphi) = \int_\Omega a(\varphi, \varphi) \mathrm{d}g, \quad a(f, h) = (1-\mu)\langle D^2 f, D^2 h\rangle_{T^2_x} + \mu\, trD^2 f\, trD^2 h. \tag{2.4}$$

Since Γ_0 is non-empty, the bilinear form $B(\,.\,,\,.\,)$ induces an equivalent norm of $H^2_{\Gamma_0}(\Omega)$. Meanwhile, $P^*\mathbf{\Pi}^*(\varphi_0, \varphi_1) = P^*(\varphi_0, \varphi_1, 0) = \left(\frac{\partial \varphi_t}{\partial \nu}\Big|_{\Gamma_1}, -\varphi_t|_{\Gamma_1}, \eta\psi|_{\Gamma_2}\right)$. Thus, the observability inequality (2.3) becomes

$$\int_0^T \left[\left\|\frac{\partial \varphi_t}{\partial \nu}\right\|^2_{L^2(\Gamma_1)} + \|\varphi_t\|^2_{H^1(\Gamma_1)}\right] \mathrm{d}t + \eta^2 \|\psi\|^2_{[H^s((0,T)\times\Gamma_2)]'} \geq C_T E_\varphi(T). \tag{2.5}$$

3. Proof of Theorem 1.2

We need to first define certain operators for the subsequent development.

Let A, A_D, A_M and $A_R : L^2(\Omega) \to L^2(\Omega)$ be given by

$$A = \mathscr{A}, \quad D(A) = \{f \in H^4(\Omega) \cap H^2_{\Gamma_0}(\Omega) : \ \Delta f + (1-\mu)B_1 f|_{\Gamma_1} = 0 \ \text{ and } \ \frac{\partial \Delta f}{\partial \nu} + (1-\mu)B_2 f|_{\Gamma_1} = 0\},$$

$$A_D = -\Delta, \quad D(A_D) = \left\{f \in H^2(\Omega) : f|_\Gamma = 0\right\},$$

$$A_M = -\Delta, \quad D(A_M) = \left\{f \in H^2(\Omega) : f|_{\Gamma_0} = \frac{\partial f}{\partial \nu}\Big|_{\Gamma_1} = 0\right\},$$

and

$$A_R = -\Delta + \frac{\sigma}{\eta}\mathbf{I}, \quad D(A_R) = \left\{f \in H^2(\Omega) : \frac{\partial f}{\partial \nu} + \lambda f|_\Gamma = 0\right\},$$

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respectively. It is easy to check that the above operators are all positive and self-adjoint on $L^2(\Omega)$. Moreover, according to the interpolation results in [24], we have the following spaces for the square-root operators:

$$D(A^{\frac{1}{2}}) = H^2_{\Gamma_0}(\Omega) \quad \text{and} \quad D(A_R^{\frac{1}{2}}) = H^1(\Omega),$$

where the inner product for the space $D(A_R^{\frac{1}{2}})$ is

$$(f,h)_{D(A_R^{\frac{1}{2}})} = \left(A_R^{\frac{1}{2}} f, A_R^{\frac{1}{2}} h\right)_{L^2(\Omega)} = \lambda (f,h)_{L^2(\Gamma)} + (f,h)_{T^1(\Omega)} + \frac{\sigma}{\eta}(f,h)_{L^2(\Omega)}. \tag{3.1}$$

Further, we define the elliptic operators $\mathbf{D}, G_1$ and G_2 as

$$\mathbf{D}f = z \Leftrightarrow \begin{cases} \Delta z = 0 \quad \text{in} \quad \Omega, \\ z|_\Gamma = f, \end{cases} \tag{3.2}$$

$$G_1 f = z \Leftrightarrow \begin{cases} \mathscr{A} z = 0 & \text{in } \Omega, \\ z = \dfrac{\partial z}{\partial \nu} = 0 & \text{on } \Gamma_0, \\ \begin{cases} \Delta z + (1-\mu) B_1 z = f \\ \dfrac{\partial \Delta z}{\partial \nu} + (1-\mu) B_2 z = 0 \end{cases} & \text{on } \Gamma_1, \end{cases} \tag{3.3}$$

and

$$G_2 f = z \Leftrightarrow \begin{cases} \mathscr{A} z = 0 & \text{in } \Omega, \\ z = \dfrac{\partial z}{\partial \nu} = 0 & \text{on } \Gamma_0, \\ \begin{cases} \Delta z + (1-\mu) B_1 z = 0 \\ \dfrac{\partial \Delta z}{\partial \nu} + (1-\mu) B_2 z = f \end{cases} & \text{on } \Gamma_1. \end{cases} \tag{3.4}$$

respectively.

It is easy to show that for all $w \in D\left(A^{\frac{1}{2}}\right)$, $i = 1, 2$, operators $G_i^* A \in \mathscr{L}\left(D\left(A^{\frac{1}{2}}\right),\ H^{i-\frac{1}{2}}(\Gamma)\right)$ satisfy, respectively,

$$G_1^* A w = \begin{cases} \dfrac{\partial w}{\partial \nu} & \text{on} \quad \Gamma_1, \\ 0 & \text{on} \quad \Gamma_0; \end{cases} \tag{3.5}$$

$$G_2^* A w = \begin{cases} -w & \text{on} \quad \Gamma_1, \\ 0 & \text{on} \quad \Gamma_0. \end{cases} \tag{3.6}$$

Denote by γ_0, γ_1 the classical Sobolev trace operators:

$$\gamma_0 f = f|_\Gamma, \quad \gamma_1 f = \frac{\partial f}{\partial \nu}|_\Gamma, \quad \text{for } f \in C^\infty(\overline{\Omega}).$$

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Finally, we define a positive and self-adjoint operator $P_\gamma : D(P_\gamma) \subset L^2(\Omega) \to L^2(\Omega)$ by

$$P_\gamma = \mathbf{I} + \gamma A_M.$$

Then $D\left(P_r^{\frac{1}{2}}\right) = H^1_{\Gamma_0,\gamma}(\Omega)$. Further, we find that $P_\gamma = \mathbf{I} - \gamma(\Delta + AG_2\gamma_1) \in \mathscr{L}\left(H^1_{\Gamma_0,\gamma}(\Omega), H^{-1}_{\Gamma_0,\gamma}(\Omega)\right)$, with $\langle P_\gamma f_1, f_2\rangle_{H^{-1}_{\Gamma_0,\gamma}(\Omega)\times H^1_{\Gamma_0,\gamma}(\Omega)} = (f_1, f_2)_{H^1_{\Gamma_0,\gamma}(\Omega)}$, for $f_1,\ f_2 \in H^1_{\Gamma_0,\gamma}(\Omega)$. It follows from the Lax-Milgram theorem that P_γ is bounded and invertible.

The operator $\mathscr{B} : \mathcal{H}_\gamma \supset D(\mathscr{B}) \to \mathcal{H}_\gamma$ for the coupled system (2.1) now can be defined as

$$\mathscr{B} = \begin{pmatrix} \mathbf{I} & 0 & 0 \\ 0 & P_\gamma^{-1} & 0 \\ 0 & 0 & \mathbf{I} \end{pmatrix} \begin{pmatrix} 0 & -\mathbf{I} & 0 \\ A & 0 & -\alpha\left(A_R - \frac{\sigma}{\eta} - AG_1\gamma_0 + \lambda AG_2\gamma_0\right) \\ 0 & \frac{\alpha}{\beta}A_D(\mathbf{I} - D\gamma_0) & -\frac{\eta}{\beta}A_R \end{pmatrix},$$

where

$$D(\mathscr{B}) = \left\{(\varphi_0, \varphi_1, \psi_0) \in D\left(A^{\frac{1}{2}}\right) \times D\left(A^{\frac{1}{2}}\right) \times D(A_R) \text{ such that } A\varphi_0 + \alpha AG_1\gamma_0\psi_0 \in H^{-1}_{\Gamma_0,\gamma}(\Omega)\right\}.$$

Then, for terminal data $(\varphi_0, \varphi_1, \psi_0) \in \mathcal{H}_\gamma$, the coupled system (2.1) can be rewritten as

$$\frac{d}{dt}\begin{bmatrix} \varphi(t) \\ \varphi_t(t) \\ \psi(t) \end{bmatrix} = \mathscr{B}\begin{bmatrix} \varphi(t) \\ \varphi_t(t) \\ \psi(t) \end{bmatrix}, \quad \begin{bmatrix} \varphi(T) \\ \varphi_t(T) \\ \psi(T) \end{bmatrix} = \begin{bmatrix} \varphi_0 \\ \varphi_1 \\ \psi_0 \end{bmatrix}. \tag{3.7}$$

From [2], we have the following.

Proposition 3.1. *Let $(\varphi_0, \varphi_1, \psi_0) \in D(\mathscr{B}^2)$ be given. Then there is a unique solution $(\varphi, \varphi_t, \psi)$ to problem* (2.1) *in $C^2([0,T]; \mathcal{H}_\gamma) \cap C^1([0,T]; D(\mathscr{B})) \cap C([0,T]; D(\mathscr{B}^2))$ with the following regularities:*

$$\varphi \in C([0,T]; H^4(\Omega)),\ \varphi_t \in C([0,T]; H^3(\Omega)),\ \varphi_{tt} \in C\left([0,T]; D\left(A^{\frac{1}{2}}\right)\right),$$

$$\psi_t \in C([0,T]; D(A_R)),\ \varphi - \gamma G_2\gamma_1\varphi_{tt} + \alpha G_1\gamma_0\psi - \alpha\lambda G_2\gamma_0\psi \in C([0,T]; D(A)).$$

We also need several multiplier identities. We can perform similar computations as in [15, Lemma 3.1, Lemma 3.2] and obtain the following two lemmas:

Lemma 3.1. *Let the boundary Γ be a closed curve. Then for $f, h \in H^4(\Omega)$, we have*

$$B(f,h) = (\mathscr{A}f, h) + \int_\Gamma [\Delta f + (1-\mu)B_1 f]\frac{\partial h}{\partial \nu}d\Gamma - \int_\Gamma \left[\frac{\partial \Delta f}{\partial \nu} + (1-\mu)B_2 f\right] h d\Gamma. \tag{3.8}$$

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Lemma 3.2. *Let H be a vector field on $\overline{\Omega}$. Then there is a $C > 0$ such that*

$$\int_\Omega a(\varphi, H(\varphi))\mathrm{d}g \le \frac{1}{2}\int_\Gamma a(\varphi,\varphi)\langle H, \nu\rangle\, d\Gamma + C\int_\Omega |D^2\varphi|^2\mathrm{d}g + L(\varphi), \tag{3.9}$$

where $L(\varphi)$ denotes some lower order terms relative to φ in the norm of $H^2(\Omega)$. Moreover, if H is the escape vector field for plate in (1.2)*, then the following identity holds true*

$$\int_\Omega a(\varphi, H(\varphi))\mathrm{d}g = \frac{1}{2}\int_\Gamma a(\varphi,\varphi)\langle H, \nu\rangle\, \mathrm{d}\Gamma + \int_\Omega b(x)a(\varphi,\varphi)\mathrm{d}g + L(\varphi). \tag{3.10}$$

Quite frequently, we will utilize the following formula derived from [30, Lemma 2.2].

Lemma 3.3. *Let f be a function on $\mathbb{R}^n$ and let H be a vector field on $\mathbb{R}^n$. Then*

$$2\langle Df, D(H(f))\rangle = 2DH(Df, Df) + \operatorname{div}\left(|Df|^2 H\right) - |Df|^2 \operatorname{div} H, \qquad x \in \mathbb{R}^n. \tag{3.11}$$

Next, we make the substitution:

$$\hat{\varphi}(t) = e^{-\xi t}\varphi(t),\ \hat{\psi}(t) = e^{-\xi t}\psi(t), \tag{3.12}$$

where $\xi = \frac{\alpha^2}{2\gamma\eta}$.

Then $(\hat{\varphi}, \hat{\varphi}_t, \hat{\psi})$ solves the following system:

$$\begin{cases} \begin{cases} (\xi^2\hat{\varphi} + 2\xi\hat{\varphi}_t + \hat{\varphi}_{tt}) - \gamma\Delta(\xi^2\hat{\varphi} + 2\xi\hat{\varphi}_t + \hat{\varphi}_{tt}) + \mathscr{A}\hat{\varphi} + \alpha\Delta\hat{\psi} = 0 \\ \beta(\xi\hat{\psi} + \hat{\psi}_t) + \eta\Delta\hat{\psi} - \sigma\hat{\psi} - \alpha\Delta(\xi\hat{\varphi} + \hat{\varphi}_t) = 0 \end{cases} & \text{on } (0,T)\times\Omega, \\ \hat{\varphi} = \dfrac{\partial\hat{\varphi}}{\partial\nu} = 0 \qquad \text{on } (0,T)\times\Gamma_0, \\ \begin{cases} \Delta\hat{\varphi} + (1-\mu)B_1\hat{\varphi} + \alpha\hat{\psi} = 0 \\ \dfrac{\partial\Delta\hat{\varphi}}{\partial\nu} + (1-\mu)B_2\hat{\varphi} - \gamma\dfrac{\partial}{\partial\nu}(\xi^2\hat{\varphi} + 2\xi\hat{\varphi}_t + \hat{\varphi}_{tt}) + \alpha\dfrac{\partial\hat{\psi}}{\partial\nu} = 0 \end{cases} & \text{on } (0,T)\times\Gamma_1, \\ \dfrac{\partial\hat{\psi}}{\partial\nu} + \lambda\hat{\psi} = 0 \qquad \text{on } (0,T)\times\Gamma, \\ \hat{\varphi}(T) = e^{-\xi T}\varphi_0,\ \hat{\varphi}_t(T) = -\xi e^{-\xi T}\varphi_0 + e^{-\xi T}\varphi_1, \hat{\psi}(T) = 0 & \text{on } \Omega. \end{cases} \tag{3.13}$$

It follows from the regularities provided by Proposition 3.1 that the above equations can be rewritten abstractly as

$$\begin{aligned}(\xi^2\hat{\varphi} + 2\xi\hat{\varphi}_t + \hat{\varphi}_{tt}) - \gamma\Delta(\xi^2\hat{\varphi} + 2\xi\hat{\varphi}_t + \hat{\varphi}_{tt}) + A\hat{\varphi} - \gamma AG_2\gamma_1(\xi^2\hat{\varphi} + 2\xi\hat{\varphi}_t + \hat{\varphi}_{tt}) \\ + \alpha AG_1\gamma_0\hat{\psi} - \alpha\lambda AG_2\gamma_0\hat{\psi} + \alpha\Delta\hat{\psi} = 0,\end{aligned} \tag{3.14}$$

$$\beta(\xi\hat{\psi} + \hat{\psi}_t) + \eta\Delta\hat{\psi} - \sigma\hat{\psi} - \alpha\Delta(\xi\hat{\varphi} + \hat{\varphi}_t) = 0, \tag{3.15}$$

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$$\hat{\varphi}(T) = e^{-\xi T}\varphi_0,\ \hat{\varphi}_t(T) = -\xi e^{-\xi T}\varphi_0 + e^{-\xi T}\varphi_1, \hat{\psi}(T) = 0. \tag{3.16}$$

Multiplying the heat equation (3.15) by $\frac{\alpha}{\eta}$ and adding it to (3.14), we obtain

$$\begin{aligned}\hat{\varphi}_{tt} - \gamma\Delta\hat{\varphi}_{tt} + A\hat{\varphi} - \gamma AG_2\gamma_1(\xi^2\hat{\varphi} + 2\xi\hat{\varphi}_t + \hat{\varphi}_{tt}) + \alpha AG_1\gamma_0\hat{\psi} - \alpha\lambda AG_2\gamma_0\hat{\psi}\\ = c_0\hat{\psi} + c_1\hat{\psi}_t + c_2\hat{\varphi} + c_3\hat{\varphi}_t + c_4\Delta\hat{\varphi},\end{aligned} \tag{3.17}$$

where $c_0 = \frac{\alpha^3\beta}{2\gamma\eta^2} - \frac{\alpha\sigma}{\eta}$, $c_1 = \frac{\alpha\beta}{\eta}$, $c_2 = -\frac{\alpha^4}{4\gamma^2\eta^2}$, $c_3 = -\frac{\alpha^2}{\gamma\eta}$ and $c_4 = -\frac{\alpha^4}{4\gamma\eta^2}$.

Consequently, we have derived the following equation similar to a Kirchoff plate model:

$$\begin{cases}\hat{\varphi}_{tt} - \gamma\Delta\hat{\varphi}_{tt} + \mathscr{A}\hat{\varphi} = c_0\hat{\psi} + c_1\hat{\psi}_t + c_2\hat{\varphi} + c_3\hat{\varphi}_t + c_4\Delta\hat{\varphi} & \text{on } (0,T)\times\Omega,\\ \hat{\varphi} = \dfrac{\partial\hat{\varphi}}{\partial\nu} = 0 \qquad \text{on } (0,T)\times\Gamma_0,\\ \begin{cases}\Delta\hat{\varphi} + (1-\mu)B_1\hat{\varphi} = -\alpha\hat{\psi}\\ \dfrac{\partial\Delta\hat{\varphi}}{\partial\nu} + (1-\mu)B_2\hat{\varphi} - \gamma\dfrac{\partial\hat{\varphi}_{tt}}{\partial\nu} = \gamma\dfrac{\partial}{\partial\nu}(\xi^2\hat{\varphi} + 2\xi\hat{\varphi}_t) - \alpha\dfrac{\partial\hat{\psi}}{\partial\nu}\end{cases} & \text{on } (0,T)\times\Gamma_1,\\ \hat{\varphi}(T) = e^{-\xi T}\varphi_0,\ \hat{\varphi}_t(T) = -\xi e^{-\xi T}\varphi_0 + e^{-\xi T}\varphi_1, \hat{\psi}(T) = 0 & \text{on } \Omega.\end{cases} \tag{3.18}$$

In what follows, we denote by *lot* (φ,ψ) the lower order terms, that is, any sum of terms that satisfy the following estimate for some constant C_T and $0 \le a < 1/2$:

$$lot\ (\varphi,\psi) \le\ C_T\left(\|\varphi\|^2_{L^\infty(0,T;H^{\frac{3}{2}+a}(\Omega))} + \|\varphi_t\|^2_{L^\infty(0,T;H^{\frac{1}{2}+a}(\Omega))} + \|\psi\|^2_{L^\infty(0,T;H^{-\frac{1}{2}+a}(\Omega))} + \|\psi\|^2_{L^2(0,T;H^{\frac{1}{2}+a}(\Omega))}\right). \tag{3.19}$$

The following is an estimate result with respect to energy $E_{\hat{\varphi}}(t)$.

Lemma 3.4. *For any $\epsilon > 0$ and terminal data in $D(\mathscr{B}^2)$, the corresponding solution $\left(\hat{\varphi},\hat{\varphi}_t,\hat{\psi}\right)$ to problem* (3.13) *satisfies the following estimate for all s and $\tau \in [0,T]$:*

$$\begin{aligned}E_{\hat{\varphi}}(t)|_{t=s}^{t=\tau} \le \epsilon\left(E_{\hat{\varphi}}(s) + E_{\hat{\varphi}}(\tau)\right) + \epsilon\int_0^T\left[E_{\hat{\varphi}}(t) + \left\|A_R^{\frac{1}{2}}\hat{\psi}\right\|^2_{L^2(\Omega)}\right]\mathrm{d}t\\ + C_\epsilon\int_0^T |||D\hat{\varphi}_t|||^2_{L^2(\Gamma_1)}\mathrm{d}t + lot\ \left(\hat{\varphi},\hat{\psi}\right).\end{aligned} \tag{3.20}$$

Proof. We take the duality pairing of the abstract equation (3.17) with $\hat{\varphi}_t$ and integrate in time and space to obtain

$$\begin{aligned}&\int_s^\tau\left[\langle\hat{\varphi}_{tt} - \gamma\Delta\hat{\varphi}_{tt} - \gamma AG_2\gamma_1\hat{\varphi}_{tt}, \hat{\varphi}_t\rangle_{H^{-1}_{\Gamma_0,\gamma}(\Omega)\times H^1_{\Gamma_0,\gamma}(\Omega)} + \langle A\hat{\varphi},\hat{\varphi}_t\rangle_{\left[D\left(A^{\frac{1}{2}}\right)\right]'\times D\left(A^{\frac{1}{2}}\right)}\right]\mathrm{d}t\\ &= \int_s^\tau\left[\left\langle\gamma AG_2\gamma_1(\xi^2\hat{\varphi} + 2\xi\hat{\varphi}_t) - \alpha AG_1\gamma_0\hat{\psi} + \alpha\lambda AG_2\gamma_0\hat{\psi}, \hat{\varphi}_t\right\rangle_{\left[D\left(A^{\frac{1}{2}}\right)\right]'\times D\left(A^{\frac{1}{2}}\right)}\right]\mathrm{d}t\\ &\quad + \int_s^\tau\left(c_0\hat{\psi} + c_1\hat{\psi}_t + c_2\hat{\varphi} + c_3\hat{\varphi}_t + c_4\Delta\hat{\varphi}, \hat{\varphi}_t\right)_{L^2(\Omega)}\mathrm{d}t.\end{aligned} \tag{3.21}$$

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According to the properties of operators $A, P_\gamma, \gamma_0, \gamma_1, G_1$ and G_2, we have

$$\begin{aligned}\text{LHS of (3.21)} &= \int_s^\tau \left\langle P_\gamma \hat{\varphi}_{tt}, \hat{\varphi}_t \right\rangle_{H^{-1}_{\Gamma_0,\gamma}(\Omega)\times H^1_{\Gamma_0,\gamma}(\Omega)} + \left(A^{\frac{1}{2}}\hat{\varphi}, A^{\frac{1}{2}}\hat{\varphi}_t\right)_{L^2(\Omega)} \mathrm{d}t \\ &= \int_s^\tau (\hat{\varphi}_{tt}, \hat{\varphi}_t)_{H^1_{\Gamma_0,\gamma}(\Omega)} + (\hat{\varphi}, \hat{\varphi}_t)_{H^2_{\Gamma_0}(\Omega)} \mathrm{d}t = E_{\hat{\varphi}}(t)|_{t=s}^{t=\tau}. \end{aligned} \tag{3.22}$$

$$\begin{aligned}\text{RHS of (3.21)} = &- \int_s^\tau \left[\left(\gamma\xi^2 \frac{\partial\hat{\varphi}}{\partial\nu} + 2\gamma\xi\frac{\partial\hat{\varphi}_t}{\partial\nu} + \alpha\lambda\hat{\psi}, \hat{\varphi}_t\right)_{L^2(\Gamma_1)} + \alpha\left(\hat{\psi}, \frac{\partial\hat{\varphi}_t}{\partial\nu}\right)_{L^2(\Gamma_1)}\right]\mathrm{d}t \\ &+ \int_s^\tau \left(c_0\hat{\psi} + c_2\hat{\varphi} + c_3\hat{\varphi}_t + c_4\Delta\hat{\varphi}, \hat{\varphi}_t\right)_{L^2(\Omega)} \mathrm{d}t + c_1\int_s^\tau \left(\hat{\psi}_t, \hat{\varphi}_t\right)_{L^2(\Omega)} \mathrm{d}t. \end{aligned} \tag{3.23}$$

Multiplying (3.18) by $-\frac{c_1}{\gamma}A_D^{-1}\hat{\psi}$ and integrating in time and space gives

$$-\frac{c_1}{\gamma}\int_s^\tau \left(\hat{\varphi}_{tt} - \gamma\Delta\hat{\varphi}_{tt} + \mathscr{A}\hat{\varphi} - \left[c_0\hat{\psi} + c_1\hat{\psi}_t + c_2\hat{\varphi} + c_3\hat{\varphi}_t + c_4\Delta\hat{\varphi}\right], A_D^{-1}\hat{\psi}\right)_{L^2(\Omega)} \mathrm{d}t = 0. \tag{3.24}$$

The use of integration by parts, Green's formula, the heat equation in system (3.13) and the definition of the operator **D** yield

$$\begin{aligned}&-\frac{c_1}{\gamma}\int_s^\tau \left(\hat{\varphi}_{tt} - \gamma\Delta\hat{\varphi}_{tt}, A_D^{-1}\hat{\psi}\right)_{L^2(\Omega)} \mathrm{d}t \\ &= -\left[\frac{c_1}{\gamma}\left(\hat{\varphi}_t, A_D^{-1}\hat{\psi}\right)_{L^2(\Omega)} + c_1\left(D\hat{\varphi}_t, DA_D^{-1}\hat{\psi}\right)_{T^1(\Omega)}\right]_s^\tau + c_1\int_s^\tau \left(\hat{\varphi}_t, \hat{\psi}_t\right)_{L^2(\Omega)} \mathrm{d}t \\ &+ \frac{c_1}{\gamma}\int_s^\tau \left(\hat{\varphi}_t, \left(\frac{\sigma}{\beta} - \xi\right)A_D^{-1}\hat{\psi} + \frac{\eta}{\beta}(\mathbf{I} - \mathbf{D}\gamma_0)\hat{\psi} - \frac{\alpha\xi}{\beta}(\mathbf{I} - \mathbf{D}\gamma_0)\hat{\varphi} - \frac{\alpha}{\beta}(\mathbf{I} - \mathbf{D}\gamma_0)\hat{\varphi}_t\right)_{L^2(\Omega)} \mathrm{d}t \\ &+ c_1\int_s^\tau \left(\hat{\varphi}_t, \left(\frac{\sigma}{\beta} - \xi\right)\frac{\partial A_D^{-1}\hat{\psi}}{\partial\nu} + \frac{\eta}{\beta}\frac{\partial(\mathbf{I} - \mathbf{D}\gamma_0)\hat{\psi}}{\partial\nu} - \frac{\alpha\xi}{\beta}\frac{\partial(\mathbf{I} - \mathbf{D}\gamma_0)\hat{\varphi}}{\partial\nu} - \frac{\alpha}{\beta}\frac{\partial(\mathbf{I} - \mathbf{D}\gamma_0)\hat{\varphi}_t}{\partial\nu}\right)_{L^2(\Gamma_1)} \mathrm{d}t. \end{aligned} \tag{3.25}$$

From the boundary conditions $\hat{\varphi} = \frac{\partial\hat{\varphi}}{\partial\nu} = 0$ on Γ_0, we find

$$D\hat{\varphi} = \tau(\hat{\varphi})\tau + \frac{\partial\hat{\varphi}}{\partial\nu}\nu = 0 \text{ on } \Gamma_0, \tag{3.26}$$

and, $D_\tau D\hat{\varphi} = \tau(\hat{\varphi})D_\tau\tau + \tau(\tau(\hat{\varphi}))\tau + \tau\left(\frac{\partial\hat{\varphi}}{\partial\nu}\right)\nu + \frac{\partial\hat{\varphi}}{\partial\nu}D_\tau\nu = 0$ on Γ_0. This gives

$$D^2\hat{\varphi}(\tau,\tau) = \langle D_\tau D\hat{\varphi}, \tau\rangle = 0, \quad D^2\hat{\varphi}(\tau,\nu) = D^2\hat{\varphi}(\nu,\tau) = \langle D_\tau D\hat{\varphi}, \nu\rangle = 0, \tag{3.27}$$

further,

$$\frac{\partial H(\hat{\varphi})}{\partial\nu} = \nu\langle H\ , D\hat{\varphi}\rangle = D^2\hat{\varphi}(H,\nu) = \langle H\ ,\nu\rangle D^2\hat{\varphi}(\nu,\nu) = \langle H\ ,\nu\rangle\Delta\hat{\varphi}, \text{ on } \Gamma_0. \tag{3.28}$$

Applying Lemma 3.1 with the above relations (3.27) and the boundary conditions of problem (3.18), we obtain

$$\begin{aligned}&-\frac{c_1}{\gamma}\int_s^\tau (\mathscr{A}\hat{\varphi}, A_D^{-1}\hat{\psi})_{L^2(\Omega)}\mathrm{d}t \\ &= -\frac{c_1}{\gamma}\int_s^\tau B\left(\hat{\varphi}, A_D^{-1}\hat{\psi}\right)\mathrm{d}t - \frac{\alpha c_1}{\gamma}\int_s^\tau \left(\hat{\psi}, \frac{\partial A_D^{-1}\hat{\psi}}{\partial\nu}\right)_{L^2(\Gamma_1)} \mathrm{d}t + \frac{c_1}{\gamma}\int_s^\tau \left(\Delta\hat{\varphi}, \frac{\partial A_D^{-1}\hat{\psi}}{\partial\nu}\right)_{L^2(\Gamma_0)} \mathrm{d}t. \end{aligned} \tag{3.29}$$

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Summing relations (3.22)-(3.25), and equality (3.29), we derive

$$
\begin{aligned}
E_{\hat{\varphi}}(t)|_{t=s}^{t=\tau} =& c_1\left[\frac{1}{\gamma}\left(\hat{\varphi}_t, A_D^{-1}\hat{\psi}\right)_{L^2(\Omega)} + \left(D\hat{\varphi}_t, DA_D^{-1}\hat{\psi}\right)_{T^1(\Omega)}\right]_s^\tau - \frac{c_1^2}{\gamma}\int_s^\tau \left(\hat{\psi}_t, A_D^{-1}\hat{\psi}\right)_{L^2(\Omega)} \mathrm{d}t \\
&+ \int_s^\tau \left(c_0\hat{\psi} + c_2\hat{\varphi} + c_3\hat{\varphi}_t + c_4\Delta\hat{\varphi}, \hat{\varphi}_t - \frac{c_1}{\gamma}A_D^{-1}\hat{\psi}\right)_{L^2(\Omega)} \mathrm{d} + \frac{c_1}{\gamma}\int_s^\tau B\left(\hat{\varphi}, A_D^{-1}\hat{\psi}\right)\mathrm{d}tt \\
&+ \frac{c_1}{\gamma}\int_s^\tau \left[\left(\hat{\varphi}_t, \frac{\alpha}{\beta}(\mathbf{I}-\mathbf{D}\gamma_0)(\xi\hat{\varphi}+\hat{\varphi}_t) - \left(\frac{\sigma}{\beta}-\xi\right)A_D^{-1}\hat{\psi} - \frac{\eta}{\beta}(\mathbf{I}-\mathbf{D}\gamma_0)\hat{\psi}\right)_{L^2(\Omega)}\right]\mathrm{d}t \\
&- \int_s^\tau \left[\left(\gamma\xi^2\frac{\partial\hat{\varphi}}{\partial\nu} + 2\gamma\xi\frac{\partial\hat{\varphi}_t}{\partial\nu} + \alpha\lambda\hat{\psi}, \hat{\varphi}_t\right)_{L^2(\Gamma_1)} + \alpha\left(\hat{\psi}, \frac{\partial\hat{\varphi}_t}{\partial\nu}\right)_{L^2(\Gamma_1)}\right]\mathrm{d}t \\
&+ c_1\int_s^\tau \left[\left(\hat{\varphi}_t, \frac{\alpha}{\beta}\frac{\partial(\mathbf{I}-\mathbf{D}\gamma_0)(\xi\hat{\varphi}+\hat{\varphi}_t)}{\partial\nu} - \left(\frac{\sigma}{\beta}-\xi\right)\frac{\partial A_D^{-1}\hat{\psi}}{\partial\nu} - \frac{\eta}{\beta}\frac{\partial(\mathbf{I}-\mathbf{D}\gamma_0)\hat{\psi}}{\partial\nu}\right)_{L^2(\Gamma_1)}\right]\mathrm{d}t \\
&+ \frac{c_1}{\gamma}\int_s^\tau \left[\alpha\left(\hat{\psi}, \frac{\partial A_D^{-1}\hat{\psi}}{\partial\nu}\right)_{L^2(\Gamma_1)} - \left(\Delta\hat{\varphi}, \frac{\partial A_D^{-1}\hat{\psi}}{\partial\nu}\right)_{L^2(\Gamma_0)}\right]\mathrm{d}t. \qquad (3.30)
\end{aligned}
$$

Using the Cauchy-Schwarz inequality (or with ϵ), the heat equation in system (3.13), Green's formula, trace theorem and the properties of operators A_D, $\mathbf{D}$ and γ_0, we estimate all the terms on the right hand side of (3.30), respectively,

$$
c_1\left[\frac{1}{\gamma}\left(\hat{\varphi}_t, A_D^{-1}\hat{\psi}\right)_{L^2(\Omega)} + \left(D\hat{\varphi}_t, DA_D^{-1}\hat{\psi}\right)_{T^1(\Omega)}\right]_s^\tau \le \epsilon\left(E_{\hat{\varphi}}(s) + E_{\hat{\varphi}}(\tau)\right) + lot\ \left(\hat{\varphi}, \hat{\psi}\right); \qquad (3.31)
$$

$$
\begin{aligned}
-\frac{c_1^2}{\gamma}\int_s^\tau \left(\hat{\psi}_t, A_D^{-1}\hat{\psi}\right)_{L^2(\Omega)}\mathrm{d}t =& -\frac{c_1^2}{\gamma}\int_s^\tau \left[\left(\left(\left(\frac{\sigma}{\beta}-\xi\right)\hat{\psi}, A_D^{-1}\hat{\psi}\right)_{L^2(\Omega)} + \left(\frac{\eta}{\beta}\hat{\psi} - \frac{\alpha}{\beta}\left(\xi\hat{\varphi}+\hat{\varphi}_t\right), \hat{\psi}\right)_{L^2(\Omega)}\right.\right. \\
&\left.\left. + \left(\frac{\eta}{\beta}\hat{\psi} - \frac{\alpha}{\beta}\left(\xi\hat{\varphi}+\hat{\varphi}_t\right), \frac{\partial A_D^{-1}\hat{\psi}}{\partial\nu}\right)_{L^2(\Gamma)}\right]\mathrm{d}t \\
\le& lot\ \left(\hat{\varphi}, \hat{\psi}\right); \qquad (3.32)
\end{aligned}
$$

$$
\begin{aligned}
&\int_s^\tau \left(c_0\hat{\psi} + c_2\hat{\varphi} + c_3\hat{\varphi}_t + c_4\Delta\hat{\varphi}, \hat{\varphi}_t - \frac{c_1}{\gamma}A_D^{-1}\hat{\psi}\right)_{L^2(\Omega)}\mathrm{d}t + \frac{c_1}{\gamma}\int_s^\tau B\left(\hat{\varphi}, A_D^{-1}\hat{\psi}\right)\mathrm{d}t \\
&+ \frac{c_1}{\gamma}\int_s^\tau \left[\left(\hat{\varphi}_t, \frac{\alpha}{\beta}(\mathbf{I}-\mathbf{D}\gamma_0)(\xi\hat{\varphi}+\hat{\varphi}_t) - \left(\frac{\sigma}{\beta}-\xi\right)A_D^{-1}\hat{\psi} - \frac{\eta}{\beta}(\mathbf{I}-\mathbf{D}\gamma_0)\hat{\psi}\right)_{L^2(\Omega)}\right]\mathrm{d}t \\
&- c_1\int_s^\tau \left(\hat{\varphi}_t, \left(\frac{\sigma}{\beta}-\xi\right)\frac{\partial A_D^{-1}\hat{\psi}}{\partial\nu}\right)_{L^2(\Gamma_1)}\mathrm{d}t + \frac{\alpha c_1}{\gamma}\int_s^\tau \left(\hat{\psi}, \frac{\partial A_D^{-1}\hat{\psi}}{\partial\nu}\right)_{L^2(\Gamma_1)}\mathrm{d}t \\
&\le \epsilon\int_0^T E_{\hat{\varphi}}(t)\mathrm{d}t + lot\ \left(\hat{\varphi}, \hat{\psi}\right); \qquad (3.33)
\end{aligned}
$$

$$
\begin{aligned}
&- \int_s^\tau \left[\left(\gamma\xi^2\frac{\partial\hat{\varphi}}{\partial\nu} + 2\gamma\xi\frac{\partial\hat{\varphi}_t}{\partial\nu} + \alpha\lambda\hat{\psi}, \hat{\varphi}_t\right)_{L^2(\Gamma_1)} + \alpha\left(\hat{\psi}, \frac{\partial\hat{\varphi}_t}{\partial\nu}\right)_{L^2(\Gamma_1)}\right]\mathrm{d}t \\
&+ c_1\int_s^\tau \left(\hat{\varphi}_t, \frac{\alpha\xi}{\beta}\frac{\partial(\mathbf{I}-\mathbf{D}\gamma_0)\hat{\varphi}}{\partial\nu} + \frac{\alpha}{\beta}\frac{\partial\hat{\varphi}_t}{\partial\nu} - \frac{\eta}{\beta}\frac{\partial\hat{\psi}}{\partial\nu}\right)_{L^2(\Gamma_1)}\mathrm{d}t \\
&\le C\int_0^T |||D\hat{\varphi}_t|||_{L^2(\Gamma_1)}^2\mathrm{d}t + lot\ \left(\hat{\varphi}, \hat{\psi}\right); \qquad (3.34)
\end{aligned}
$$

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$$
\begin{aligned}
& c_1 \int_s^\tau \left(\hat{\varphi}_t, \frac{\eta}{\beta} \frac{\partial \mathbf{D}\gamma_0 \hat{\psi}}{\partial \nu} - \frac{\alpha}{\beta} \frac{\partial \mathbf{D}\gamma_0 \hat{\varphi}_t}{\partial \nu} \right)_{L^2(\Gamma_1)} \mathrm{d}t \\
& \leq C \int_0^T \|\hat{\varphi}_t\|_{H^{\frac{1}{2}}(\Gamma_1)} \left\| \frac{\eta}{\beta} \frac{\partial \mathbf{D}\gamma_0 \hat{\psi}}{\partial \nu} - \frac{\alpha}{\beta} \frac{\partial \mathbf{D}\gamma_0 \hat{\varphi}_t}{\partial \nu} \right\|_{H^{-\frac{1}{2}}(\Gamma_1)} \mathrm{d}t \\
& \leq \epsilon \int_0^T \left(E_{\hat{\varphi}}(t) + \left\| A_R^{\frac{1}{2}} \hat{\psi} \right\|^2_{L^2(\Omega)} \right) \mathrm{d}t + C_\epsilon \int_0^T |||D\hat{\varphi}_t|||^2_{L^2(\Gamma_1)} \mathrm{d}t; \qquad (3.35)
\end{aligned}
$$

$$
\begin{aligned}
-\frac{c_1}{\gamma} \int_s^\tau \left(\Delta\hat{\varphi}, \frac{\partial A_D^{-1} \hat{\psi}}{\partial \nu} \right)_{L^2(\Gamma_0)} \mathrm{d}t \leq\ & C \int_s^\tau \|\triangle\hat{\varphi}\|_{L^2(\Gamma_0)} \left\| A_D^{-1} \hat{\psi} \right\|_{H^{\frac{3}{2}}(\Omega)} \mathrm{d}t \leq\ \epsilon \int_s^\tau \|\triangle\hat{\varphi}\|^2_{L^2(\Gamma_0)} \mathrm{d}t + lot\ \left(\hat{\varphi}, \hat{\psi} \right) \\
\leq\ & \epsilon \left(\int_0^T E_{\hat{\varphi}}(t) \mathrm{d}t + E_{\hat{\varphi}}(s) + E_{\hat{\varphi}}(\tau) \right) + lot\ \left(\hat{\varphi}, \hat{\psi} \right). \qquad (3.36)
\end{aligned}
$$

Here, we need the trace result of Lemma 3.5 below to obtain the last inequality (3.36).

Combining relations (3.30)-(3.36), we obtain the desired inequality (3.20). □

Lemma 3.5. *Let* $\left(\hat{\varphi}, \hat{\varphi}_t, \hat{\psi} \right)$ *be a solution to problem* (3.13) *with terminal data in* $D(\mathscr{B}^2)$. *Then there is a* $C > 0$ *such that the component* $\hat{\varphi}$ *satisfies the following estimate for all* s *and* $\tau \in [0, T]$:

$$
\int_s^\tau \|\triangle\hat{\varphi}\|^2_{L^2(\Gamma_0)} \mathrm{d}t \leq C \left(\int_0^T E_{\hat{\varphi}}(t) \mathrm{d}t + E_{\hat{\varphi}}(s) + E_{\hat{\varphi}}(\tau) \right) + lot\ \left(\hat{\varphi}, \hat{\psi} \right). \qquad (3.37)
$$

Proof. We choose a vector field H on $\overline{\Omega}$ such that

$$
H = \begin{cases} \nu & \text{on } \Gamma_0, \\ 0 & \text{on } \Gamma_1. \end{cases} \qquad (3.38)
$$

Multiplying the equation in (3.18) by $H(\hat{\varphi})$ and integrating in time and space gives

$$
\int_s^\tau (\hat{\varphi}_{tt} - \gamma\Delta\hat{\varphi}_{tt} + \mathscr{A}\hat{\varphi}, H(\hat{\varphi}))_{L^2(\Omega)} \mathrm{d}t = \int_s^\tau \left(c_0\hat{\psi} + c_1\hat{\psi}_t + c_2\hat{\varphi} + c_3\hat{\varphi}_t + c_4\Delta\hat{\varphi}, H(\hat{\varphi}) \right)_{L^2(\Omega)} \mathrm{d}t. \quad (3.39)
$$

First, integrating by parts, and then applying divergence theorem with the boundary conditions of problem (3.18) and relation (3.38), we have

$$
\int_s^\tau (\hat{\varphi}_{tt}, H(\hat{\varphi}))_{L^2(\Omega)} \mathrm{d}t = (\hat{\varphi}_t, H(\hat{\varphi}))_{L^2(\Omega)}|_s^\tau + \frac{1}{2} \int_s^\tau \int_\Omega \hat{\varphi}_t^2 \operatorname{div} H \mathrm{d}g\mathrm{d}t. \qquad (3.40)
$$

Second, Green's formula and an integration by parts, plus Lemma 3.3 with relations (3.38) and (3.26) give

$$
\begin{aligned}
& \int_s^\tau (-\gamma\Delta\hat{\varphi}_{tt}, H(\hat{\varphi}))_{L^2(\Omega)} \mathrm{d}t \\
=& \gamma (D(H(\hat{\varphi})), D\hat{\varphi}_t)_{T^1(\Omega)} \Big|_s^\tau - \gamma \int_s^\tau \int_\Omega \left[DH(D\hat{\varphi}_t, D\hat{\varphi}_t) - \frac{1}{2} |D\hat{\varphi}_t|^2 \operatorname{div} H \right] \mathrm{d}g\mathrm{d}t. \qquad (3.41)
\end{aligned}
$$

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Then, using Lemma 3.1 with relation (3.38), relations (3.26) and (3.27), relation (3.28) with $H = \nu$, relation (3.9) in Lemma 3.2, we obtain

$$\begin{aligned}\int_s^\tau (\mathscr{A}\hat{\varphi}, H(\hat{\varphi}))_{L^2(\Omega)}\,\mathrm{d}t &= \int_s^\tau \int_\Omega a(\hat{\varphi}, H(\hat{\varphi}))\mathrm{d}g\mathrm{d}t - \int_s^\tau \int_{\Gamma_0} (\Delta\hat{\varphi})^2 \mathrm{d}\Gamma\mathrm{d}t \\ &\le \int_s^\tau \int_{\Gamma_0} \left[\frac{a(\hat{\varphi},\hat{\varphi})}{2} - (\Delta\hat{\varphi})^2\right]\mathrm{d}\Gamma\mathrm{d}t + C\int_s^\tau \int_\Omega |D^2\varphi|^2 \mathrm{d}g\mathrm{d}t + L(\hat{\varphi}). \quad (3.42)\end{aligned}$$

Again, using relation (3.27), we have

$$a(\hat{\varphi},\hat{\varphi}) = (D^2\hat{\varphi}(\nu,\nu))^2 = (\Delta\hat{\varphi})^2 \qquad \text{on } \Gamma_0. \quad (3.43)$$

From (3.39), integration by parts yields

$$\begin{aligned}\text{RHS of (3.39)} = c_1\left[\left(\hat{\psi}, H(\hat{\varphi})\right)_{L^2(\Omega)}\right]_s^\tau - c_1\int_s^\tau \left(\hat{\psi}, H(\hat{\varphi}_t)\right)_{L^2(\Omega)}\mathrm{d}t \\ + \int_s^\tau \left(c_0\hat{\psi} + c_2\hat{\varphi} + c_3\hat{\varphi}_t + c_4\Delta\hat{\varphi}, H(\hat{\varphi})\right)_{L^2(\Omega)}\mathrm{d}t. \quad (3.44)\end{aligned}$$

By combining relations (3.39)-(3.44), we have

$$\begin{aligned}&\frac{1}{2}\int_s^\tau \int_{\Gamma_0} (\Delta\hat{\varphi})^2 \mathrm{d}g\mathrm{d}t \\ &\le \left[(\hat{\varphi}_t, H(\hat{\varphi}))_{L^2(\Omega)} + \gamma\,(D(H(\hat{\varphi})), D\hat{\varphi}_t)_{T^1(\Omega)} - c_1\left(\hat{\psi}, H(\hat{\varphi})\right)_{L^2(\Omega)}\right]_s^\tau + C\int_s^\tau \int_\Omega |D^2\varphi|^2 \mathrm{d}g\mathrm{d}t \\ &\quad + \frac{1}{2}\int_s^\tau \int_\Omega \hat{\varphi}_t^2 \operatorname{div} H \mathrm{d}g\mathrm{d}t - \gamma\int_s^\tau \int_\Omega DH(D\hat{\varphi}_t, D\hat{\varphi}_t)\mathrm{d}g\mathrm{d}t + \frac{\gamma}{2}\int_s^\tau \int_\Omega |D\hat{\varphi}_t|^2 \operatorname{div} H \mathrm{d}g\mathrm{d}t \\ &\quad + c_1\int_s^\tau \left(\hat{\psi}, H(\hat{\varphi}_t)\right)_{L^2(\Omega)}\mathrm{d}t - \int_s^\tau \left(c_0\hat{\psi} + c_2\hat{\varphi} + c_3\hat{\varphi}_t + c_4\Delta\hat{\varphi}, H(\hat{\varphi})\right)_{L^2(\Omega)}\mathrm{d}t + L(\hat{\varphi}). \quad (3.45)\end{aligned}$$

Majorizing the right side of the above inequality mainly by the Cauchy-Schwarz inequality, we obtain the desired estimate (3.37). □

Next, we introduce an important boundary trace estimate result for the Kirchoff plate in [20].

Lemma 3.6. *([20]) Let u satisfies the following Kirchoff equation:*

$$\begin{cases} u_{tt} - \gamma\Delta u_{tt} + \mathscr{A}u = f & \text{on } (0,T)\times\Omega, \\ u = \dfrac{\partial u}{\partial \nu} = 0 & \text{on } (0,T)\times\Gamma_0, \\ \begin{cases} \Delta\hat{\varphi} + (1-\mu)B_1 u = g_1 \\ \dfrac{\partial \Delta u}{\partial \nu} + (1-\mu)B_2 u - \gamma\dfrac{\partial u_{tt}}{\partial \nu} = g_2 \end{cases} & \text{on } (0,T)\times\Gamma_1, \end{cases} \quad (3.46)$$

where $\Omega \subset \mathbb{R}^n$ is an open, bounded domain with smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, and $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$, Γ_0 is possibly empty while Γ_1 is nonempty. Let $0 < \varepsilon < T$ and $\epsilon > 0$ be arbitrary. Then the following

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inequality holds for the solution u:

$$\int_{\varepsilon}^{T-\varepsilon}\int_{\Gamma_1}|D^2u|^2_{T_x^2}\mathrm{d}\Gamma\mathrm{d}t \le C_{T,\varepsilon,\gamma,\epsilon}\Bigg[\int_0^T\Big(\|f\|^2_{[H^{\frac{3}{2}-\epsilon}(\Omega)]'}+\|g_1\|^2_{L^2(\Gamma_1)}+\|u\|^2_{H^{\frac{3}{2}+\epsilon}(\Omega)} +\|u_t\|^2_{L^2(\Gamma_1)}+|||Du_t|||^2_{L^2(\Gamma_1)}\Big)\mathrm{d}t+\|g_2\|^2_{H^{-1}([0,T]\times\Gamma_1)}\Bigg]. \tag{3.47}$$

By virtue of the above result, we further have the following energy estimate.

Lemma 3.7. *Let a vector field H be the escape vector field for plate. For arbitrary* $\epsilon_0 \in [0,T)$, $\epsilon_1>0$ *and terminal data in* $D(\mathscr{B}^2)$*, the corresponding solution* $\left(\hat{\varphi},\hat{\varphi}_t,\hat{\psi}\right)$ *to problem (3.18) satisfies the following estimate:*

$$\int_{\epsilon_0}^{T-\epsilon_0}E_{\hat{\varphi}}(t)\mathrm{d}t \le C^*\left(E_{\hat{\varphi}}(T-\epsilon_0)+E_{\hat{\varphi}}(\epsilon_0)\right)+C_T\int_0^T|||D\hat{\varphi}_t|||^2_{L^2(\Gamma_1)}\mathrm{d}t +\epsilon_1\int_0^T\left\|A_R^{\frac{1}{2}}\hat{\psi}\right\|^2_{L^2(\Omega)}\mathrm{d}t+lot\ \left(\hat{\varphi},\hat{\psi}\right), \tag{3.48}$$

where the (time independent) constant $C^*\ge 1+\frac{8sup\ |H|^2\gamma}{1-\mu}$.

Proof. Since H is the escape vector field for plate, it follows that

$$\operatorname{div}H = trDH = DH(e_1,e_1)+DH(e_2,e_2)=2b(x). \tag{3.49}$$

We multiply the equation in (3.18) by $H(\hat{\varphi})-\frac{\hat{\varphi}}{2}$ and integrate in time and space to obtain

$$\int_{\epsilon_0}^{T-\epsilon_0}\left(\hat{\varphi}_{tt}-\gamma\Delta\hat{\varphi}_{tt}+\mathscr{A}\hat{\varphi}-\left[c_0\hat{\psi}+c_1\hat{\psi}_t+c_2\hat{\varphi}+c_3\hat{\varphi}_t+c_4\Delta\hat{\varphi}\right],H(\hat{\varphi})-\frac{\hat{\varphi}}{2}\right)_{L^2(\Omega)}\mathrm{d}t=0. \tag{3.50}$$

Applying integration by parts and divergence theorem with the boundary conditions of problem (3.18), Lemma 3.3 and relation (3.49), we have

$$\begin{aligned}&\int_{\epsilon_0}^{T-\epsilon_0}\left(\hat{\varphi}_{tt}-\gamma\Delta\hat{\varphi}_{tt},H(\hat{\varphi})-\frac{\hat{\varphi}}{2}\right)_{L^2(\Omega)}\mathrm{d}t\\ =&\gamma\int_{\epsilon_0}^{T-\epsilon_0}\int_{\Gamma_1}\left(\frac{\hat{\varphi}}{2}-H(\hat{\varphi})\right)\frac{\partial\hat{\varphi}_{tt}}{\partial\nu}\mathrm{d}\Gamma\mathrm{d}t-\frac{1}{2}\int_{\epsilon_0}^{T-\epsilon_0}\int_{\Gamma_1}(\hat{\varphi}_t^2+\gamma|D\hat{\varphi}_t|^2)\langle H\ ,\nu\rangle\mathrm{d}\Gamma\mathrm{d}t\\ &-\gamma\int_{\epsilon_0}^{T-\epsilon_0}\int_{\Omega}b(x)|D\hat{\varphi}_t|^2\mathrm{d}g\mathrm{d}t+\int_{\epsilon_0}^{T-\epsilon_0}\int_{\Omega}\left(\frac{1}{2}+b(x)\right)\left(\hat{\varphi}_t^2+\gamma|D\hat{\varphi}_t|^2\right)\mathrm{d}g\mathrm{d}t\\ &+\left[\left(\hat{\varphi}_t,H(\hat{\varphi})-\frac{\hat{\varphi}}{2}\right)_{L^2(\Omega)}+\gamma\left(D(H(\hat{\varphi}))-\frac{D\hat{\varphi}}{2},D\hat{\varphi}_t\right)_{T^1(\Omega)}\right]_{\epsilon_0}^{T-\epsilon_0}.\end{aligned} \tag{3.51}$$

Using Lemma 3.1 with relations (3.26) and (3.27), the boundary conditions of problem (3.18), equality (3.10) in Lemma 3.2 with relations (3.28), (3.43) and (3.49), we obtain

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$$
\begin{aligned}
&\int_{\epsilon_0}^{T-\epsilon_0}\left(\mathscr{A}\hat{\varphi}, H(\hat{\varphi})-\frac{\hat{\varphi}}{2}\right)_{L^2(\Omega)}\mathrm{d}t\\
&=\int_{\epsilon_0}^{T-\epsilon_0}\int_{\Gamma_1}\left[\gamma\frac{\partial\hat{\varphi}_{tt}}{\partial\nu}+\gamma\frac{\partial}{\partial\nu}(\xi^2\hat{\varphi}+2\xi\hat{\varphi}_t)-\alpha\frac{\partial\hat{\psi}}{\partial\nu}\right]\left(H(\hat{\varphi})-\frac{\hat{\varphi}}{2}\right)\mathrm{d}\Gamma\mathrm{d}t\\
&\quad+\int_{\epsilon_0}^{T-\epsilon_0}\int_{\Gamma_1}\alpha\hat{\psi}\left(\frac{\partial H(\hat{\varphi})}{\partial\nu}-\frac{1}{2}\frac{\partial\hat{\varphi}}{\partial\nu}\right)\mathrm{d}\Gamma\mathrm{d}t+\frac{1}{2}\int_{\epsilon_0}^{T-\epsilon_0}\int_{\Gamma_1}a(\hat{\varphi},\hat{\varphi})\langle H\ ,\nu\rangle\mathrm{d}\Gamma\mathrm{d}t\\
&\quad-\frac{1}{2}\int_{\epsilon_0}^{T-\epsilon_0}\int_{\Gamma_0}(\Delta\hat{\varphi})^2\langle H\ ,\nu\rangle\mathrm{d}\Gamma\mathrm{d}t+\int_{\epsilon_0}^{T-\epsilon_0}\int_{\Omega}\left(b(x)-\frac{1}{2}\right)a(\hat{\varphi},\hat{\varphi})\mathrm{d}g\mathrm{d}t+L(\hat{\varphi}).
\end{aligned}
\tag{3.52}
$$

Integrating by parts and applying Green's formula with the boundary conditions of problem (3.18), Lemma 3.3 with relations (1.2),(3.26) and (3.49), we have

$$
\begin{aligned}
&\int_{\epsilon_0}^{T-\epsilon_0}\left(c_0\hat{\psi}+c_1\hat{\psi}_t+c_2\hat{\varphi}+c_3\hat{\varphi}_t+c_4\Delta\hat{\varphi}, H(\hat{\varphi})-\frac{\hat{\varphi}}{2}\right)_{L^2(\Omega)}\mathrm{d}t\\
&=\int_{\epsilon_0}^{T-\epsilon_0}\left(c_0\hat{\psi}+c_2\hat{\varphi}+c_3\hat{\varphi}_t, H(\hat{\varphi})-\frac{\hat{\varphi}}{2}\right)_{L^2(\Omega)}\mathrm{d}t+c_1\left[\left(\hat{\psi}, H(\hat{\varphi})-\frac{\hat{\varphi}}{2}\right)_{L^2(\Omega)}\right]_{\epsilon_0}^{T-\epsilon_0}\\
&\quad-c_1\int_{\epsilon_0}^{T-\epsilon_0}\left(\hat{\psi}, H(\hat{\varphi}_t)-\frac{\hat{\varphi}_t}{2}\right)_{L^2(\Omega)}\mathrm{d}t+c_4\int_{\epsilon_0}^{T-\epsilon_0}\int_{\Gamma_1}\left(H(\hat{\varphi})-\frac{\hat{\varphi}}{2}\right)\frac{\partial\hat{\varphi}}{\partial\nu}\\
&\quad-\frac{|D\hat{\varphi}|^2}{2}\langle H,\ \nu\rangle\mathrm{d}\Gamma\mathrm{d}t+\frac{c_4}{2}\int_{\epsilon_0}^{T-\epsilon_0}\int_{\Omega}|D\hat{\varphi}|^2\mathrm{d}g\mathrm{d}t.
\end{aligned}
\tag{3.53}
$$

Combining equalities (3.50)-(3.53), we obtain

$$
\begin{aligned}
&\int_{\epsilon_0}^{T-\epsilon_0}\int_{\Omega}\left[\frac{1}{2}\left(\hat{\varphi}_t^2+\gamma|D\hat{\varphi}_t|^2\right)+\left(b(x)-\frac{1}{2}\right)a(\hat{\varphi},\hat{\varphi})\right]\mathrm{d}g\mathrm{d}t\\
&=\int_{\epsilon_0}^{T-\epsilon_0}\left[\left(c_0\hat{\psi}+c_2\hat{\varphi}+c_3\hat{\varphi}_t, H(\hat{\varphi})-\frac{\hat{\varphi}}{2}\right)_{L^2(\Omega)}-c_1\left(\hat{\psi}, H(\hat{\varphi}_t)-\frac{\hat{\varphi}_t}{2}\right)_{L^2(\Omega)}\right]\mathrm{d}t\\
&\quad+\int_{\epsilon_0}^{T-\epsilon_0}\int_{\Omega}\left(\frac{c_4}{2}|D\hat{\varphi}|^2-b(x)\hat{\varphi}_t^2\right)\mathrm{d}g\mathrm{d}t+\frac{1}{2}\int_{\epsilon_0}^{T-\epsilon_0}\int_{\Gamma_0}(\Delta\hat{\varphi})^2\langle H\ ,\nu\rangle\mathrm{d}\Gamma\mathrm{d}t\\
&\quad-\alpha\int_{\epsilon_0}^{T-\epsilon_0}\left(\hat{\psi}, \frac{\partial H(\hat{\varphi})}{\partial\nu}\right)_{L^2(\Gamma_1)}\mathrm{d}t+\frac{1}{2}\int_{\epsilon_0}^{T-\epsilon_0}\int_{\Gamma_1}\left(\hat{\varphi}_t^2+\gamma|D\hat{\varphi}_t|^2-c_4|D\hat{\varphi}|^2\right)\langle H\ ,\nu\rangle\mathrm{d}\Gamma\mathrm{d}t\\
&\quad-\frac{1}{2}\int_{\epsilon_0}^{T-\epsilon_0}\int_{\Gamma_1}a(\hat{\varphi},\hat{\varphi})\langle H\ ,\nu\rangle\mathrm{d}\Gamma\mathrm{d}t+\int_{\epsilon_0}^{T-\epsilon_0}\left[\left(\frac{\alpha}{2}\hat{\psi}+c_4\left(H(\hat{\varphi})-\frac{\hat{\varphi}}{2}\right), \frac{\partial\hat{\varphi}}{\partial\nu}\right)_{L^2(\Gamma_1)}\right.\\
&\quad\left.-\left(\gamma\frac{\partial}{\partial\nu}(\xi^2\hat{\varphi}+2\xi\hat{\varphi}_t)-\alpha\frac{\partial\hat{\psi}}{\partial\nu}, H(\hat{\varphi})-\frac{\hat{\varphi}}{2}\right)_{L^2(\Gamma_1)}\right]dt\\
&\quad-\left[\left(\hat{\varphi}_t-c_1\hat{\psi}, H(\hat{\varphi})-\frac{\hat{\varphi}}{2}\right)_{L^2(\Omega)}+\gamma\left(D(H(\hat{\varphi}))-\frac{D\hat{\varphi}}{2}, D\hat{\varphi}_t\right)_{T^1(\Omega)}\right]_{\epsilon_0}^{T-\epsilon_0}+lot\ \left(\hat{\varphi},\hat{\psi}\right).
\end{aligned}
\tag{3.54}
$$

Since $\min_{x\in\Omega} b(x)\geq 1$, it follows that

$$
\text{LHS of (3.54)}\geq\int_{\epsilon_0}^{T-\epsilon_0}E(t)\mathrm{d}t. \tag{3.55}
$$

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To estimate the terms on the right hand side of (3.54), we proceed by separating the work into the following segments (i)-(vi).

(i) Application of Cauchy-Schwarz inequality gives

$$\int_{\epsilon_0}^{T-\epsilon_0}\left(c_0\hat{\psi}+c_2\hat{\varphi}+c_3\hat{\varphi}_t, H(\hat{\varphi})-\frac{\hat{\varphi}}{2}\right)_{L^2(\Omega)}dt \leq \ lot\ \left(\hat{\varphi},\hat{\psi}\right). \tag{3.56}$$

(ii) Applying divergence theorem, Cauchy-Schwarz inequality, the fact that $\frac{\partial\hat{\varphi}}{\partial\nu}|_\Gamma = -\lambda\hat{\varphi}$ and trace theorem, we have

$$\begin{aligned}
&\int_{\epsilon_0}^{T-\epsilon_0}\int_\Omega\left(\frac{c_4}{2}|D\hat{\varphi}|^2-b(x)\hat{\varphi}_t^2\right)\mathrm{d}g\mathrm{d}t-\int_{\epsilon_0}^{T-\epsilon_0}c_1\left(\hat{\psi},H(\hat{\varphi}_t)-\frac{\hat{\varphi}_t}{2}\right)_{L^2(\Omega)}\mathrm{d}t\\
&+\int_{\epsilon_0}^{T-\epsilon_0}\left[\left(\frac{\alpha}{2}\hat{\psi}+c_4\left(H(\hat{\varphi})-\frac{\hat{\varphi}}{2}\right),\frac{\partial\hat{\varphi}}{\partial\nu}\right)_{L^2(\Gamma_1)}-\left(\gamma\frac{\partial}{\partial\nu}(\xi^2\hat{\varphi}+2\xi\hat{\varphi}_t)-\alpha\frac{\partial\hat{\psi}}{\partial\nu},H(\hat{\varphi})-\frac{\hat{\varphi}}{2}\right)_{L^2(\Gamma_1)}\right]dt\\
&\leq c_1\int_{\epsilon_0}^{T-\epsilon_0}\int_\Omega\hat{\varphi}_t\,\mathrm{div}\,(\hat{\psi}H)\mathrm{d}g\mathrm{d}t-c_1\int_{\epsilon_0}^{T-\epsilon_0}\int_{\Gamma_1}\hat{\varphi}_t\hat{\psi}\langle H\ ,\nu\rangle\mathrm{d}\Gamma\mathrm{d}t+C\int_0^T|||D\hat{\varphi}_t|||^2_{L^2(\Gamma_1)}\mathrm{d}t+lot\ (\hat{\varphi},\hat{\psi})\\
&\leq\epsilon_1\int_0^T\left\|A_R^{\frac{1}{2}}\hat{\psi}\right\|^2_{L^2(\Omega)}\mathrm{d}t+C\int_0^T|||D\hat{\varphi}_t|||^2_{L^2(\Gamma_1)}\mathrm{d}t+lot\ \left(\hat{\varphi},\hat{\psi}\right).
\end{aligned}\tag{3.57}$$

(iii) Since $\langle H\ ,\nu\rangle\leq 0$ on Γ_0, $\langle H\ ,\nu\rangle>0$ on Γ_1, it follows that

$$\frac{1}{2}\int_{\epsilon_0}^{T-\epsilon_0}\int_{\Gamma_0}(\Delta\hat{\varphi})^2\langle H\ ,\nu\rangle\mathrm{d}\Gamma\mathrm{d}t\leq 0, \tag{3.58}$$

$$\frac{1}{2}\int_{\epsilon_0}^{T-\epsilon_0}\int_{\Gamma_1}\left(\hat{\varphi}_t^2+\gamma|D\hat{\varphi}_t|^2-c_4|D\hat{\varphi}|^2\right)\langle H\ ,\nu\rangle\mathrm{d}\Gamma\mathrm{d}t\leq C\int_0^T|||D\hat{\varphi}_t|||^2_{L^2(\Gamma_1)}\mathrm{d}t+lot\ \left(\hat{\varphi},\hat{\psi}\right). \tag{3.59}$$

(iv) Note that

$$\frac{\partial H(\hat{\varphi})}{\partial\nu}=\langle D_\nu H\ ,D\hat{\varphi}\rangle+\langle H\ ,\nu\rangle D^2\hat{\varphi}(\nu,\nu)+\langle H,\tau\rangle D^2\hat{\varphi}(\tau,\nu)\quad\text{on }\Gamma_1, \tag{3.60}$$

we use Cauchy-Schwarz inequality, trace theorem and Lemma 3.6 with problem (3.18) to obtain

$$\begin{aligned}
&-\alpha\int_{\epsilon_0}^{T-\epsilon_0}\left(\hat{\psi},\frac{\partial H(\hat{\varphi})}{\partial\nu}\right)_{L^2(\Gamma_1)}\mathrm{d}t\leq C\int_{\epsilon_0}^{T-\epsilon_0}\left[\|\hat{\psi}\|^2_{L^2(\Gamma_1)}+\left\|\frac{\partial H(\hat{\varphi})}{\partial\nu}\right\|^2_{L^2(\Gamma_1)}\right]\mathrm{d}t\\
&\leq C\int_{\epsilon_0}^{T-\epsilon_0}\left[\|\hat{\psi}\|^2_{H^{\frac{1}{2}}(\Omega)}+2C_1|||D\hat{\varphi}|||^2_{L^2(\Gamma_1)}+2C_2\int_{\Gamma_1}\left(D^2\hat{\varphi}(\nu,\nu)+D^2\hat{\varphi}(\tau,\nu)\right)^2\mathrm{d}\Gamma\right]\mathrm{d}t\\
&\leq C\int_{\epsilon_0}^{T-\epsilon_0}|||D^2\hat{\varphi}|_{T^2_{\hat{x}}}||^2_{L^2(\Gamma_1)}\mathrm{d}t+lot\ \left(\hat{\varphi},\hat{\psi}\right)\\
&\leq C_{T,\epsilon_0,\gamma,\epsilon}\left[\int_0^T\left(\|c_0\hat{\psi}+c_1\hat{\psi}_t+c_2\hat{\varphi}+c_3\hat{\varphi}_t+c_4\Delta\hat{\varphi}\|^2_{H^{-\frac{3}{2}+\epsilon}(\Omega)}+\|\alpha\hat{\psi}\|^2_{L^2(\Gamma_1)}+\|\hat{\varphi}\|^2_{H^{\frac{3}{2}+\epsilon}(\Omega)}\right.\right.\\
&\left.\left.+\|\hat{\varphi}_t\|^2_{L^2(\Gamma_1)}+|||D\hat{\varphi}_t|||^2_{L^2(\Gamma_1)}\right)\mathrm{d}t+\left\|\gamma\frac{\partial}{\partial\nu}\left(\xi^2\hat{\varphi}+2\xi\hat{\varphi}_t\right)-\alpha\frac{\partial\hat{\psi}}{\partial\nu}\right\|^2_{H^{-1}([0,T]\times\Gamma_1)}\right]+lot\ (\hat{\varphi},\hat{\psi})\\
&\leq C_T\int_0^T\left(|||D\hat{\varphi}_t|||^2_{L^2(\Gamma_1)}+\|c_1\hat{\psi}_t\|^2_{H^{-\frac{3}{2}+\epsilon}(\Omega)}\right)\mathrm{d}t+lot\ \left(\hat{\varphi},\hat{\psi}\right).
\end{aligned}\tag{3.61}$$

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Since $\hat{\psi}(t) = e^{-\xi t}\psi(t)$, we have $\hat{\psi}_t = -\xi\hat{\psi} + e^{-\xi t}\psi_t$. This together with problem (2.1) gives

$$\begin{aligned}\int_0^T \left\|c_1\hat{\psi}_t\right\|^2_{H^{-\frac{3}{2}+\epsilon}(\Omega)} \mathrm{d}t \leq\ & C\int_0^T \left(\left\|\hat{\psi}\right\|^2_{H^{-\frac{3}{2}+\epsilon}(\Omega)} + \|\psi_t\|^2_{H^{-\frac{3}{2}+\epsilon}(\Omega)}\right)\mathrm{d}t\\ \leq & C\int_0^T \left\|\frac{\alpha}{\beta}\Delta\varphi_t + \frac{\sigma}{\beta}\psi - \frac{\eta}{\beta}\Delta\psi\right\|^2_{H^{-\frac{3}{2}+\epsilon}(\Omega)} \mathrm{d}t + lot\ (\hat{\varphi},\hat{\psi}) \leq lot\ \left(\hat{\varphi},\hat{\psi}\right). \end{aligned} \tag{3.62}$$

By combining (3.61) with (3.62), we obtain

$$-\alpha\int_{\epsilon_0}^{T-\epsilon_0}\left(\hat{\psi}, \frac{\partial H(\hat{\varphi})}{\partial\nu}\right)_{L^2(\Gamma_1)} \mathrm{d}t \leq C_T\int_0^T |||D\hat{\varphi}_t|||^2_{L^2(\Gamma_1)}\mathrm{d}t + lot\ \left(\hat{\varphi},\hat{\psi}\right). \tag{3.63}$$

(v) Applying Lemma 3.6 with problem (3.18), we have

$$\begin{aligned} & -\frac{1}{2}\int_{\epsilon_0}^{T-\epsilon_0}\int_{\Gamma_1} a(\hat{\varphi},\hat{\varphi})\langle H,\nu\rangle \mathrm{d}\Gamma\mathrm{d}t\\ & \leq C\int_{\epsilon_0}^{T-\epsilon_0}\int_{\Gamma_1}\left[(1+\mu)\left(|D^2\hat{\varphi}(\nu,\nu)|^2 + |D^2\hat{\varphi}(\tau,\tau)|^2\right) + 2(1-\mu)|D^2\hat{\varphi}(\nu,\tau)|^2\right]\mathrm{d}\Gamma\mathrm{d}t\\ & \leq C\int_{\epsilon_0}^{T-\epsilon_0}\int_{\Gamma_1}(1+\mu)|D^2\hat{\varphi}|^2_{T_x^2}\mathrm{d}\Gamma\mathrm{d}t\\ & \leq C_T\int_0^T |||D\hat{\varphi}_t|||^2_{L^2(\Gamma_1)}\mathrm{d}t + lot\ (\hat{\varphi},\hat{\psi}). \end{aligned} \tag{3.64}$$

(vi) The use of Cauchy-Schwarz inequality yields

$$\begin{aligned} & -\left[\left(\hat{\varphi}_t - c_1\hat{\psi}, H(\hat{\varphi}) - \frac{\hat{\varphi}}{2}\right)_{L^2(\Omega)} + \gamma\left(D(H(\hat{\varphi})) - \frac{D\hat{\varphi}}{2}, D\hat{\varphi}_t\right)_{T^1(\Omega)}\right]_{\epsilon_0}^{T-\epsilon_0}\\ & \leq\left\{2\gamma\int_\Omega\left[(DH(D\hat{\varphi},e_1))^2 + (DH(D\hat{\varphi},e_2))^2\right] + \left[\langle H,e_1\rangle D^2\hat{\varphi}(e_1,e_1) + \langle H,e_2\rangle D^2\hat{\varphi}(e_2,e_1)\right]^2\right.\\ & \quad\left. + \left[\langle H,e_1\rangle D^2\hat{\varphi}(e_1,e_2) + \langle H,e_2\rangle D^2\hat{\varphi}(e_2,e_2)\right]^2 \mathrm{d}g + E(t)\right\}_{\epsilon_0}^{T-\epsilon_0} + lot\ \left(\hat{\varphi},\hat{\psi}\right)\\ & \leq\left(1 + \frac{8\underset{x\in\Omega}{sup}\,|H|^2\gamma}{1-\mu}\right)\left(E_{\hat{\varphi}}(T-\epsilon_0) + E_{\hat{\varphi}}(\epsilon_0)\right) + lot\ \left(\hat{\varphi},\hat{\psi}\right). \end{aligned} \tag{3.65}$$

Combining (3.54)-(3.65), we obtain the desired inequality (3.48). □

Lemma 3.8. *Let $\left(\hat{\varphi},\hat{\varphi}_t,\hat{\psi}\right)$ be a solution to problem (3.18) with terminal data in $D(\mathscr{B}^2)$. Then for $T > 2C^*$(see Lemma 3.7 for C^*), there is a $C_T > 0$ such that*

$$E_{\hat{\varphi}}(T) + \int_0^T E_{\hat{\varphi}}(t)\mathrm{d}t + \int_0^T \left\|A_R^{\frac{1}{2}}\hat{\psi}\right\|^2_{L^2(\Omega)} \mathrm{d}t \leq C_T\int_0^T |||D\hat{\varphi}_t|||^2_{L^2(\Gamma_1)}\mathrm{d}t + lot\ \left(\hat{\varphi},\hat{\psi}\right). \tag{3.66}$$

Proof. For any $\epsilon_0 \in (0,T)$, we have

$$\int_0^T E_{\hat{\varphi}}(t)\mathrm{d}t = \int_0^{\epsilon_0} E_{\hat{\varphi}}(t)\mathrm{d}t + \int_{\epsilon_0}^{T-\epsilon_0} E_{\hat{\varphi}}(t)\mathrm{d}t + \int_{T-\epsilon_0}^T E_{\hat{\varphi}}(t)\mathrm{d}t. \tag{3.67}$$

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According to Lemma 3.4, we have for all s and $\tau \in [0, T]$,

$$E_{\hat{\varphi}}(\tau) \leq \frac{1+\epsilon}{1-\epsilon} E_{\hat{\varphi}}(s) + \frac{\epsilon}{1-\epsilon} \int_0^T \left[E_{\hat{\varphi}}(t) + \left\| A_R^{\frac{1}{2}} \hat{\psi} \right\|^2_{L^2(\Omega)} \right] \mathrm{d}t + C_\epsilon \int_0^T |||D\hat{\varphi}_t|||^2_{L^2(\Gamma_1)} \mathrm{d}t + lot\ \left(\hat{\varphi}, \hat{\psi}\right). \tag{3.68}$$

Using the above relation with $\tau = t$, $s = T$, and integrating in time twice from 0 to ϵ_0 and $T - \epsilon_0$ to T, respectively, then substituting them into equality (3.67), we obtain

$$\begin{aligned} \int_0^T E_{\hat{\varphi}}(t) \mathrm{d}t \leq & \int_{\epsilon_0}^{T-\epsilon_0} E_{\hat{\varphi}}(t) \mathrm{d}t + \frac{2\epsilon_0(1+\epsilon)}{1-\epsilon} E_{\hat{\varphi}}(T) + 2\epsilon_0 C_\epsilon \int_0^T |||D\hat{\varphi}_t|||^2_{L^2(\Gamma_1)} \mathrm{d}t \\ & + \frac{2\epsilon_0 \epsilon}{1-\epsilon} \int_0^T \left[E_{\hat{\varphi}}(t) + \left\| A_R^{\frac{1}{2}} \hat{\psi} \right\|^2_{L^2(\Omega)} \right] \mathrm{d}t + lot\ \left(\hat{\varphi}, \hat{\psi}\right). \end{aligned} \tag{3.69}$$

Applying Lemma 3.7 with $\epsilon_1 = \frac{2\epsilon}{1-\epsilon}$ and relation (3.68) twice with $s = T$, $\tau = T - \epsilon_0$ and ϵ_0, respectively, we get

$$\begin{aligned} \int_{\epsilon_0}^{T-\epsilon_0} E_{\hat{\varphi}}(t) \mathrm{d}t \leq & \ \frac{2(1+\epsilon)C^*}{1-\epsilon} E_{\hat{\varphi}}(T) + C_{\epsilon,T} \int_0^T |||D\hat{\varphi}_t|||^2_{L^2(\Gamma_1)} \mathrm{d}t + \frac{2\epsilon C^*}{1-\epsilon} \int_0^T E_{\hat{\varphi}}(t) \mathrm{d}t \\ & + \frac{2\epsilon(1+C^*)}{1-\epsilon} \int_0^T \left\| A_R^{\frac{1}{2}} \hat{\psi} \right\|^2_{L^2(\Omega)} \mathrm{d}t + lot\ \left(\hat{\varphi}, \hat{\psi}\right). \end{aligned} \tag{3.70}$$

Combining the above two inequalities, we have

$$\begin{aligned} \int_0^T E_{\hat{\varphi}}(t) \mathrm{d}t \leq & \frac{2\epsilon(1+\epsilon_0+C^*)}{1-\epsilon} \int_0^T \left\| A_R^{\frac{1}{2}} \hat{\psi} \right\|^2_{L^2(\Omega)} \mathrm{d}t + \frac{2\epsilon(\epsilon_0+C^*)}{1-\epsilon} \int_0^T E_{\hat{\varphi}}(t) \mathrm{d}t \\ & + \frac{2(\epsilon_0+C^*)(1+\epsilon)}{1-\epsilon} E_{\hat{\varphi}}(T) + C_{\epsilon,T} \int_0^T |||D\hat{\varphi}_t|||^2_{L^2(\Gamma_1)} \mathrm{d}t + lot\ \left(\hat{\varphi}, \hat{\psi}\right). \end{aligned} \tag{3.71}$$

Let $F(s, \tau) = E_{\hat{\varphi}}(s) - E_{\hat{\varphi}}(\tau)$, then $E_{\hat{\varphi}}(t) = E_{\hat{\varphi}}(T) + F(t, T)$. We integrate both sides of it in time from 0 to T to obtain

$$\int_0^T E_{\hat{\varphi}}(t) \mathrm{d}t = T E_{\hat{\varphi}}(T) + \int_0^T F(t, T) \mathrm{d}t. \tag{3.72}$$

Using Lemma 3.4 with $\tau = t, s = T$, and integrating both sides of it in time from 0 to T, we derive

$$\begin{aligned} \left| \int_0^T F(t, T) \mathrm{d}t \right| \leq & \ \epsilon T E_{\hat{\varphi}}(T) + \epsilon(T+1) \int_0^T E_{\hat{\varphi}}(t) \mathrm{d}t + \epsilon T \int_0^T \left\| A_R^{\frac{1}{2}} \hat{\psi} \right\|^2_{L^2(\Omega)} \mathrm{d}t \\ & + C_{\epsilon,T} \int_0^T |||D\hat{\varphi}_t|||^2_{L^2(\Gamma_1)} \mathrm{d}t + lot\ \left(\hat{\varphi}, \hat{\psi}\right). \end{aligned} \tag{3.73}$$

Relations (3.71) and (3.72) together with (3.73) give

$$\begin{aligned} T E_{\hat{\varphi}}(T) \leq & \left[\frac{2(\epsilon_0+C^*)(1+\epsilon)}{1-\epsilon} + \epsilon T \right] E_{\hat{\varphi}}(T) + \epsilon \left[\frac{2(\epsilon_0+C^*)}{1-\epsilon} + (T+1) \right] \int_0^T E_{\hat{\varphi}}(t) \mathrm{d}t \\ & + \epsilon \left[\frac{2(1+\epsilon_0+C^*)}{1-\epsilon} + T \right] \int_0^T \left\| A_R^{\frac{1}{2}} \hat{\psi} \right\|^2_{L^2(\Omega)} \mathrm{d}t + C_{\epsilon,T} \int_0^T |||D\hat{\varphi}_t|||^2_{L^2(\Gamma_1)} \mathrm{d}t + lot\ \left(\hat{\varphi}, \hat{\psi}\right), \end{aligned} \tag{3.74}$$

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then if $T > \frac{2(\epsilon_0+C^*)(1+\epsilon)}{(1-\epsilon)^2}$, sufficiently large, or equivalently, $T > 2C^*$ for ϵ_0 and ϵ small enough, we obtain

$$E_{\hat{\varphi}}(T) \leq \epsilon \tilde{C}_T \int_0^T \left[E_{\hat{\varphi}}(t) + \left\| A_R^{\frac{1}{2}} \hat{\psi} \right\|^2_{L^2(\Omega)} \right] \mathrm{d}t + C_{\epsilon,T} \int_0^T |||D\hat{\varphi}_t|||^2_{L^2(\Gamma_1)} \mathrm{d}t + lot\ \left(\hat{\varphi}, \hat{\psi}\right), \tag{3.75}$$

where throughout $\tilde{C}_T$ will denote a constant independent of ϵ.

Applying the above relation to inequality (3.71), we obtain

$$\begin{aligned} \int_0^T E_{\hat{\varphi}}(t)\mathrm{d}t \leq & C_{\epsilon,T} \int_0^T |||D\hat{\varphi}_t|||^2_{L^2(\Gamma_1)} \mathrm{d}t + \frac{2\epsilon \left[(\epsilon_0 + C^*)(1+\epsilon)\tilde{C}_T + (1+\epsilon_0+C^*) \right]}{1-\epsilon} \int_0^T \left\| A_R^{\frac{1}{2}} \hat{\psi} \right\|^2_{L^2(\Omega)} \mathrm{d}t \\ & + \frac{2\epsilon(\epsilon_0 + C^*) \left[(1+\epsilon)\tilde{C}_T + 1 \right]}{1-\epsilon} \int_0^T E_{\hat{\varphi}}(t)\mathrm{d}t + lot\ \left(\hat{\varphi}, \hat{\psi}\right). \end{aligned} \tag{3.76}$$

Then for $\epsilon > 0$ small enough, we have

$$\int_0^T E_{\hat{\varphi}}(t)\mathrm{d}t \leq \epsilon \tilde{C}_T \int_0^T \left\| A_R^{\frac{1}{2}} \hat{\psi} \right\|^2_{L^2(\Omega)} \mathrm{d}t + C_{\epsilon,T} \int_0^T |||D\hat{\varphi}_t|||^2_{L^2(\Gamma_1)} \mathrm{d}t + lot\ \left(\hat{\varphi}, \hat{\psi}\right). \tag{3.77}$$

Summing inequalities (3.75) and (3.77), we derive the following preliminary inequality for the mechanical energy

$$E_{\hat{\varphi}}(T) + \int_0^T E_{\hat{\varphi}}(t)\mathrm{d}t \leq \epsilon \tilde{C}_T \int_0^T \left\| A_R^{\frac{1}{2}} \hat{\psi} \right\|^2_{L^2(\Omega)} \mathrm{d}t + C_{\epsilon,T} \int_0^T |||D\hat{\varphi}_t|||^2_{L^2(\Gamma_1)} \mathrm{d}t + lot\ \left(\hat{\varphi}, \hat{\psi}\right). \tag{3.78}$$

Next, we need to estimate the thermal component $\hat{\psi}$.

We multiply the heat equation in problem (3.13) by $\hat{\psi}$ and integrate in time and space to obtain

$$\int_0^T \int_\Omega \left\langle \beta(\xi\hat{\psi} + \hat{\psi}_t) + \eta\Delta\hat{\psi} - \sigma\hat{\psi} - \alpha\Delta(\xi\hat{\varphi} + \hat{\varphi}_t), \hat{\psi} \right\rangle \mathrm{d}g\mathrm{d}t = 0. \tag{3.79}$$

Applying Green's formula with the boundary conditions of problem (3.13) and relation (3.1), we find

$$\int_0^T \int_\Omega \left\langle \eta\Delta\hat{\psi} - \sigma\hat{\psi}, \hat{\psi} \right\rangle \mathrm{d}g\mathrm{d}t = -\eta \int_0^T \left\| A_R^{\frac{1}{2}} \hat{\psi} \right\|^2_{L^2(\Omega)} \mathrm{d}t. \tag{3.80}$$

As a result of (3.79) and (3.80), we further use Green's formula with the boundary conditions of problem (3.13), Cauchy-Schwarz inequality (with $\tilde{\epsilon}$) and trace theorem to obtain

$$\begin{aligned} \eta \int_0^T \left\| A_R^{\frac{1}{2}} \hat{\psi} \right\|^2_{L^2(\Omega)} \mathrm{d}t = & \left[\frac{\beta}{2} \left\| e^{-\xi t} \psi(t) \right\|^2_{L^2(\Omega)} \right]_{t=0}^{t=T} + \xi \int_0^T (\beta\hat{\psi} - \alpha\Delta\hat{\varphi}, \hat{\psi})_{L^2(\Omega)} \mathrm{d}t \\ & + \alpha \int_0^T \left[(D\hat{\psi}, D\hat{\varphi}_t)_{T^1(\Omega)} - \left(\hat{\psi}, \frac{\partial \hat{\varphi}_t}{\partial \nu} \right)_{L^2(\Gamma_1)} \right] \mathrm{d}t \\ & \leq C_{\tilde{\epsilon}} \left(\int_0^T |||D\hat{\varphi}_t|||^2_{L^2(\Gamma_1)} \mathrm{d}t + \int_0^T E_{\hat{\varphi}}(t)\mathrm{d}t \right) + \tilde{\epsilon} \int_0^T \left\| A_R^{\frac{1}{2}} \hat{\psi} \right\|^2_{L^2(\Omega)} \mathrm{d}t + lot\ \left(\hat{\varphi}, \hat{\psi}\right). \end{aligned} \tag{3.81}$$

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Then for $\tilde{\epsilon} > 0$ small enough, we have

$$\int_0^T \left\| A_R^{\frac{1}{2}} \hat{\psi} \right\|^2_{L^2(\Omega)} \mathrm{d}t \le C_1 \left(\int_0^T |||D\hat{\varphi}_t|||^2_{L^2(\Gamma_1)} \mathrm{d}t + \int_0^T E_{\hat{\varphi}}(t) \mathrm{d}t \right) + lot\ \left(\hat{\varphi}, \hat{\psi} \right), \tag{3.82}$$

where $C_1 = \frac{C_{\tilde{\epsilon}}}{\eta - \tilde{\epsilon}}$.

Combining (3.78) with (3.82), we obtain

$$E_{\hat{\varphi}}(T) + \int_0^T E_{\hat{\varphi}}(t) \mathrm{d}t \le \epsilon \tilde{C}_T C_1 \int_0^T E_{\hat{\varphi}}(t) \mathrm{d}t + C_{\epsilon,T} \int_0^T |||D\hat{\varphi}_t|||^2_{L^2(\Gamma_1)} \mathrm{d}t + lot\ \left(\hat{\varphi}, \hat{\psi} \right). \tag{3.83}$$

Then for $\epsilon > 0$ small enough, we have

$$E_{\hat{\varphi}}(T) + \int_0^T E_{\hat{\varphi}}(t) \mathrm{d}t \le C_{\epsilon,T} \int_0^T |||D\hat{\varphi}_t|||^2_{L^2(\Gamma_1)} \mathrm{d}t + lot\ \left(\hat{\varphi}, \hat{\psi} \right). \tag{3.84}$$

Finally, combining (3.82) and (3.84) with $1 - C_1 > 0$, we obtain the desired estimate (3.66). □

We are now in a position to prove the observability result (2.5).

Proof of Theorem 1.2. Since $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ and Γ is smooth, $D(\mathscr{B})$ is dense in the graph topology of $D(P^*)$. Accordingly, it suffices to show inequality (2.5) for solutions $(\varphi, \varphi_t, \psi)$ to problem (2.1) with terminal data in $D(\mathscr{B}^2)$.

Using relation (3.12), Lemma 3.8 and the trace theorem, we have

$$\begin{aligned}
&E_\varphi(T) + \int_0^T E_\varphi(t) \mathrm{d}t + \int_0^T \left\| A_R^{\frac{1}{2}} \psi \right\|^2_{L^2(\Omega)} \mathrm{d}t \\
=&\frac{1}{2} \left[B\left(e^{\xi T} \hat{\varphi}(T), e^{\xi T} \hat{\varphi}(T) \right) + \left\| e^{\xi T} \hat{\varphi}_t(T) + \xi e^{\xi T} \hat{\varphi}(T) \right\|^2_{H^1_{\Gamma_{0,\gamma}}(\Omega)} \right] \\
&+ \int_0^T \frac{1}{2} \left[B\left(e^{\xi t} \hat{\varphi}, e^{\xi t} \hat{\varphi} \right) + \left\| e^{\xi t} \hat{\varphi}_t + \xi e^{\xi t} \hat{\varphi} \right\|^2_{H^1_{\Gamma_{0,\gamma}}(\Omega)} \right] + \left\| A_R^{\frac{1}{2}} e^{\xi t} \hat{\psi} \right\|^2_{L^2(\Omega)} \mathrm{d}t \\
\le& C_T \int_0^T \left\| \left| D\left(e^{-\xi t} \varphi_t - \xi e^{-\xi t} \varphi \right) \right| \right\|^2_{L^2(\Gamma_1)} \mathrm{d}t + lot\ \left(e^{-\xi t} \varphi, e^{-\xi t} \psi \right) \\
\le& C_T \int_0^T |||D\varphi_t|||^2_{L^2(\Gamma_1)} \mathrm{d}t + lot\ (\varphi, \psi).
\end{aligned} \tag{3.85}$$

By virtue of Lemma 3.9 below, the lower terms in the above inequality can be absorbed. Therefore, we further obtain

$$E_\varphi(T) + \int_0^T E_\varphi(t) \mathrm{d}t + \int_0^T \left\| A_R^{\frac{1}{2}} \psi \right\|^2_{L^2(\Omega)} \mathrm{d}t \le C_T \left[\int_0^T |||D\varphi_t|||^2_{L^2(\Gamma_1)} \mathrm{d}t + \|\psi\|^2_{[H^s((0,T)\times\Gamma_2)]'} \right], \tag{3.86}$$

which immediately implies the observability inequality (2.5). Accordingly, we have completed the proof of Theorem 1.2. □

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Lemma 3.9. *For $(\varphi_0, \varphi_1) \in D(P^*\Pi^*)$ and $T > T^*$, there exists a $C_T > 0$ such that the following estimate holds true for the corresponding solution $(\varphi, \varphi_t, \psi)$ to system (2.1):*

$$\begin{aligned}&\|\varphi\|^2_{L^\infty\left(0,T;H^{\frac{3}{2}+a}(\Omega)\right)} + \|\varphi_t\|^2_{L^\infty\left(0,T;H^{\frac{1}{2}+a}(\Omega)\right)} + \|\psi\|^2_{L^\infty\left(0,T;H^{-\frac{1}{2}+a}(\Omega)\right)} + \|\psi\|^2_{L^2\left(0,T;H^{\frac{1}{2}+a}(\Omega)\right)}\\ &\le C_T\left(\int_0^T |||D\varphi_t|||^2_{L^2(\Gamma_1)}\mathrm{d}t + \|\psi\|^2_{[H^s((0,T)\times\Gamma_2)]'}\right).\end{aligned} \tag{3.87}$$

Proof. We assume that inequality (3.87) does not hold. Then there exists a sequence $\left\{\left(\varphi_0^{(n)}, \varphi_1^{(n)}\right)\right\}_{n=1}^\infty \subseteq D(P^*\Pi^*)$ such that the corresponding solutions $\left\{\left(\varphi^{(n)}, \varphi_t^{(n)}, \psi^{(n)}\right)\right\}_{n=1}^\infty$ to system (2.1) satisfy

$$\|\varphi^{(n)}\|^2_{L^\infty\left(0,T;H^{\frac{3}{2}+a}(\Omega)\right)} + \|\varphi_t^{(n)}\|^2_{L^\infty\left(0,T;H^{\frac{1}{2}+a}(\Omega)\right)} + \|\psi^{(n)}\|^2_{L^\infty\left(0,T;H^{-\frac{1}{2}+a}(\Omega)\right)} + \int_0^T \|\psi^{(n)}\|^2_{H^{\frac{1}{2}+a}(\Omega)}\mathrm{d}t = 1 \ \ \forall n, \tag{3.88}$$

$$\lim_{n\to\infty}\int_0^T |||D\varphi_t^{(n)}|||^2_{L^2(\Gamma_1)}\mathrm{d}t + \|\psi^{(n)}\|^2_{[H^s((0,T)\times\Gamma_2)]'} = 0. \tag{3.89}$$

Then relations (3.86), (3.88) and (3.89) imply that the sequence

$$\left\{\int_0^T \left\|(\varphi_0^{(n)}, \varphi_1^{(n)})\right\|^2_{H^2_{\Gamma_0}(\Omega)\times H^1_{\Gamma_{0,\gamma}}(\Omega)}\mathrm{d}t + \int_0^T \left[\left\|\left(\varphi^{(n)}(t), \varphi_t^{(n)}(t)\right)\right\|^2_{H^2_{\Gamma_0}(\Omega)\times H^1_{\Gamma_{0,\gamma}}(\Omega)} + \left\|A_R^{\frac{1}{2}}\psi^{(n)}(t)\right\|^2_{L^2(\Gamma)}\right]\mathrm{d}t\right\}_{n=1}^\infty \tag{3.90}$$

is bounded. Thus, there exists a subsequence, still denoted as $\left\{\left(\varphi_0^{(n)}, \varphi_1^{(n)}\right)\right\}_{n=1}^\infty$, and $(\tilde{\varphi}_0, \tilde{\varphi}_1) \in H^2_{\Gamma_0}(\Omega) \times H^1_{\Gamma_{0,\gamma}}(\Omega)$, such that

$$\varphi_0^{(n)} \rightharpoonup \tilde{\varphi}_0 \text{ in } H^2_{\Gamma_0}(\Omega), \qquad \varphi_1^{(n)} \rightharpoonup \tilde{\varphi}_1 \text{ in } H^1_{\Gamma_{0,\gamma}}(\Omega)\,. \tag{3.91}$$

Further, let $\left(\tilde{\varphi}, \tilde{\varphi}_t, \tilde{\psi}\right)$ be the solution to system (2.1) with initial data $(\tilde{\varphi}_0, \tilde{\varphi}_1, 0)$. Then

$$\left(\varphi^{(n)}, \varphi_t^{(n)}, \psi^{(n)}\right) \to (\tilde{\varphi}, \tilde{\varphi}_t, \tilde{\psi}) \quad \text{in } L^\infty(0,T;\mathcal{H}_\gamma) \text{ weak star.} \tag{3.92}$$

Besides, $\left\{\varphi_{tt}^{(n)}\right\}_{n=1}^\infty$ is bounded in $L^\infty\left(0,T;\left[D\left(A^{\frac{1}{2}}P_\gamma^{-1}\right)\right]'\right)$, where $D\left(A^{\frac{1}{2}}P_\gamma^{-1}\right) = \{\varphi \in L^2(\Omega) : P_\gamma^{-1}\varphi \in D(A^{\frac{1}{2}}) \cap D(A_M)\}$. The reason is as follows: For terminal data $\left(\varphi_0^{(n)}, \varphi_1^{(n)}, 0\right) \in D(\mathscr{B})$, we have

$$\varphi_{tt}^{(n)} - \gamma\Delta\varphi_{tt}^{(n)} + A\varphi^{(n)} - \gamma AG_2\gamma_1\varphi_{tt}^{(n)} + \alpha AG_1\gamma_0\psi^{(n)} - \alpha\lambda AG_2\gamma_0\psi^{(n)} + \alpha\Delta\psi^{(n)} = 0, \tag{3.93}$$

$$\beta\psi_t^{(n)} + \eta\Delta\psi^{(n)} - \sigma\psi^{(n)} - \alpha\Delta\varphi_t^{(n)} = 0. \tag{3.94}$$

Then for all $w \in L^1\left(0,T;D\left(A^{\frac{1}{2}}P_\gamma^{-1}\right)\right)$, by (3.93), (3.5) and (3.6), Green's formula and Cauchy-Schwarz inequality with (3.90), we have

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$$\begin{aligned}\int_0^T \left(w, \varphi_{tt}^{(n)}\right)_{L^2(\Omega)} \mathrm{d}t &= \int_0^T \left(w, P_\gamma^{-1}\left[-A\varphi^{(n)} - \alpha\Delta\psi^{(n)} - \alpha AG_1\gamma_0\psi^{(n)} + \alpha\lambda AG_2\gamma_0\psi^{(n)}\right]\right)_{L^2(\Omega)} \mathrm{d}t \\ &= \int_0^T \Big[-\left(A^{\frac{1}{2}}P_\gamma^{-1}w, A^{\frac{1}{2}}\varphi^{(n)}\right)_{L^2(\Omega)} - \alpha\left(P_\gamma^{-1}w, \Delta\psi^{(n)}\right)_{L^2(\Omega)} \\ &\quad -\alpha\left(G_1^*AP_\gamma^{-1}w, \gamma_0\psi^{(n)}\right)_{L^2(\Gamma)} + \alpha\left(G_2^*AP_\gamma^{-1}w, \lambda\gamma_0\psi^{(n)}\right)_{L^2(\Gamma)}\Big]\mathrm{d}t \\ &= \int_0^T \left[-\left(A^{\frac{1}{2}}P_\gamma^{-1}w, A^{\frac{1}{2}}\varphi^{(n)}\right)_{L^2(\Omega)} - \alpha\left(\Delta P_\gamma^{-1}w, \psi^{(n)}\right)_{L^2(\Omega)}\right]\mathrm{d}t \\ &\le C\int_0^T \|w\|_{D\left(A^{\frac{1}{2}}P_\gamma^{-1}\right)}\left(\left\|A^{\frac{1}{2}}\varphi^{(n)}\right\|_{L^2(\Omega)} + \|\psi^{(n)}\|_{L^2(\Omega)}\right)\mathrm{d}t \le C\|w\|_{L^1\left(0,T;D\left(A^{\frac{1}{2}}P_\gamma^{-1}\right)\right)}.\end{aligned} \tag{3.95}$$

Then, we obtain the boundedness of $\varphi_{tt}^{(n)}$ after a standard density argument.

Similarly, for all $w \in L^2(0,T;H^{\frac{3}{2}-a}(\Omega))$, using (3.94), Green's formula, (3.1), Cauchy-Schwarz inequality, trace theorem and Sobolev embedding theorem, we have

$$\begin{aligned}\|\psi_t^{(n)}\|^2_{L^2(0,T;H^{-\frac{3}{2}+a}(\Omega))} &= \sup_w \frac{\int_0^T (w,\psi_t^{(n)})^2_{L^2(\Omega)}\mathrm{d}t}{\|w\|^2_{L^2(0,T;H^{\frac{3}{2}-a}(\Omega))}} = \sup_w \frac{\int_0^T \left(w, \eta A_R\psi^{(n)} + \alpha\Delta\varphi_t^{(n)}\right)^2_{L^2(\Omega)}\mathrm{d}t}{\beta^2\|w\|^2_{L^2(0,T;H^{\frac{3}{2}-a}(\Omega))}} \\ =&\sup_w \left\{\left(\beta^2\|w\|^2_{L^2(0,T;H^{\frac{3}{2}-a}(\Omega))}\right)^{-1}\int_0^T \Big[\eta\left(A_R^{\frac{3}{4}-\frac{a}{2}}w, A_R^{\frac{1}{4}+\frac{a}{2}}\psi^{(n)}\right)_{L^2(\Omega)} - \alpha\left(A_R^{\frac{3}{4}-\frac{a}{2}}w, A_R^{\frac{1}{4}+\frac{a}{2}}\varphi_t^{(n)}\right)_{L^2(\Omega)} \\ &+\alpha\lambda\left(w,\varphi_t^{(n)}\right)_{L^2(\Gamma)} + \frac{\alpha\sigma}{\eta}\left(w,\varphi_t^{(n)}\right)_{L^2(\Omega)} + \alpha\left(w, \frac{\partial\varphi_t^{(n)}}{\partial\nu}\right)_{L^2(\Gamma_1)}\Big]^2\mathrm{d}t\right\} \\ \le& \int_0^T \left(\|\psi^{(n)}\|^2_{H^{\frac{1}{2}+a}(\Omega)} + \|\varphi_t^{(n)}\|^2_{H^{\frac{1}{2}+a}(\Omega)} + \|\varphi_t^{(n)}\|^2_{H^{\frac{1}{2}}(\Omega)} + \|\varphi_t^{(n)}\|^2_{L^2(\Omega)} + |||D\varphi_t^{(n)}|||^2_{L^2(\Gamma_1)}\right)\mathrm{d}t \\ \le& lot\ \left(\varphi^{(n)},\psi^{(n)}\right) + |||D\varphi_t^{(n)}|||^2_{L^2(0,T;L^2(\Gamma_1))}.\end{aligned} \tag{3.96}$$

By (3.88), (3.89) and a standard density argument, we obtain that $\left\{\psi_t^{(n)}\right\}_{n=1}^{\infty}$ is bounded in $L^2(0,T;H^{-\frac{3}{2}+a}(\Omega))$.

Then, by Aubin's compactness Theorem in [1], we derive

$$\varphi^{(n)} \to \tilde{\varphi} \text{ strongly in } L^\infty(0,T;H^{\frac{3}{2}+a}(\Omega)),\quad \varphi_t^{(n)} \to \tilde{\varphi}_t \text{ strongly in } L^\infty(0,T;H^{\frac{1}{2}+a}(\Omega)),$$

$$\psi^{(n)} \to \tilde{\psi} \text{ strongly in } L^\infty(0,T;H^{-\frac{1}{2}+a}(\Omega)),\quad \psi^{(n)} \to \tilde{\psi} \text{ strongly in } L^2(0,T;H^{\frac{1}{2}+a}(\Omega)).$$

Combining these convergences with (3.88), we obtain

$$\|\tilde{\varphi}\|^2_{L^\infty(0,T;H^{\frac{3}{2}+a}(\Omega))} + \|\tilde{\varphi}_t\|^2_{L^\infty(0,T;H^{\frac{1}{2}+a}(\Omega))} + \|\tilde{\psi}\|^2_{L^\infty(0,T;H^{-\frac{1}{2}+a}(\Omega))} + \int_0^T \|\tilde{\varphi}\|^2_{H^{\frac{1}{2}+a}(\Omega)}\mathrm{d}t = 1. \tag{3.97}$$

Furthermore, according to (2.2), (3.89) and (3.91), we have

$$\int_0^T |||D\tilde{\varphi}_t|||^2_{L^2(\Gamma_1)}\mathrm{d}t + \|\tilde{\psi}\|^2_{[H^s((0,T)\times\Gamma_2)]'} = 0. \tag{3.98}$$

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Next, let $y = \tilde{\varphi}_t,\ z = \tilde{\psi}_t$. Then (y, z) solve the following problem

$$\begin{cases} \begin{cases} y_{tt} - \gamma\Delta y_{tt} + \mathscr{A}y + \alpha\Delta z = 0 \\ \beta z_t + \eta\Delta z - \sigma z - \alpha\Delta y_t = 0 \end{cases} & \text{on } (0,T)\times\Omega, \\ y = \dfrac{\partial y}{\partial \nu} = 0 \qquad \text{on } (0,T)\times\Gamma_0, \\ \begin{cases} \Delta y + (1-\mu)B_1 y + \alpha z = 0 \\ \dfrac{\partial \Delta y}{\partial \nu} + (1-\mu)B_2 y - \gamma\dfrac{\partial y_{tt}}{\partial \nu} + \alpha\dfrac{\partial z}{\partial \nu} = 0 \end{cases} & \text{on } (0,T)\times\Gamma_1, \\ \dfrac{\partial z}{\partial \nu} + \lambda z = 0 \qquad \text{on } (0,T)\times\Gamma, \\ z = 0 \qquad\qquad \text{on } \Gamma_2. \end{cases} \tag{3.99}$$

Using the uniqueness theorem of Isakov in [16], we have for $T > 2\sqrt{\gamma}\sup\limits_{x\in\Omega}\rho(x,\Gamma_2)$ that the solution (y, z) to problem (3.99) is zero, which implies both $\tilde{\varphi}$ and $\tilde{\psi}$ are constants. Further, by the boundary condition of system (2.1) on Γ_0, we obtain that $\tilde{\varphi} = 0$ in $(0,T)\times\Omega$. Moreover, the free boundary conditions of system (2.1) on Γ_1 give that $\tilde{\psi} = 0$ *in* $(0,T)\times\Omega$. Consequently, $(\tilde{\varphi}, \tilde{\psi}) = (0, 0)$, it contradicts (3.97). This concludes the proof of Lemma 3.9. □

4. Proof of Theorem 1.1

We proceed in four steps.

Step 1. We assert that the original system (1.1) is approximately controllable with the boundary controls $(u_1, u_2, u_3) \in \mathcal{U}_{r+1}$ for $T > T^*$. Indeed, given any $(\varphi_0, \varphi_1, \psi_0) \in ker(P^*)$, we obtain $\frac{\partial\varphi_t}{\partial\nu}\big|_{\Gamma_1} = \varphi_t|_{\Gamma_1} = \psi|_{\Gamma_2} = 0$, where $(\varphi, \varphi_t, \psi)$ is the corresponding solution to system (2.1). Similarly, by the uniqueness theorem of Isakov in [16], we have $(\varphi, \varphi_t, \psi) = (0, 0, 0)$ in $(0,T)\times\Omega$. Then necessarily $(\varphi_0, \varphi_1, \psi_0) = (0, 0, 0)$.

Step 2. Based on (2.5), we have obtained the exact controllability of the mechanical component of the original system (1.1)(i.e., without the thermal effects). Then proceeding as in Appendix B of [21], we obtain that

$$\Pi PP^*\Pi^* \text{ is an isomorphism from } D(P^*\Pi^*) \text{ into } [D(P^*\Pi^*)]'.$$

Therefore, $\Pi PP^*\Pi^*(\Pi PP^*\Pi^*)^{-1}\Pi \in \mathscr{L}\left(\mathcal{H}_\gamma, H^2_{\Gamma_0}(\Omega)\times H^1_{\Gamma_{0,\gamma}}(\Omega)\right)$. What's more, $(\mathbb{I} - \Pi^*\Pi)PP^*\Pi^*(\Pi PP^*\Pi^*)^{-1}\Pi \in \mathscr{L}(\mathcal{H}_\gamma)$, where $\mathbb{I} : \mathcal{H}_\gamma \to \mathcal{H}_\gamma$ is the identity mapping.

For any $\epsilon > 0$ and terminal data $\left(w_0^T, w_1^T, \theta_0^T\right) \in \mathcal{H}_\gamma$, because the entire system is approximately controllable in $\mathcal{U}_{r+1}$, we can select a control triple $\bar{u}_1 = \left(u_1^{(1)}, u_2^{(1)}, u_3^{(1)}\right) \in D(P) \subset \mathcal{U}_{r+1}$, such that

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the corresponding solution $\left(w^{(1)}, w_t^{(1)}, \theta^{(1)}\right)$ to system (1.1) satisfies

$$\begin{aligned}&\left\|\left(w^{(1)}(T)-w_0^T, w_t^{(1)}(T)-w_1^T, \beta\theta^{(1)}(T)-\beta\theta_0^T\right)\right\|_{\mathcal{H}_\gamma}\\ <&\frac{\epsilon}{1+\|(\mathbb{I}-\Pi^*\Pi)PP^*\Pi^*(\Pi PP^*\Pi^*)^{-1}\Pi\|_{\mathscr{L}(\mathcal{H}_\gamma)}}.\end{aligned} \tag{4.1}$$

Step 3. Since the mechanical component of the original system (1.1) is exactly controllable, we can select a control triple $\bar{u}_2 = \left(u_1^{(2)}, u_2^{(2)}, u_3^{(2)}\right) \in D(P) \subset \mathcal{U}_{r+1}$ to be the minimal norm steering control with respect to the partial terminal data $\left(w_0^T - w^{(1)}(T),\ \ w_1^T - w_t^{(1)}(T)\right)$, namely, $\bar{u}_2$ satisfies

$$\Pi P\bar{u}_2 = \left(w_0^T - w^{(1)}(T), w_1^T - w_t^{(1)}(T)\right) \quad \text{and} \quad \|\bar{u}_2\|^2_{\mathcal{U}_{r+1}} = \min_{\bar{u}\in\mathcal{U}_{r+1}} \|\bar{u}\|^2_{\mathcal{U}_{r+1}},$$

where $\bar{u}$ steer $(0,0)$ to $\left(w_0^T - w^{(1)}(T), w_1^T - w_t^{(1)}(T)\right)$.

By convex optimization theory, the minimizer $\bar{u}_2$ can be given explicitly as

$$\bar{u}_2 = P^*\Pi^*(\Pi PP^*\Pi^*)^{-1}\Pi\left(w_0^T - w^{(1)}(T), w_1^T - w_t^{(1)}(T), \beta\theta_0^T - \beta\theta^{(1)}(T)\right). \tag{4.2}$$

Combining it with inequality (4.1) gives the following estimate

$$\|(\mathbb{I}-\Pi^*\Pi)P\bar{u}_2\|_{\mathcal{H}_\gamma} < \frac{\|(\mathbb{I}-\Pi^*\Pi)PP^*\Pi^*(\Pi PP^*\Pi^*)^{-1}\Pi\|_{\mathscr{L}(\mathcal{H}_\gamma)}\,\epsilon}{1+\|(\mathbb{I}-\Pi^*\Pi)PP^*\Pi^*(\Pi PP^*\Pi^*)^{-1}\Pi\|_{\mathscr{L}(\mathcal{H}_\gamma)}}. \tag{4.3}$$

Step 4. Set the control $u^* = \bar{u}_1 + \bar{u}_2$, then

$$\begin{aligned}Pu^* &= P\bar{u}_1 + P\bar{u}_2 = \left(w^{(1)}(T), w_t^{(1)}(T), \beta\theta^{(1)}(T)\right) + \left(w_0^T - w^{(1)}(T), w_1^T - w_t^{(1)}(T), 0\right) + (\mathbb{I}-\Pi^*\Pi)P\bar{u}_2\\ &= \left(w_0^T, w_1^T, \beta\theta^{(1)}(T)\right) + (\mathbb{I}-\Pi^*\Pi)P\bar{u}_2.\end{aligned} \tag{4.4}$$

Let (w^*, w_t^*, θ^*) be the solution to system (1.1) with the chosen control u^*, then

$$(w^*(T), w_t^*(T)) = \left(w_0^T, w_1^T\right); \tag{4.5}$$

$$\begin{aligned}\|\beta\theta^*(T) - \beta\theta_0^T\|_{L^2(\Omega)} &\le \left\|Pu^* - \left(w_0^T, w_1^T, \beta\theta_0^T\right)\right\|_{\mathcal{H}_\gamma}\\ &\le \left\|\left(0,0,\beta\theta^{(1)}(T) - \beta\theta_0^T\right)\right\|_{\mathcal{H}_\gamma} + \|(\mathbb{I}-\Pi^*\Pi)P\bar{u}_2\|_{\mathcal{H}_\gamma}\\ &< \frac{\epsilon}{1+\|(\mathbb{I}-\Pi^*\Pi)PP^*\Pi^*(\Pi PP^*\Pi^*)^{-1}\Pi\|_{\mathscr{L}(\mathcal{H}_\gamma)}} + \|(\mathbb{I}-\Pi^*\Pi)P\bar{u}_2\|_{\mathcal{H}_\gamma}\\ &< \epsilon.\end{aligned} \tag{4.6}$$

Consequently, the constructed control triple $u^* = \left(u_1^*, u_2^*, u_3^*\right) \in D(P) \subset \mathcal{U}_{r+1}$ satisfies the desired exact-approximate controllability property (1.3). Further, $u_3^* \in C^r(\Sigma_{2,T})$ owing to the Sobolev embedding theorem. This concludes the proof of Theorem 1.1. □

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5. Concluding Remarks

Usually, mechanics researchers classify a *flat* two-dimensional structure as plate while a *curved* structure as shell. Here in our work, we allow the plate to be curved. Thus, the model considered here may be viewed either as a *transition* model (from plate to shell), or a *hybrid* plate/shell model. The thermoelastic curved plate model and its generalizations are a potential candidate for the tectonic plate model for the earth's crust. The control theoretic results in this paper can have potential applications for the analysis and data assimilation of earthquakes, for example. Such applications will be ultimate objectives of our research in the future.

Appendix: Derivation of the Plate Model (1.1)

In this appendix we mainly show how to use the Principle of Virtual Work to derive the equations of motion for the vertical displacement w of the specific model of the plate under study in this paper. Let a plate have a curved middle surface Ω, which is a part of a surface in R^3. Then the curved plate is given by

$$\{\, x + t\,\vec{n} \,|\, x \in \Omega,\ \ t \in (-h/2, h/2)\,\},$$

where $\vec{n}$ is the normal of Ω and $h > 0$ is the thickness of the plate along its normal. We assume that the external deformation along the normal of the middle surface is small, i.e., the strain tensor of the middle surface satisfies $\Psi(\zeta) = 0$. Then the strain energy of the bending of the plate is ([17])

$$V = \frac{\mathsf{D}}{2}\int_\Omega a(w,w) + (1+\mu)\theta\Delta w \mathrm{d}x,$$

where $\mathsf{D} = \frac{Eh^3}{12(1-\mu^2)}$ is called the modulus of flexural rigidity, E and μ denote Young's modulus and Poisson's ratio, respectively. The kinetic energy of the plate is assumed to be

$$K = \frac{\rho h}{2}\int_\Omega \left(w_t^2 + \gamma |Dw_t|^2\right)\mathrm{d}x.$$

where $\gamma = \frac{h^2}{12}$, ρ denotes mass density.

Then, the Lagrangian for the plate $L = \int_0^T (K - V)\mathrm{d}t$. And the usual calculus of variations (Principle of Virtual Work) for a stationary point of L yields

$$\begin{cases} \rho h w_{tt} - \rho h\gamma \Delta w_{tt} + \mathsf{D}\left[\Delta^2 w + (1-\mu)\,\mathrm{div}\,(kDw) + \dfrac{1+\mu}{2}\Delta\theta\right] = 0 & \text{on } (0,T)\times\Omega, \\ w = \dfrac{\partial w}{\partial \nu} = 0 & \text{on } (0,T)\times\Gamma_0, \\ \begin{cases} \Delta w + (1-\mu)B_1 w + \dfrac{1+\mu}{2}\theta = 0 \\ \mathsf{D}\left[\dfrac{\partial \Delta w}{\partial \nu} + (1-\mu)B_2 w + \dfrac{1+\mu}{2}\dfrac{\partial \theta}{\partial \nu}\right] - \rho h\gamma \dfrac{\partial w_{tt}}{\partial \nu} = 0 \end{cases} & \text{on } (0,T)\times\Gamma_1, \\ w(0) = w_0, w_t(0) = w_1 & \text{on } \Omega. \end{cases} \tag{.1}$$

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As in [17], by laws in thermodynamics (primarily the Fourier conduction law), we derive the following equation for the temperature θ of the plate:

$$\begin{cases} \frac{1}{\kappa}\theta_t - \Delta\theta + \frac{12}{h^2}\left(1 + \frac{h\lambda_1}{2}\right)\theta - \mathbb{Q}\mathbb{n}\Delta w_t = 0 & \text{on } (0,T)\times\Omega, \\ \frac{\partial\theta}{\partial\nu} + \lambda_2\theta = 0 & \text{on } (0,T)\times\Gamma, \\ \theta(0) = \theta_0 & \text{on } \Omega. \end{cases} \tag{.2}$$

where $\kappa > 0, \lambda_1 \geq 0, \lambda_2 \geq 0, \mathbb{Q} > 0$, and $\mathbb{n} > 0$ are various independent thermal parameters.

Combining (.1) with (.2) and making the change $t \to t\sqrt{\frac{\mathsf{D}}{\rho h}}$ in the time scale, we derive the bending model described as (1.1).

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