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Gradient estimates via the Wolff potentials for a class of quasilinear elliptic equations

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In this paper we obtain the pointwise gradient estimates via the nonlinear Wolff potentials for weak solutions of a class of non-homogeneous quasilinear elliptic equations with measure data.

1. Introduction

In this paper we are mainly concerned with the pointwise gradient estimates for weak solutions of a class of non-homogeneous quasilinear elliptic equations with measure data

$$\operatorname{div}\left(a\left(|Du|\right)Du\right)=\mu \quad \text{in } \Omega, \tag{1.1}$$

where μ is a Borel measure with finite mass and $a:(0,\infty)\to(0,\infty)\in C^1(0,\infty)$ satisfies

$$0\le i_a=:\inf_{t>0}\frac{ta'(t)}{a(t)}\le\sup_{t>0}\frac{ta'(t)}{a(t)}=:s_a<\infty. \tag{1.2}$$

We define

$$g(t)=ta(t) \tag{1.3}$$

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and

$$G(t) = \int_0^t g(\tau)\, d\tau = \int_0^t \tau a(\tau)\, d\tau \quad \text{for } t \geq 0. \tag{1.4}$$

From (1.2) we observe that

$$g(t) \text{ is strictly increasing and continuous over } [0, +\infty), \tag{1.5}$$

and then

$$G(t) \text{ is increasing over } [0, +\infty) \text{ and strictly convex with } G(0) = 0. \tag{1.6}$$

For example we see that

$$G(t) = t^p \quad \text{and} \quad G(t) = t^p \log\left(1 + t\right) \quad \text{for any } p \geq 2 \tag{1.7}$$

satisfy the condition (1.2). Especially when $a(t) = t^{p-2}$ (and then $G(t) = t^p/p$), (1.1) is reduced to the p-Laplacian equation

$$\operatorname{div}\left(|Du|^{p-2} Du\right) = \mu \quad \text{for } p \geq 2.$$

Many authors [7–9,13,14,18,19,22,28,30,32] have extensively studied regularity estimates for p-Laplacian elliptic equation and the general case. Moreover, Cianchi and Maz'ya [10,11] proved global Lipschitz regularity for the Dirichlet and Neumann elliptic boundary value problems of

$$\operatorname{div}\left(a\left(|Du|\right) Du\right) = f \tag{1.8}$$

with the condition (1.2). Furthermore, Cianchi and Maz'ya [12] obtained a sharp estimate for the decreasing rearrangement of the length of the gradient for the Dirichlet and Neumann elliptic boundary value problems of (1.8) with (1.2) and

$$Ct^{p-1} \leq ta(t) \leq C(t^{p-1} + 1) \quad \text{for any } t > 0 \text{ and } p \in [2, n).$$

The pointwise estimates of the weak solution u via the Wolff potential $W^{\mu}_{\beta,p}(x, R)$ for nonlinear elliptic equations with right-hand side measure are developed by [20,21,31]. We recall that the classical non-linear Wolff potential is defined by

$$W^{\mu}_{\beta,p}(x, R) := \int_0^R \left(\frac{|\mu|\left(B(x,\varrho)\right)}{\varrho^{n-\beta p}}\right)^{\frac{1}{p-1}} \frac{d\varrho}{\varrho} \quad \text{for } \beta \in \left(0, \frac{n}{p}\right].$$

Furthermore, the classical case of general p-Laplacian equations (even with coefficients $a(x, \cdot)$ depending on the space variable x) was treated before by Mingione [29] in the case that $p = 2$, and Duzaar and Mingione [15,17] in the case that $p \neq 2$. In the singular subquadratic case that $p < 2$ even a linear gradient potential estimate was established by Duzaar and Mingione [16], i.e. the nonlinear Wolff-potential is replaced by $I_1^{|\mu|}(x, 2R)^{\frac{1}{p-1}}$, where

$$I_1^{|\mu|}(x, R) = \int_0^R \frac{|\mu|(B(x,\varrho))}{\varrho^{n-1}} \frac{d\varrho}{\varrho} \quad \text{and} \quad |\mu|(B(x,\varrho)) = \int_{B(x,\varrho)} |\mu| dy.$$

Later, Korte, Kuusi and Mingione [23,26] extended these results in different directions. Recently, Baroni [2] proved the following pointwise gradient estimates via the linear Riesz potential $I_1^{|\mu|}(x,2R)$ for weak solution of (1.1)

$$g(|Du(x)|) \leq Cg\left(\fint_{B(x,R)} |Du|dx \right) + CI_1^{|\mu|}(x,2R).$$

Furthermore, based on the original Duzaar and Mingione's paper [15], many authors [3–6,24,25,27] have studied a number of results concerning regularity theory for degenerate parabolic equations with measure data. The main challenge here is the inhomogeneous nature of the nonlinear parabolic equations, which leads to the lack of scaling. We mention here that, because of the approach we will use, it appears possible to extend this present result to the parabolic case. The implementation of this project will be our next one in the near future.

Definition 1.1. A function $B : [0,+\infty) \to [0,+\infty)$ is said to be a Young function if it is convex and $B(0) = 0$. Moreover, a Young function B is called an N-function if $0 < B(t) < \infty$ for $t > 0$ and

$$\lim_{t\to+\infty} \frac{B(t)}{t} = \lim_{t\to 0+} \frac{t}{B(t)} = +\infty. \tag{1.9}$$

Definition 1.2. A Young function B is said to satisfy the global Δ_2 condition, denoted by $B \in \Delta_2$, if there exists a positive constant C such that for every $t > 0$,

$$B(2t) \leq CB(t). \tag{1.10}$$

Furthermore, a Young function B is said to satisfy the global ∇_2 condition, denoted by $B \in \nabla_2$, if there exists a number $\theta > 1$ such that for every $t > 0$,

$$B(t) \leq \frac{B(\theta t)}{2\theta}. \tag{1.11}$$

Lemma 1.3 *(see [10], Proposition 2.9). If $a(t)$ satisfies (1.2) and $G(t)$ is defined in (1.4), then we have $G(t) \in \Delta_2$.*

Lemma 1.4. *Assume that $a(t)$ satisfies (1.2) and $G(t)$ is defined in (1.4). Then we have*

(1) *For any $t > 0$ we find that*

$$\theta^{i_a} \leq \frac{a(\theta t)}{a(t)} \leq \theta^{s_a} \quad \text{and} \quad \theta^{2+i_a} \leq \frac{G(\theta t)}{G(t)} \leq \theta^{2+s_a} \quad \text{for any } \theta \geq 1. \tag{1.12}$$

(2) $G(t) \in \nabla_2$.

Proof. (1) From (1.2) we find that

$$\frac{i_a}{\theta t} \leq \frac{a'(\theta t)}{a(\theta t)} \leq \frac{s_a}{\theta t} \quad \text{for any } \theta, t > 0.$$

By integrating the above inequality over $[1,\theta]$ for any $\theta \geq 1$, we have

$$\int_1^\theta \frac{i_a}{\theta t} d\theta \leq \int_1^\theta \frac{a'(\theta t)}{a(\theta t)} d\theta \leq \int_1^\theta \frac{s_a}{\theta t} d\theta,$$

which implies that

$$a(t)\theta^{i_a} \le a(\theta t) \le a(t)\theta^{s_a}.$$

Moreover, (1.2) implies that

$$i_a a(t) \le t a'(t).$$

Then we have

$$i_a \int_0^t \tau a(\tau)\, d\tau \le \int_0^t \tau^2 a'(\tau)\, d\tau \le t^2 a(t) - 2\int_0^t \tau a(\tau)\, d\tau,$$

which implies that

$$\frac{tG'(t)}{G(t)} = \frac{t^2 a(t)}{\int_0^t \tau a(\tau)\, d\tau} \ge 2 + i_a > 1.$$

Therefore, we obtain

$$(\log G(t))' \ge (2 + i_a)(\log t)'.$$

By integrating the above inequality over $[t, \theta t]$ for any $t > 0$, we conclude that

$$G(\theta t) \ge \theta^{2+i_a} G(t). \tag{1.13}$$

Similarly, we have

$$G(\theta t) \le \theta^{2+s_a} G(t),$$

which implies that (1) is true.

(2) Especially when $\theta = 2^{\frac{1}{1+i_a}} > 1$, from (1.13) we have

$$G(\theta t) \ge \theta^{2+i_a} G(t) = 2\theta G(t),$$

which implies that $G \in \nabla_2$. This finishes our proof. □

Definition 1.5. The Orlicz class $K^G(\Omega)$ is the set of all measurable functions $u : \Omega \to \mathbb{R}$ satisfying

$$\int_\Omega G(|u|)\, dz < \infty.$$

The Orlicz space $L^G(\Omega)$ is the linear hull of $K^G(\Omega)$. Furthermore, we define $W^{1,G}(\Omega)$ as

$$W^{1,G}(\Omega) = \left\{ u \in L^G(\Omega) \,\middle|\, Du \in L^G(\Omega) \right\}.$$

The space $W_0^{1,G}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,G}(\Omega)$.

In general, $K^G \subset L^G$. If $G \in \Delta_2 \cap \nabla_2$, then $K^G = L^G$ and C_0^∞ is dense in L^G (see [1]). Moreover, from Lemma 1.4 we have

$$L^{2+s_a}(\Omega) \subset L^G(\Omega) \subset L^{2+i_a}(\Omega) \subset L^1(\Omega).$$

As usual, the solutions of (1.1) are taken in a weak sense. We now state the definition of weak solutions.

Definition 1.6. A function $u \in W^{1,G}_{loc}(\Omega)$ is a local weak solution of (1.1) if for any $\varphi \in W^{1,G}_0(\Omega) \cap L^\infty(\Omega)$ we have

$$\int_\Omega a\left(|Du|\right) Du \cdot D\varphi dx = \int_\Omega \varphi d\mu.$$

Now let us state the main result of this work.

Theorem 1.7. *If $u \in W^{1,G}_{loc}(\Omega)$ is a local weak solution of (1.1) with $\mu \in L^1(\Omega)$ and (1.2), then there exists a constant $C = C(n, i_a, s_a) > 1$ such that*

$$|Du(x_0)| \leq C \fint_{B(x_0,R)} |Du| dx + CW^\mu_{i_a}(x_0, 2R), \tag{1.14}$$

where $B(x_0, 2R) \subseteq \Omega$ and

$$W^\mu_{i_a}(x_0, R) := \int_0^R \left(\frac{|\mu| \left(B(x_0, \varrho)\right)}{\varrho^{n-1}} \right)^{\frac{1}{1+i_a}} \frac{d\varrho}{\varrho}.$$

Remark 1.8. From (1.3) we find that the condition (1.2) is equivalent to the condition (1.6) in [2]. Moreover, Baroni [2] posed an additional degeneracy hypothesis

$$\lim_{t\to 0^+} a(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} a(t) = \infty. \tag{1.15}$$

Actually, from (1.12) in Lemma 1.4 we can know that the condition (1.2) can cover this hypothesis (1.15). In a sense, this paper is just a comment on Baroni's paper [2]. Baroni [2] proved the following pointwise gradient estimates via the linear Riesz potential $I_1^{|\mu|}(x, 2R)$ for weak solution of (1.1)

$$g(|Du(x)|) \leq Cg\left(\fint_{B(x,R)} |Du| dx \right) + CI_1^{|\mu|}(x, 2R).$$

One of the natural examples of functions g is $g(t) = t^{p-1} \log(1 + t)$ for $p \geq 2$. In this paper we use the strategy of the original Duzaar and Mingione's paper [15]. Although we employ the non-linear Wolff type potential, we obtain the homogeneous linear pointwise gradient estimates if we focus on the first two terms.

2. Proof of the main result

In this section we will finish the proof of Theorem 1.7. We first give the following result.

Lemma 2.1. *Assume that $a(t)$ satisfies (1.2) and $G(t)$ is defined in (1.4). Then there exists $C = C(n, i_a, s_a) > 0$ we have*

$$[\xi a(|\xi|) - \eta a(|\eta|)] \cdot (\xi - \eta) \geq CG(|\xi - \eta|) \quad \textit{for any } \xi, \eta \in \mathbb{R}^n. \tag{2.1}$$

Especially, we have

$$a(|\xi|)\, \xi \cdot \xi \geq CG(|\xi|) \quad \textit{for any } \xi \in \mathbb{R}^n. \tag{2.2}$$

Proof. We first find that

$$\begin{aligned}\xi a(|\xi|) - \eta a(|\eta|) &= (\xi - \eta) \int_0^1 a(|s\xi + (1-s)\eta|)\, ds \\ &\quad + \int_0^1 \frac{a'(|s\xi + (1-s)\eta|)}{|s\xi + (1-s)\eta|} (s\xi + (1-s)\eta)\, [s\xi + (1-s)\eta] \cdot (\xi - \eta)\, ds,\end{aligned}$$

which implies that

$$\begin{aligned}&[\xi a(|\xi|) - \eta a(|\eta|)] \cdot (\xi - \eta) \\ &= |\xi - \eta|^2 \int_0^1 a(|s\xi + (1-s)\eta|)\, ds + \int_0^1 \frac{a'(|s\xi + (1-s)\eta|)}{|s\xi + (1-s)\eta|} \left|[s\xi + (1-s)\eta] \cdot (\xi - \eta)\right|^2 ds \\ &\geq |\xi - \eta|^2 \int_0^1 a(|s\xi + (1-s)\eta|)\, ds + i_a \int_0^1 a(|s\xi + (1-s)\eta|) \left| \frac{[s\xi + (1-s)\eta] \cdot (\xi - \eta)}{|s\xi + (1-s)\eta|} \right|^2 ds \\ &\geq |\xi - \eta|^2 \int_0^1 a(|s\xi + (1-s)\eta|)\, ds \end{aligned} \tag{2.3}$$

in view of (1.2). Without loss of generality we may as well assume that $|\xi| \geq |\eta| > 0$. Then we have $|s\xi + (1-s)\eta| \leq |\xi|$ for any $0 \leq s \leq 1$. Therefore, from (1.12) we find that

$$\int_0^1 a(|s\xi + (1-s)\eta|)\, ds \geq \int_{\frac{3}{4}}^1 a(|s\xi + (1-s)\eta|)\, ds \geq \int_{\frac{3}{4}}^1 \frac{1}{2} a\left(\frac{1}{2}|\xi|\right) ds \geq \left(\frac{1}{2}\right)^{s_a+3} a(|\xi|), \tag{2.4}$$

since

$$|s\xi + (1-s)\eta| \geq s|\xi| - (1-s)|\eta| \geq (2s-1)|\xi| \geq \frac{1}{2}|\xi| \quad \text{for any } |\xi| \geq |\eta| > 0 \ \text{ and } \ \frac{3}{4} \leq s \leq 1$$

and

$$a(|s\xi + (1-s)\eta|) = \frac{a(|s\xi + (1-s)\eta|)\, |s\xi + (1-s)\eta|}{|s\xi + (1-s)\eta|} \geq \frac{a\left(\frac{1}{2}|\xi|\right) \frac{1}{2}|\xi|}{|\xi|} = \frac{1}{2} a\left(\frac{1}{2}|\xi|\right)$$

in view of (1.5). Therefore, from (2.3) and (2.4) we conclude that

$$[\xi a(|\xi|) - \eta a(|\eta|)] \cdot (\xi - \eta) \geq \left(\frac{1}{2}\right)^{s_a+3} a(|\xi|)\, |\xi - \eta|^2 \quad \text{for } |\xi| \geq |\eta| > 0,$$

which implies that

$$G(|\xi-\eta|) = \int_0^{|\xi-\eta|} ta(t)dt$$

$$\leq a(|\xi-\eta|)|\xi-\eta|^2 \leq 2^{s_a+3}\frac{a(|\xi-\eta|)}{a(|\xi|)}[\xi a(|\xi|)-\eta a(|\eta|)]\cdot(\xi-\eta) \tag{2.5}$$

in view of (1.5). Since $|\xi-\eta| \leq |\xi|+|\eta| \leq 2|\xi|$ for $|\xi| \geq |\eta| > 0$, from (1.12) we have

$$\frac{a(|\xi-\eta|)}{a(|\xi|)} = \frac{a(|\xi-\eta|)}{a(2|\xi|)}\frac{a(2|\xi|)}{a(|\xi|)} \leq 2^{s_a}\left(\frac{|\xi-\eta|}{2|\xi|}\right)^{i_a} \leq 2^{s_a},$$

which finishes the proof by using (2.5). □

Similarly to the proof of Theorem 3.1 in [15], we can obtain the following lemma.

Lemma 2.2. *If $u \in W^{1,G}_{loc}(\Omega)$ is a local weak solution of*

$$div\,(a(|Dv|)Dv) = 0 \quad in\ \Omega \tag{2.6}$$

under the assumptions (1.2), then there exist constants $\beta \in (0,1]$ and $C_0 = C_0(n,i_a,s_a) > 1$ such that

$$\fint_{B_\varrho}\left|Dv-(Dv)_{B_\varrho}\right|dx \leq C_0\left(\frac{\varrho}{R}\right)^{\beta}\fint_{B_R}\left|Dv-(Dv)_{B_R}\right|dx,$$

where $B_\varrho \subseteq B_R \subseteq \Omega$ and $B_R = B(x_0,R)$.

Furthermore, we shall prove the following important result.

Lemma 2.3. *Assume that $u \in W^{1,G}_{loc}(\Omega)$ is a local weak solution of (1.1) with $B_{2R} \subset \Omega$, $\mu \in L^1(\Omega)$ and (1.2). If $v \in W^{1,G}_0(B_R)$ is the weak solution of*

$$\begin{cases} div\,(a(|Dv|)Dv) = 0 & in\ B_R, \\ v = u & on\ \partial B_R, \end{cases} \tag{2.7}$$

then there exists a constant $C_1 = C_1(n,i_a,s_a) > 1$ such that

$$\fint_{B_R}|Du-Dv|\,dx \leq C_1\left[\frac{|\mu|(B_R)}{R^{n-1}}\right]^{\frac{1}{1+i_a}}.$$

Proof. Without loss of generality we may as well assume that $R=1$ by defining

$$\tilde{u}(x) = R^{-1}u(Rx), \quad \tilde{v}(x) = R^{-1}v(Rx) \quad \text{and} \quad \tilde{\mu}(x) = R\mu(Rx).$$

For $k \geq 1$ we define the following truncation operators (see [26,30])

$$T_k(s) := \max\{-k, \min\{k,s\}\} \quad \text{and} \quad \Phi_k(s) := T_1(s-T_k(s)),\ s \in \mathbb{R}.$$

Since u and v are weak solutions of (1.1) and (2.7) respectively, then we have

$$\int_{B_1} [a(|Du|)\,Du - a(|Dv|)\,Dv] \cdot D\varphi dx = \int_{B_1} \varphi d\mu \tag{2.8}$$

for any $\varphi \in L^\infty(B_1) \cap W_0^{1,G}(B_1)$. We divide into two cases.

Case 1: $|\mu|(B_1) \le 1$. If $2 + i_a > n$ (recall that $u - v \in W_0^{1,G}(B_1)$ and then $u - v \in W_0^{1,2+i_a}(B_1)$), then from Sobolev's inequality we have $u - v \in L^\infty(B_1)$. By selecting $\varphi = u - v \in L^\infty(B_1) \cap W_0^{1,G}(B_1)$, from Lemma 2.1 and Sobolev's inequality we find that

$$\int_{B_1} |G(Du - Dv)|\, dx \le C\|u - v\|_{L^\infty(B_1)} |\mu|(B_1) \le C\|Du - Dv\|_{L^{2+i_a}(B_1)}.$$

Then from (1.12) we have

$$\|Du - Dv\|_{L^{2+i_a}(B_1)}^{2+i_a} = \int_{B_1} |Du - Dv|^{2+i_a}\, dx \le C \int_{B_1} |G(Du - Dv)| + 1 dx \le C\|Du - Dv\|_{L^{2+i_a}(B_1)} + C,$$

which implies that

$$\|Du - Dv\|_{L^{2+i_a}(B_1)} \le C \quad \text{and then} \quad \int_{B_1} |Du - Dv|\, dx \le C$$

by using Hölder's inequality. Therefore, we may as well assume that $2 + i_a \le n$. Then by selecting the test function $\varphi = T_k(u - v) \in L^\infty(B_1) \cap W_0^{1,G}(B_1)$, from (2.8) and Lemma 2.1 we have

$$\int_{D_k} |G(Du - Dv)|\, dx \le Ck|\mu|(B_1) \le Ck,$$

where $D_k := \{x \in B_1 : |u(x) - v(x)| \le k\}$, which implies that

$$\int_{D_k} |Du - Dv|^{2+i_a}\, dx \le \int_{D_k} |G(Du - Dv)| + 1 dx \le Ck$$

and then

$$\int_{D_k} |Du - Dv|\, dx \le Ck \tag{2.9}$$

by using Young's inequality. Moreover, testing (2.8) again with $\varphi = \Phi_k(u - v) \in L^\infty(B_1) \cap W_0^{1,G}(B_1)$ and using Lemma 2.1, we find that

$$\int_{C_k} |G(Du - Dv)|\, dx \le C \int_{B_1} |\mu| dx \le C,$$

where $C_k := \{x \in B_1 : k < |u(x) - v(x)| \le k + 1\}$, which implies that

$$\int_{C_k} |Du - Dv|^{2+i_a}\, dx \le \int_{C_k} |G(Du - Dv)| + 1 dx \le C. \tag{2.10}$$

From (1.12), Hölder's inequality, (2.9), (2.10) and the definition of C_k we find that

$$\begin{aligned}\int\limits_{C_k} |Du - Dv|\,dx &\le C\,|C_k|^{1-\frac{1}{2+i_a}} \left(\int\limits_{C_k} |Du - Dv|^{2+i_a}\,dx\right)^{\frac{1}{2+i_a}} \\ &\le C\,|C_k|^{1-\frac{1}{2+i_a}} \le \frac{C}{k^{\frac{n}{n-1}(1-\frac{1}{2+i_a})}} \left(\int\limits_{C_k} |u-v|^{\frac{n}{n-1}}\,dx\right)^{1-\frac{1}{2+i_a}},\end{aligned}$$

which implies that

$$\begin{aligned}\int\limits_{B_1} |Du - Dv|\,dx &= \int\limits_{D_{k_0}} |Du - Dv|\,dx + \sum_{k=k_0}^{\infty} \int\limits_{C_k} |Du - Dv|\,dx \\ &\le Ck_0 + C\sum_{k=k_0}^{\infty} \frac{1}{k^{\frac{n}{n-1}(1-\frac{1}{2+i_a})}} \left(\int\limits_{C_k} |Du - Dv|\,dx\right)^{\frac{n}{n-1}(1-\frac{1}{2+i_a})} \\ &\le Ck_0 + C\left[\sum_{k=k_0}^{\infty} \frac{1}{k^{\frac{n(1+i_a)}{n-1}}}\right]^{\frac{1}{2+i_a}} \left(\sum_{k=k_0}^{\infty} \int\limits_{C_k} |Du - Dv|\,dx\right)^{\frac{n}{n-1}(1-\frac{1}{2+i_a})} \\ &\le Ck_0 + C\left[\sum_{k=k_0}^{\infty} \frac{1}{k^{\frac{n(1+i_a)}{n-1}}}\right]^{\frac{1}{2+i_a}} \left(\int\limits_{B_1} |Du - Dv|\,dx\right)^{\frac{n}{n-1}(1-\frac{1}{2+i_a})}\end{aligned}$$

in view of Sobolev's inequality. Considering the fact that

$$\frac{n(1+i_a)}{n-1} > 1 \quad \text{and} \quad \frac{n}{n-1}\left(1 - \frac{1}{2+i_a}\right) \le 1$$

in view of $2 + i_a \le n$ and then choosing $k_0 \in \mathbb{N}$ large enough such that

$$C\left[\sum_{k=k_0}^{\infty} \frac{1}{k^{\frac{n(1+i_a)}{n-1}}}\right]^{\frac{1}{2+i_a}} \le \frac{1}{2},$$

we obtain

$$\int\limits_{B_1} |Du - Dv|\,dx \le C.$$

Case 2: $|\mu|(B_1) > 1$. Let

$$\tilde{u}(x) = A^{-1}u(x), \quad \tilde{v}(x) = A^{-1}v(x),$$

and

$$\tilde{\mu}(x) = A^{-1-i_a}\mu(x) \quad \text{and} \quad \tilde{a}\,(t) = A^{-i_a}a(At),$$

where

$$A = (|\mu|(B_1))^{\frac{1}{1+i_a}} > 1. \tag{2.11}$$

Then it is easy to check that

$$|\tilde{\mu}|(B_1) = 1.$$

Moreover, $\tilde{a}(t)$ satisfies (1.2) and $\tilde{u}(x) \in W^{1,G}_{loc}(\Omega)$ is a local weak solution of

$$\text{div}\,(\tilde{a}(|D\tilde{u}|)\,D\tilde{u}) = \tilde{\mu}$$

If $2 + i_a > n$ (recall that $u - v \in W_0^{1,G}(B_1)$ and then $\tilde{u} - \tilde{v} \in W_0^{1,2+i_a}(B_1)$), then from Sobolev's inequality we have $\widetilde{u} - \widetilde{v} \in L^\infty$. Moreover, testing (2.8) again with $\varphi = \tilde{u} - \tilde{v} \in L^\infty(B_1) \cap W_0^{1,G}(B_1)$, and using Lemma 2.1 and Sobolev's inequality, we find that

$$A^{-2-i_a} \int_{B_1} |G\,(Du - Dv)|\,dx \le C\|\tilde{u} - \tilde{v}\|_{L^\infty(B_1)} |\tilde{\mu}|(B_1) \le C\|D\tilde{u} - D\tilde{v}\|_{L^{2+i_a}(B_1)}. \tag{2.12}$$

Then from (1.2), (1.12), (2.11) and (2.12) we have

$$\begin{aligned}\|D\tilde{u} - D\tilde{v}\|^{2+i_a}_{L^{2+i_a}(B_1)} &= \int_{B_1} |D\tilde{u} - D\tilde{v}|^{2+i_a}\,dx \\ &\le CA^{-2-i_a} \int_{B_1} |G\,(Du - Dv)| + 1dx \le C\|D\tilde{u} - D\tilde{v}\|_{L^{2+i_a}(B_1)} + C,\end{aligned}$$

which implies that

$$\|D\tilde{u} - D\tilde{v}\|_{L^{2+i_a}(B_1)} \le C \quad \text{and then} \quad \int_{B_1} |D\tilde{u} - D\tilde{v}|\,dx \le C$$

by using Hölder's inequality. Therefore, we may as well assume that $2 + i_a \le n$. Furthermore, similarly to Case 1 we find that

$$\int_{B_1} |D\tilde{u} - D\tilde{v}|\,dx \le C,$$

which finishes our proof. □

Define

$$E(Du, B) := \fint_B |Du - (Du)_B|\,dy.$$

Therefore, it is easy to check that

$$E(Du, B) \le 2 \fint_B |Du - \gamma|\,dy \quad \text{for any } \gamma \in \mathbb{R}^n. \tag{2.13}$$

Moreover, we consider

$$B^i := B(x_0, r_i) \quad \text{and} \quad r_i = \delta_1^i R,$$

where $i \geq 0$ is integer and $B(x_0, 2R) \subset \Omega$. Fix a positive constant δ_1 such that

$$\delta_1 \leq \min \left\{ \left(\frac{1}{10^8 C_0} \right)^{1/\beta} \left(\frac{1}{10^8 C_1} \right), \frac{1}{4} \right\}. \tag{2.14}$$

Moreover, we consider the related comparison solution $v_i \in u + W_0^{1,G}(B^i)$ satisfying

$$\begin{cases} \operatorname{div} \left(a \left(|Dv_i| \right) Dv_i \right) = 0 & \text{in } B^i, \\ v_i = u & \text{on } \partial B^i \end{cases} \tag{2.15}$$

in weak sense. Furthermore, from Lemma 2.3 we find that

$$\fint_{B^i} |Du - Dv_i| \, dx \leq C_1 \left[\frac{|\mu|(B^i)}{r_i^{n-1}} \right]^{\frac{1}{1+i_a}},$$

which implies that

$$\begin{aligned} &\fint_{B^{i+1}} |Du - Dv_i| \, dx \\ &\leq \delta_1^{-n} \fint_{B^i} |Du - Dv_i| \, dx \leq C_1 \delta_1^{-n} \left[\frac{|\mu|(B^i)}{r_i^{n-1}} \right]^{\frac{1}{1+i_a}} \leq C_1 \delta_1^{-2n} \left[\frac{|\mu|(B^{i-1})}{r_{i-1}^{n-1}} \right]^{\frac{1}{1+i_a}}. \end{aligned} \tag{2.16}$$

Lemma 2.4. *Assume that $u \in W_{loc}^{1,G}(\Omega)$ is a local weak solution of (1.1) with $B_{2R} \subset \Omega$, $\mu \in L^1(\Omega)$ and (1.2). Then we have*

$$E(Du, B^{i+2}) \leq \frac{E(Du, B^{i+1})}{4} + 4\delta_1^{-3n} C_1 \left[\frac{|\mu|(B^{i-1})}{r_{i-1}^{n-1}} \right]^{\frac{1}{1+i_a}}.$$

Proof. If $v_i \in W_0^{1,G}(B_R)$ is the weak solution of (2.15), then from Lemma 2.2 we have

$$E(Dv_i, B^{i+2}) = \fint_{B^{i+2}} |Dv_i - (Dv_i)_{B^{i+2}}| \, dy \leq C_0 \delta_1^\beta \fint_{B^{i+1}} |Dv_i - (Dv_i)_{B^{i+1}}| \, dy \leq \frac{E(Dv_i, B^{i+1})}{2^6}.$$

Therefore, from (2.13) we obtain

$$\begin{aligned} E(Du, B^{i+2}) &= \fint_{B^{i+2}} |Du - (Du)_{B^{i+2}}| \, dy \\ &\leq 2 \fint_{B^{i+2}} |Du - (Dv_i)_{B^{i+2}}| \, dy \\ &\leq 2E(Dv_i, B^{i+2}) + 2 \fint_{B^{i+2}} |Du - Dv_i| \, dy, \end{aligned}$$

which implies that

$$
\begin{aligned}
E(Du, B^{i+2}) &\le \frac{E(Dv_i, B^{i+1})}{2^5} + 2\delta_1^{-n} \fint_{B^{i+1}} |Du - Dv_i|\, dy \\
&\le \frac{E(Du, B^{i+1})}{2^4} + 4\delta_1^{-n} \fint_{B^{i+1}} |Du - Dv_i|\, dy \\
&\le \frac{E(Du, B^{i+1})}{2^4} + 4\delta_1^{-3n} C_1 \left[\frac{|\mu|(B^{i-1})}{r_{i-1}^{n-1}}\right]^{\frac{1}{1+i_a}},
\end{aligned}
$$

in view of (2.16). Thus, we finish the proof. □

For simplicity we denote

$$
A_i := E(Du, B^i) \quad \text{and} \quad a_i = \fint_{B^i} |Du|\, dy.
$$

Then we define the composite quantity

$$
M_i := \sum_{j=i-2}^{i} \fint_{B^j} |Du|\, dy + \delta_1^{-n} E(Du, B^i) = \sum_{j=i-2}^{i} a_j + \delta_1^{-n} A_i \tag{2.17}
$$

for every inter $i \ge 2$ and

$$
\lambda := H_1 \fint_{B(x_0,R)} |Du|\, dy + H_2 \int_0^{2R} \left(\frac{|\mu|(B(x_0,\varrho))}{\varrho^{n-1}}\right)^{\frac{1}{1+i_a}} \frac{d\varrho}{\varrho} = H_1 a_0 + H_2 W^{\mu}_{i_a}(x_0, R), \tag{2.18}
$$

where

$$
H_1 := 10^8 \delta_1^{-3n} \quad \text{and} \quad H_2 := 10^8 \delta_1^{-(k+3)n-2} \quad k \ge 3. \tag{2.19}
$$

Then from (2.18) and (2.19) we find that

$$
M_i \le 5\delta_1^{-3n} \fint_{B(x_0,R)} |Du|\, dy = 5\delta_1^{-3n} \fint_{B(x_0,R)} |Du|\, dy \le \frac{\lambda}{10^7} \quad \text{for } i = 2, 3. \tag{2.20}
$$

Moreover, we know that

$$
\begin{aligned}
W^{\mu}_{i_a}(x_0, 2R) &= \int_0^{2R} \left(\frac{|\mu|\,(B(x_0,\varrho))}{\varrho^{n-1}}\right)^{\frac{1}{1+i_a}} \frac{d\varrho}{\varrho} \\
&= \sum_{i=0}^{\infty} \int_{r_{i+1}}^{r_i} \left(\frac{|\mu|(B(x_0,\varrho))}{\varrho^{n-1}}\right)^{\frac{1}{1+i_a}} \frac{d\varrho}{\varrho} + \int_R^{2R} \left(\frac{|\mu|(B(x_0,\varrho))}{\varrho^{n-1}}\right)^{\frac{1}{1+i_a}} \frac{d\varrho}{\varrho} \\
&\ge \sum_{i=0}^{\infty} \left(\frac{|\mu|(B^{i+1})}{r_i^{n-1}}\right)^{\frac{1}{1+i_a}} \int_{r_{i+1}}^{r_i} \frac{d\varrho}{\varrho} + \left(\frac{|\mu|(B(x_0,R))}{(2R)^{n-1}}\right)^{\frac{1}{1+i_a}} \int_R^{2R} \frac{d\varrho}{\varrho}
\end{aligned}
$$

$$= \delta_1^{\frac{n-1}{1+i_a}} \log\left(\frac{1}{\delta_1}\right) \sum_{i=0}^{\infty} \left(\frac{|\mu|(B^{i+1})}{r_{i+1}^{n-1}}\right)^{\frac{1}{1+i_a}} + \log 2 \left(\frac{|\mu|(B(x_0,R))}{(2R)^{n-1}}\right)^{\frac{1}{1+i_a}}$$
$$\geq \sum_{i=0}^{\infty} \left(\delta_1^{n-2} \frac{|\mu|(B^i)}{r_i^{n-1}}\right)^{\frac{1}{1+i_a}}. \tag{2.21}$$

Here without loss of generality we may as well assume that the above inequality holds by choosing $\delta_1 > 0$ small enough. Therefore, from (2.18), (2.19) and (2.21) we have

$$10^8 \delta_1^{-4n} \sum_{i=0}^{\infty} \left(\frac{|\mu|(B^i)}{r_i^{n-1}}\right)^{\frac{1}{1+i_a}} \leq \lambda \tag{2.22}$$

and

$$10^8 \delta_1^{-4n} \left(\frac{|\mu|(B^i)}{r_i^{n-1}}\right)^{\frac{1}{1+i_a}} \leq \lambda \quad \text{for any } i \geq 0. \tag{2.23}$$

In view of (2.20), without loss of generality we may as well assume that there exists $i_e \geq 3$ such that

$$M_{i_e} \leq \frac{\lambda}{100} \quad \text{and} \quad M_j > \frac{\lambda}{100} \quad \text{for any } j > i_e. \tag{2.24}$$

If not, we deduce that $M_{j_i} \leq \lambda/100$ for every $i \in \mathbb{N}$ and an increasing subsequence $\{j_i\}$, and then for a Lebesgue point x_0 of Du

$$|Du(x_0)| \leq \lim_{i\to\infty} \fint_{B^{j_i}} |Du|\, dy \leq \frac{\lambda}{100},$$

which implies that the desired result is true.

Now we are ready to finish the proof of the main result, Theorem 1.7.

Proof. From (2.17) and (2.24) we have

$$\sum_{j=i_e-2}^{i_e} a_j + \delta_1^{-n} A_{i_e} = M_{i_e} \leq \frac{\lambda}{100}. \tag{2.25}$$

We prove it by induction. We shall prove that

$$a_j + A_j \leq \lambda \quad \text{for any } j \geq i_e. \tag{2.26}$$

From (2.25) we can know that (2.26) is true for $j = i_e$. Now we assume by induction that (2.26) is true for $i_e \leq j \leq i$. From Lemma 2.4 we obtain

$$A_{j+2} \leq \frac{1}{4} A_{j+1} + 4\delta_1^{-3n} C_1 \left(\frac{|\mu|(B^{j-1})}{r_{j-1}^{n-1}}\right)^{\frac{1}{1+i_a}}. \tag{2.27}$$

From (2.23), (2.26) and (2.27) we find that

$$A_{i+1} \leq \frac{1}{4} A_i + 4\delta_1^{-3n} C_1 \left(\frac{|\mu|(B^{i-2})}{r_{i-2}^{n-1}}\right)^{\frac{1}{1+i_a}} \leq \frac{1}{4}\lambda + \frac{\lambda}{10^3} \leq \frac{\lambda}{3}. \tag{2.28}$$

Furthermore, by summing up (2.27) for $j = i_e - 1, i_e, ..., i - 1$ we find that

$$\sum_{j=i_e}^{i+1} A_j \leq A_{i_e} + \frac{1}{4}\sum_{j=i_e}^{i} A_j + 4\delta_1^{-3n} C_1 \sum_{j=0}^{\infty}\left(\frac{|\mu|(B^j)}{r_j^{n-1}}\right)^{\frac{1}{1+i_a}}, \tag{2.29}$$

which implies that

$$\sum_{j=i_e}^{i+1} A_j \leq 2A_{i_e} + 8\delta_1^{-3n} C_1 \sum_{j=0}^{\infty}\left(\frac{|\mu|(B^j)}{r_j^{n-1}}\right)^{\frac{1}{1+i_a}}. \tag{2.30}$$

Therefore, we deduce that

$$a_{i+1} - a_{i_e} \leq \sum_{j=i_e}^{i}(a_{j+1} - a_j) \leq \sum_{j=i_e}^{i} \fint_{B^{j+1}} |Du - (Du)_{B^j}|\, dy = \delta_1^{-n}\sum_{j=i_e}^{i} E(Du, B^j) = \delta_1^{-n}\sum_{j=i_e}^{i} A_j,$$

which implies that

$$a_{i+1} \leq a_{i_e} + 2\delta_1^{-n} A_{i_e} + 8\delta_1^{-4n} C_1 \sum_{j=0}^{\infty}\frac{|\mu|(B^j)}{r_j^{n-1}} \leq 2M_{i_e} + \frac{\lambda}{100} \leq \frac{\lambda}{3},$$

in view of (2.22) and (2.25). This inequality together with (2.28) implies that $a_{i+1} + A_{i+1} \leq \lambda$. Therefore, (2.26) holds for any $i \geq i_e$. Finally, since x_0 is a Lebesgue point of Du, we have

$$|Du(x_0)| = \lim_{i\to\infty} a_i \leq \lambda.$$

Thus, we complete the proof. □

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